Hw3

Monday, March 14, 2022 5:42 PM

Problem Set 3

AS.171.402: Applied Quantum Information [Spring 2022]

Due Date: Mar 15, 2022 (11:59 pm)

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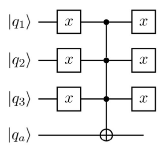
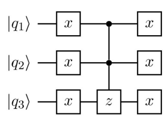
1. Grover's Search Oracles

In this problem, we explore oracle operators and how to create them. The purpose of this problem is to help build intuition in the construction, implementation, and verification of oracle operators for Grover's Search.

(a) Consider a three-qubit Grover's Search problem where the marked state is $|000\rangle$. Show that the associated Boolean oracle given below performs the desired phase flip operation. More specifically, start with the state $|\psi\rangle|-\rangle$, where $|\psi\rangle = 1/\sqrt{N}\sum_{x=0}^{N-1}|x\rangle$, $N = 2^3$, and apply the oracle operator.



(b) Verify that the phase oracle associated with the marked state $|000\rangle$ is given by the circuit below. Hint: start with $|\psi\rangle$ as defined in part (a).



(c) Using the oracles in part (a) and (b), construct a Boolean and phase oracle for the marked state $|\phi\rangle = |101\rangle$.

(d) In class, we showed analytically that Grover's Search exhibits a periodic behavior. Let's now verify this result via numerical experiments. In the Jupyter notebook pro-

1) a $|\psi-\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \otimes |-\rangle$

$X|\psi\rangle = \frac{1}{\sqrt{8}}(|111\rangle + |110\rangle + |101\rangle + |100\rangle + |011\rangle + |010\rangle + |001\rangle + |000\rangle) \otimes |-\rangle$

$\hat{C}XX|\psi-\rangle = \frac{1}{\sqrt{8}}(|1111\rangle + |110-\rangle + |101-\rangle \ldots)$

$\hat{X}\hat{C}X|\psi-\rangle = \frac{1}{\sqrt{8}}(|0001\rangle + |110-\rangle + |110-\rangle \ldots)$

ancilla bit flipped on marked state

b) $|\psi\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$

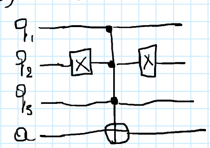
$X|\psi\rangle = \frac{1}{\sqrt{8}}(|111\rangle + |110\rangle + |101\rangle + |100\rangle + |011\rangle + |010\rangle + |001\rangle + |000\rangle)$

$C\hat{X}\hat{Z}|\psi\rangle = \frac{1}{\sqrt{8}}(-|111\rangle + |110\rangle + |101\rangle + |100\rangle + |011\rangle + |010\rangle + |001\rangle + |000\rangle)$

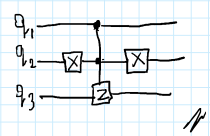
$\hat{X}C\hat{X}\hat{Y}|\psi\rangle = \frac{1}{\sqrt{8}}(-|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$

marked

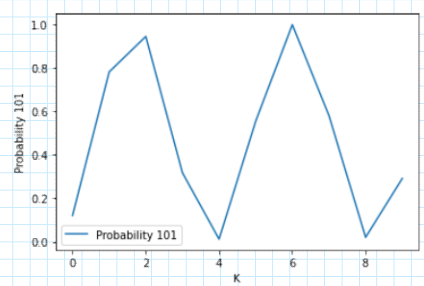
c) Boolean:



Phase:



d) 'Probability of Measuring Marked state as a function of K iterations of Grovers Search'



vided for this homework assignment, construct the full Grover's Search algorithm using an oracle of your choice from parts (a)-(c). Plot the probability of being in the marked state as a function of k, the number of iterations of the Grover operator. Consider $k = 1, 2, \ldots, 10$ for this analysis. Hint: The example notebook provided on Blackboard will be helpful here.

