Quantum Noise and Quantum Operations

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Topics

- System-environment interactions — NdC: 8.2
- Operator-sum representation
- Axiomatic view of general quantum operations — Preskill: 3.2

Quantum Operations: An Overview

- Quantum Operation Formalism
  - general tool for describing the evolution of a quantum system in various scenarios
  - thus far, you have learned
    1) time evolution of a closed system

$$\rho \mapsto \rho(t) = U(t)\,\rho\,U^{\dagger}(t)$$

    2) Measurement

$$\rho \mapsto M_m \rho M_m^{\dagger}$$

  - More generally,

$$\rho \mapsto \rho' = \mathcal{E}(\rho) \qquad \mathcal{E} = \text{quantum operation / "map" / "channel"}$$

  - 3 separate ways of understanding quantum operations

System interacting with environment	Operator Sum Representation	Physically Motivated Axioms
• concrete • easy to relate to real world • not mathematically convenient	• math. convenient • more abstract • useful theoretical tool	• a set of axioms we would expect a dynamical map to satisfy • very general • not math. convenient

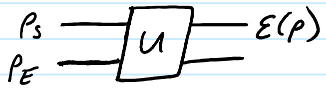
together, very powerful!

Open Quantum System

- Closed system: dynamics of system described by unitary transformation

$$\rho_s \;-\!\!\boxed{U}\!\!-\; U\rho_s U^{\dagger}$$

- Open system: interaction between system and its environment



* together, S+E form a closed system
* in general, $\mathcal{E}$ is <u>not</u> unitary

How do we calculate $\mathcal{E}(\rho)$ formally?

→ Let S+E be initially uncoupled: $\rho_{SE} = \rho_S \otimes \rho_E$
→ U couples them, but no interaction thereafter
→ Examining the state of the system thereafter requires tracing over E degrees of freedom

$$\mathcal{E}(\rho) = Tr_E\left[U(\rho_S \otimes \rho_E)U^+\right]$$

* Note that if U does not induce interactions, i.e., $U = U_S \otimes U_E$

$$\begin{aligned}\mathcal{E}(\rho) &= Tr_E\left[(U_S \otimes U_E)(\rho_S \otimes \rho_E)(U_S \otimes U_E)^+\right] \\ &= Tr_E\left[U_S \rho_S U_S^+ \otimes U_E \rho_E U_E^+\right] \\ &= U_S \rho_S U_S^+ \, Tr_E(U_E \rho_E U_E^+) \\ &= U_S \rho_S U_S^+\end{aligned}$$

– A couple of things to note:
1) Initial state: system can be assumed to be uncoupled from E when experimentalist prepares an initial state.
→ Usually involves projective measurement

2) Defining U: How can U be specified if E could have nearly infinite degrees of freedom?
→ Turns out, for a system of dimension d, we only need to model E as having no more than d^2 dims.!

Operator – Sum Representation

- S+E Evolution Revisited:

Consider a system initially in the state ρ and its env. in the state $\rho_E = |e_0\rangle\langle e_0|$, where $\{|e_k\rangle\}$ form an orthonormal basis. The state of the system after the application of unitary U is

$$\begin{aligned}\mathcal{E}(\rho) &= Tr_E\left[U(\rho \otimes |e_0\rangle\langle e_0|)U^+\right] \\ &= \sum_k \langle e_k| U(\rho \otimes |e_0\rangle\langle e_0|)U^+ |e_k\rangle \\ &= \sum_k E_k \rho E_k^+ \, , \quad E_k = \langle e_k|U|e_0\rangle.\end{aligned}$$

← OSR

The OSR satisfies

$$1 = Tr(\mathcal{E}(\rho))$$

$$= Tr\left[\sum_k E_k \rho E_k^\dagger \right]$$

$$= \sum_k Tr(E_k \rho E_k^\dagger)$$

$$= \sum_k Tr(\rho E_k^\dagger E_k)$$

$$= Tr(\rho \sum_k E_k^\dagger E_k) \Rightarrow \sum_k E_k^\dagger E_k = I$$

- OSR allows for a description of a system w/o needing to describe an env., just need to define operators E_k.

