pHw 2

Sunday, February 27, 2022 4:12 PM

Problem Set 2

AS.171.402: Applied Quantum Information [Spring 2022]

Due Date: Mar 1, 2022 (11:59 pm)

---

1. Gate Synthesis

In this problem, we study how some current generation quantum computers (for e.g. IBM) implements the standard universal quantum computing gatesets such as $\{\hat{CX}, \hat{H}, \hat{T}\}$. Quantum computers based on *fixed-frequency transmons* (such as IBM) natively implements the following single and two qubit gates,

$$\text{Arbitrary rotation about } z\ :\ \hat{R}_z(\theta) = \exp\left(-i\frac{\theta}{2}\hat{Z}\right) \tag{1}$$

$$\frac{\pi}{2} \text{ rotation about } x\ :\ \hat{R}_x(\pi/2) = \exp\left(-i\frac{\pi}{4}\hat{X}\right) \tag{2}$$

$$\text{Two-qubit entangling gate} : \hat{ZX}_{12}(\pi/2) = \exp\left(-i\frac{\pi}{4}\hat{Z}\otimes\hat{X}\right) \tag{3}$$

By restricting to these limited set of gates, one can focus on calibrating these gates for maximum accuracy for greater accuracy in eventual computation. In the following, we would like to show that this set of gates in Eqs. (1)-(3) is also universal by showing that the $\hat{CX}$, $\hat{H}$ and $\hat{T}$ gates are realizable from these three gates mentioned above.

(a) $\hat{T}$ *gate*: Show that the T gate can be realized (up to an overall phase) from one of the single qubit gates.

(b) $\hat{H}$ *gate*: Show that the Hadamard gate can be obtained from the following combination of $\hat{R}_z(\theta)$ and $\hat{R}_x(\pi/2)$ gates,

$$\hat{H} = e^{i\phi}\hat{R}_z(\theta)\,\hat{R}_x\left(\frac{\pi}{2}\right)\hat{R}_z(\theta) \tag{4}$$

What value of θ, and ϕ gives the Hadamard gate?

(c) *Arbitrary single qubit gate*: In fact, we can rewrite any arbitrary single qubit unitary gate using $R_x(\pi/2)$ and $R_z(\theta)$ gate. Rewriting an arbitrary unitary gate as,

$$\hat{U}(\theta,\phi,\lambda) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -ie^{i\lambda}\sin\left(\frac{\theta}{2}\right) \\ -ie^{i\phi}\sin\left(\frac{\theta}{2}\right) & e^{i(\phi+\lambda)}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{5}$$

(i) Show that $\hat{U}$ can represent an arbitrary single qubit gate.
(Hint: Show that $\hat{U}$ is unitary, and one can prepare arbitrary single qubit states using $\hat{U}|0\rangle$ and $\hat{U}|1\rangle$.)

(ii) Decompose $\hat{U}$ in terms of $R_x(\theta)$ and $R_z(\phi)$ gates. Show that,

$$\hat{U}(\theta,\phi,\lambda) = \hat{R}_z(\phi)\hat{R}_x(\theta)\hat{R}_z(\lambda) \tag{6}$$

This expression is consistent with the Euler decomposition, that states that any arbitrary rotation matrix can be decomposed into three elementary rotations.

(iii) Show that the following identity is true,

$$\hat{R}_x(\theta) = \hat{R}_z\left(-\frac{\pi}{2}\right)\hat{R}_x\left(\frac{\pi}{2}\right)\hat{R}_z(\pi-\theta)\,\hat{R}_x\left(\frac{\pi}{2}\right)\hat{R}_z\left(-\frac{\pi}{2}\right) \tag{7}$$

Since arbitrary $\hat{R}_z$ gates are implemented natively, one can use this identity to implement an arbitrary $R_x(\theta)$ gate using only $R_x(\pi/2)$ gates.

