Problem Set 4

AS.171.402: Applied Quantum Information [Spring 2022]

Due Date: April 5, 2022 (11:59 pm)

---

1. Quantum Approximate Optimization

The Quantum Approximate Optimization Algorithm (QAOA) is a near-term quantum algorithm designed to find approximate solutions to binary optimization problems. In this homework problem, you will translate a classical optimization problem into a quantum problem and utilize QAOA to find an approximate solution. In particular, we will seek to maximize the function

$$E(\vec{x}) = x_1 x_2 + x_2 x_3 + x_1 x_3, \tag{1}$$

where $x_j = 0, 1$ is a binary variable. For simplicity, we will focus on a single-layer QAOA of the form

$$U_{QAOA}(\beta, \gamma) = e^{-i\beta H_M} e^{-i\gamma H_C}, \tag{2}$$

with H_M denoting the mixer Hamiltonian and H_C representing the cost (problem) Hamiltonian. By the Variational Principle of Quantum Mechanics, a maximum of $E(\vec{x})$ can be found by maximizing

$$\langle H_C \rangle = \langle \psi(\beta, \gamma) | H_C | \psi(\beta, \gamma) \rangle \,, \tag{3}$$

where $|\psi(\beta, \gamma)\rangle = U_{QAOA}(\beta, \gamma) \left|+++\right\rangle$, $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, is the state resulting from evolving the equal superposition state according to the QAOA unitary.

(a) *Translating to a Quantum Problem.* First, we must translate $E(\vec{x})$ to a quantum Hamiltonian. We will perform this task by first making a change of variables of $x_i = (1 + s_i)/2$, where $s_j = \pm 1$. Show that the resulting energy function can be written as

$$E(\vec{s}) = \sum_i h_i s_i + \sum_{i,j} J_{ij} s_i s_j + C, \tag{4}$$

where h_i and J_{ij} are real numbers and C is a real constant. By taking $s_j \mapsto \sigma_j^z$ and dropping the constant, show that the function $E(\vec{s})$ can be cast as a quantum Hamiltonian of the Ising form.

(b) *Constructing the QAOA circuits.* Let $H_M = \sum_i \sigma_i^x$ and H_C be the quantum Hamiltonian found in part (a). Using Qiskit and the notebook provided, build quantum circuits that implement, $e^{-i\beta H_M}$, $e^{-i\beta H_C}$, and $U_{QAOA}(\beta, \gamma)$. Plot the circuit diagrams.

(c) *Plotting the function landscape.* Construct the full QAOA algorithm with the $U_{QAOA}(\beta, \gamma)$ circuit from part (b) and the correct state preparation. Using the functions provided in the example notebook, estimate and plot the expectation value $\langle H_C \rangle$ as a function of β and γ. Assume $\beta \in [0, 2\pi)$ and $\gamma \in [0, \pi]$.

(d) *Finding optimal variational parameters.* Using the results from part (c), identify the β and γ values that maximize $E(\vec{x})$.