Continuity and Differentiability

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In the real world, households and firms general consume and use discrete units of goods. Households can buy *N* apples, and firms can hire *M* numbers of workers. The world is full of discreteness. To derive mathmatical expressions that summarize the aggregate behavior of economic agents, we generally approximate our discrete world with continuous functions.

Definition Continuous

Visually, "a function is **continuous** if its graph has no breaks" (SB). "The graph of a function cannot have a tangent line at a point of discontinuity"

This function, for example is not continuous. Note that we can not use both ≤ and ≥, otherwise this would no longer be a function:

- $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$

This is the simplest continuous function

- $f(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{cases}$

The more formal definition of continuity is based on convergence of sequences, which you do not need to remember (SB P32):

- Continuous at a **point**: A function $f : D \to \mathbf{R}^1$ is **continuous** at $x_0 \in D$ if for *any* sequence $\{x_n\}$ which converges to x_0 in *D*, $f(x_n)$ converges to $f(x_0)$.
- Continuous on a **set**: A function is **continuous on a set** $U \in D$ if it is continuous at every $x \in U$.
- **Whole function** is continuous: Finally, we say that a function is **continuous** if it is continuous at every point in its domain.

Often, if you write down an economic model where functions have discontinuity, you might need to rely on brute-force type solution method to solve for household and firm maximization problems, and can not take advantage of derivatives.

Definition Continuously Differentiable

- As stated before, if the following limit exists, then the function *f* is **differentiable** at x_0:

  $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- *f* is a **differentiable function**, if "it is differentiable at every point x_0 in its domain *D*" (SB P29), which means "its derivative $f'(x)$ is another function of *x*" (SB P32):

- If $f'(x)$ is a continuous function of *x*, we say that the original function *f* is **continuously differentiable**, or C^1

The 2 period savings problem involved a utility maximization equation that was continuous over the domain, and that was differentiable everywhere over the domain. The derivative we obtained was also continuous. Hence we were dealing with a continuously differentiable function. With that function, we were able to easily find the optimal savings choice