Equilibrium Interest Rate and Wage Rate

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We have solved for the problem with constrained labor and saving/borrowing choice, and the problem with saving/borrowing and tax.

Household and Firm's Problem

Following our previous discussions, the household's borrowing constrained problem is:

- specifically: $\max_{b,\text{work},\text{leisure}} \log(Z_1 + w \cdot \text{work} - b) + \psi \log(\text{leisure}) + \beta \cdot \log(Z_2 + b \cdot (1 + r))$

And the constraints are:

1. $b \geq \overline{b}$
2. $\text{work} \geq 0$
3. $\text{leisure} \geq 0$
4. $\text{work} + \text{leisure} \leq T$, where T is total time available

There are $N = 3$ households, each with a different β_i.

For the firm, we have solved previously for the firm's optimal choices given w and r:

- $\max_{K,L} (p \cdot A \cdot K^{\alpha} \cdot L^{\beta} - r \cdot K - w \cdot L)$

Setting Up Parameters

Solve with three different discount rates, and different r and w. First, let's set up some parameters. The firm here has decreasing return to scale, let's ignore the issue of profit when looking for equilibrium.

```
clear all

% Parameters for the Household
psi = 0.5;
z1 = 1;
z2 = 2;
b_bar_num = -1; % borrow up to 1 dollar
T = 1; % think about time as share of time in a year

% Parameters for the firm
p = 1;
alpha = 0.3;
beta = 0.5;
Aproductivity = 2.0;

% Vector of 3 betas
beta_vec = [0.85 0.90 0.95];
```

```
% Vector of interest rates
R_vec = linspace(0.60, 2.50, 30);
% Vector of wage rates, 3 wage rates for now
W_vec = linspace(0.6, 2, 15);

% What we had from before to use fmincon
A = [-1,0,0;0,0,-1;0,-1,0;0,1,1];
q = [-b_bar_num;0;0;T];
b0 = [0,0.5,0.5]; % starting value to search for optimal choice
```

Household Labor Supply and Borrow/Save with different β and r ?

In the problem without labor supply I showed different excess supply of credit for each β_i household, we can do the same here for excess credit supply, but that is too much to show. I will just sum up the total across the households for both excress credit supply and total work hours:

- **Aggregate Household Excess Supply**: $B^{*}_{hh}(r,w) = \sum_{i=1}^{3} b^{*}(r,w,\beta_i)$
- **Aggregate Household Labor Supply**: $\mathrm{WORK}^{*}_{hh}(r,w) = \sum_{i=1}^{3} \mathrm{work}^{*}(r,w,\beta_i)$

I store results in a matrix where each row correspond to an interest rate level and each color a wage rate.

```
% Various Matrixes to store optimal choices
rows = length(R_vec);
cols = length(W_vec);
wage_dim_len = length(W_vec);
b_opti_mat = zeros(rows, cols);
worKOpti_mat = zeros(rows, cols);
leisure_opti_mat = zeros(rows, cols);
c1_opti_mat = zeros(rows, cols);
c2_opti_mat = zeros(rows, cols);

% Solving for optimal choices as we change Z2
for i=1:1:length(R_vec)
    for j=1:1:length(W_vec)

        % Initialize aggregate household statistics given r and w
        agg_b_supply_at_w_r = 0;
        agg_work_at_w_r = 0;
        agg_leisure_at_w_r = 0;
        agg_c1_at_w_r = 0;
        agg_c2_at_w_r = 0;

        for h=1:1:length(beta_vec)
            % Solve
            U_neg = @(x) -1*(log(z1 + W_vec(j)*x(2) - x(1)) + psi*log(x(3)) + beta_vec(h)*log(z
            options = optimoptions('FMINCON','Display','off');
            [x_opti,U_at_x_opti] = fmincon(U_neg, b0, A, q, [], [], [], [], [], options);
            % Sum up at current r and w for all households
            agg_b_supply_at_w_r = agg_b_supply_at_w_r + x_opti(1);
            agg_work_at_w_r = agg_work_at_w_r + x_opti(2);
            agg_leisure_at_w_r = agg_leisure_at_w_r + x_opti(3);
```

```
            agg_c1_at_w_r = agg_c1_at_w_r + z1 + W_vec(j)*x_opti(2) - x_opti(1);
            agg_c2_at_w_r = agg_c2_at_w_r + z2 + x_opti(1)*(R_vec(i));
        end

        % Store aggregate Household statistics
        b_opti_mat(i, j)  = agg_b_supply_at_w_r;
        workOpti_mat(i, j)  = agg_work_at_w_r;
        leisure_opti_mat(i, j)  = agg_leisure_at_w_r;
        c1_opti_mat(i, j) = agg_c1_at_w_r;
        c2_opti_mat(i, j) = agg_c2_at_w_r;
    end
end
```

