Firm's Profit Maximization Problem over Outputs, Marginal Costs and Profits given Constant Returns to Scale

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We have already solved the firm's maximization problem before given decreasing return to scale: Firm Maximization Problem with Capital and Labor (Decreasing Return to Scale). We have also solved the constrained profit maximization or cost minimization problem as well: Profit Maximize and Cost Minimize.

What is the Profit of the firm at Constrained Optimal Choices?

We derived the optimal constrained *K* and *L* equations here: Profit Maximize and Cost Minimize. The constrained profit equation given, p, q, w, r, is:

- $\Pi^{*\text{cost minimize}}(p, q, w, r) = p \cdot q - w \cdot L^*(w, r, q) - r \cdot K^*(w, r, q)$

Profit Maximization and Marginal Cost

Imagine a firm is now trying to decide how much to produce, given our cost minimization problem, now rather than thinking about the firm directly choosing *K* and *L* to maximize profit, we can think of the marginal cost and marginal profit of the firm as *q* changes for the firm. If the firm can choose *q*, it will want to choose the *q* that maximizes profit.

$$\max_q \left(p \cdot q - w \cdot L^*(w, r, q) - r \cdot K^*(w, r, q)\right)$$

The solution to this problem has to be the same as the problem we solved earlier where we directly chose *K* and *L*, but now when formulated this way, we can think about the marginal cost and marginal revenue for the firm when *q* changes:

- $\text{MC(w,r)} = \frac{\partial(w \cdot L^*(w, r, q) + r \cdot K^*(w, r, q))}{\partial q}$
- $\text{MR}(w, r) = p$

Marginal revenue is of course constant at *p* and marginal cost is the derivative of the cost minimizing *L* and *K* choices multiplied by respective prices with respect to *q*. Note we drived previously that these are functions of *q*. Together with what we derived here: Firm Maximization Problem with Capital and Labor (Decreasing Return to Scale), the Cobb-Douglas Production function firm's problem has given us on the return side: marginal productivity of capital, marginal productivity of labor, and marginal revenue. On the cost side: marginal cost of capital, marginal cost of labor, marginal cost of additional output (given cost minimization). These six marginal ideas are crucial to any firm's problem, the specific functional forms differ depending on our production function specifications, but formulating how firms operate with these marginal ideas is at the heart of economic analysis.

Constant Return to Scale

In our previous exercise decreasing return to scale, Firm Maximization Problem with Capital and Labor (Decreasing Return to Scale), firms chose optimal *K* and *L* to maximize profit. We showed that the log linearized coefficient matrix is not full rank and invertible with constant return to scale, and hence firms did not have unconstrained profit maximizing *K* and *L* choices. Why is that?

Formulating the problem with marginal cost and marginal revenue helps us to understand what is going on.

It turns out that if $\alpha + \beta = 1$, that is, the firm has constant return to scale (CRS)--the elasticities of inputs sum up to 1--the cost minimizing optimal *K* and *L* choices are **linear** in terms of *q*. The equations we derived in Profit Maximize and Cost Minimize, become, with CRS:

- $K^*(w, r, q) = q \cdot \left\{ \frac{1}{A} \cdot \left[\frac{\alpha}{1-\alpha} \cdot \frac{w}{r} \right]^{1-\alpha} \right\}$
- $L^*(w, r, q) = q \cdot \left\{ \frac{1}{A} \cdot \left[\frac{\alpha}{1-\alpha} \cdot \frac{w}{r} \right]^{-\alpha} \right\}$

These equations mean that the marginal cost of producing one more unit of *q*, given CRS, is not impacted by *q*, hence, it is a constant (determined by A, α, w, r):

- $\text{MC(w,r)} = r \cdot \left\{ \frac{1}{A} \cdot \left[\frac{\alpha}{1-\alpha} \cdot \frac{w}{r} \right]^{1-\alpha} \right\} + w \cdot \left\{ \frac{1}{A} \cdot \left[\frac{\alpha}{1-\alpha} \cdot \frac{w}{r} \right]^{-\alpha} \right\}$

With CRS, this means that if a firm makes $q = 1$, the cost would be MC(w,r), if the firm makes $q = 10$, the marginal cost for making the 10th good, given that the firm is cost minimizing by choosing optimal bundle of capital and labor, is just MC(w,r), and the total cost is also $10 \cdot \text{MC(w,r)}$.

When will the Firm produce, and what is its Profit?

With decreasing return to scale, for any prices, there will be profit maximizing *K* and *L* choices that lead to some profit maximizing output, as shown here: Firm Maximization Problem with Capital and Labor (Decreasing Return to Scale).

With CRS:

- if $p < \text{MC}(w, r)$, the firm does not produce, supply is perfectly inelastic
- if $p > \text{MC}(w, r)$, the firm produces infinity, every additional unit brings $p - \text{MC(w,r)}$ unit of profit, so the firm would want to produce up to infinity
- if $p = \text{MC}(w, r)$. There is no profit, but there is also no loss. Households can also produce any amount, because there is nothing lost from producing.

So the firm makes a profit when: $p > \text{MC}(w, r)$,

Given perfect competition, firms do not have pricing power, and take *p* as given, at equilibrium, $p = \text{MC}(w, r)$. With CRS and perfect competition, firms will not make a profit. The fact that marginal cost is constant and profit is linear in *q* lead to this result. If there is monopolistic competition, there could be profits given CRS because firms would then be able to shift price as they shift quantity.