

# Natural Logarithm and Exponential

back to [Fan's Intro Math for Econ](#), [Matlab Examples](#), or [Dynamic Asset Repositories](#)

We use log for log utility in our [household maximization problems](#), and we use exponential functions with other bases for [production functions](#).

See also: [Exponential and Infinitely Compounding Interest Rate](#).

## Log and Exponential

If the natural log of  $x$  is  $y$  (in economics we generally just write  $\ln$  and  $\log$  interchangeably, be careful though, google thinks function  $\log$  means log with base 10, matlab thinks function  $\log$  means base  $e$ , you will get different numbers typing in  $\log(10)$  in google and matlab).

- $\ln(x) = y$

then

- $e^y = x$

we can write:

- $e^x = \exp(x)$ , writing  $\exp(x)$  is a little easier to read, means just  $e$  to the power of  $x$

because of this:

- since  $e^0 = 1$ ,  $\log(1) = 0$
- since  $e^1 \approx 2.718$ ,  $\log(2.718) \approx 1$

The natural log is just the inverse of the exponential function,

We use log to linearize exponential functions, which allows us to do regressions afterwards for example.

## Log Rules

Suppose we have:  $\log\left(\frac{\exp(A + \epsilon) \cdot a^\alpha \cdot b^\beta}{c^\theta \cdot d^\phi}\right)$

This looks complicated, but because there is log, we can take the equation apart:

$$\log\left(\frac{\exp(A + \epsilon) \cdot a^\alpha \cdot b^\beta}{c^\theta \cdot d^\phi}\right) = (A + \epsilon) + \alpha \cdot \log(a) + \beta \cdot \log(b) - \theta \cdot \log(c) - \phi \cdot \log(d)$$

Generally (:

- $\log(\exp(A)) = A$
- $\log(x^\alpha) = \alpha \cdot \log(x)$
- $\log(x \cdot y) = \log(x) + \log(y)$

- $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

## Why does $\log(x \cdot y) = \log(x) + \log(y)$ ?

Why is the log of the product of two numbers the same as the sum of the log of each of the two numbers?

Intuitively, because we can write  $x \cdot y$  as the exponential of a sum: when  $e^a \cdot e^b$ , even though it's multiplication, it is also just  $e^{a+b}$ , the exponential of a sum.

- **Rule:**  $\log(x \cdot y) = \log(x) + \log(y)$

We can write separately what each term equals to as:

1.  $\log(x \cdot y) = z$
2.  $\log(x) = z_x$
3.  $\log(y) = z_y$

By definition, for each of the three terms above:

1.  $x \cdot y = \exp(z)$
2.  $x = \exp(z_x)$
3.  $y = \exp(z_y)$

So:

- $\log(x \cdot y) = \log(\exp(z_x) \cdot \exp(z_y))$

Given that:  $e^a \cdot e^b = e^{a+b}$ , and  $\log(\exp(x)) = x$ :

- $\log(x \cdot y) = \log(\exp(z_x) \cdot \exp(z_y)) = \log(\exp(z_x + z_y)) = (z_x + z_y)$

Hence:

- $\log(x \cdot y) = z = (z_x + z_y) = \log(x) + \log(y)$

## Why does $\log(x^a) = a \cdot \log(x)$ ?

Why is the log of an exponential term equal to the power times the log of the base of the exponential?

Intuitively, because we can re-write any positive number as base e to the power of a coefficient:

We start with:

- $\log(x^a) = z$

Note that  $x$  must be positive, otherwise log of zero or negative numbers are undefined. Hence, let  $x = e^b$ , by shifting  $b$ ,  $e^b$  can be equal to any positive number  $x$ . Then we have:

- $\log((e^b)^a) = z$

Given that  $\log(x^a) = z$  can be rewritten as  $x^a = e^z$

1.  $\log(x^a) = z, x = e^b$

2.  $\log((e^b)^a) = z$

3.  $x^a = e^{b \cdot a} = e^z$

4.  $b \cdot a = z$

Having defined  $x = e^b$ , that means  $\log(x) = b$ . Hence  $b \cdot a = z$  means that:

- $a \cdot \log(x) = z = \log(x^a)$

## For Variables that Grow, Log difference is close to rate of change

Suppose that growth rate is  $x$  percent per year, after 5 years, the gdp will be:

- $Y_{1995} = Y_{1990} \cdot (1 + x)^5$

We can take log on both sides:

- $\ln(Y_{1995}) = \ln(Y_{1990}) + 5 \cdot \ln(1 + x)$

Which says that the difference in GDP between these two years divided by 5 is equal to the log of 1 plus the growth rate.

Approximately, for  $x$  small:

- $\frac{\ln(Y_{1995}) - \ln(Y_{1990})}{5} = \ln(1 + x) \approx x$

For example:

```
xVec = linspace(0,0.10,10);
log(1+ xVec)
```

```
ans = 1x10
      0      0.0110      0.0220      0.0328      0.0435      0.0541      0.0645      0.0749 ...
```

```
xVec
```

```
xVec = 1×10
      0      0.0111      0.0222      0.0333      0.0444      0.0556      0.0667      0.0778 ...
```

**Note:** This is a bad approximation if  $x$  is large. For example, we know that  $\ln(2.718) = \ln(1 + 1.718) \approx 1$  is almost exact. But the approximation here would have said  $\ln(1 + 1.718) \approx 1.718$ , which is very incorrect.