Appendix A Optimization process

Fixing U and V optimizing P, we have

$$\min_{P} \sum_{i=1}^{N} \sum_{k=1}^{C} \left\| x_i P - v_k P \right\|_2 u_{ik} + \mu \sum_{i,j} \left\| x_i P - x_j P \right\|_2 w_{ij} + \eta \|P\|_{2,1} - \xi \sum_{k=1}^{C} \left\| R_k v_k P - \bar{v} P \right\|_2^2$$
$$s.t. : \mathrm{P}^T P = I \tag{A1}$$

We calculate the four parts in formula A1 separately.First of all:

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{k=1}^{C} \left\| x_i P - v_k P \right\|_2 u_{ik} = 2 \sum_{i=1}^{N} \sum_{k=1}^{C} \left\| x_i P - v_k P \right\|_2^2 g_{ik} \\
&= 2 \sum_{i=1}^{N} \sum_{k=1}^{C} \left(x_i P P^T x_i^T g_{ik} - 2 v_k P P^T x_i^T g_{ik} + v_k P P^T v_k^T g_{ik} \right) \\
&= 2 \sum_{i=1}^{N} x_i P P^T x_i^T \left(\sum_{k=1}^{C} g_{ik} \right) - 2 \sum_{i=1}^{N} \sum_{k=1}^{C} \left(2 v_k P P^T x_i^T g_{ik} \right) + 2 \sum_{k=1}^{C} \left(v_k P P^T v_k^T \sum_{i=1}^{N} g_{ik} \right) \\
&= 2tr \left(X P P^T X G_1 \right) - 2tr \left(V P P^T X^T G \right) + 2tr \left(V P P^T V^T G_2 \right)
\end{aligned} \tag{A2}$$

where tr(.) represents the trace of a matrix.$g_{ik} = \frac{u_{ik}}{2\left\| x_i P - v_k P \right\|_2}$is an element of row i, column k of the auxiliary matrix G.

$$G_1 = \begin{bmatrix} g_{1_{11}} & 0 & \cdots & 0 \\ 0 & g_{1_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{1_{NN}} \end{bmatrix}$$
$$g_{1_{ii}} = \sum_{k=1}^{C} g_{ik} \, (i = 1, 2, \ldots M) \tag{A3}$$

$$G_2 = \begin{bmatrix} g_{2_{11}} & 0 & \cdots & 0 \\ 0 & g_{2_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{2_{NN}} \end{bmatrix}$$
$$g_{2_{ii}} = \sum_{\mathrm{i}=1}^{N} g_{ik} \, (k = 1, 2, \ldots C) \tag{A4}$$

And then, by introducing an auxiliary variable $Q = [q_{ij}]_{N \times N}$, where

$$q_{ij} = \begin{cases} \frac{1}{2\left\| x_i P - v_k P \right\|_2} & \left\| x_i P - v_k P \right\|_2 \neq 0 \\ 0 & \left\| x_i P - v_k P \right\|_2 = 0 \end{cases} \tag{A5}$$

the second part can be rewritten as:

$$\begin{aligned}
&\sum_{i,j} \left\| x_i P - x_j P \right\|_2 w_{ij} \\
&= 2 \sum_{i,j} \left\| x_i P - x_j P \right\|_2^2 q_{ij} w_{ij} \\
&= 2tr \left(P^T X^T L X P \right)
\end{aligned} \tag{A6}$$

$L = R - W \otimes Q$ is the Laplacian matrix according to $W \otimes Q$, where $\otimes$ expresses the Hadamard product of two matrices.

$$\begin{aligned} R &= \begin{bmatrix} R_{11} & 0 & \cdots & 0 \\ 0 & R_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{NN} \end{bmatrix} \\ R_{ii} &= \sum_j (W \otimes Q)_{ij} \end{aligned} \tag{A7}$$

The auxiliary variable q_{ij} can be considered as a weighted value for w_{ij}w . The weight is small for the two samples located far away, which further reduces the affinity between these two samples. If $q_{ij} = 0$, it means that the representations of x_i and x_j in the projected space have the same position. In this case, the second part in (A7) with respect to these samples tends to be zero which is reasonable for minimizing the proposed objective function. The third part is:

$$\|P\|_{2,1} = 2tr\left(P^T D P\right) \tag{A8}$$

where

$$\begin{aligned} D &= \begin{bmatrix} D_{11} & 0 & \cdots & 0 \\ 0 & D_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{NN} \end{bmatrix} \\ D_{ii} &= \frac{1}{2\|p_i\|_2} \end{aligned} \tag{A9}$$

The fourth part is

$$\sum_{k=1}^{C} \left\|R_k V_k P - \bar{V} P\right\|_2^2 = 2tr\left(P^T S P\right) \tag{A10}$$

Where $S = \left(V - \bar{V}\right)^T \left(V - \bar{V}\right)$.

After combining the above four parts, we get formula(10).