Firm's Demand for Capital and Labor

The firm's problem loops over *r* and *w*, do not need to loop over β_i. We get here:

- **Firm Demand For Capital**: $K^*_{firm}(r, w)$
- **Firm Demand For Labor**: $L^*_{firm}(r, w)$

```
% Various Matrixes to store optimal choices
rows = length(R_vec);
cols = length(W_vec);
K_demand_mat = zeros(rows, cols);
L_demand_mat = zeros(rows, cols);

% We solved before optimal choices
syms w r
% Matrix Form of linear system, same as before
B = [log(r/(p*Aproductivity*alpha)); log(w/(p*Aproductivity*beta))];
A = [(alpha-1), beta;alpha, beta-1];
% Solve linear equations, and then exponentiate, same as before
% We can use the simplify command to simplify this solution, get rid of exp and log:
lin_solu = simplify(exp(linsolve(A, B)));
KOpti = lin_solu(1)
```

KOpti =

$$\frac{9\,\sqrt{15}}{125\,r^{5/2}\,w^{5/2}}$$

```
LOpti = lin_solu(2)
```

LOpti =

$$\frac{3\,\sqrt{15}}{25\,r^{3/2}\,w^{7/2}}$$

```
% Solving for optimal choices as we change Z2
for i=1:1:length(R_vec)
    for j=1:1:length(W_vec)
        K_demand_mat(i,j) = subs(KOpti,{r,w},{R_vec(i), W_vec(j)});
        L_demand_mat(i,j) = subs(LOpti,{r,w},{R_vec(i), W_vec(j)});
```

```
    end
end
```