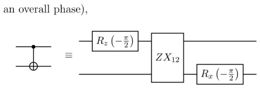
(iv) Taking into account the results from parts (i)-(iii), What is the gate sequence that needs to be implemented to implement an arbitrary single qubit gate, $\hat{U}$?

(d) $\hat{CX}$ *gate* - Finally, we would like to show that CX (or CNOT) gate can be obtained from the ZX_{12} gate for Eq. 3.

(i) Show that $\hat{ZX}_{12}$ is a unitary matrix.

(ii) Using the two-qubit computational states as a basis, obtain the matrix representation of the $\hat{ZX}_{12}$ gate.

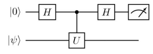
(iii) Show that the CX gate is obtained from the ZX_{12} gate with the following circuit (up to an overall phase),



Therefore, we have shown that the standard universal gate set $\{\hat{H}, \hat{T}, \hat{CX}\}$ can be generated from these gates.

2. Hadamard Test

The Hadamard test is a method used to create a random variable whose expected value is determined by a quantum expectation value of a unitary operator U with respect to a state $|\psi\rangle$. The Hadamard test is a key component of many algorithms as it can be used to calculate overlaps between states and expectation values of operators. Here, we will examine the Hadamard test in detail.



The circuit above describes the Hadamard test. The first qubit is initialized in the $|0\rangle$ state, while the second is initialized in the state $|\psi\rangle$. Following a Hadamard operation on the first qubit, the two-qubit system is subject to a controlled-U gate. This operation is followed by an additional $\hat{H}$ operation and Z-basis measurement on the first qubit. In the

a) $\hat{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$ while $R_z = \begin{bmatrix} e^{-i\frac{\theta}{4}} & 0 \\ 0 & e^{i\theta/4} \end{bmatrix}$

So $\hat{T} = R_z(\frac{\pi}{4})$ when $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

ie $\hat{T} = R_z(\frac{\pi}{4})$ when Given Phase $= |1\rangle$

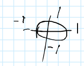
b) Matix vise: $H = e^{i\phi}\hat{R}_z(\theta)\hat{R}_x(\frac{\pi}{2})\hat{R}_z(\theta)$

$H|0\rangle = |+\rangle$

$$= e^{i\phi}\begin{bmatrix} e^{i\theta/4} & 0 \\ 0 & e^{-i\theta/4} \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -i\frac{\sqrt{2}}{2} \\ -i\frac{\sqrt{2}}{2} & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} e^{i\theta/4} & 0 \\ 0 & e^{-i\theta/4} \end{bmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{i\phi}\begin{pmatrix} e^{-i\theta/4}\frac{\sqrt{2}}{2}e^{-i\theta/4} \\ -i\frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix} e^{i(\phi+\theta/2)} \\ -ie^{i\phi} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$e^{i(\phi+\theta/2)} = 1$ & $-ie^{i\phi} = 1$ $\Rightarrow$ $\phi = \frac{\pi}{2} + 2\pi K$, $\theta = \frac{3\pi}{2} + 2\pi K$ where K is any Integer

c) i) $U \cdot U^{\dagger} = I$

$$\begin{pmatrix} \cos(\frac{\theta}{2}) & -ie^{i\lambda}\sin(\frac{\theta}{2}) \\ -ie^{i\phi}\sin(\frac{\theta}{2}) & e^{i(\phi+\lambda)}\cos(\frac{\theta}{2}) \end{pmatrix}\cdot\begin{pmatrix} \cos(\frac{\theta}{2}) & -ie^{i\lambda}\sin(\frac{\theta}{2}) \\ -ie^{i\phi}\sin(\frac{\theta}{2}) & e^{i(\phi+\lambda)}\cos(\frac{\theta}{2}) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos^2(\frac{\theta}{2}) - e^{i(\lambda+\phi)}\sin^2(\theta/2) & 0 \\ -ie^{i\phi}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) - ie^{i(2\phi+\lambda)}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $e^{i(\phi+\lambda)} = -1$

$\phi + \lambda = \pi$ then $\hat{U}$ is unitary.

