Hw 4

Problem Set 4

AS.171.402: Applied Quantum Information [Spring 2022]

Due Date: April 5, 2022 (11:59 pm)

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1. Quantum Approximate Optimization

The Quantum Approximate Optimization Algorithm (QAOA) is a near-term quantum algorithm designed to find approximate solutions to binary optimization problems. In this homework problem, you will translate a classical optimization problem into a quantum problem and utilize QAOA to find an approximate solution. In particular, we will seek to maximize the function

$$E(\vec{x}) = x_1x_2 + x_2x_3 + x_1x_3, \tag{1}$$

where $x_j = 0, 1$ is a binary variable. For simplicity, we will focus on a single-layer QAOA of the form

$$U_{QAOA}(\beta, \gamma) = e^{-i\beta H_M} e^{-i\gamma H_C}, \tag{2}$$

with H_M denoting the mixer Hamiltonian and H_C representing the cost (problem) Hamiltonian. By the Variational Principle of Quantum Mechanics, a maximum of $E(\vec{x})$ can be found by maximizing

$$\langle H_C \rangle = \langle \psi(\beta, \gamma) | H_C | \psi(\beta, \gamma) \rangle, \tag{3}$$

where $|\psi(\beta,\gamma)\rangle = U_{QAOA}(\beta,\gamma)\,|+++\rangle$, $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, is the state resulting from evolving the equal superposition state according to the QAOA unitary.

(a) *Translating to a Quantum Problem.* First, we must translate $E(\vec{x})$ to a quantum Hamiltonian. ~~We will perform~~ this task by first making a change of variables of $x_i = (1 + s_i)/2$, where $s_j = \pm 1$. Show that the resulting energy function can be written as

$$E(\vec{s}) = \sum_i h_i s_i + \sum_{i,j} J_{ij} s_i s_j + C, \tag{4}$$

where h_i and J_{ij} are real numbers and C is a real constant. By taking $s_j \mapsto \sigma_j^z$ and dropping the constant, show that the function $E(\vec{s})$ can be cast as a quantum Hamiltonian of the Ising form.

Truth Table

X_1	X_2	X_3	$E(\vec{X})$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	3

a) $E(x_1) = x_1x_2 + x_2x_3 + x_1x_3$

$$= \frac{(1+s_1)(1+s_2)}{4} + \frac{(1+s_2)(1+s_3)}{4} + \frac{(1+s_1)(1+s_3)}{4}$$

$$= \frac{1}{4}(1 + s_1 + s_2 + s_1s_2 + 1 + s_2 + s_3 + s_2s_3 + 1 + s_1 + s_3 + s_1s_3)$$

$$= \frac{1}{4}(3 + 2s_1 + 2s_2 + 2s_3 + s_1s_2 + s_2s_3 + s_1s_3)$$

$$= \underbrace{\left(\tfrac{1}{2}s_1 + \tfrac{1}{2}s_2 + \tfrac{1}{2}s_3\right)}_{\sum_i h_i s_i} + \underbrace{\left(\tfrac{1}{4}s_1s_2 + \tfrac{1}{4}s_2s_3 + \tfrac{1}{4}s_1s_3\right)}_{\sum_{i,j,\, i\neq j} J_{ij} s_i s_j} + \underbrace{3/4}_{C}$$

$$= \sum_i h_i s_i + \sum_{\substack{i,j \\ i \neq j}} J_{ij} s_i s_j + C \quad \text{where } h_i = \{\tfrac{1}{2}\},\; J_{ij} = \{\tfrac{1}{4}\},\; C = 3/4$$

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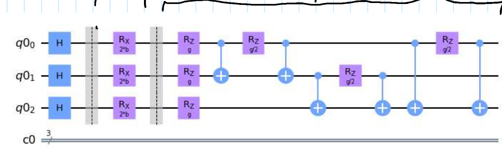
(b) *Constructing the QAOA circuits.* Let $H_M = \sum_i \sigma_i^x$ and H_C be the quantum Hamiltonian found in part (a). Using Qiskit and the notebook provided, build quantum circuits that implement, $e^{-i\beta H_M}$, $e^{-i\beta H_C}$, and $U_{QAOA}(\beta, \gamma)$. Plot the circuit diagrams.

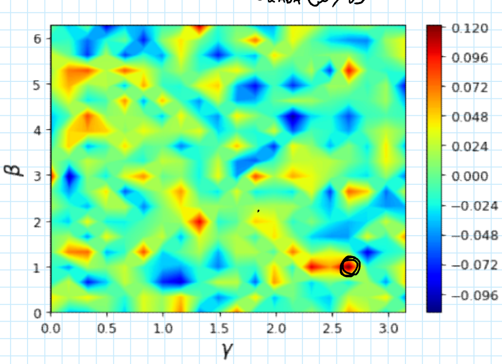
b) $U_m(\beta)$ $\quad U_c(\gamma)$



$U_{QAOA}(\beta, \gamma)$

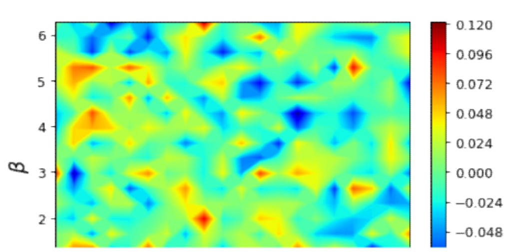
(c) *Plotting the function landscape.* Construct the full QAOA algorithm with the $U_{QAOA}(\beta, \gamma)$ circuit from part (b) and the correct state preparation. Using the functions provided in the example notebook, estimate and plot the expectation value $\langle H_C \rangle$ as a function of β and γ. Assume $\beta \in [0, 2\pi)$ and $\gamma \in [0, \pi]$.

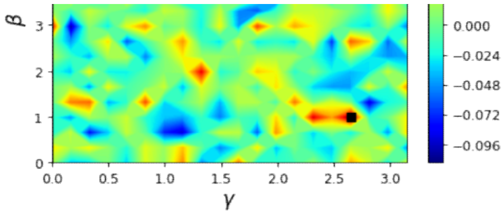
c)

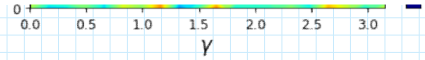


(d) *Finding optimal variational parameters.* Using the results from part (c), identify the β and γ values that maximize $E(\vec{x})$.

d)





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Optimal beta = 0.9921,  Optimal gamma = 2.6456
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