Borrowing Constrained Firm's Profit Maximization Problem with Cobb Douglas Production Function

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In this problem, we solve the constrained firm's profit maximization problem with decreasing returns to scale. This continues from the unconstrained profit maximization problem from Firm's Profit Maximization Problem with Cobb Douglas Production Function (Decreasing Returns to Scale).

Firm and Capital and Labor

The problem is the same as before, the profit maximization problem is:

- $\max_{K,L} (p \cdot A \cdot K^{\alpha} \cdot L^{\beta} - r \cdot K - w \cdot L)$

The constraint is such that the firm can not borrow more than $\bar{K}$

- $K \leq \bar{K}$

To find optimal choices, we will assume that $\alpha + \beta < 1$

Lagrangian and First Order Conditions

$$\mathcal{L} = (p \cdot A \cdot K^{\alpha} \cdot L^{\beta} - r \cdot K - w \cdot L) - \lambda(K - \bar{K})$$

- $\frac{\partial \mathcal{L}}{\partial K} : \alpha \cdot p \cdot A \cdot K^{\alpha - 1} \cdot L^{\beta} - r = \lambda$
- $\frac{\partial \mathcal{L}}{\partial L} : \beta \cdot p \cdot A \cdot K^{\alpha} \cdot L^{\beta - 1} - w = \lambda$
- $\lambda(K - \bar{K}) = 0$
- $\lambda \geq 0$
- $K < \bar{K}$

If the optimal unconstrained capital choice is less than $\bar{K}$, then the inequality constraint can not impact optimal choices. The inequality constraint should disappear from the lagrangian, which is achieved with $\lambda = 0$.

If the optimal constrained capital choice would have been greater than $\bar{K}$, then the constraint is binding, in the sense that the $\bar{K}$ bound will limit the firm from borrowing optimally. The firm will borrow as much as it can so that $K = \bar{K}$. Since $K - \bar{K} = 0$, $\lambda \geq 0$. Note that the larger λ is, the greater the gap between marginal productivity and marginal cost.

Solving for Different Cases

When faced with inequality constrained problems, we have to solve the problem in different possible cases in which different combinations of the inequality constraints present would be binding. Then we compare across cases to find the case that maximized the objective.

Our problem here is simpler, we only have two cases:

1. The inequality constraint does not bind, which means we can use the optimal unconstrained capital and labor choices we found previously in Firm's Profit Maximization Problem with Cobb Douglas Production Function (Decreasing Returns to Scale).
2. The inequality constraint does bind for capital, which means we solve for optimal labor choice given fixed level of capital. This is exactly what we did in Firm's Profit Maximization Problem and Optimal Capital Choice, except there we solve for optimal capital fixing labor. Now we need to solve for optimal labor fixing capital at the constraint.

Solution

With *con* denoting constrained, *unc* denoting unconstrained, we have:

- $K^{\text{con}} = \begin{cases} K^{\text{unc}}, \text{ if } K^{\text{unc}} < \bar{K} \\ \bar{K}, \text{ otherwise} \end{cases}$
- $L^{\text{con}} = \begin{cases} L^{\text{unc}}, \text{ if } K^{\text{unc}} < \bar{K} \\ \arg\max\limits_{L} \Pi(\bar{K}, L; r, w), \text{ otherwise} \end{cases}$