- Physical Interpretation of OSR:

Consider $\rho_{SE} = \rho \otimes |e_0\rangle\langle e_0|$. Apply U, but now measure $|e_k\rangle$.

$$\rho \mapsto \rho'_{SE} = U \rho_{SE} U^\dagger$$

$$\mapsto \rho''_{SE,k} = \frac{M_k U \rho_{SE} U^\dagger M_k^\dagger}{p_k}, \quad M_k = I_S \otimes |e_k\rangle\langle e_k|$$

$$p_k = Tr(M_k U \rho_{SE} U^\dagger M_k^\dagger) = Tr_S(E_k \rho_S E_k^\dagger)$$

$$= \frac{E_k \rho E_k^\dagger \otimes |e_k\rangle\langle e_k|}{p_k}$$

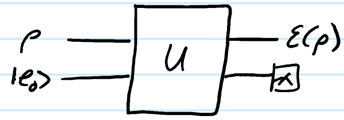
$$\rho''_{S,k} = Tr_E(\rho''_{SE,k}) = \frac{E_k \rho E_k^\dagger}{p_k}$$

Relating this back to $\mathcal{E}$, we find

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k p_k \rho''_{S,k}.$$

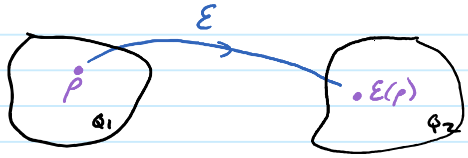
What does it all mean?

The action of the quantum system is equivalent to taking state ρ and randomly taking it to $\rho''_{S,k}$.



Axiomatic Approach

Let $\mathcal{E}$ be a quantum operation that maps density operators to density operators



1) $Tr[\mathcal{E}(\rho)]$ is the probability of $\mathcal{E}$ occurring; thus, $0 \leq Tr(\mathcal{E}(\rho)) \leq 1$

1) $Tr[\mathcal{E}(\rho)]$ is the probability of $\mathcal{E}$ occurring; thus $0 \le Tr(\mathcal{E}(\rho)) \le 1$ for any ρ

2) $\mathcal{E}$ is convex-linear map. Let $\rho = \sum_i p_i \rho_i$ then

$$\mathcal{E}(\rho) = \mathcal{E}\left(\sum_i p_i \rho_i\right)$$
$$= \sum_i p_i \, \mathcal{E}(\rho_i)$$

3) $\mathcal{E}$ is completely positive map.

If $\mathcal{E}$ maps density operators to density operators then $\mathcal{E}(\rho)$ must be positive

• Interpretations:

1) Follows from $Tr(\rho) = 1$ for trace-preserving $\mathcal{E}$.
Also, implies:
given $\mathcal{E}(\rho)$ the correct normalized state is $\mathcal{E}'(\rho) = \dfrac{\mathcal{E}(\rho)}{Tr(\mathcal{E}(\rho))}$

2) Non linear evolution would not be compatible with interpreting the density operator as an ensemble of states

3) ρ is positive; thus, $\mathcal{E}$ must be positive. Furthermore, if space is extended to $\rho_s \otimes \rho_A$ then $\mathcal{E} \otimes I_A$ is positive. Only true is $\mathcal{E}$ is completely positive.

Quantum Noise

• Depolarizing Channel

$$\mathcal{E}(\rho) = \left(1 - \tfrac{3p}{4}\right)\rho + \tfrac{p}{4}\left(\sigma^x \rho \sigma^x + \sigma^y \rho \sigma^y + \sigma^z \rho \sigma^z\right)$$

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_I = \sqrt{1 - \tfrac{3p}{4}}\, I, \quad E_\mu = \sqrt{\tfrac{p}{4}}\, \sigma^\mu, \quad \mu = x, y, z$$

General single qubit state: $\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$

$$\vec{r} = (r_x, r_y, r_z)$$
$$\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

$$\left(1 - \tfrac{3p}{4}\right)\rho = \left(1 - \tfrac{3p}{4}\right)\tfrac{1}{2}\left(I + r_x\sigma^x + r_y\sigma^y + r_z\sigma^z\right)$$

$$\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$$

$$\left(1-\tfrac{3p}{4}\right)\rho = \left(1-\tfrac{3p}{4}\right)\tfrac{1}{2}(I + r_x\sigma^x + r_y\sigma^y + r_z\sigma^z)$$

$$\tfrac{p}{4}\sigma^x\rho\sigma^x = \tfrac{p}{4}\tfrac{1}{2}(I + r_x\sigma^x - r_y\sigma^y - r_z\sigma^z)$$