2. Grover's Search and 3-Bit Satisfiability

In class, we discussed the 2-Bit Satisfiability (2-SAT) problem and how to construct a quantum search algorithm to find a satisfying assignment for a given 2-SAT Boolean formula. Here, we expand on these concepts and consider the 3-bit Satisfiability (3-SAT) problem. In particular, we will focus on a 3-SAT problem defined by the Boolean function

$$f(x_1, x_2, x_3) = (x_0 \vee x_1 \vee x_2) \wedge (x_0 \vee x_1 \vee \neg x_2) \wedge (x_0 \vee \neg x_1 \vee x_2) \wedge (x_0 \vee \neg x_1 \vee \neg x_2) \wedge (\neg x_0 \vee x_1 \vee x_2), \quad (1)$$

consisting of three clauses, each with three variables. The logical operations of OR, AND, and NOT are denoted by $\vee$, $\wedge$, $\neg$, respectively. Let's now work through a series of steps to find satisfying assignments for $f(x_1, x_2, x_3)$ using Grover's Search Algorithm.

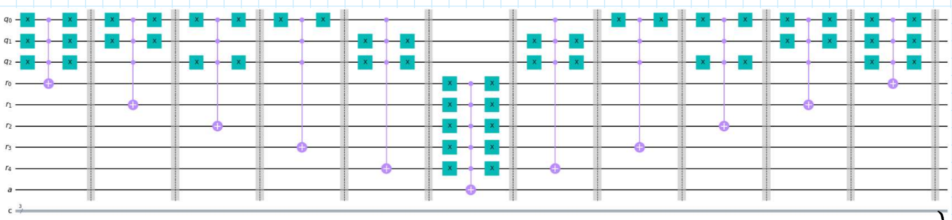
(a) First, we must construct an oracle function for the search problem. Using your knowledge of 2-SAT oracles and the results from Problem 1 as inspiration, devise an oracle for the Boolean formula in Eq. (1). Draw the circuit diagram in Jupyter notebook provided using Qiskit.

(b) Using Qiskit's QASM Simulator, construct the full Grover's Search using the oracle constructed in part (a). Verify that the solution ~~states~~ resulting from the search matches the satisfying assignments expected from Eq. (1). Use more than 1000 shots to gain good statistics ~~from the simulations~~ and consider only one Grover iteration. Plot your results in the ~~Jupyter notebook provided.~~

(c) Let's now examine the performance of this search algorithm over ~~multiple~~ iterations of the Grover operator. Apply the Grover operator k times and plot the resulting cumulative probability of being in the solution states of f in the Jupyter notebook. Find the k value that gives the largest cumulative probability. This is your optimal number of Grover iterations. How does this compare to the analytical upper bound derived in class?

(d) Now, let's examine how noise changes the results from part (c). Using the example Jupyter notebook as inspiration, add a depolarizing error to all single qubit gates. Assume the depolarizing error is $p = 0.004$. Find the optimal number of Grover iterations for the noisy implementation. Does the result differ from part (c)?

2. a)



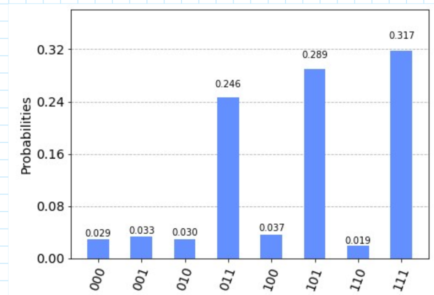
UnComPutation

* when this is not included Solution Probability is actually minimized

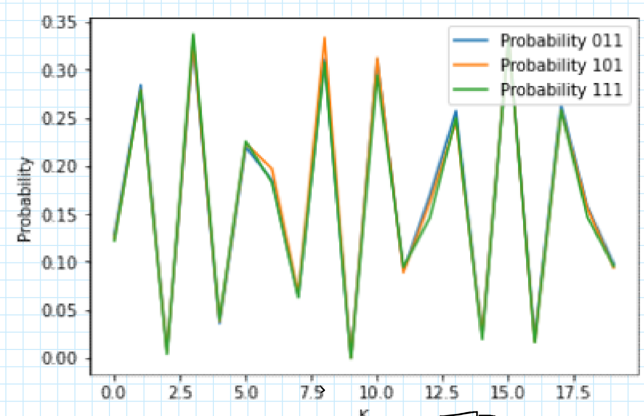
b)

q_1	q_2	q_3	k_1	k_2	k_3	k_4	k_5	S
0	0	0	0	1	1	1	1	✗
0	0	1	1	0	1	1	1	✗
0	1	0	1	1	0	1	1	✗
0	1	1	1	1	1	0	1	✗
1	0	0	1	1	1	1	0	✗
1	0	1	1	1	1	1	1	✓
1	1	0	1	1	1	1	1	✓
1	1	1	1	1	1	1	1	✓

