Definition of Differentials

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In economics papers, we often see these symbols: Δ, *d*, ∂

Δy and Δx are changes along the function graph: given some x_0:

- $\Delta y = f(x_0 + \Delta x) - f(x_0)$

$\mathrm{d}y$ and $\mathrm{d}x$ are **differentials**, which are, at each point $(x, f(x))$, the changes in y for the tangent line given a change in *x*:

- $\mathrm{d}x = \Delta x$
- $\mathrm{d}y = f'(x_0) \cdot \mathrm{d}x$

we have seen that the tangent line to $f(x)$ at x_0 approximates the function $f(x)$ around x_0 (and is identical at x_0), so approximately, for small Δx:

- $\Delta y \approx \mathrm{d}y$

Cobb-Douglas Production Function Marginal Productivity of Labor

MPL for Cobb-Douglas

With this Cobb-Douglas production function:

$$F(K, L) = K^{\alpha} \cdot L^{1-\alpha}$$

As derived earlier, the derivative with respect to labor is (MPL=marginal product of labor):

$$\mathrm{MPL}(K, L) = (1-\alpha) \cdot K^{\alpha} \cdot L^{-\alpha}$$

Interpreting MPL

In the above problem, suppose $K_0 = 1$ and $L_0 = 1$, and $\alpha = 0.5$. Without a calculator, we can calculate what output and marginal product of labor is:

$$F(K_0 = 1, L_0 = 1) = 1$$

$$\mathrm{MPL}(K_0 = 1, L_0 = 1) = 0.5$$

This means the total output with one unit of worker and one unit of capital is 1.

Becareful about interpreting the MPL term (we are treating it as a function of continuous *L*, some define MPL in terms of discrete increases in *L*), it is a derivative, which as we have discussed is the slope of the tangent line to the production function line with fixed *K* and *L* along the x-axis. Which means if you increase labor by a infinitestimally small amount when existing $K = 1$ and $L = 1$, the **slope** of output increase will be 0.5. The actual output increase is that infinitestimally small increase in labor multiplied by 0.5. It is perhaps difficult to conceptualize what it means to multiply something infinitely small by another number. To make the idea more conconcret, we will think using MPL to approximate the increase in output given a small increase in labor.

Exact Output Calculated with Matlab

Continuing with the two numbers we can calculate without a calculator:

$$F(K_0 = 1, L_0 = 1) = 1$$

$$\text{MPL}(K_0 = 1, L_0 = 1) = 0.5$$

Suppose we are interested in the increase in output when labor increases from $L_0 = 1$ to $L_1 = 1.03$, what is the new output? What is the increase in output? (You can think of this as increasing the number of workers by 3 percentage points.)

Exact** Solution:** We can directly calculate this, very hard by hand, but using matlab:

```
% Define parameters, fixed K0
alpha = 0.5;
K0 = 1;
% Define equation with L is unknown
syms L
f(L) = K0^(alpha)*L^(1-alpha);
% two different L levels
L0 = 1;
L1 = 1.03;
% Fill the L0 and L1 values into the symbolic function
YL0 = subs(f, L0)
```

YL0(L) = 1

```
YL1 = subs(f, L1)
```

YL1(L) =

$$\frac{\sqrt{103}}{10}$$

```
% Take difference
increase_in_output = YL1 - YL0
```

increase_in_output(L) =

$$\frac{\sqrt{103}}{10} - 1$$

```
% Turn symbolic answer to double (easier to read), increase in output
increase_in_output = double(increase_in_output)
```

increase_in_output = 0.0149

```
% new level of output
new_level_of_output = double(YL1)
```

```
new_level_of_output = 1.0149
```

Approximate Output Increase with Derivative (MPL)

Remember as we have seen, the slope of the tangent line at L_0 is similar to the slope of the line between $L_0 + h$ and L_0, from the definition of derivative, for *h* small, the following should be true:

- $F'_L(K_0, L_0) \approx \dfrac{F(K_0, L_0 + h) - F(K_0, L_0)}{h}$

Just move the *h* from the right to the left, the **increase in output** is *approximately*:

- $F(K_0, L_0 + h) - F(K_0, L_0) \approx F'_L(K_0, L_0) \cdot h$

Furthermore, the **level of output** is *approximately*:

- $F(K_0, L_0 + h) \approx F(K_0, L_0) + F'_L(K_0, L_0) \cdot h$

In our case above, we can now approximate output levels using the two numbers we calculated by hand, with $K_0 = 1$ and $L_0 = 1$:

- $F(K_0, L_0 + h) \approx F(K_0, L_0) + F'_L(K_0, L_0) \cdot h = 1 + 0.5 \cdot h$

Now with $\mathbf{1 + 0.5 \cdot h}$, that is something we can use very easily, back to 1st grade math. We calculated previously that if $h = 0.03$, the exact new level of output is 1.0149:

```
new_level_of_output
```

```
new_level_of_output = 1.0149
```

What is our approximated increase that we can calculate by hand? It is 1.015

```
new_approximated_level_of_output = 1 + 0.5 * 0.03
```

```
new_approximated_level_of_output = 1.0150
```

These are almost identical.

First Order Taylor Polynomial Approximation

What we have just done is called **First Order Taylor Polynomial Approximation**, which can be written more generally as:

- $F(X_0 + h) \approx F(X_0) + F'(X_0) \cdot h$

Often you see this written as below, these are equivalent:

- $f(x) \approx f(a) + f'(a) \cdot (x - a)$

This is just another way to write down the differential formula described at the beginning

- $F(X_0 + \Delta x) - F(X_0) = \Delta Y \approx \mathrm{d}Y = F'(X_0) \cdot \mathrm{d}X$

When solving economics problems, we often end up with functions that takes too much time to evaluate. To save time, we often approximate functions by the first order taylor approximation. We do this when we are solving for points around a point where we have already evaluated (a point where perhaps it is easier to evaluate the function). We just demonstrated this idea using the MPL example here, where we used something we can approximate using 1st grade algebra something that we would need a calculator (matlab) to compute accurately for us.

Analyze the functional form of MPL, what accurate is the 1st order taylor approximation or differential approximation for the same *h* increase in *L* if existing *L* is high vs if it is low?