Borrowing Constrained Utility Maximization with Borrow/Savings Choices

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We previously solved for the unconstrained household's savings and borrowing problem: unconstrained problem.

What is the constrained borrowing problem?

Imagine if endowment in the first period is Z_1, but now endowment in the second period is Z_2.

- **Utility**: $U(c_{today}, c_{tomrrow}) = \log(c_{today}) + \beta \cdot \log(c_{tomorrow})$
- **Budget Today**: $c_{today} + b = Z_1$
- **Budget Tomorrow**: $c_{tomorrow} = b \cdot (1 + r) + Z_2$

Now *b* can be positive or negative. Generally, if you go to a bank, they let you save however much you want to deposit there, but you don't usually get to borrow any amount you would like to borrow. Remember we discussed before there is the natural borrowing constriant in this model, which restricts borrowing what what we can repay in the worst state of shock tomorrow (there is only one state in this case), so borrowing is already naturally constrained by the household's optimization problem.

If the borrowing constraint of the bank is lower than the natural borrowing constraint, it is irrelevant, but if it is tighter than the natural borrowing constraint, then it becomes relevant.

Inequality Constraint

We can formulate the problem above as having 1 savings choice that is constrained.

The objective function is :

- generally: $\max_b f(b)$
- specifically: $\max_b \log(Z_1 - b) + \beta \cdot \log(Z_2 + b \cdot (1 + r))$

And the constraint is:

- $b \geq \overline{b}$

$\overline{b}$ is the borrowing limit. Note that because *b* is negative when we are borrowing, so a higher upper bound on how much you can borrow is represented by a more negative $\overline{b}$.

We can think of the inequality constraint more generally as a function:

- $g(b) \leq q$

Where g is some function of b, and q is just a number, note that we want to write this as the function of the choice is less than or equal to something. For our example here, you can think of function g as: $g(b) = -b$ and $q = -\overline{b}$; or $g(b) = \overline{b} - b$, $q = 0$. They of course are the same:

- $\overline{b} - b \leq 0$

Lagrangian with Inequality Constraint

When we write the lagrangian, we have to be careful about the signs, writing the inequality constraint as we do above, we will do the "double negative" as we did with equality constraint when we add in the lagrange multiplier term, the lagrangian is:

- $\mathscr{L} = \left\{ \log(Z_1 - b) + \beta \cdot \log(Z_2 + b \cdot (1 + r)) \right\} - \lambda \cdot (\overline{b} - b - 0)$

For inequality constraint, we follow SB and use λ for the lagrange multiplier.

Derivative with Respect to b

The key thing to understand about inequality constraint is that the first order condition that we had from the unconstrained problem no longer holds. Specifically, the uncontrained problem's derivative with respect to b set equal to 0 would be:

- $\frac{1}{Z_1 - b} = \beta \frac{1 + r}{Z_2 + b(1 + r)}$

Which means the Marginal Utility of Consumption today must be equal to the Marginal Utility of Consumption tomorrow. The household will use saving and borrowing as a mechanism to smooth their consumption given their endowment in each period, the interest rate, and discount factor. But now, with the inequlaity constraint, the derivative of the lagrangian with respect to b set equal to 0 is:

- $\frac{1}{Z_1 - b} = \beta \frac{1 + r}{Z_2 + b(1 + r)} + \lambda$

We gained an extra λ term. Given that we can not adjust (borrow) b freely now, we might have too little consumption today, leading to high marginal utility of consumption today, and too much consumption tomorrow (due to higher endowment), leading to lower marginal utility of consumption tomorrow. Without constraint, we would have chosen, $b^{*,\text{unconstrained}}$: borrowing today to reduce marginal utilty today and increase marginal utility tomorrow until consumption is smoothed over the two periods. With constraint, we would chose, $b^{*,\text{constrained}}$. If $b^{*,\text{unconstrained}} \geq \overline{b}$, then the constraint does not matter, and $\lambda = 0$, if $b^{*,\text{unconstrained}} < \overline{b}$, then the constraint does matter, and $\lambda > 0$ in this case to account for the marginal utility cost of the borrowing constraint.

First Order Conditions with Inequality Constraint

Following our discussion above, what are the conditionals that the optimal choice must satisfy in the prescence of inequality constraint?

The general problem here is:

- $\max_{b} f(b)$
- such that: $g(b) \leq q$

With Lagrangian:

- $\mathscr{L} = f(b) - \lambda \cdot (g(b) - g)$

Suppose that *f* and *g* functions are both continuously differentiable, and b^* maximizes *f* given the constraint, then there exists λ^*, such that:

1. $\frac{\partial \mathscr{L}}{\partial b}(b^*, \lambda^*) = 0$
2. $\lambda^* \cdot [g(b^*) - q] = 0$
3. $\lambda^* \geq 0$
4. $g(b^*) \leq q$

When the constraint does not bind, $\lambda^* = 0$, satisfying the second and third conditions, and the fourth condition is a strict inequality, and the first condition's derivative is the same as the one in the unconstrained problem. When the constraint does bind, the fourth condition is an equality constraint, λ is a postive number as in the example above.

Solving the Problem

How do we solve this problem? Given that the problem here only has one choice, and given the concavity of log utility, and the linear constraints, we can solve the unconstrained problem first, if the optimal unconstrained choice is less than the constraint bound, then the optimal choice with be the $b^* = \overline{b}$, if the optimal unconstrained choice is greater than the constraint bound, then the $b^* = b^{*,\text{unconstrained}}$.