$\sigma^x\sigma^y\sigma^x = -\sigma^y$

$\sigma^x\sigma^z\sigma^x = -\sigma^z$

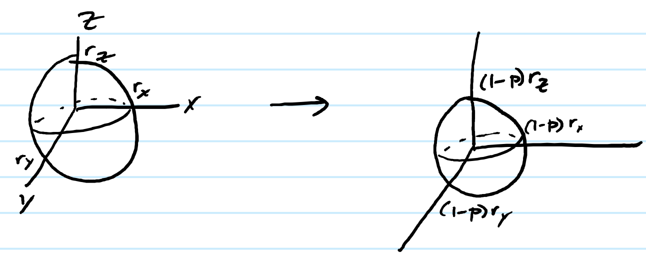
$$\tfrac{p}{4}\sigma^y\rho\sigma^y = \tfrac{p}{4}\tfrac{1}{2}(I - r_x\sigma^x + r_y\sigma^y - r_z\sigma^z)$$

$$+\quad \tfrac{p}{4}\sigma^z\rho\sigma^z = \tfrac{p}{4}\tfrac{1}{2}(I - r_x\sigma^x - r_y\sigma^y + r_z\sigma^z)$$

$$\left[\tfrac{1}{2}\left(1-\tfrac{3p}{4}\right)+\tfrac{3p}{8}\right]I + \left[\tfrac{1}{2}\left(1-\tfrac{3p}{4}\right)-\tfrac{1}{2}\tfrac{p}{4}\right](r_x\sigma^x + r_y\sigma^y + r_z\sigma^z)$$

$$\mathcal{E}(\rho) = \tfrac{1}{2}I + \tfrac{1}{2}(1-p)(\vec{r}\cdot\vec{\sigma})$$

$$= \tfrac{1}{2}(I + \vec{r}\,'\cdot\vec{\sigma}) \qquad \vec{r}\,' = [(1-p)r_x, (1-p)r_y, (1-p)r_z]$$



What does this mean from the prespective of the density matrix elements?

$$\rho = \tfrac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$$

$$= \tfrac{1}{2}(I + r_x\sigma^x + r_y\sigma^y + r_z\sigma^z)$$

$$= \tfrac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & r_x\\ r_x & 0\end{pmatrix} + \begin{pmatrix}0 & -ir_y\\ ir_y & 0\end{pmatrix} + \begin{pmatrix}r_z & 0\\ 0 & -r_z\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix}1+r_z & r_x - ir_y\\ r_x + ir_y & 1-r_z\end{pmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\begin{pmatrix}1+(1-p)r_z & (1-p)(r_x-ir_y)\\ (1-p)(r_x+ir_y) & 1-(1-p)r_z\end{pmatrix}$$

$$\mathcal{E}\circ\mathcal{E}(\rho) = \mathcal{E}(\mathcal{E}(\rho)) = \frac{1}{2}\begin{pmatrix}1+(1-p)^2 r_z & (1-p)^2(r_x-ir_y)\\ (1-p)^2(r_x+ir_y) & 1-(1-p)^2 r_z\end{pmatrix}$$

$$\mathcal{E}^n\circ\rho = \frac{1}{2}\begin{pmatrix}1+(1-p)^n r_z & (1-p)^2(r_x-ir_y)\\ (1-p)^n(r_x+ir_y) & 1-(1-p)^2 r_z\end{pmatrix}$$

Let's assume each map is applied over a small interval δt.

Γ = prob of depolarizing per unit time

$p = \Gamma \delta t$ = prob of depolarizing $\ll 1$

N applications: $(1-p)^N = (1-\Gamma\delta t)^N = (1-\frac{\Gamma t}{N}) = e^{-\Gamma t}$

$t = N\delta t$

$$\mathcal{E}^N \circ \rho = \frac{1}{2}\begin{pmatrix} 1+e^{-\Gamma t} r_z & (r_x - i r_y) e^{-\Gamma t} \\ (r_x + i r_y) e^{-\Gamma t} & 1 - e^{-\Gamma t} r_z \end{pmatrix}$$

$N \to \infty$: $\mathcal{E}^N \circ \rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ fully mixed state
classical!