Intertemporal Expenditure Minimization

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We previously solved for the unconstrained household's savings and borrowing problem: unconstrained problem. And also the constrained optimization problem with asset choice.

Utility Maximization over Consumption in Two Periods

We solved the constrained utility maximization probem already.

- **Utility**: $U(c_1, c_2) = \log(c_1) + \beta \cdot \log(c_2)$
- **Budget Today and Tomorrow Together**: $c_1 \cdot (1+r) + c_2 = Z_1 \cdot (1+r) + Z_2$

We found the indirect utility given optimal choices:

```
clear all
% previous solution, indirect uitlity
U_at_c_opti = 0.75563;
c1_opti = 1.5288;
c2_opti = 1.4447;
% parameters
beta = 0.90;
z1 = 1;
z2 = 2;
r = 0.05;
```

The Expenditure Minimization Problem

We can represent the budget constraint and utility function (objective function) graphically. When we plug the optimal choices back into the utility function, we have the ***indirect utility***.

- **Indirect Utility function**: $V(c_1^*(r, Z_1, Z_2), c_2^*(r, Z_1, Z_2)) = V(r, Z_1, Z_2)$

Note that the indirect utility is a function of the price *r* that households face, and the resources the have available--their income--Z_1, Z_2. We can also write:

- $V(r, Z_1, Z_2) = \max_{c_1, c_2} U(c_1, c_2; r, Z_1, Z_2)$

Similar to the firm's profit maximization and cost minimization problems, which gave us similar optimality conditions, we can solve the household's expenditure minimization problem given V^*, and the optimal choices will be the same choices that gave us V^* initially. Specifically:

$$\min_{c1, c2} (c_2 + (1+r)c_1)$$

- such that:

$$\log(c_1) + \beta \log(c_2) = V^*(r, Z_1, Z_2)$$

First Order Conditions of the Constrained Consumption Problem

Note again we already know the solution of this problem from: unconstrained problem. What we are doing here is to resolve the problem, but now directly for c_1 and c_2, rather than *b*. But the results are the same because once you know *b* you know the consumption choices from the budget, and vice-versa. The solution method here is more complicated because we went from an one-choice problem in unconstrained problem to a three choice problem below. But the solution here is more general, allowing us to have addition constraints that can not be easily plugged directly into the utility function.

To solve the problem, we write down the Lagrangian, and solve a problem with three choices.

- $\mathscr{L} = c_2 + (1+r)c_1 - \mu\big(\log(c_1) + \beta \log(c_2) - V^*(r)\big)$

We have three partial derivatives of the lagrangian, and at the optimal choices, these are true:

- $\frac{\partial \mathscr{L}}{\partial c_1} = 0$, then, $\frac{\mu}{c_1^*} = (1+r)$
- $\frac{\partial \mathscr{L}}{\partial c_2} = 0$, then, $\frac{\beta \cdot \mu}{c_2^*} = 1$
- $\frac{\partial \mathscr{L}}{\partial \mu} = 0$, then, $\log(c_1^*) + \beta \log(c_2^*) = V^*(r, Z_1, Z_2)$

Optimal Relative Allocations of Consumptions in the First and Second Periods

Bringing the firs two conditions together, we have:

- $\frac{\beta}{c_2^*} = \frac{1}{c_1^* \cdot (1+r)}$
- $\frac{c_1^*}{c_2^*} = \frac{1}{\beta \cdot (1+r)}$
- $c_1^* = \frac{1}{\beta \cdot (1+r)} \cdot c_2^*$

This is the same as for the constrained utility maximization problem: constrained utility maximization probem.

Optimal Expenditure Minimization Consumption Choices

Using the third first order condition, and the optimal consumption ratio, we have:

- $\log(\frac{1}{\beta \cdot (1+r)} \cdot c_2^*) + \beta \log(c_2^*) = V^*(r, Z_1, Z_2)$

- $c_2^* = \exp\left(\left(V^*(r, Z_1, Z_2) + \log(\beta \cdot (1+r))\right) \cdot \frac{1}{1+\beta}\right)$

Subsequently, we can obtain optimal expenditure minimization c_1 and *b*.

