BE Kriging approach - 3 hours

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Expectations for the session

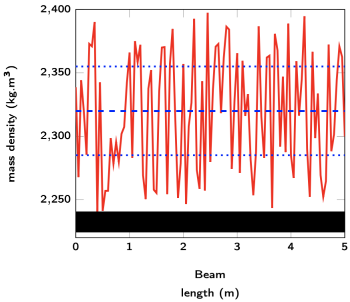
- This 3-hour session is to offer you the opportunity to learn more about Gaussian fields and kriging approach.
- No report, no evaluation.
- Individual work and active session, but assistance to help you at best. At any moment, you can raise your hand and contact me to support your progress.
- To support you individually, for each question of the first part, a set of slides is offered:
  a. Repeat the question and provide more detail about the context of the question
  b. Give a set of possible pictures obtained from the investigation. They don't have to be reproduced exactly nor exhaustively, but give perspectives on the outcome.
  c. Give the code lines in Matlab to obtain pictures provided in Part b.

Random fields, correlation and auto-correlation

1a. Create a random field, where the value for each point is considered as an independent random variable.

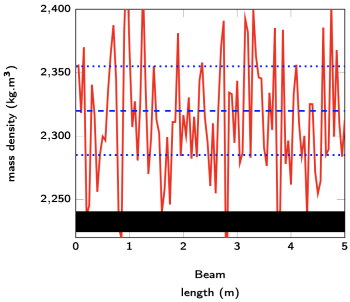
Many strategies exist to represent a random field. Among them, an easy way is just to consider each point, for instance, each Gauss point or each node, or each element, as an independent random variable.

Let consider a one-dimensional random field, where the random field has a random value in each node which satisfies a uniform distribution between 0 and 1.



One realization of the random mass density random field uniformly distributed between 2 240 kg.m^3 and 2 400 kg.m^3 - mean = 2 320 kg.m^3 and standard deviation = 46 kg.m^3

- Influence of the discretisation
- Meaning of such local fluctuations of the properties?
- Meaning of coupling between fluctuations and spatial discretisation?



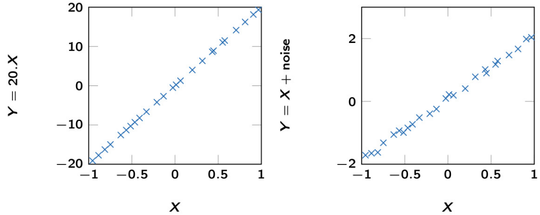
One realization of the random field representing a Gaussian distributed random mass density with mean = 2 320 kg.m^3 and standard deviation = 46 kg.m^3

- Influence of the discretisation
- Meaning of such local fluctuations of the properties?
- Meaning of coupling between fluctuations and spatial discretisation?

```
dx = 0.1;
x = [0:dx:1];

% Uniform distribution between 0 and 1:
y = rand(1,length(x));

% Gaussian distribution with mean = 0.5 and std = 0.25:
mu = 0.5;
sigma = 0.25;
x = [0:dx:1];
y = normrnd(mu,sigma,[1,length(x)]);

% Plot:
plot(x,y);
xlabel('x');
ylabel('y');
```

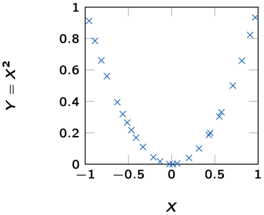
2a. Evaluate correlation between two random variables.

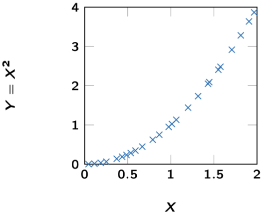
Let consider samples for two random variables x and y.