$\hat{U}|0\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ -ie^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix}$ $\hat{U}|1\rangle = \begin{pmatrix} -ie^{i\lambda}\sin(\frac{\theta}{2}) \\ e^{i(\phi+\lambda)}\cos(\frac{\theta}{2}) \end{pmatrix}$

ii) $U = R_z(\phi)R_x(\theta)R_z(\lambda)$

$$= \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}\begin{bmatrix} \cos(\frac{\theta}{2}) & -i\sin(\theta/2) \\ -i\sin(\frac{\theta}{2}) & \cos(\theta/2) \end{bmatrix}\begin{bmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{bmatrix}$$

$$= \ldots \begin{bmatrix} e^{i\frac{\lambda}{2}}\cos(\frac{\theta}{2}) & -ie^{i\lambda/2}\sin(\theta/2) \\ -ie^{i\lambda/2}\sin(\frac{\theta}{2}) & e^{i\lambda/2}\cos(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{i}{2}(\phi+\lambda)}\cos(\frac{\theta}{2}) & -ie^{\frac{i}{2}(\lambda-\phi)}\sin(\theta/2) \\ -ie^{\frac{i}{2}(\lambda-\phi)}\sin(\frac{\theta}{2}) & e^{\frac{i}{2}(\lambda+\phi)}\cos(\theta/2) \end{bmatrix} \div e^{-\frac{i}{2}(\lambda+\phi)}$$

$$= e^{-\frac{i}{2}(\lambda+\phi)}\begin{bmatrix} \cos(\frac{\theta}{2}) & -ie^{i\lambda}\sin(\theta/2) \\ -ie^{i\phi}\sin(\frac{\theta}{2}) & e^{i(\lambda+\phi)}\cos(\theta/2) \end{bmatrix}$$

So when $e^{-\frac{i}{2}(\lambda+\phi)} = 1$ or $(\lambda+\phi) = 0$ then $U = R_z(\phi)R_x(\theta)R_z(\lambda)$

iii) So we are saying:

$$\begin{bmatrix} \cos(\theta/2) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\begin{bmatrix} -e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \ldots \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \ldots \begin{bmatrix} 0 & e^{i\theta} \\ e^{i\theta} & 0 \end{bmatrix}$$

$$= \ldots \begin{bmatrix} -ie^{-i\theta} & 0 \\ 0 & -ie^{i\theta} \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & -e^{i\theta} \end{bmatrix}$$?

iv) You can exclusively use $R_z(\theta)$ & $R_x(\pi/2)$

d) i) $ZX_{12} = Z \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \rightarrow ZX_{12}\cdot ZX_{12}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

ii)

? how are you supposed to prove unitary without matrix representation.

iii) $\hat{CX} = [R_z(-\frac{\pi}{2}) \otimes I][Z \otimes X][I \otimes R_z(-\frac{\pi}{2})]$

$$= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \hat{CX}$$

2. a) $S = |\psi\rangle\langle 0|$

$H|0\rangle \xrightarrow[H]{} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|\psi\rangle \longrightarrow \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\ \alpha|00\rangle + \beta|01\rangle + \alpha|10\rangle + \beta|11\rangle) = (|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\psi\rangle)\cdot\frac{1}{\sqrt{2}}$

- $\hat{U}$ $\quad = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes \hat{U}|\psi\rangle)$
- H $\quad = \frac{1}{2}([|0\rangle + |1\rangle] \otimes |\psi\rangle + [|0\rangle - |1\rangle] \otimes U|\psi\rangle)$

$$= \frac{1}{2}(|0\rangle \otimes (I+U)|\psi\rangle + |1\rangle \otimes (I-U)|\psi\rangle)$$

