Lec 1 of AMA1110

Lecturer: Yijun Lou

Department of Applied Mathematics, PolyU

Sep 7, 2020

Today's topics

(1) Go through the course outline (The outline file can be downloaded from BLACKBOARD system)

(2) Real numbers and set notations

(3) Basic inequalities (with quadratics)

(4) Polynomial (functions)

(5) Rational functions

(6) Operations of functions

(7) Inverse function

Welcome to AMA 1110

About Me and the subject

– Lecture Time: Mon, 13:30-15:20

– Online-teaching platform: mainly use the Blackboard Collaborate Ultra, with Zoom as a backup (you are welcome to raise all concerns with the platform in the semester)

– Lecturer: Dr. Yijun LOU; Office: TU826; Tel: 3400-3980; Email: yijun.lou@polyu.edu.hk

– Consultation: You are welcome to contact me through email to raise your questions or make arrangements. We may arrange online meetings

– Course website: Blackboard system (https://learn.polyu.edu.hk/) for important announcements, assignment questions, exercise questions and lecture notes.

– Textbook: “Foundation Mathematics and Statistics" by KF Hung, Wilson CK Kwan and Glory TY Pong (Second edition 2013)

Assessment

- Continuous Assessment & Final Assessment
- 70% continuous assessment. 30% final exam. Weighting may be subject to adjustment
- The Continuous assessment: 3 assignments (30%); 2 midterm tests (15%+25%).
- Midterm test (tentative): Test I: (Online test) Sunday Oct 11; Test II: (Online test) Sunday, Nov 8
- Assignments: 3 assignments due on Fridays (tentatively and assignment questions will be announced around 10 days before the due date).
- Final Assessment: Time will be announced later.

Teaching Platform

- Lectures: mainly use Blackboard Collaborate Ultra, with Zoom as a backup. The links will be sent through email before lectures
- Tutorials: the platforms will be decided by the tutorial coordinators
- Supplementary exercises: questions will be uploaded on BLACKBOARD after the lecture for your practice
- Lecture notes: posted on BLACKBOARD before each lecture
- The second page of the outline contains the teaching topics and a tentative teaching schedule

Real Numbers and Set Notation

- Set: a collection of elements
- $\mathbb{N}$: the set of all natural numbers (also called positive integers). $\mathbb{N}$=$\{1, 2, 3, 4, \ldots\}$
- $\mathbb{Z}$: set of all integers. $\mathbb{Z}$=$\{0, 1, -1, 2, -2, 3, -3, \ldots\}$
- $\mathbb{Q}$: set of all rational numbers. Rational numbers can be written as ratio or quotient $\dfrac{p}{q}$, where p, q are integers and $q \neq 0$, for example, $\dfrac{1}{2}, \dfrac{11}{5}, -\dfrac{7}{29}$
- $\mathbb{R}$: set of all real numbers:
  set of all rational numbers + set of all irrational numbers (union of these two sets)

Set notation: represent a set

Method 1:
List all elements in the set

Example: $S = \left\{\sqrt{2}, \sqrt{3}, \pi\right\}$ denotes the set containing three elements.

Example: Suppose there are three students Stu1, Stu2 and Stu3 in the working group A. List all elements in the set (Group) A:
Solution: A={Stu1,Stu2,Stu3}

Set notation: represent a set

Method 2:
List all elements in the set that satisfies some properties: $\{x : p(x)\}$ is the set of all those x for which the statement $p(x)$ is true.

Example: $S = \left\{x : 2x^2 - 5x - 3 = 0\right\}$ is the set of real numbers x such that $2x^2 - 5x - 3 = 0$ is true; that is, S is the set consisting of the real roots of

$$2x^2 - 5x - 3 = 0.$$

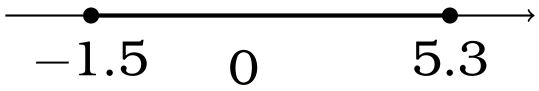
Solving the above equation, we get $x_1 = 3$ and $x_2 = -\frac{1}{2}$. Therefore, the solution set is $\{3, -\dfrac{1}{2}\}$.