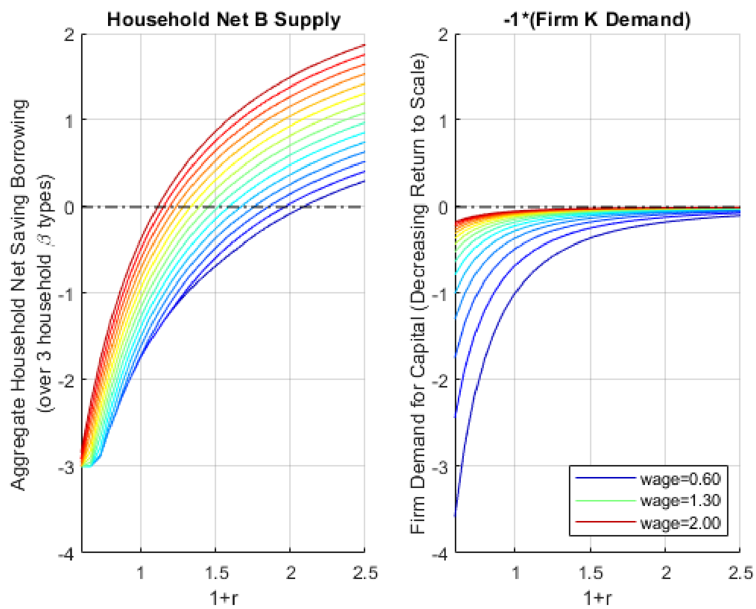
Demand and Supply for Capital

We can graph out from the firm and household problem demand and supply for capital

```
figure();

% Household b (some borrow some save added up)
subplot(1,2,1);
hold on;
chart = plot(R_vec, b_opti_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
plot(R_vec,ones(size(R_vec)) * 0, 'k-.');
xlim([min(R_vec) max(R_vec)]);
ylim([-4, 2]);
grid on;
title('Household Net B Supply')
ylabel({['Aggregate Household Net Saving Borrowing'], ['(over 3 household \beta types)']})
xlabel('1+r')

% Firm's Graph
subplot(1,2,2)
hold on;
chart = plot(R_vec, -K_demand_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
plot(R_vec,ones(size(R_vec)) * 0, 'k-.');
xlim([min(R_vec) max(R_vec)]);
ylim([-4, 2]);
grid on;
title('-1*(Firm K Demand)')
ylabel('Firm Demand for Capital (Decreasing Return to Scale)')
xlabel('1+r')
legend2plot = [1 round(numel(chart)/2) numel(chart)];
legendCell = cellstr(num2str(W_vec', 'wage=%3.2f'));
legend(chart(legend2plot), legendCell(legend2plot), 'Location','southeast');
```



Demand and Supply for Labor Demand and Supply

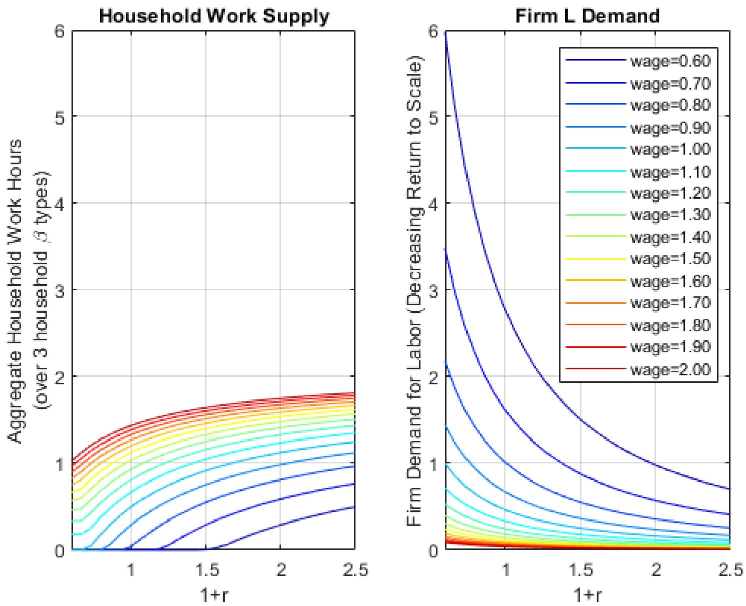
We now generate the same graphs for Labor

```
figure();

% Household b (some borrow some save added up)
subplot(1,2,1);
chart = plot(R_vec, worKOpti_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
xlim([min(R_vec) max(R_vec)]);
ylim([0,6]);
grid on;
title('Household Work Supply')
ylabel({['Aggregate Household Work Hours'], ['(over 3 household \beta types)']})
xlabel('1+r')

% Firm's Graph
subplot(1,2,2)
chart = plot(R_vec, L_demand_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
xlim([min(R_vec) max(R_vec)]);
ylim([0,6]);
grid on;
title('Firm L Demand')
```

```
ylabel('Firm Demand for Labor (Decreasing Return to Scale)')
xlabel('1+r')
legendCell = cellstr(num2str(W_vec', 'wage=%3.2f'));
legend(legendCell, 'Location','northeast');
```



Excess Demand for Capital and Labor

We can sum up the firm and household sides to try to find the *r* and *w* where demand and supply are equalized.

