Problem Set 1

AS.171.402: Applied Quantum Information [Spring 2022]

Due Date: Feb 8, 2022

---

1. Driving a Qubit: Rabi Oscillations

In this problem, we study the evolution of a single qubit. Later in class we will see that this type of Hamiltonian is typically used to engineer single qubit gates for quantum computing. We are given the following single qubit Hamiltonian and the initial state $|\psi_0\rangle$,

$$\hat{H} = \frac{\omega}{2}\hat{Z} + \frac{\Omega_x(t)}{2}\hat{X} + \frac{\Omega_y(t)}{2}\hat{Y}, \tag{1}$$

where, $\Omega_x(t) \in \{0, \Omega\}$ and $\Omega_y(t) \in \{0, \Omega\}$ are piecewise constant and at any given time they can be either a constant or zero. For a typical qubit, ω is the frequency of the qubit that is determined by the details of the qubit platform. $\Omega_x(t)$ and $\Omega_y(t)$ are external drives applied to control the qubit.

(a) Starting with the initial state, $|\psi_0\rangle = |0\rangle$, we apply a drive along the y axis for a time t_g i.e. $\Omega_y(t) = \Omega$ for $0 \le t < t_g$ and 0 otherwise. For this part, set $\Omega_x = \omega = 0$.

  (i) Calculate the matrix expression for the time-evolution operator at time t_g.

  (ii) Obtain the state of the qubit as a function of time up to t_g.

  (iii) Obtain an expression for t_g in terms of Ω that creates the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

  (iv) Plot the trajectory of the qubit on the Bloch sphere for creating the $|+\rangle$ state. Clearly identify the direction the qubit evolves, and the direction of the applied drive.

(b) Having created the $|+\rangle$ state, we consider the evolution of a qubit in the $|+\rangle$ state, under $\hat{H}$ $(\omega \neq 0)$ and without any drives $(\Omega_x(t) = \Omega_y(t) = 0)$ for a time of t_f.

  (i) Obtain the state of the system as a function of the time t_f.

  (ii) Plot the overlap with the $|+\rangle$ state as a function of t_f. Use $\Omega = 1$ if needed.

2

(c) Now, consider the scenario, where the $|+\rangle$ state evolves in the presence of an additional constant drive along the x-direction, for a time t_f. So, $\Omega_x = \Omega \neq 0$, $\omega \neq 0$ and $\Omega_y = 0$.

  (i) Obtain the state of the system as a function of the time t_f.

  (ii) Plot the probabilities of $|0\rangle$ and $|1\rangle$ as a function of time.

(d) In this part, we will simulate the qubit dynamics using quantum circuits and the statevector simulator on `qiskit`. Reproduce the results of parts (a) and (b) on the accompanying jupyter notebook.

(Tip: Make sure that the simulations agree with your analysis.)

2. Spin Dephasing

Here, we will examine quantum channels. In particular, we will focus on the so-called dephasing channel described by $\mathcal{E}$, where

$$\mathcal{E}(\rho) = \hat{E}_0\rho\hat{E}_0^\dagger + \hat{E}_1\rho\hat{E}_1^\dagger, \tag{2}$$

with

$$\hat{E}_0 = \frac{\sqrt{p}}{2}(I + \hat{\sigma}^z) \quad \text{and} \quad \hat{E}_1 = \frac{\sqrt{p}}{2}(I - \hat{\sigma}^z) \tag{3}$$

representing the Kraus operators. Let p represent the probability of dephasing, and I and $\hat{\sigma}^z$ denote the identity operator and Pauli Z operator, respectively.

(a) Check that the Kraus operators defining the dephasing channel are indeed trace-preserving, i.e., satisfy $\sum_i \hat{E}_i\hat{E}_i^\dagger = I$.

(b) Consider the Bloch representation of a single qubit, $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$, where $\vec{r} = (r_x, r_y, r_z)$ and $\vec{\sigma} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$. Apply the channel to this state to determine $\rho' = \mathcal{E}(\rho)$. Letting $\rho' = \frac{1}{2}(I + \vec{r}'\cdot\vec{\sigma})$, find a mapping between $\vec{r}$ and $\vec{r}'$.

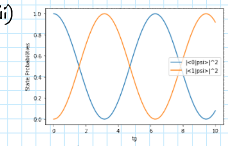
(c) How does the mapping of the Bloch vector $\vec{r}$ change if the channel is applied twice: $\mathcal{E}^2 \circ \rho = \mathcal{E}(\mathcal{E}(\rho))$? Generalize this mapping to n applications of the channel: $\mathcal{E}^n \circ \rho$.