The solutions here are **Hicksian**, these are the dual version of the Marshallian problem from: constrained utility maximization probem

Compuational Solution

Solving the problem with the same parameters as before given V^*, we will get the same solutions that we got above:

```
syms c1 c2 lambda
% The Lagrangian given U_at_c_opti found earlier
lagrangian = (c2 + (1+r)*c1 - lambda*( log(c1) + beta*log(c2) - U_at_c_opti));
% Derivatives
d_lagrangian_c1 = diff(lagrangian, c1);
d_lagrangian_c2 = diff(lagrangian, c2);
d_lagrangian_mu = diff(lagrangian, lambda);
GRADIENTmin = [d_lagrangian_c1; d_lagrangian_c2; d_lagrangian_mu]
```

```
GRADIENTmin =
```

$$\begin{pmatrix} \frac{21}{20} - \frac{\lambda}{c_1} \\ 1 - \frac{9\,\lambda}{10\,c_2} \\ \frac{75563}{100000} - \frac{9\log(c_2)}{10} - \log(c_1) \end{pmatrix}$$

```
solu_min = solve(GRADIENTmin(1)==0, GRADIENTmin(2)==0, GRADIENTmin(3)==0, c1, c2, lambda, 'Real
soluMinC1 = double(solu_min.c1);
soluMinC2 = double(solu_min.c2);
soluMinLambda = double(solu_min.lambda);
disp(table(soluMinC1, soluMinC2, soluMinLambda));
```

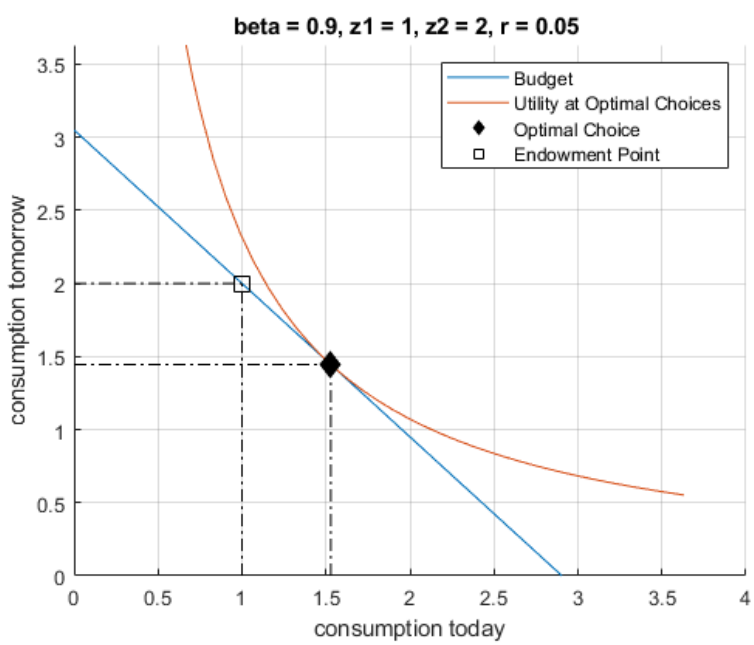
soluMinC1	soluMinC2	soluMinLambda
1.5288	1.4447	1.6053

Graphical Representation

At a particular r, Z_1, Z_2, we have a specific numerical value for V^*. We have different bundles of c_1, c_2 that can all achieve this particular utility level V^*. We can draw these and see visually where the optimal choices are. So we have two equations that we can draw:

- Budget: $(1)c_2 + (1+r)c_1 \leq Z_1(1+r) + Z_2$

- Indifference at V^*: $V^* = \log(c_1) + \beta \log(c_2)$, which is: $c_2 = \exp\left(\frac{V^* - \log(c_1)}{\beta}\right)$

Think of c_1 as the x-axis variable, and c_2 as the y-axis variable, we can plot them together

```
% Numbers defined before, and U_at_c_opti found before
syms c1
% The Budget Line
f_budget = z1*(1+r) + z2 - (1+r)*c1;
% Indifference at V*
f_indiff = exp((U_at_c_opti-log(c1))/(beta));
% Graph
figure();
hold on;
% Main Lines
fplot(f_budget, [0, (z1 + z2/(1+r))*1.25]);
fplot(f_indiff, [0, (z1 + z2/(1+r))*1.25]);
% Endowment Point
scatter(c1_opti, c2_opti, 100, 'k', 'filled', 'd');
plot(linspace(0,c1_opti,10),ones(10,1) * c2_opti, 'k-.', 'HandleVisibility','off');
plot(ones(10,1) * c1_opti, linspace(0,c2_opti,10), 'k-.', 'HandleVisibility','off');
% Optimal Choices Point
scatter(z1, z2, 100, 'k', 's');
plot(linspace(0,z1,10),ones(10,1) * z2, 'k-.', 'HandleVisibility','off');
plot(ones(10,1) * z1, linspace(0,z2,10), 'k-.', 'HandleVisibility','off');
% Labeling
ylim([0, (z1 + z2/(1+r))*1.25])
title(['beta = ' num2str(beta) ', z1 = ' num2str(z1) ', z2 = ' num2str(z2) ', r = ' num2str(r)
xlabel('consumption today');
ylabel('consumption tomorrow');
legend({'Budget', 'Utility at Optimal Choices', 'Optimal Choice', 'Endowment Point'})
grid on;
```