Cov (X, Y) :	$6,67$	$0,67$
ρ_{XY} :	$1,00$	$0,99$

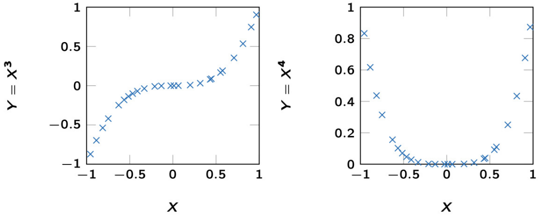
- Possible values of correlation?
- Independent variables $\Rightarrow \rho = 0$
- $\rho = 0 \not\Rightarrow$ Independent variables





$\mathrm{Cov}(X, Y)$:	$0,00$	$0,67$
ρ_{XY} :	$0,00$	$0,97$

- Possible values of correlation?
- Independent variables $\Rightarrow \rho = 0$
- $\rho = 0 \not\Rightarrow$ Independent variables



$\mathrm{Cov}\,(X, Y)$:	$0,20$	$0,00$
ρ_{XY} :	$0,92$	$0,00$

- Possible values of correlation?
- Independent variables $\Rightarrow \rho = 0$
- $\rho = 0 \nRightarrow$ Independent variables

```
x = -1+2*rand(1,20);
y = 20.*x;

% Plot:
plot(x,y,'x');

% Covariance:
C = cov(x,y);
cc = C(1,2);

% Correlation:
R = corrcoef(x,y);
rho = R(1,2);
```

3a. Investigate the influence of various correlation structures.

It does not seem realistic to consider within a material that each point is an independent random variable. It is also problematic to fix the fluctuation with the FEM discretization, for instance.

Let consider analytical expression to describe the fall of the correlation with distance:
The exponential correlation function reads

$$R(x - x', \theta) = \exp\left(-\frac{|x - x'|}{\theta}\right).$$

The squared exponential correlation function reads

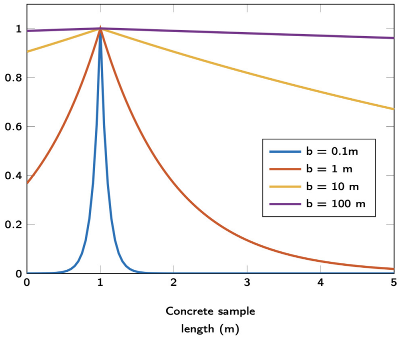
$$R(x - x', \theta) = \exp\left(-\frac{|x - x'|^2}{\theta}\right).$$

The scale parameters θ is called the correlation length.

$\Rightarrow$ Investigate and understand the influence of the correlation length.

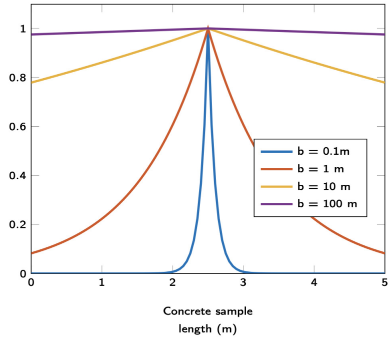
Correlation structure

3b. Possible plots for exponential auto-correlation



Evolution of the autocorrelation function of the one-dimensional random field over the beam for different values of the correlation length with respect to the point $x_0 = 1.0$ m

- Influence of the correlation length
- Influence of the correlation structure
- Possible values of correlation?



Evolution of the autocorrelation function of the one-dimensional random field over the beam for different values of the correlation length with respect to the point $x_0 = 2.5$ m

- Influence of the correlation length
- Influence of the correlation structure
- Possible values of correlation?

```
xR = 0.3;
Lc = 0.2;
x = [0:dx:1];

% Exponential auto-correlation:
c = exp(-abs(x-xR)/Lc);

% Squared Exponential auto-correlation:
c = exp(-abs(x-xR).^2/Lc);

% Plot:
plot(x,c);
xlabel('x');
ylabel(strcat('correlation with point x = ',num2str(xR)));
ylim([0 1]);
title(strcat('correlation length: Lc = ',num2str(Lc)));
```

4a. Create various realisations of random fields assuming a correlation structure between the points.