(d) Consider a case where the channel acts over a very small interval of time δt. Let Γ represent the probability of dephasing per unit time such that the probability during the application of the channel is $p = \Gamma\delta t$. Since δt is very small, $p \ll 1$. Now, let's

1.a) i) $R_y = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$

where $\theta = \Omega \cdot t_g$

ii)

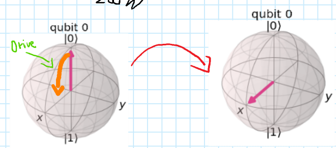


iii) $|+\rangle$ is a $\frac{\pi}{2}$ rotation from $|0\rangle$ or $|1\rangle$

so $t_g = \pi/2 \div \omega = (\theta) \div \left(\frac{\theta}{S}\right) = S$

$t_g = \frac{\pi}{2\omega}$

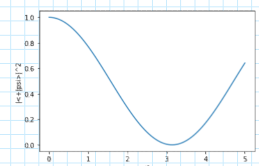
iv)



b) i) $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{R_z} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\left(\frac{\omega t_f}{2}\right)} \\ e^{i\left(\omega\frac{t_f}{2}\right)} \end{pmatrix}$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\omega t_f}{2}} \begin{pmatrix} e^{-i\omega t_f} \\ 1 \end{pmatrix}$

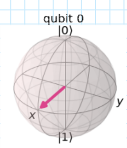
ii)

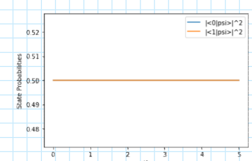


c) i) $R_x = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \rightarrow |+\rangle \xrightarrow{R_x} |+\rangle$ at all t_f

$\theta = \omega t_f$

ii) Since the plus state is along the x axis any rotations with an x drive will essentially be identity meaning that the state should not change





2) a) $\sum_i E_i E_i^\dagger$

$= \frac{p}{4}(I^2 + I\sigma + \sigma_z^2) + \frac{p}{4}(I^2 - \sigma I + \sigma_z^2)$; $\sigma_z^2 = I$

$= \frac{p}{4}(I^2 + I^2) = \frac{p}{2} I$

b) $\mathcal{E}(\rho) = \frac{\sqrt{p}}{2}(I + \sigma_z)\cdot\frac{1}{2}(I + r\cdot\sigma)\frac{\sqrt{p}}{2}(I + \sigma_z) + \frac{\sqrt{p}}{2}(I - \sigma_z)\cdot\frac{1}{2}(I + r\cdot\sigma)\frac{\sqrt{p}}{2}(I - \sigma_z)_p$

$= \frac{p}{8}\ I^3 + \cancel{\sigma_z I \sigma_z} + I(r\cdot\sigma)I + \sigma_z(r\cdot\sigma)\sigma_z + \frac{p}{8}\ I^3 + \cancel{\sigma_z I \sigma_z} + {}_p I(r\cdot\sigma)I - \sigma_z(r\cdot\sigma)\sigma_z$

$= \frac{p}{8}\left[2I + 2(r\cdot\sigma) + 2\sigma_z(r\cdot\sigma)\sigma_z\right)$

$= \frac{p}{4}\left[I + \sigma_x r_x - \sigma_x r_x + \sigma_y r_y - \sigma_y r_y + \sigma r_z + \sigma r_z\right)$

$= \frac{p}{4}\left[I + 2r_z\cdot\sigma_z\right] = \frac{1}{2}\left[I + p r_z\right.$

$r' = p r_x$ ie $r = (x, y, z) \rightarrow r' = (0, 0, pz)$

c) well $r'^n = (0, 0, p^n z)$ $\quad p : 0 \le p \le 1$

so $\lim_{n\to\infty}(r^n) = 0$

d) ?

e) very confused

(d) Consider a case where the channel acts over a very small interval of time δt. Let Γ represent the probability of dephasing per unit time such that the probability during the application of the channel is $p = \Gamma \delta t$. Since δt is very small, $p \ll 1$. Now, let's

focus on the case where the channel is applied n times such that the total evolution time is $t = n\delta t$. Using your result from (c) and

$$(1-p)^n = \left(1 - \frac{\Gamma t}{n}\right)^n \to e^{-\Gamma t} \tag{4}$$

find a mapping between the initial Bloch vector $\vec{r}$ and the Bloch vector resulting from $\mathcal{E}^n \circ \rho$. What happens to the quantum state (and Bloch vector) as $t \to \infty$.

(e) As we learned in class, when a unitary is applied to a joint quantum system and its environment, a quantum channel can arise from partial tracing over the environment degrees of freedom. Here, we will explore this further and examine a case where a unitary on the joint system-environment produces the dephasing channel. Consider a unitary operator that produces the mapping:

$$U(|0\rangle_S \otimes |0\rangle_E) \mapsto \sqrt{1-p}\,|0\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E \tag{5}$$

$$U(|1\rangle_S \otimes |0\rangle_E) \mapsto \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|1\rangle_S \otimes |2\rangle_E\,. \tag{6}$$

The subscript S denotes the system subspace, while E denotes the environment. Start with the density operator $\rho_{SE} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = (\alpha\,|0\rangle_S + \beta\,|1\rangle_S) \otimes |0\rangle_E$. Apply the unitary to produce $\rho'_{SE} = U\rho_{SE}U^\dagger$. Show that by applying a partial trace over the environment, $\rho_S = \mathrm{Tr}_E\,[\rho'_{SE}]$, the resulting state matches $\mathcal{E}(\rho)$ found in (a).