Example: How to describe the collection of real numbers between 1 and 2 (1 is included but 2 is not in the collection) by a set?
Solution: A=$\{x : 1 \leqslant x < 2\}$

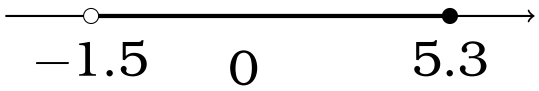
Intervals: a piece on the real line

Four types of intervals

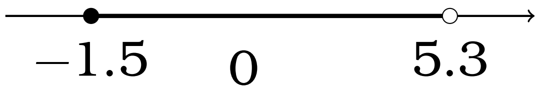
- $[-1.5, 5.3]$=$\{x : -1.5 \leqslant x \leqslant 5.3\}$



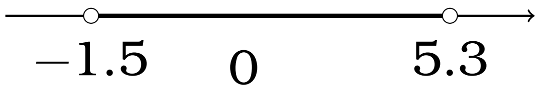
- $(-1.5, 5.3]$=$\{x : -1.5 < x \leqslant 5.3\}$



- $[-1.5, 5.3)$=$\{x : -1.5 \leqslant x < 5.3\}$

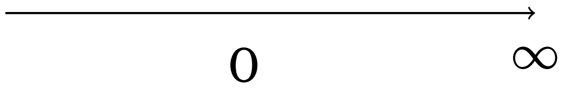


- $(-1.5, 5.3)$=$\{x : -1.5 < x < 5.3\}$

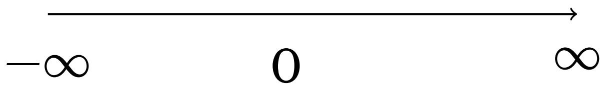


Notation ∞

- Notation ∞ (read as infinity, it is just a notation, not a real number but is greater than any real number)
- $\mathbb{R} = (-\infty, \infty)$
- The real line can also be represented as:

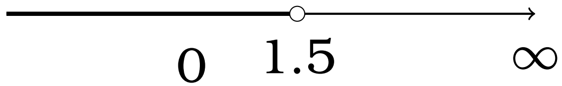


or



- $[1.5, \infty) = \{x : 1.5 \leqslant x\}$

- $(-\infty, 1.5) = \{x : x < 1.5\}$



Element of the set

$x \in S$: x is an element of the set S
$x \notin S$: x is not an element of the set S

Example: "$x \in \mathbb{N}$" means "x is an element of $\mathbb{N}$ (the set of positive integers), that is, "x is a positive integer"

Example: Suppose that $A = \{x : 1 \leqslant x < 2\}$. It is easy to see that: $1.5 \in A$ and $3 \notin A$

Set relations

A is a subset of B if and only if every element of A is an element of B. Denoted by $A \subset B$. If a set A is a subset of B and B is not a subset of A, then A is said to be a proper subset of B

Example: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Example: Suppose there are three students Ho Hei Chun, Lau Chi Kit and Yuen Chen Kit in the working group A. Then

$$A = \{\text{Ho Hei Chun, Lau Chi Kit and Yuen Chen Kit}\}$$

If another group B contains two students Ho Hei Chun and Lau Chi Kit, that is B={Ho Hei Chun, Lau Chi Kit}, then It is easy to see that:

$$B \subset A$$

Set intersection

$A \cap B$: intersection of two sets A and B (contain all elements in both A and B). That is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Example: If $A = \left\{\sqrt{2}, \pi\right\}$ and $B = \left\{\sqrt{2}, 1, 0\right\}$, then $A \cap B$ =

Venn Diagram for the set intersection

$\emptyset$:

empty set (a set with no element).

For example, if $A = \{1, 2\}$ and $B = \{3, 4\}$, then $A \cap B = \emptyset$.

Set union

$A \cup B$:
the union of two sets A and B (contain all elements in A or in B)

Venn Diagram

Example: If A is the set of odd positive integers and B is the set of even positive integers, then $A \cup B = \mathbb{N}$.

Absolute values

Definition:

If $a \in \mathbb{R}$, we define $|a|$, called the absolute value of a, by

$$|a| = \begin{cases} a, & \text{if } a \geqslant 0 \\ -a, & \text{if } a < 0 \end{cases}$$

– Real line representation of the absolute value $|a|$: the distance between a and 0.

– For instance, $|3| = 3$, $|-4| = 4$, $|8 - 17|$ = $|-9|$ = 9.

– The notation $|\cdot|$ can also be used to denote a function, such as $f(x) = |x|$

Solve inequalities with quadratics

Two basic methods: (1) Factorization; (2) Graphic approach
Example: $x^2 + 2x - 1 > 2$.

