

Derivative Definition and Rules

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Linear and Non-linear Functions

Linear functions have a constant slope, but what is the rate of change for a non-linear function as we shift along its domain?

Definition

(SB) Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$.

The **derivative** of f at x_0 is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$.

There are some common ways of denoting derivative of function f at x_0 :

- $f'(x_0)$
- $\frac{df}{dx}(x_0)$
- $\frac{dy}{dx}(x_0)$
- $f_x(x_0)$: this is popular in economics

We write this analytically as:

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists, then the function f is **differentiable** at x_0 .

We will use this formula to derive first order Taylor approximation. And this will also appear when we derive the formula for point elasticity.

Derivative Rules--Constant Rule

given constant k ,

- $f(x) = a \cdot x$
- $f'(x_0) = a$

```
syms x a
f(x, a) = a*x
```

```
f(x, a) = a x
```

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a) = a$$

Derivative Rules--Power Rule (Polynomial Rule)

(SB) For any positive integer k (or real number k), the derivative of $f(x) = x^k$ at x_0 is:

- $f(x) = x^k$
- $f'(x_0) = k \cdot x_0^{k-1}$

```
syms x a k
f(x, a, k) = a*x^k
```

$$f(x, a, k) = a x^k$$

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a, k) = a k x^{k-1}$$

Derivative Rules--Chain Rule

- $f(x) = p(q(x))$
- $f'(x_0) = p'(q(x_0)) \cdot q'(x_0)$

```
syms x a k
f(x, a, k) = (a*x)^k
```

$$f(x, a, k) = (a x)^k$$

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a, k) = a k (a x)^{k-1}$$

Derivative Rules--Sum (and difference) Rule

Given functions p and q that are differentiable at x , then:

- $f(x) = p(x) + q(x)$
- $f'(x) = p'(x) + q'(x)$

```
syms x a b c d
f(x, a, b, c, d) = a*x^b + c*x^d
```

$$f(x, a, b, c, d) = a x^b + c x^d$$

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a, b, c, d) = a b x^{b-1} + c d x^{d-1}$$

Derivative Rules--Product Rule

Given functions p and q that are differentiable at x , then:

- $f(x) = p(x) \cdot q(x)$
- $f'(x) = p'(x) \cdot q(x) + p(x) \cdot q'(x)$

```
syms x a b c d
f(x, a, b, c) = (a*x^b)*(c*x^d)
```

$$f(x, a, b, c) = a c x^b x^d$$

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a, b, c) = a b c x^d x^{b-1} + a c d x^b x^{d-1}$$

Derivative Rules--Quotient Rule

Given functions p and q that are differentiable at x , then:

- $f(x) = \frac{p(x)}{q(x)}$
- $f'(x) = \frac{p'(x) \cdot q(x) - p(x) \cdot q'(x)}{(q(x))^2}$

Note that the quotient rule is based on the product rule, because:

- $f(x) = \frac{p(x)}{q(x)} = p(x) \cdot \frac{1}{q(x)}$

So you can derive the quotient rule formula based on the product rule where the first term is $p(x)$ and the second term is $\frac{1}{q(x)}$.

```
syms x a b c d
f(x, a, b, c) = (a*x^b)/(c*x^d)
```

$$f(x, a, b, c) =$$

$$\frac{a x^b}{c x^d}$$

```
dfk = diff(f,x)
```

$$\text{dfk}(x, a, b, c) =$$

$$\frac{a b x^{b-1}}{c x^d} - \frac{a d x^b}{c x^{d+1}}$$

Derivative Rules--Exponential

We use exponential functions in economics a lot:

- $f(x) = \exp(a \cdot x)$
- $f'(x) = a \cdot \exp(a \cdot x)$

```
syms x a
f(x, a) = exp(a*x)
```

$$f(x, a) = e^{a \cdot x}$$

```
dfk = diff(f,x)
```

$$dfk(x, a) = a e^{a \cdot x}$$

This is a special case of any power function

- $f(x) = c^{a \cdot x}$
- $f'(x) = a \cdot (\log c) \cdot c^{a \cdot x}$

note that $\log(\exp(c)) = c$

```
syms x a c
f(x, a, c) = c^(a*x)
```

$$f(x, a, c) = c^{a \cdot x}$$

```
dfk = diff(f,x)
```

$$dfk(x, a, c) = a c^{a \cdot x} \log(c)$$

Derivative Rules--Log

We use Log functions in economics a lot:

- $f(x) = \log(a \cdot x)$
- $f'(x) = \frac{1}{x}$

note that the c cancels out.

```
syms x a
f(x, a) = log(a*x)
```

$$f(x, a) = \log(ax)$$

$$dfk = \text{diff}(f, x)$$

$$dfk(x, a) =$$

$$\frac{1}{x}$$