CPS 2.

1. (a) $W_{t_i^n} - W_{t_{i-1}^n} \sim N(0, \Delta T) = Z_i\sqrt{\Delta T}$

Alors, $W_{t_i^n} = W_{t_{i-1}^n} + W_{t_i^n} - W_{t_{i-1}^n} = W_{t_{i-1}^n} + Z_i\sqrt{\Delta T}$

2. (a) On sait que $\begin{pmatrix} W_{s_1} \\ W_{s_2} \end{pmatrix} \sim N(0, \Sigma)$ où $\Sigma = \begin{pmatrix} s_1 & s_1 \\ s_1 & s_2 \end{pmatrix}$

La distribution conditionel de $W_{s_1} | W_{s_2}$ suit encore une loi nomale $N(\mu, \sigma^2)$

où $\mu = \frac{s_1}{s_2} W_{s_2}$ $\quad \sigma^2 = s_1 - s_1 \cdot \frac{1}{s_2} \cdot s_1 = s_1(1 - \frac{s_1}{s_2})$

Donc $E[W_{s_1} | W_{s_2}] = \frac{s_1}{s_2} W_{s_2}$

$Var[W_{s_1} | W_{s_2}] = s_1(1 - \frac{s_1}{s_2})$

(b) On sait que $W_{\bar{s}} | W_{s_1} = x_1 = (W_{\bar{s}} - W_{s_1}) + W_{s_1} | W_{s_1} = x_1$

$= x_1 + \sqrt{\bar{s} - s_1}\, N(0,1) = x_1 + W_{\bar{s} - s_1}$

$W_{s_2} | W_{s_1} = x_1 = (W_{s_2} - W_{s_1}) + W_{s_1} | W_{s_1} = x_1$

$= x_1 + \sqrt{s_2 - s_1}\, N(0,1) = x_1 + W_{s_2 - s_1}$

Alors, $\begin{pmatrix} W_{\bar{s}} | W_{s_1} = x_1 \\ W_{s_2} | W_{s_1} = x_1 \end{pmatrix} \sim N(\mu, \Sigma)$ où $\mu = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

$\Sigma = \begin{pmatrix} \bar{s} - s_1 & \bar{s} - s_1 \\ \bar{s} - s_1 & s_2 - s_1 \end{pmatrix}$

Donc, selon la loi conditionel d'un vecteur gaussian,

$W_{\bar{s}} | W_{s_1} = x_1 \,\big|\, W_{s_2} | W_{s_1} = x_1 = W_{\bar{s}} | W_{s_1} = x_1, W_{s_2} = x_2$

$\sim N(\bar{\mu}, \bar{\sigma}^2)$ où $\bar{\mu} = x_1 + \frac{\bar{s} - s_1}{s_2 - s_1}(x_2 - x_1) = \frac{x_1 + x_2}{2}$

$$\bar{\sigma}^2 = \bar{s} - s_1 - \frac{(\bar{s}-s_1)^2}{s_2 - s_1} = \frac{s_2 - s_1}{4}$$

Donc, $W_{\bar{s}} \mid W_{s_1} = x_1, W_{s_2} = x_2 \sim N(\bar{x}, \frac{s_2 - s_1}{4})$

(b) si $u < s_1$, alors, $F_u \subset F_{s_1}$

si $u > s_2$, on peut construire $B_t = W_{T-t} - W_T$ où $T > u, s_2, s_1$

alors, $F_u \subset F_{s_2}$.

De cette façon, $E[f(W_{\bar{s}}) \mid F_{s_1}, F_{s_2}, F_u] = E[f(W_{\bar{s}}) \mid F_{s_1}, F_{s_2}]$

Donc, $W_{\bar{s}} \mid W_{s_1} = x_1, W_{s_2} = x_2, W_u, u \notin [s_1, s_2]$ a la même distribution que $W_{\bar{s}} \mid W_{s_1} = x_1, W_{s_2} = x_2$

Alors, $W_{\bar{s}} \mid W_{s_1} = x_1, W_{s_2} = x_2, W_u, u \notin [s_1, s_2] \sim N(\bar{x}, \frac{s_2 - s_1}{4})$

(c)