Example: $-x^2 + 5x + 14 > 0$.

Polynomials (polynomial functions)

If a function $f : \mathbb{R} \to \mathbb{R}$ has this form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where a_0, a_1, $\cdots$, a_n are real numbers with $a_n \neq 0$, then we call f is a polynomial function with degree n.

Examples of polynomials:

- $f(x) = 3x^2 + 6x + 7$ (degree 2 polynomial)
- $f(x) = 9x^3 + 7$ (degree 3 polynomial)

The polynomials are well defined for all $x \in \mathbb{R}$. (The domain of a polynomial is $\mathbb{R}$)

Rational Functions

The quotient of two polynomials are called a rational function

$$f(x) = \frac{p_n(x)}{q_m(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m}$$

– $p_n(x)$: numerator with degree n
– $q_m(x)$: denominator with degree m

The rational function f is defined for all values of x for which the denominator $q_m(x) \neq 0$.

Basic concepts of functions

Definition:

A function $f : A \to B$ is a rule that assigns a value to each $x \in A$ (in the domain) **a uniquely determined** element $y \in B$ (in the range).

In the high school, we may be familiar with this $y = x^2$, then this rule (taking the square) assigns to $x = -1$ a y value $y = 1$. Now we can rewrite a function as $f(x) = x^2$ and $f(-1) = 1$.

Definition:

A function $f : A \to B$ is a rule that assigns a value to each $x \in A$ (in the domain) a uniquely determined element $y \in B$ (in the range).

- Notation of a rule (the function): $y = f(x)$.
- A: domain of the function f, denoted by $Dom(f)$.
- The set of all elements in B that can occur as values of f is called the range of f, denoted by $Range(f)$. That is, $Range(f)$ =

$$\{y : y \in B \text{ and } y = f(x) \text{ for some } x \in A\}.$$

- x: independent variable of the function.
- y: dependent variable of the function.

Graph of a function (for any x, there is a unique y)

Example: $y = x^2 - 2x + 1$ is a function of x since for each x in the domain, there is a unique y. However, $y^2 = x$ is not a function of x since more than one y for some x.

Vertical line test:
The graph of a function can only intersect with any vertical line at most one point.

Find the Largest Possible Domain ($\frac{\text{Numerator}}{\text{Denominator}}$ and $\sqrt{\ }$)

Example: Find the largest possible domain for $f(x) = \frac{2}{x-3}$.
Solution: This function is not defined when $x - 3 = 0$, that is, the function is defined only when $x \neq 3$. Hence, $Dom(f) = \{x : x \neq 3\}$ or $Dom(f) = \mathbb{R}\backslash\{3\}$.
Note that the range Range(f) =$\{y : y \neq 0\}$.

Remark: set minus $\backslash$: $B\backslash A$=$\{x : x \in B$ but $x \notin A\}$.

Example: Find the largest possible domain and range for $y = g(x) = \sqrt{6-2x}$
Solution: We need $6 - 2x \geqslant 0$ for the square root "$\sqrt{\ }$" to be defined. Then $6 - 2x \geqslant 0$ $\iff$ $x \leqslant 3$. So Dom(g) = $(-\infty, 3]$. Note that the range Range(g) =$[0, \infty)$.

Example: Find the domain of the function $f(x) = \dfrac{x^2 + x - 2}{x^2 + 5x - 6}$.
Solutions: We need the denominator $x^2 + 5x - 6 \neq 0$. Then

$$(x+6)(x-1) \neq 0 \implies x \neq -6 \text{ and } x \neq 1.$$

The domain is

$$\{x : x \neq -6 \text{ and } x \neq 1\}$$

Operations on Functions

If f and g are given functions, then their sum, difference, product and quotient are the functions defined respectively by

$$(f+g)(x) = f(x) + g(x) \quad (f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Remark:

Domains of $f+g$, $f-g$ and fg are all equal to $Dom(f) \cap Dom(g)$, except that the domain of $\frac{f}{g}$ is given by $\{x : x \in Dom(f) \cap Dom(g) \text{ and } g(x) \neq 0\}$.

Example:

Suppose $f(x) = \dfrac{x-1}{x-3}$ and $g(x) = \sqrt{x}$. Then $Dom(f) = \mathbb{R}\backslash\{3\}$ and $Dom(g) = \{x : x \geqslant 0\}$. Find $f+g$ and gf and their domains.