Our brute force method also works well in this case, we simply limit the grid of feasible *b* choices to be within the constraint set, and find the point along the grid where utility is the highest.

Matlab has a conveninent function that solves any constrained maximization problem, ***fmincon***, we will use it here. First, let's write our constraint like this:

- we had: $\overline{b} - b \leq 0$
- this is also: $[-1] \cdot [b] \leq [-\overline{b}]$
- we can think of this as: $A \cdot b \leq q$. The *A* matrix and *q* vector represent the set of linear constraints.

Define the parameters and the equations

```
clear all
% Parameters
beta_num = 0.95;
z1_num = 10;
z2_num = 20;
r_num = 1.05;
b_bar_num = -1; % borrow up to 1 dollar

% Write down the objective function, we will define it as a function handle, negative utility f
syms beta z1 z2 r
U_neg = @(b) -1*(log(z1 - b) + beta*log(z2 + b*(1+r)))
```

```
U_neg = function_handle with value:
    @(b)-1*(log(z1-b)+beta*log(z2+b*(1+r)))
```

```
% Constraint
A = [-1];
q = -b_bar_num;
```

Now call fminunc to solve

```
b0 = [0] % starting value to search for optimal choice
```

```
b0 = 0
```

```
U_neg_num = matlabFunction(subs(U_neg, {beta, z1, z2, r}, {beta_num, z1_num, z2_num, r_num}));
[b_opti,U_at_b_opti] = fmincon(U_neg_num, b0, A, q);
```

```
Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
b_opti
```

```
b_opti = -0.1313
```

```
U_at_b_opti
```

```
U_at_b_opti = -5.1487
```

Effects of Z_2 on optimal choices

How does optimal choice change if the household has more endowment tomorrow?

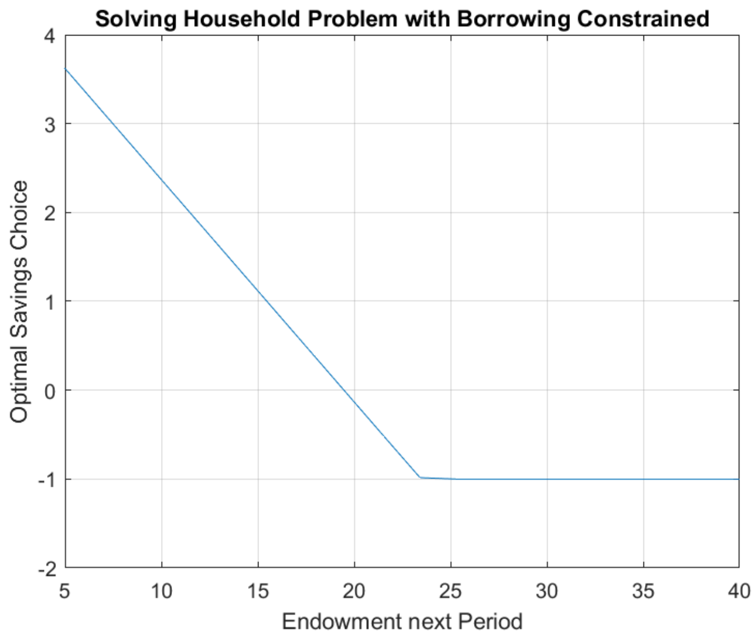
```
% Create a vector of Z2, so Z2 fector starts at the same value as Z1*0.5 going up to 4 times Z1
Z2_vec = linspace(z1_num*0.5, z1_num*4, 20)
```

```
Z2_vec = 1×20
    5.0000    6.8421    8.6842   10.5263   12.3684   14.2105   16.0526   17.8947 ···
```

```
% A vector to store optimal choices
b_opti_vec = zeros(size(Z2_vec));
```

```
% Solving for optimal choices as we change Z2
for i=1:1:length(Z2_vec)
    U_neg_num = matlabFunction(subs(U_neg, {beta, z1, z2, r}, {beta_num, z1_num, Z2_vec(i), r_n
    options = optimoptions('FMINCON','Display','off');
    [b_opti,U_at_b_opti] = fmincon(U_neg_num, b0, A, q, [], [], [], [], [], options);
    b_opti_vec(i) = b_opti;
end

% Plot Results
figure()
plot(Z2_vec, b_opti_vec)
grid on;
ylim([-2 4]);
title('Solving Household Problem with Borrowing Constrained')
ylabel('Optimal Savings Choice')
xlabel('Endowment next Period')
```



Effects of *r* on optimal choices

How does optimal choice change if the household has more endowment tomorrow?

```
% Vector of interest rates
r_vec = linspace(0.5, 1.50, 50);
% A vector to store optimal choices
b_opti_vec = zeros(size(r_vec));
% Solving for optimal choices as we change Z2
for i=1:1:length(r_vec)
    U_neg_num = matlabFunction(subs(U_neg, {beta, z1, z2, r}, {beta_num, z1_num, z2_num, r_vec(
    options = optimoptions('FMINCON','Display','off');
    [b_opti,U_at_b_opti] = fmincon(U_neg_num, b0, A, q, [], [], [], [], [], options);
    b_opti_vec(i) = b_opti;
end
```

```
% Plot Results
figure()
plot(r_vec, b_opti_vec)
ylim([-1.5 1]);
grid on;
title('Solving Household Problem with Borrowing Constrained')
ylabel('Optimal Savings Choice')
xlabel('interest rate (1+r)')
```