CPS 3

1. $I_n = \sum_{i=1}^{n} W_{t_{i-1}^n} \Delta W_{t_i^n}$ $\quad J_n = \sum_{i=1}^{n} W_{t_i^n} \Delta W_{t_i^n}$

$$K_n = \sum_{i=1}^{n} \frac{W_{t_i^n} + W_{t_{i-1}^n}}{2} \Delta W_{t_i^n}$$

donc, $K_n = \frac{1}{2}(I_n + J_n)$

$$\frac{1}{2}W_T^2 - K_n = \frac{1}{2}\left(\frac{1}{2}W_T^2 - I_n + \frac{1}{2}W_T^2 - J_n\right)$$

On sait aussi $K_n = \sum_{i=1}^{n} \frac{1}{2}(W_{t_i^n} + W_{t_{i-1}^n})(W_{t_i^n} - W_{t_{i-1}^n})$

$$= \sum_{i=1}^{n} \frac{1}{2}(W_{t_i^n}^2 - W_{t_{i-1}^n}^2) = \frac{1}{2}W_T^2$$

Alors, $\frac{1}{2}W_T^2 - K_n = 0$ et $\frac{1}{2}W_T^2 - I_n = -\left(\frac{1}{2}W_T^2 - J_n\right)$

$$\frac{1}{2}W_T^2 - I_n - \left(\frac{1}{2}W_T^2 - J_n\right) = \sum_{i=1}^{n} \Delta W_{t_i^n}^2$$

$E[\sum_{i=1}^{n} \Delta W_{t_i^n}^2] = T$, alors, $E[\frac{1}{2}W_T^2 - I_n] = \frac{T}{2} = 1$

$$E[\frac{1}{2}W_T^2 - J_n] = -\frac{T}{2} = 1$$

2. $A_n = \sum_{i=1}^{n} e^{t_{i-1}^n} \Delta W_{t_i^n}$ $\quad B_n = \sum_{i=1}^{n} e^{t_i^n} \Delta W_{t_i^n}$ $\quad C_n = \sum_{i=1}^{n} e^{\frac{t_i^n + t_{i-1}^n}{2}} \Delta W_{t_i^n}$

Comme, $e^{t_{i-1}^n}$, $e^{t_i^n}$ et $e^{\frac{t_i^n + t_{i-1}^n}{2}}$ sont deterministes,

$$E[A_n] = E[B_n] = E[C_n] = 0.$$

Alors, $E[\frac{1}{2}W_T^2 - A_n] = E[\frac{1}{2}W_T^2 - B_n] = E[\frac{1}{2}W_T^2 - C_n] = \frac{1}{2}T = 1$

3. $A_n = \sin(W_T) + \frac{1}{2n}\sum_{i=1}^{n} \sin(W_{t_{i-1}^n})$

Comme sin est une fonction impaire, donc, pour chaque

$W_{t_{i-1}^n}$, $E[\sin(W_{t_{i-1}^n})] = \int_{-\infty}^{+\infty} \sin(x) \frac{1}{\sqrt{2\pi t_{i-1}^n}} \exp(-\frac{x^2}{2t_{i-1}^n})dx = 0.$

Alors, $E[A_n] = 0$.