

# Continuity and Differentiability

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In the real world, households and firms general consume and use discrete units of goods. Households can buy  $N$  apples, and firms can hire  $M$  numbers of workers. The world is full of discreteness. To derive mathematical expressions that summarize the aggregate behavior of economic agents, we generally approximate our discrete world with continuous functions.

## Definition Continuous

Visually, "a function is **continuous** if its graph has no breaks" (SB). "The graph of a function cannot have a tangent line at a point of discontinuity"

This function, for example is not continuous. Note that we can not use both  $\leq$  and  $\geq$ , otherwise this would no longer be a function:

- $$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

This is the simplest continuous function

- $$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

The more formal definition of continuity is based on convergence of sequences, which you do not need to remember (SB P32):

- Continuous at a **point**: A function  $f : D \rightarrow \mathbf{R}^1$  is **continuous** at  $x_0 \in D$  if for *any* sequence  $\{x_n\}$  which converges to  $x_0$  in  $D$ ,  $f(x_n)$  converges to  $f(x_0)$ .
- Continuous on a **set**: A function is **continuous on a set**  $U \in D$  if it is continuous at every  $x \in U$ .
- **Whole function** is continuous: Finally, we say that a function is **continuous** if it is continuous at every point in its domain.

Often, if you write down an economic model where functions have discontinuity, you might need to rely on brute-force type solution method to solve for household and firm maximization problems, and can not take advantage of derivatives.

## Definition Continuously Differentiable

- As stated before, if the following limit exists, then the function  $f$  is **differentiable** at  $x_0$ :

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- $f$  is a **differentiable function**, if "it is differentiable at every point  $x_0$  in its domain  $D$ " (SB P29), which means "its derivative  $f'(x)$  is another function of  $x$ " (SB P32):
- If  $f'(x)$  is a continuous function of  $x$ , we say that the original function  $f$  is **continuously differentiable**, or  $C^1$

The 2 period savings problem involved a utility maximization equation that was continuous over the domain, and that was differentiable everywhere over the domain. The derivative we obtained was also continuous. Hence we were dealing with a continuously differentiable function. With that function, we were able to easily find the optimal savings choice