– 101, 110, 111



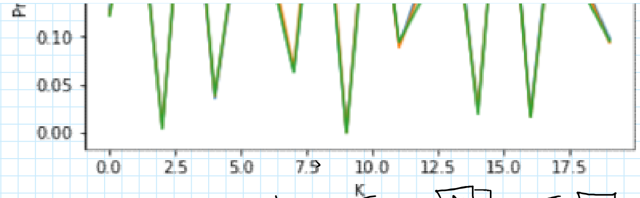
c)



0	0.1228
1	0.279
2	0.0048
3	0.3374
4	0.0398
5	0.226
6	0.183
7	0.0638
8	0.3078
9	0.0004
10	0.2948
11	0.0952
12	0.1462
13	0.25
14	0.02
15	0.3336
16	0.0168
17	0.2582
18	0.1472
19	0.0962

Analytically $K \leq \left[\frac{\pi}{4}\sqrt{\frac{N}{M}}\right] = \frac{\pi}{4}\sqrt{\frac{8}{3}} = 1.2825$

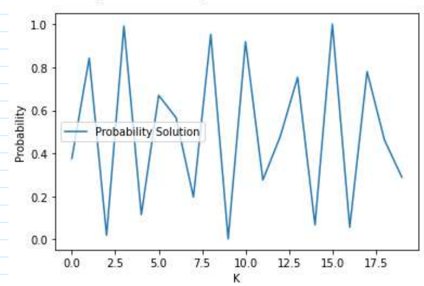
(d) Now, let's examine how noise changes the results from part (c). Using the example Jupyter notebook as inspiration, add a depolarizing error to all single qubit gates. Assume the depolarizing error is $p = 0.004$. Find the optimal number of Grover iterations for the noisy implementation. Does the result differ from part (c)?



11	0.0952
12	0.1462
13	0.25
14	0.02
15	0.3336
16	0.0168
17	0.2582
18	0.1472
19	0.0962

Analytically $K \leq \left\lfloor \frac{\pi}{4}\sqrt{\frac{N}{M}} \right\rfloor = \frac{\pi}{4}\sqrt{\frac{8}{3}} = 1.2825$

'Probability of Measuring solution as a function of K iterations of Grovers Search'

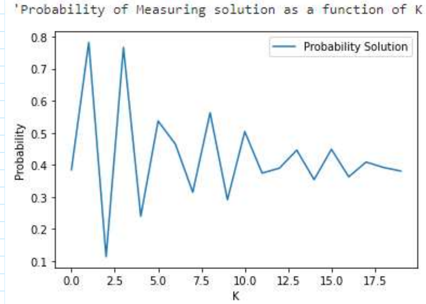


0	0.3764
1	0.8430000000000001
2	0.0194
3	0.9913999999999998
4	0.1154
5	0.6698
6	0.5658
7	0.19699999999999998
8	0.9530000000000001
9	0.0026
10	0.9185999999999999
11	0.2764
12	0.4784
13	0.7534000000000001
14	0.0672
15	1.0
16	0.0562
17	0.7797999999999999
18	0.4618
19	0.2894

Computationally: 3 is max & 1 is a local max

d)

'Probability of Measuring solution as a function of K iterations of Grovers Search'



0	0.38480000000000003
1	0.7824
2	0.1136
3	0.7666
4	0.2398
5	0.537
6	0.46440000000000003
7	0.3146
8	0.5626
9	0.2912
10	0.5038
11	0.3744
12	0.38980000000000004
13	0.44599999999999995
14	0.35400000000000004
15	0.4486
16	0.3628
17	0.4088
18	0.392
19	0.38079999999999997

This is as expected because we can see that as time goes on and the circuit has more chance to depolarize with the longer iterations it is more likely to skew the solution form the expected