Solution: The addition $(f+g)(x) = \dfrac{x-1}{x-3} + \sqrt{x}$ and its domain $Dom(f+g)$=$Dom(f) \cap Dom(g)$=$\{x : x \geqslant 0 \text{ and } x \neq 3\}$. The quotient

$$gf(x) = \frac{x-1}{x-3}\sqrt{x}.$$

The domain

$$Dom(gf) = Dom(f) \cap Dom(g) = \{x : x \geqslant 0 \text{ and } x \neq 3\}$$

Composite function

Definition of Composite function

If f and g are functions with domains $Dom(f)$ and $Dom(g)$, then the composite function $g \circ f$ is defined by

$$(g \circ f)(x) = g(f(x))$$

with the domain $Dom(g \circ f) =$

$$\{x : x \in Dom(f) \text{ and } f(x) \in Dom(g)\}$$

Remark: Be careful with the notations of composite and product of two functions: (a) fg is the product of two functions; (b) $f \circ g$ is the composite of two functions.

Example: Suppose the functions $f(x) = \sqrt{x-1}$ and $g(x) = \dfrac{1}{x}$, find expressions for $g \circ f$ and $f \circ g$.

Inverse Function

Definition of inverse function

Let $f : A \to \mathbb{R}$ be a function with $Dom(f) = A$ and

$$Range(f) = \{y : y = f(x) \text{ for some } x \in A\}.$$

Then an inverse function of f is a function g such that $g : Range(f) \to A$ and

$$g(y) = x \text{ if and only if } f(x) = y$$

for every $y \in Range(f)$ and every $x \in Dom(f)$.

Notation: Normally, the inverse function of f is denoted by f^{-1}.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be the function which converts degrees Celsius to degrees Fahrenheit, i.e., $y = f(x) = \frac{9}{5}x + 32$, where $Dom(f) = [-273.15, \infty)$. Try to find its inverse.

Solution: Solving the equation $y = f(x)$ for x, we obtain

$$x = f^{-1}(y) = \frac{5}{9}(y - 32)$$

where f^{-1} is the inverse function converting degrees Fahrenheit to degrees Celsius.

One-to-one function

Not every function has its inverse. Only one-to-one function has inverse function.

Definition of One-to-one function

Let $f : A \to B$ be a given function. f is one-to-one if and only if for any x_1 and x_2 in A such that $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Horizontal line test:

The graph of a **one-to-one** function can only intersect with any horizontal line at most one point.

Example:

Consider the inverse functions for the following:

(a) Let f be the function defined by $f(x) = x^2$ with $Dom(f) = \mathbb{R}$.

(b) Let f_+ be the function defined by $f_+(x) = x^2$ with $Dom(f_+) = [0, \infty)$.

(c) Let f_- be the function defined by $f_-(x) = x^2$ with domain $Dom(f_-) = (-\infty, 0]$.

Solution: f is not one-to-one while f_+ and f_- are. There is no inverse for f. The inverse functions for f_+ and f_- exist.

A special type of one-to-one functions: strictly increasing/decreasing

(1) $f(x) = x^3$ defined for any $x \in \mathbb{R}$ is a strictly increasing function.
(2) $g(x) = -3x$ defined for $x \in \mathbb{R}$ is strictly decreasing.

These two functions are one-to-one and inverse function exists for each of them.

Remark: We will learn methods to determine whether or not a function is strictly increasing/decreasing later

How to find the inverse function?

The inverse function f^{-1} can be calculated by the following procedure:

(a) check whether $y = f(x)$ is one-to-one (at this stage, just state that the function is one-to-one if you are asked to fine its inverse)

(b) solve x in terms of y

Example: Find the inverse of $f(x) = 3 - 4x$.

Solution: Step 1: As $y = 3 - 4x$ is one-to-one, it has an inverse function f^{-1}.

Step 2: Solving the equality $y = 3 - 4x$ in terms of x, we have $x = \dfrac{3-y}{4}$. Therefore,

$$x = f^{-1}(y) = \frac{3-y}{4}.$$

Example: Find the inverse function g^{-1} for $g(x) = -(x-1)^2$ for $x \geqslant 1$.

Example: Find the inverse function h^{-1} for $h(x) = -(x-1)^2$ for $x \leqslant 1$.