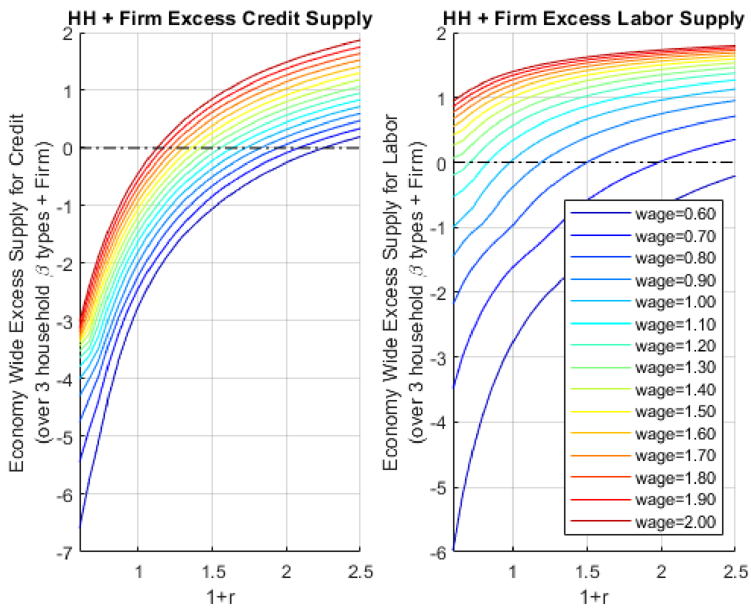
- **Economy-wide excess supply of Credit**: $\text{ExcesCreditSupply}(r, w) = B^*_{hh}(r, w) - K^*_{firm}(r, w)$
- **Economy-wide excess supply of Credit**: $\text{ExcesLaborSupply}(r, w) = \text{WORK}^*_{hh}(r, w) - L^*_{firm}(r, w)$

```
figure();

% Household and Firm Excess Credit Supply, aggregated together
subplot(1,2,1);
hold on;
chart = plot(R_vec, b_opti_mat-K_demand_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
plot(R_vec,ones(size(R_vec)) * 0, 'k-.');
xlim([min(R_vec) max(R_vec)]);
grid on;
title('HH + Firm Excess Credit Supply')
ylabel({['Economy Wide Excess Supply for Credit'], ['(over 3 household \beta types + Firm)']})
xlabel('1+r')

% Firm's Graph
subplot(1,2,2);
```

```
hold on;
chart = plot(R_vec,  worKOpti_mat - L_demand_mat);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
plot(R_vec,ones(size(R_vec)) * 0, 'k-.');
xlim([min(R_vec) max(R_vec)]);
grid on;
title('HH + Firm Excess Labor Supply')
ylabel({['Economy Wide Excess Supply for Labor'], ['(over 3 household \beta types + Firm)']})
xlabel('1+r')
legendCell = cellstr(num2str(W_vec', 'wage=%3.2f'));
legend(legendCell, 'Location','southeast');
```



w and *r* Equilibrium

Now let's do a final sum we want to find where both aggregate labor and capital clear.

```
figure();

% Aggregate Excess Supplies
excess_credit_supply = abs(b_opti_mat - K_demand_mat);
excess_labor_supply = abs(worKOpti_mat - L_demand_mat);
```

We need to take the absolute values of the two differences above and sum them up. The equilibrium is approximately where the sum of the two matrixes is the closest to 0.

```
DS_KL_DIFF = excess_credit_supply + excess_labor_supply;
[DS_KL_diff_EQUI_val, EQUI_IDX]  = min(min(DS_KL_DIFF));
[r_idx, w_idx]=find(DS_KL_DIFF==DS_KL_diff_EQUI_val);
equi_r = R_vec(r_idx);
```

```
equi_w = W_vec(w_idx);
equi_price = table(equi_r, equi_w);
disp(equi_price);
```

equi_r	equi_w
2.0414	0.7

```
% Both should be zero (if the scale of L and K are very different this would not work well)
% We can sum up the two and look for r and w closest to zero
subplot(1,2,1);
chart = plot(R_vec, DS_KL_DIFF);
% Show smoother colors
clr = jet(numel(chart));
for m = 1:numel(chart)
   set(chart(m),'Color',clr(m,:))
end
xlim([min(R_vec) max(R_vec)]);
grid on;
title('abs(Excess K) + abs(Excess L)')
ylabel({['Economy Wide Excess Supply for Credit'], ['(over 3 household \beta types + Firm)']})
xlabel('1+r')
legendCell = cellstr(num2str(W_vec', 'wage=%3.2f'));
legend(legendCell, 'Location','northeast');

% Firm's Graph
subplot(1,2,2)
mesh(R_vec, W_vec, DS_KL_DIFF');
view([30.1 3.6]);
title('abs(Excess K) + abs(Excess L)')
ylabel('wage')
xlabel('1+r')
```