Exponentiation and Compounding Interest Rate

back to Fan's Intro Math for Econ, Matlab Examples, or MEconTools Repositories

See also: Exponential Function and Log Function.

Exponential Function

- **Exopential Function:** Functions where the variable x appears as an *exponent*: a^x
- a is the base of Exponential function.

Remember that

- $a^0 = 1$
- $a^{\frac{1}{2}} = \sqrt{a}$
- if $a^b = c$, we can also write, $a = c^{\frac{1}{b}}$, for example, $2^3 = 8$, and $2 = 8^{\frac{1}{3}}$
- $a^{-b} = \dfrac{1}{a^b}$
- $x^a \cdot x^b = x^{a+b}$
- $x^{a \cdot b} = (x^a)^b$

Exponential Function Graphs?

- Note that the domain of exponential function includes positive and negative x, and the exponential function will always be positive.
- If base is below 1, then the curve is monotonically downward sloping
- If base is above 1, then the curve is monotonically upwards sloping
- If base is above 1, higher base leads to steeper curvature.

```
syms x
a1 = 0.5;
f_a1 = a1^(x);
a2 = 1.5;
f_a2 = a2^(x);
a3 = 2.5;
f_a3 = a3^(x);
figure();
hold on;
fplot(f_a1, [-2, 2]);
fplot(f_a2, [-2, 2]);
fplot(f_a3, [-2, 2]);
line([0,0],ylim);
line(xlim, [0,0]);
title('Exponential Function Graph with different bases')
legend(['base=',num2str(a1)], ['base=',num2str(a2)],['base=',num2str(a3)]);
```

```
grid on;
```


Exponential Function Graph with different bases
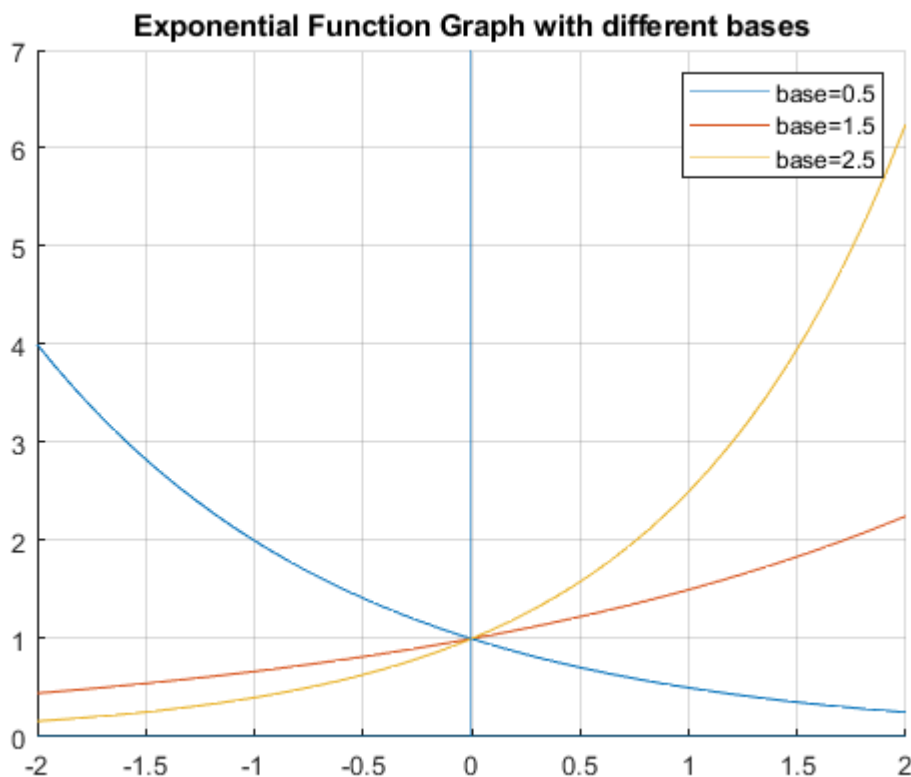
Infinitely Compounding Interest rate

with 100 percent interest rate (APR), if we compound N times within a year, interest we pay at the end of the year is

- $(1 + \frac{1}{N})^N - 1$

Suppose $N = 5$ (You can also think of this as a loan with interest rate of 20% for every 73 days), then we pay 159% interest rate by the end of the year.

```
r = 1.05;
N = 5;
(1 + r/N)^N - 1
```

```
ans = 1.5937
```

What if we do more and more compounding, if we say interest rate compounds 10, 50, 100 times over the year, what happens? With APR at 100%, the total interest rate you pay at the end of the year does not go to infinity, rather, it converges to this special number e, Euler's number, 2.7182818. This means if every second the interest rate is compounding, with an APR of 100%, you end up paying 272% of what you borrowed by the end of the year, which is 172% interest rate.