The Karhunen-Loève expansion, also called the principal component analysis, approximates $H(x, \theta)$ by a finite series:

$$H(x, \theta) \approx \hat{H}^{KLE} = \mu(x) + \sum_{i=0}^{M} \sqrt{\lambda_i} f_i(x) \xi_i(\theta), \qquad (10)$$

- $f_i(x)$, expansion basis of deterministic functions which depends on the process itself,
- uncorrelated random variables $\xi_i(\theta)$ with a mean equal to zero,
- set of constants λ_i.

$\rightarrow$ $f_i(x)$ are determined to produce the best possible basis, *i.e.* to minimize the total mean squared error integrated over $\mathcal{D}$.

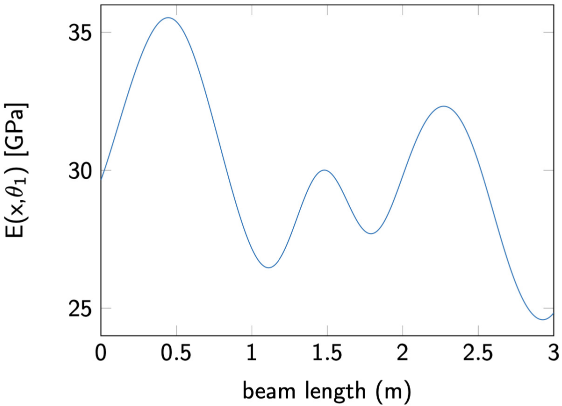
$$\begin{array}{lll} \text{Beam} & x & = [0, 2, 4, 6] \\ \text{Correlation length} & l & = 5 \\ \text{Variance} & C_0 & = 1 \end{array}$$

$$\text{Distance matrix} \quad x = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 2 & 0 & 2 & 4 \\ 4 & 2 & 0 & 2 \\ 6 & 4 & 2 & 0 \end{bmatrix}$$

$$\text{Covariance matrix} \quad C = \begin{bmatrix} 1 & 0.852 & 0.527 & 0.237 \\ 0.852 & 1 & 0.852 & 0.527 \\ 0.527 & 0.852 & 1 & 0.852 \\ 0.237 & 0.527 & 0.852 & 1 \end{bmatrix}$$

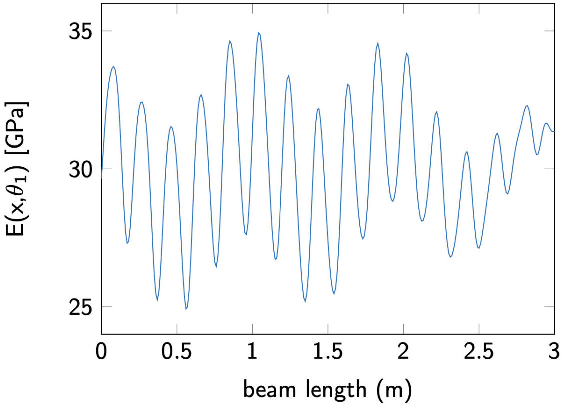
Random field based on an auto-correlation structure

considering $\mu_E = 30$ GPa, $\sigma_E = 3$ GPa
and an exponential correlation with a correlation length of 1 m



Random field based on an auto-correlation structure

considering $\mu_E = 30$ GPa, $\sigma_E = 3$ GPa
and an exponential correlation with a correlation length of 0.2 m



```
x = [0:0.002:1];
Lc=0.01;

for i=1:length(x)
    for j=1:length(x)
        D(i,j) = abs(x(i)-x(j));
        C(i,j) = exp(-D(i,j).^2/Lc);
    end
end

[a,b,c] = svd(C);

mu = 3;
field = mu;

for i=1:20
    sum = normrnd(0,1)*sqrt(b(i,i))*a(:,i);
    field = field+sum;
end

plot(x,field);
```