B) $P_0 = \frac{1}{4}\langle\psi|(I+U^{\dagger})(I+U)|\psi\rangle$ $\quad P_1 = \frac{1}{4}\langle\psi|(I-U^{\dagger})(I-U)|\psi\rangle$

The circuit above describes the Hadamard test. The first qubit is initialized in the $|0\rangle$ state, while the second is initialized in the state $|\psi\rangle$. Following a Hadamard operation on the first qubit, the two-qubit system is subject to a controlled-U gate. This operation is followed by an additional $\hat{H}$ operation and Z-basis measurement on the first qubit. In the following, we use the measurement outcomes to determine $\langle\psi|\hat{U}|\psi\rangle$.

(a) What is the state of the two-qubit system prior to measurement? Write the state in terms of $\hat{U}|\psi\rangle$

(b) What are the probabilities of measuring $|0\rangle$ and $|1\rangle$ on the first qubit?

(c) Using the probabilities in part (b) expectation value $\langle\hat{Z}\rangle$ for qubit 1, where $\hat{Z}$ is the Pauli Z operator.

(d) How does part (c) change if the first Hadamard gate is replaced by $R_x(\pi/2)$? The definition of $R_x(\pi/2)$ is given by Eq. (2) from Problem 1.

(f) Using your results from part (c) and (d), obtain an expression for the expectation value $\langle\psi|U|\psi\rangle$ in terms of the measured probabilities.

$= \frac{1}{2}\left(|0\rangle\otimes(I+U)|\psi\rangle + |1\rangle\otimes(I-U)|\psi\rangle\right)$

B) $P_0 = \frac{1}{4}\langle\psi|(I+U^\dagger)(I+U)|\psi\rangle \quad P_1 = \frac{1}{4}\langle\psi|(I-U^\dagger)(I-U)|\psi\rangle$

C) if $\hat{U}=\hat{Z} \rightarrow \langle\hat{Z}\rangle = \binom{1}{0}\cdot\frac{1}{4}\langle\psi|\begin{smallmatrix}4&0\\0&0\end{smallmatrix}|\psi\rangle + \binom{0}{1}\cdot\frac{1}{4}\langle\psi|\begin{smallmatrix}0&0\\0&4\end{smallmatrix}|\psi\rangle \; : \psi=\binom{\alpha}{\beta}$

$\binom{1}{0}\cdot\frac{1}{4}4\alpha + \binom{0}{1}\cdot\frac{1}{4}4\beta = \binom{\alpha}{\beta} = |\psi\rangle$

$\langle\hat{Z}\rangle = P_0 - P_1 = \alpha - \beta$

D) There should be a phase difference in the |1> component by I meaning that the expectation value should now be:

System after last Hadamard: $\frac{1}{2}\left([|0\rangle+|1\rangle]\otimes|\psi\rangle + i[|0\rangle-|1\rangle]\otimes U|\psi\rangle]\right)$

$= \frac{1}{2}\left(|0\rangle(1+i)\otimes(I+U)|\psi\rangle + |1\rangle(1-i)\otimes(I-U)|\psi\rangle\right)$

$= \frac{1}{2}\left(|0\rangle\otimes[I+iI+U+Ui]|\psi\rangle + |1\rangle[I-iI-U+iU]|\psi\rangle\right)$

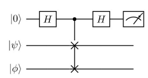
f) well $\langle\psi|U|\psi\rangle = P_0 - P_1 = \frac{1}{4}\langle\psi|[I^2+U+U^\dagger+U^\dagger U - I^2 - U - U^\dagger - U^\dagger U]|\psi\rangle$

$= \frac{1}{2}\langle\psi|(U+U^\dagger)|\psi\rangle$

3. SWAP Test

The SWAP Test is a variant of the Hadamard Test that can be used to measure the overlap between two quantum states. The circuit below describes the SWAP Test for a two qubit system. First, the second and third qubits are prepared in arbitrary states $|\psi\rangle$ and $|\phi\rangle$, respectively. Following a Hadamard gate on the first qubit, all three qubits are subject to a controlled SWAP gate. As a reminder, a SWAP gate performs the operation:

$SWAP|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$. As in the Hadamard Test, the SWAP Test ends with a Hadamard on the first qubit, followed by a measurement in the Z basis.