- $$\lim_{N \to \inf} (1 + \frac{1}{N})^N = e \approx 2.7182818$$

We can visualize this limit below

```
r = 1;
syms N
f_compoundR = (1 + r/N)^N;
figure();
fplot(f_compoundR, [1,100])
ylabel({'Principle and Interests at End of Year Given 100% APR' 'for 1 dollar Borrowed, given i
xlabel('Number of Evenly-divided Times to Compound Interest Rate in a Year')
grid on;
grid minor
```
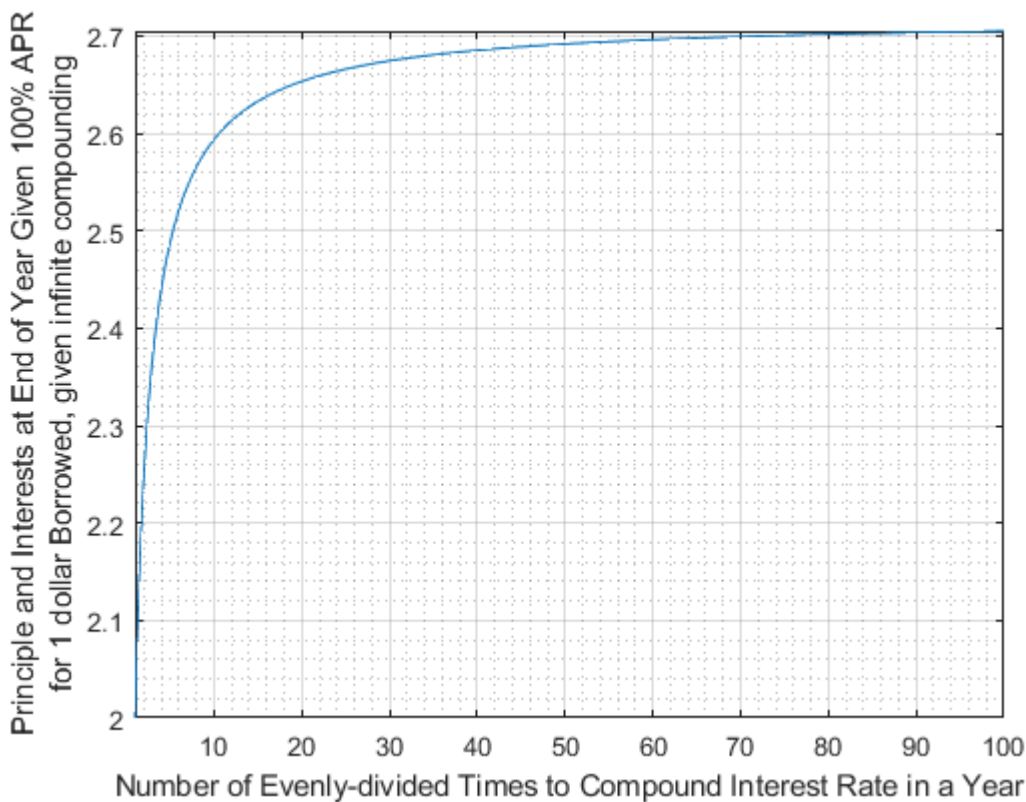


```
double(subs(f_compoundR,[1,2,3,4,5,6,7,8,9,10]))
```

ans = 1×10
   2.0000   2.2500   2.3704   2.4414   2.4883   2.5216   2.5465   2.5658 ···

Infinitely compounding Interest rate, different r (APR r)

Given:

- $$\lim_{N \to \inf} (1 + \frac{1}{N})^N = e \approx 2.7182818$$

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What is

- $$\lim_{N \to \inf} (1 + \frac{r}{N})^N ?$$

We can replace N by $N = r \cdot M$

- $$\lim_{N \to \inf} (1 + \frac{r}{N})^N = \lim_{M \to \inf} (1 + \frac{r}{r \cdot M})^{r \cdot M} = \left(\lim_{M \to \inf} (1 + \frac{1}{M})^M \right)^r = e^r$$

This gives the base e exponential function a financial interpretation.

```
syms x
f_e = exp(x);
figure();
hold on;
fplot(f_e, [-3, 3]);
line([0,0],ylim);
line(xlim, [0,0]);
title('Exponential Function Graph with base e')
xlabel('r = interest rate');
ylabel({'Principle and Interests at End of Year' 'for 1 dollar Borrowed, given infinite compour
grid on;
```