Exponential and Infinitely Compounding Interest Rate

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See also: Exponential Function and Log Function.

Exponential Function

- **Exopential Function:** Functions where the variable *x* appears as an *exponent:* a^x
- *a* is the base of Exponential function.

Remember that

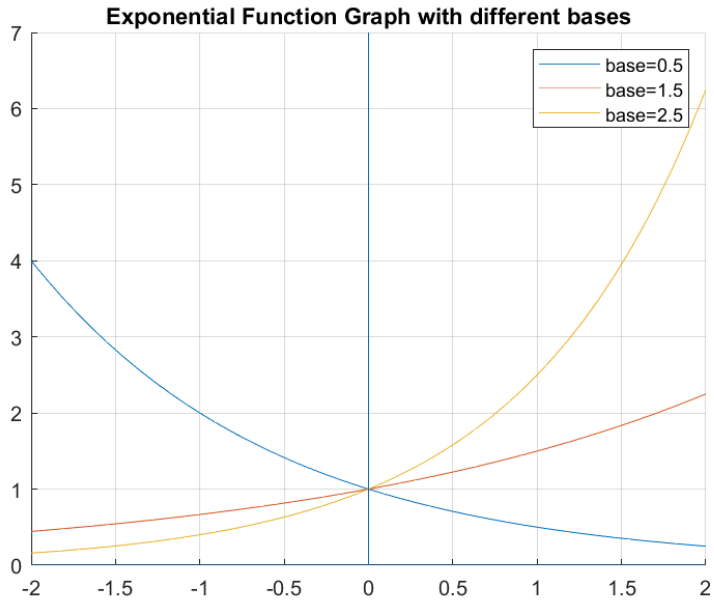
- $a^0 = 1$
- $a^{\frac{1}{2}} = \sqrt{a}$
- if $a^b = c$, we can also write, $a = c^{\frac{1}{b}}$, for example, $2^3 = 8$, and $2 = 8^{\frac{1}{3}}$
- $a^{-b} = \frac{1}{a^b}$
- $x^a \cdot x^b = x^{a+b}$
- $x^{a \cdot b} = (x^a)^b$

Exponential Function Graphs?

- Note that the domain of exponential function includes positive and negative *x*, and the exponential function will always be positive.
- If base is below 1, then the curve is monotonically downward sloping
- If base is above 1, then the curve is monotonically upwards sloping
- If base is above 1, higher base leads to steeper curvature.

```
syms x
a1 = 0.5;
f_a1 = a1^(x);
a2 = 1.5;
f_a2 = a2^(x);
a3 = 2.5;
f_a3 = a3^(x);
figure();
hold on;
fplot(f_a1, [-2, 2]);
fplot(f_a2, [-2, 2]);
fplot(f_a3, [-2, 2]);
line([0,0],ylim);
line(xlim, [0,0]);
title('Exponential Function Graph with different bases')
legend(['base=',num2str(a1)], ['base=',num2str(a2)],['base=',num2str(a3)]);
```

```
grid on;
```



Infinitely Compounding Interest rate

with 100 percent interest rate (APR), if we compound *N* times within a year, interest we pay at the end of the year is

- $(1 + \frac{1}{N})^N - 1$

Suppose $N = 5$ (You can also think of this as a loan with interest rate of 20% for every 73 days), then we pay 159% interest rate by the end of the year.

```
r = 1.05;
N = 5;
(1 + r/N)^N - 1
```

```
ans = 1.5937
```

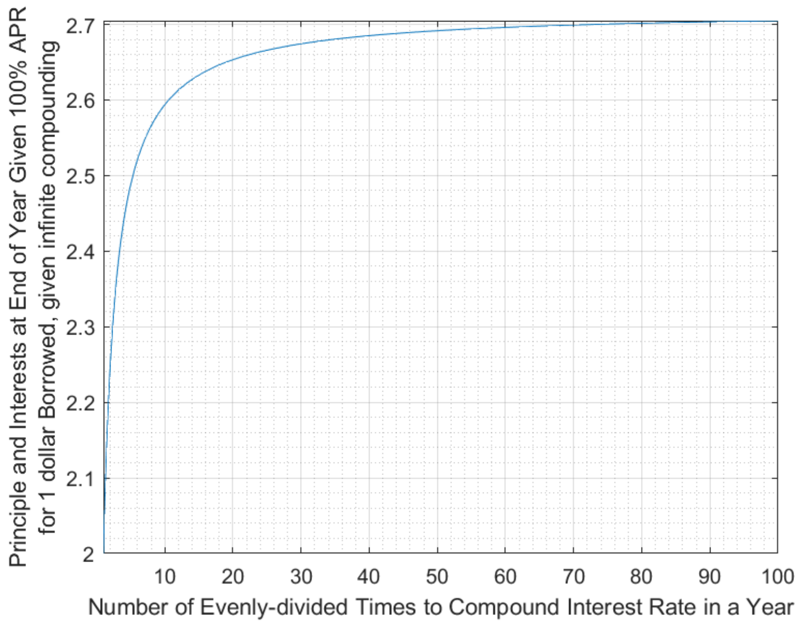
What if we do more and more compounding, if we say interest rate compounds 10, 50, 100 times over the year, what happens? With APR at 100%, the total interest rate you pay at the end of the year does not go to infinity, rather, it converges to this special number *e*, the Exponential number, 2.7182818, it is a magical number like π. This means if every second the interest rate is compounding, with an APR of 100%, you end up paying 272% of what you borrowed by the end of the year, which is 172% interest rate.

- $\lim_{N \to \inf} (1 + \frac{1}{N})^N = e \approx 2.7182818$

We can visualize this limit below

```
r = 1;
```

```
syms N
f_compoundR = (1 + r/N)^N;
figure();
fplot(f_compoundR, [1,100])
ylabel({'Principle and Interests at End of Year Given 100% APR' 'for 1 dollar Borrowed, given i
xlabel('Number of Evenly-divided Times to Compound Interest Rate in a Year')
grid on;
grid minor
```




```
double(subs(f_compoundR,[1,2,3,4,5,6,7,8,9,10]))
```

```
ans = 1×10
    2.0000    2.2500    2.3704    2.4414    2.4883    2.5216    2.5465    2.5658 ···
```

Infinitely compounding Interest rate, different *r* (APR *r*)

Given:

- $$\lim_{N \to \inf} (1 + \frac{1}{N})^N = e \approx 2.7182818$$

What is

- $$\lim_{N \to \inf} (1 + \frac{r}{N})^N ?$$

We can replace *N* by $N = r \cdot M$

- $$\lim_{N \to \inf} (1 + \frac{r}{N})^N = \lim_{M \to \inf} (1 + \frac{r}{r \cdot M})^{r \cdot M} = \left(\lim_{M \to \inf} (1 + \frac{1}{M})^M \right)^r = e^r$$

This gives the base *e* exponential function a financial interpretation.

```
syms x
f_e = exp(x);
figure();
hold on;
fplot(f_e, [-3, 3]);
line([0,0],ylim);
line(xlim, [0,0]);
title('Exponential Function Graph with base e')
xlabel('r = interest rate');
ylabel({'Principle and Interests at End of Year Given 100% APR' 'for 1 dollar Borrowed, given i
grid on;
```