Now, let's examine the SWAP Test a bit further.

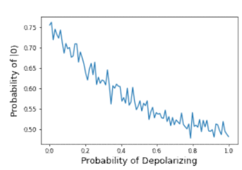
(a) What is the state of the three-qubit system prior to measurement?

(b) Show that the probability of measuring the first qubit in the zero state gives information about the overlap between $|\psi\rangle$ and $|\phi\rangle$.
(Hint: The probability should depend on $\langle\phi|\psi\rangle$.)

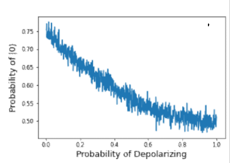
(c) Using Qiskit, construct the SWAP Test circuit for states $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |+\rangle$. Using the QASM Simulator, show that the probability of the first qubit being in $|0\rangle$ agrees with expected value.
(Hint: Use the results from part (b) for comparison.)

(d) Now, let's assume that the Hadamard gates are faulty and subject to depolarizing noise. Using the example code provided in the Jupyter notebook, plot the probability of $|0\rangle$ for the first qubit as a function of the probability of depolarization.

d)



100 points



1000 Points

a) $|0\rangle, |\psi\rangle \,\&\, |\phi\rangle = |0\rangle\otimes|\psi\rangle\otimes|\phi\rangle$

$H: |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|\psi\rangle\otimes|\phi\rangle + |1\rangle\otimes|\psi\rangle\otimes|\phi\rangle\right)$

$\text{swap}H|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|\psi\rangle\otimes|\phi\rangle + |1\rangle\otimes|\phi\rangle\otimes|\psi\rangle\right)$

$H\text{swap}H|\Psi\rangle = \frac{1}{2}\left(|0\rangle\otimes(|\psi\rangle\otimes|\phi\rangle)\otimes(|\phi\rangle\otimes|\psi\rangle) + |1\rangle\otimes(|\psi\rangle\otimes|\phi\rangle)\otimes-(|\phi\rangle\otimes|\psi\rangle)\right)$

$= \frac{1}{2}\left(|0\rangle\otimes[|\psi,\phi\rangle+|\phi,\psi\rangle] + |1\rangle\otimes[|\psi,\phi\rangle-|\phi,\psi\rangle]\right)$

b) $P_0 = |\psi\rangle^2 \text{ in } |0\rangle = \frac{1}{4}[|\psi,\phi\rangle+|\phi,\psi\rangle]\cdot[\langle\psi,\phi| + |\phi,\psi\rangle]$

$= \frac{1}{4}[\langle\psi,\phi|\phi,\psi\rangle + \langle\psi,\phi|\psi,\phi\rangle + \langle\phi,\psi|\psi,\phi\rangle + \langle\phi,\psi|\phi,\psi\rangle]$

$= \frac{1}{4}[2+2|\langle\psi|\phi\rangle|^2]$

$= \frac{1}{2} + |\langle\psi|\phi\rangle|^2 = \frac{1}{2} + \frac{1}{2}|\langle\phi|\psi\rangle|^2$ overlap

C) expected: $P_0 = \frac{1}{2} + \frac{1}{2}|\langle+|0\rangle|^2 \quad \langle+|0\rangle = \frac{1}{\sqrt{2}}(1,1)\binom{1}{0} = \frac{1}{\sqrt{2}}$

$= \frac{1}{2} + \frac{1}{2}\left|\frac{1}{\sqrt{2}}\right|^2$

$= \frac{1}{2} + \frac{1}{4} = 3/4 = 0.75$

Simulated: