# Compute Nested CES MPL Given Demand (CRS)

Testing the **bfw\_crs\_nested\_ces\_mpl** function from the **PrjLabEquiBFW Package**. Given labor quantity demanded, using first-order relative optimality conditions, find the marginal product of labor given CES production function. Results match up with correct relative wages, but not wage levels. Takes as inputs share and elasticity parameters across layers of sub-nests, as well as quantity demanded at each bottom-most CES nest layer. Works with Constant Elasticity of Substitution problems with constant returns, up to four nest layers, and two inputs in each sub-nest. Allows for uneven branches, so that some branches go up to four layers, but others have less layers, works with BFW (2022) nested labor input problem.

#### Key Inputs and Outputs for bfw\_crs\_nested\_ces\_mpl

Here are the key inputs for the CES demand solver function:

- CL\_MN\_PRHO cell array of rho (elasticity) parameter between negative infinity and 1. For example, suppose there are four nest layers, and there are two branches at each layer, then we have 1, 2, 4, and 8 ρ parameter values at the 1st, 2nd, 3rd, and 4th nest layers: size(CL\_MN\_PRHO{1})= [1,1], size(CL\_MN\_PRHO{2}) = [1,2], size(CL\_MN\_PRHO{3}) = [2,2], size(CL\_MN\_PRHO{4}) = [2,2,2]. Note that if the model has 4 nest layers, not all cells need to be specified, some branches could be deeper than others.
- CL\_MN\_PSHARE cell array of share (between 0 and 1) for the first input of the two inputs for each nest. The structure for this is similar to CL\_MN\_PRHO.
- CL\_MN\_YZ\_CHOICES cell array of quantity demanded for the first and second inputs of the bottom-most layer of sub-nests. The last index in each element of the cell array indicates first (1) or second (2) quantities. For example, suppose we have four layers, with 2 branches at each layer, as in the example for CL\_MN\_PRHO, then we have 2, 4, 8, and 16 quantity values at the 1st, 2nd, 3rd, and 4th nest layers: size(CL\_MN\_YZ\_CHOICES{1})= [1,2], size(CL\_MN\_YZ\_CHOICES{2})= [2,2], size(CL\_MN\_YZ\_CHOICES{3})= [2,2,2], size(CL\_MN\_YZ\_CHOICES{4})= [2,2,2,2]. Note that only the last layer of quantities needs to be specified, in this case, the 16 quantities at the 4th layer. Given first order conditions, we solve for the 2, 4, and 8 aggregate quantities at the higher nest layers. If some branches are deeper than other branches, then can specific NA for non-reached layers along some branches.
- **BL\_BFW\_MODEL** boolean true by default if true then will output outcomes specific to the BFW 2022 problem.

Here are the key outputs for the CES demand solver function:

- CL\_MN\_MPL\_PRICE has the same dimension as CL\_MN\_YZ\_CHOICES, suppose there are four layers, the CL\_MN\_MPL\_PRICE{4} results at the lowest layer includes wages that might be observed in the data. CL\_MN\_MPL\_PRICE cell values at non-bottom layers include aggregate wages.
- CL\_MN\_YZ\_CHOICES includes at the lowest layer observed wages, however, also includes higher layer aggregate solved quantities. CL\_MN\_PRHO and CL\_MN\_PSHARE are identical to inputs.

# Single Nest Layer Two Inputs CES Problem (MPL)

In this first example, we solve a constant returns to scale problem with a single nest, meaning just two inputs and a single output.

```
clc;
close all;
clear all;
% rho = 0.5, 1/(1-0.5)=2, elasticity of substitution of 2
cl mn prho = \{[0.1]\};
% equal share, similar "productivity"
cl mn pshare = \{[0.5]\};
% levels of the two inputs, Values picked from demand problem parallel
% example.
cl mn yz choices = \{[0.67537, 1.4589]\};
% print option
bl_verbose = true;
mp func = bfw mp func demand();
bl bfw model = false;
[cl mn_yz_choices, cl_mn_mpl_price] = ...
    bfw_crs_nested_ces_mpl(cl_mn_prho, cl_mn_pshare, cl_mn_yz_choices, ...
    mp_func, bl_verbose, bl_bfw_model);
```

	i	idx	ndim	numel	rowN	colN	sum	mean	std	coefvari	min
	_							<del></del>			
mpl_price_c1	1	1	2	2	1	2	1.0678	0.53388	0.25168	0.47141	0.35592
yz_c1	2	4	2	2	1	2	2.1343	1.0671	0.55404	0.51918	0.67537

r1 0.67537 1.4589

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# Single Nest Layer Two Inputs CES Problem, Vary Share and Elasticity (MPL)

In this second example, we test over different rho values, explore optimal relative choices, as share and elasticity change. In this exercise, we also check, at every combination of rho and share parameter, whether the

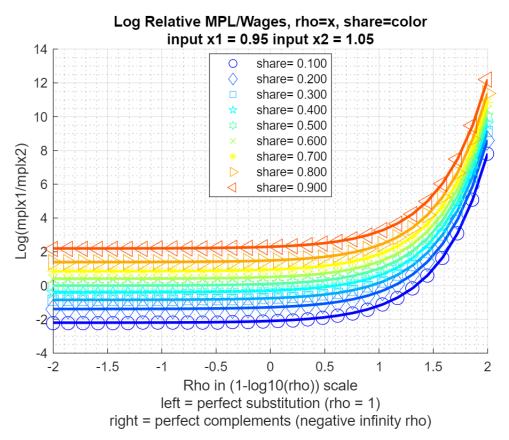
FOC condition is satisfied by the optimal choices. Also check if at the optimal choices, the minimization output requirement is met.

```
% Approximately close function
rel tol=1e-09;
abs tol=0.0;
if_is_close = @(a,b) (abs(a-b) <= max(rel_tol * max(abs(a), abs(b)), abs_tol));
% Input 1 and 2 fixed
fl_x_1 = 0.95;
fl \times 2 = 1.05;
% Define share and rho arrays
ar pshare = linspace(0.1, 0.9, 9);
ar prho = 1 - 10.^{(linspace(-2, 2, 30))};
% Loop over share and rho values
mt rela opti = NaN([length(ar pshare), length(ar prho)]);
mt_rela_wage = NaN([length(ar_pshare), length(ar_prho)]);
for it_pshare_ctr = 1:length(ar_pshare)
    for it prho ctr = 1:length(ar prho)
        % A. Parameters
        % rho
        fl_prho = ar_prho(it_prho_ctr);
        cl_mn_prho = {[fl_prho]};
        % share
        fl_pshare = ar_pshare(it_pshare_ctr);
        cl_mn_pshare = {[fl_pshare]};
        % wages for the two inputs, identical wage
        % Note that if chosee \{[1,1]\} below, \log(1/1) = \log(1) = 0,
        % elasticity does not matter.
        cl_mn_yz_choices = \{[fl_x_1, fl_x_2]\};
        % print option
        bl_verbose = false;
        % B. Call function
        [cl_mn_yz_choices, cl_mn_mpl_price] = ...
            bfw_crs_nested_ces_mpl(cl_mn_prho, cl_mn_pshare, cl_mn_yz_choices, ...
            mp func, bl verbose, bl bfw model);
        % Store results for mpl given input choices
        fl mpl x1 = cl mn mpl price\{1\}(1);
        fl mpl x2 = cl mn mpl price\{1\}(2);
        mt_rela_wage(it_pshare_ctr, it_prho_ctr) = log(fl_mpl_x1/fl_mpl_x2);
    end
end
```

Key results: (1) As share parameter of input 1 goes to zero, input 1 is less productive, and the log(mplx1/mplx2) ratio is lower. (2) Becaus x2 input in this example is larger than x1 input, so as two inputs become more inelastic (more leontief), relative MPL for the lower level input is now larger. At the Leontief extreme, the MPL of the input provided at lower level is infinity.

```
% Visualize
% Generate some Data
```

```
rng(456);
ar row grid = ar pshare;
ar_col_grid = log(1-ar_prho)/log(10);
rng(123);
mt_value = mt_rela_wage;
% container map settings
mp_support_graph = containers.Map('KeyType', 'char', 'ValueType', 'any');
mp_support_graph('cl_st_graph_title') = {...
    ['Log Relative MPL/Wages, rho=x, share=color'] ...
    ['input x1 = ' num2str(fl_x_1) ' input x2 = ' num2str(fl_x_2)]
    };
mp support graph('cl st ytitle') = {'Log(mplx1/mplx2)'};
mp_support_graph('cl_st_xtitle') = {'Rho in (1-log10(rho)) scale', ...
    'left = perfect substitution (rho = 1)', ...
    'right = perfect complements (negative infinity rho)'};
mp support graph('st legend loc') = 'best';
mp_support_graph('bl_graph_logy') = false; % do not log
mp support graph('st rowvar name') = 'share=';
mp support graph('it legend select') = 5; % how many shock legends to show
mp_support_graph('st_rounding') = '6.3f'; % format shock legend
mp support graph('cl colors') = 'jet'; % any predefined matlab colormap
% Call function
ff_graph_grid(mt_value, ar_row_grid, ar_col_grid, mp_support_graph);
```



#### **Doubly Nest Layer Two Inputs Each Sub-nest CES Problem**

In this third example, solve for optimal choices for a doubly nested problem. Below, we first solve for the optimal choices, then we do a number of checks, to make sure that the solutions are correct, as expected.

```
% output requirement
fl_yz = 2.1;
% upper nest 0.1, lower nests 0.35 and -1 separately for rho values
cl_mn_prho = \{[0.1], [0.35, -1]\};
% unequal shares of share values
cl_mn_pshare = \{[0.4], [0.3, 0.88]\};
% differential wages
% in lower-left nest, not productive and very expensive, not very elastic
% last index for left or right. Values picked from demand problem parallel
% example.
cl mn yz choices = {[nan, nan], [0.04789, 6.0934; 2.2044, 0.70496]};
% print option
bl_verbose = true;
[cl mn yz choices, cl mn mpl price] = ...
    bfw_crs_nested_ces_mpl(cl_mn_prho, cl_mn_pshare, cl_mn_yz_choices, ...
    mp_func, bl_verbose, bl_bfw_model);
```

CONTAINER NAME: mp\_container\_map ND Array (Matrix etc)

	i	idx	ndim	numel	rowN	colN	sum	mean	std	coefvari	min
	-										
mpl_price_c1	1	1	2	2	1	2	1.0206	0.51032	0.27499	0.53886	0.31587
mpl_price_c2	2	2	2	4	2	2	2.3618	0.59045	0.5082	0.86069	0.13121
prho_c2	3	4	2	2	1	2	-0.65	-0.325	0.95459	-2.9372	-1
pshare_c2	4	6	2	2	1	2	1.18	0.59	0.41012	0.69512	0.3
yz_c1	5	7	2	2	1	2	4.4862	2.2431	0.68863	0.307	1.7562
yz_c2	6	8	2	4	2	2	9.0507	2.2627	2.7086	1.1971	0.04789

```
xxx TABLE:mpl_price_c1 xxxxxxxxxxxxxxxxxx
           c1
                    c2
```

0.31587 0.70476

xxx TABLE:mpl\_price\_c2 xxxxxxxxxxxxxxxxx c1 c2

1.3121 0.13121 r1 0.39362 0.52484

xxx TABLE:prho c2 xxxxxxxxxxxxxxxxxx

c1 c2 0.35 -1

r2

r1

r1

xxx TABLE:pshare\_c2 xxxxxxxxxxxxxxxxxxxx

c1 c2 0.3 0.88

xxx TABLE:yz\_c1 xxxxxxxxxxxxxxxxxxx

```
c1
            c2
  r1
      2.73
           1.7562
xxx TABLE:yz_c2 xxxxxxxxxxxxxxxxxxx
       c1
              c2
  r1
     0.04789
             6.0934
  r2
             0.70496
      2.2044
CONTAINER NAME: mp_container_map Scalars
idx value
  prho c1
          1 3
                   0.1
  pshare_c1 2
             5
                   0.4
```

```
% there are four optimal choices, they are
fl mpl x11 = cl mn mpl price\{2\}(1,1);
fl mpl x12 = cl mn mpl price\{2\}(1,2);
fl_mpl_x21 = cl_mn_mpl_price{2}(2,1);
fl mpl x22 = cl mn mpl price\{2\}(2,2);
% display
st_print = strjoin(...
    ["completed double nest test:", ...
    ['nest 1 input 1, fl_mpl_x11=' num2str(fl_mpl_x11)], ...
    ['nest 1 input 2, fl_mpl_x12=' num2str(fl_mpl_x12)], ...
    ['nest 2 input 1, fl mpl x21=' num2str(fl mpl x21)], ...
    ['nest 2 input 2, fl_mpl_x22=' num2str(fl_mpl_x22)], ...
    ['nest 1 input 1, fl_mpl_x11/fl_mpl_x11=' num2str(fl_mpl_x11/fl_mpl_x11)], ...
    ['nest 1 input 2, fl_mpl_x12/fl_mpl_x11=' num2str(fl_mpl_x12/fl_mpl_x11)], ...
    ['nest 2 input 1, fl_mpl_x21/fl_mpl_x11=' num2str(fl_mpl_x21/fl_mpl_x11)], ...
    ['nest 2 input 2, fl_mpl_x22/fl_mpl_x11=' num2str(fl_mpl_x22/fl_mpl_x11)], ...
    ], ";");
st_out = st_print;
ar_ch_out = char(strsplit(st_print,";")');
disp(ar_ch_out);
```

```
completed double nest test:
nest 1 input 1, fl_mpl_x11=1.3121
nest 1 input 2, fl_mpl_x12=0.13121
nest 2 input 1, fl_mpl_x21=0.39362
nest 2 input 2, fl_mpl_x22=0.52484
nest 1 input 1, fl_mpl_x11/fl_mpl_x11=1
nest 1 input 2, fl_mpl_x12/fl_mpl_x11=0.099995
nest 2 input 1, fl_mpl_x21/fl_mpl_x11=0.29998
nest 2 input 2, fl_mpl_x22/fl_mpl_x11=0.39999
```

## BFW (2022) Nested Three Branch (Four Layer) Problem (MPL)

The model BFW 2022 has three branches and four layers. one of the branches go down only three layers, the other two branches go down four layers.

First, we prepare the various inputs:

```
% Controls
bl_verbose = true;
bl bfw model = true;
% Given rho and beta, solve for equilibrium quantities
mp_func = bfw_mp_func_demand();
% Following instructions in: PrjFLFPMexicoBFW\solvedemand\README.md
% Nests/layers
it nests = 4;
% Input cell of mn matrixes
it_prho_cl = 1;
it pshare cl = 2;
it yz share cl = 3;
for it_cl_ctr = [1,2,3]
    cl mn cur = cell(it nests,1);
    % Fill each cell element with NaN mn array
    for it cl mn = 1:it nests
        bl yz share = (it cl ctr == it yz share cl);
        if (~bl_yz_share && it_cl_mn == 1)
            mn nan = NaN;
        elseif (~bl_yz_share && it_cl_mn == 2) || (bl_yz_share && it_cl_mn == 1)
            mn_nan = [NaN, NaN];
        elseif (~bl_yz_share && it_cl_mn == 3) || (bl_yz_share && it_cl_mn == 2)
            mn_n = NaN(2,2);
        elseif (~bl yz share && it_cl_mn == 4) || (bl_yz share && it_cl_mn == 3)
            mn nan = NaN(2,2,2);
        elseif (~bl_yz_share && it_cl_mn == 5) || (bl_yz_share && it_cl_mn == 4)
            mn nan = NaN(2,2,2,2);
        elseif (~bl_yz_share && it_cl_mn == 6) || (bl_yz_share && it_cl_mn == 5)
            mn_n = NaN(2,2,2,2,2);
        cl_mn_cur{it_cl_mn} = mn_nan;
    end
    % Name cell arrays
    if (it_cl_ctr == it_prho_cl)
        cl_mn_prho = cl_mn_cur;
    elseif (it_cl_ctr == it_pshare_cl)
        cl mn pshare = cl mn cur;
    elseif (it_cl_ctr == it_yz_share_cl)
        cl_mn_yz_choices = cl_mn_cur;
    end
end
% Initialize share matrix
```

```
rng(123);
for it_cl_mn = 1:it_nests
    mn_pshare = cl_mn_pshare{it_cl_mn};
    if it cl mn == 4
        mn_pshare(2,:,:) = rand(2,2);
    else
        mn pshare = rand(size(mn pshare));
    cl_mn_pshare{it_cl_mn} = mn_pshare;
end
% Initialize rho matrix
rng(456);
for it_cl_mn = 1:it_nests
    mn_prho = cl_mn_prho{it_cl_mn};
    if it cl mn == 4
        mn_prho(2,:,:) = rand(2,2);
    else
        mn_prho = rand(size(mn_prho));
    end
   % Scalling rho between 0.7500 and -3.0000
    % 1 - 2.^(linspace(-2,2,5))
    mn_prho = 1 - 2.^{mn_prho*(4)} - 2);
    cl_mn_prho{it_cl_mn} = mn_prho;
end
% Initialize quantities matrix
rng(789);
for it_cl_mn = 1:it_nests
    mn_yz_choices = cl_mn_yz_choices{it_cl_mn};
    if it_cl_mn == 3
        mn_yz_choices(1,:,:) = rand(2,2);
    elseif it_cl_mn == 4
        mn_yz_choices(2,:,:,:) = rand(2,2,2);
    end
    % Scalling quantities between 3 amd 5
    mn yz choices = mn yz choices*(2) + 3;
    cl_mn_yz_choices{it_cl_mn} = mn_yz_choices;
end
% Initialize yz matrix
rng(101112);
```

Second, display created inputs:

```
celldisp(cl_mn_prho);
```

```
cl_mn_prho{1} =
    0.5017

cl_mn_prho{2} =
```

```
0.6071 -1.1955

cl_mn_prho{3} =
    -1.3523    -0.3346
    -0.4167    -1.9136

cl_mn_prho{4} =

(:,:,1) =
    NaN     NaN
    -1.0512    0.5869

(:,:,2) =
    NaN     NaN
```

0.6209

### celldisp(cl\_mn\_pshare);

0.1633

cl\_mn\_pshare{1} = 0.6965 cl\_mn\_pshare{2} = 0.2861 0.2269  $cl_mn_pshare{3} =$ 0.5513 0.4231 0.7195 0.9808  $cl_mn_pshare{4} =$ (:,:,1) = NaN NaN 0.6848 0.4809 (:,:,2) = NaN NaN 0.3921 0.3432

### celldisp(cl\_mn\_yz\_choices);

```
cl_mn_yz_choices{1} =
  NaN NaN
cl_mn_yz_choices{2} =
  NaN
        NaN
  NaN
        NaN
cl_mn_yz_choices{3} =
(:,:,1) =
   3.6467
           3.4605
               NaN
      NaN
(:,:,2) =
   4.5876
           4.2488
      NaN
               NaN
cl_mn_yz_choices{4} =
(:,:,1,1) =
      NaN
              NaN
   4.9508
            4.5178
(:,:,2,1) =
                NaN
      NaN
   3.0212
             3.0495
(:,:,1,2) =
      NaN
               NaN
   3.2221
             4.0763
(:,:,2,2) =
                NaN
             4.1031
   3.0909
```

#### Third, call function and solve for optimal demand:

```
% Call function
[cl_mn_yz_choices, cl_mn_mpl_price] = ...
  bfw_crs_nested_ces_mpl(cl_mn_prho, cl_mn_pshare, cl_mn_yz_choices, ...
  mp_func, bl_verbose, bl_bfw_model);
```

-----

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

CONTAINER NAME: mp container map ND Array (Matrix etc)

	i	idx	ndim	numel	rowN	colN	sum	mean	std	coefvari
	_									
mpl_price_c1	1	1	2	2	1	2	1.0002	0.5001	0.28686	0.57362
mpl_price_c2	2	2	2	4	2	2	1.0009	0.25022	0.17949	0.71731
mpl_price_c3	3	3	3	8	2	4	1.0088	0.1261	0.10191	0.80822
mpl_price_c4	4	4	4	16	2	8	NaN	NaN	NaN	NaN
prho_c2	5	6	2	2	1	2	-0.58844	-0.29422	1.2746	-4.3323
prho_c3	6	7	2	4	2	2	-4.0173	-1.0043	0.76195	-0.75868
prho_c4	7	8	3	8	2	4	NaN	NaN	NaN	NaN
pshare_c2	8	10	2	2	1	2	0.51299	0.2565	0.041923	0.16344
pshare_c3	9	11	2	4	2	2	2.6747	0.66866	0.24087	0.36023
pshare_c4	10	12	3	8	2	4	NaN	NaN	NaN	NaN
yz_c1	11	13	2	2	1	2	8.0897	4.0448	0.173	0.04277
yz_c2	12	14	2	4	2	2	16.015	4.0039	0.19166	0.04787
yz_c3	13	15	3	8	2	4	31.235	3.9044	0.51337	0.13149
yz_c4	14	16	4	16	2	8	NaN	NaN	NaN	NaN

c1 c2

**r1** 0.70294 0.29725

xxx TABLE:mpl\_price\_c2 xxxxxxxxxxxxxxxxxx

c1 c2

r1 0.19946 0.50351

r2 0.080381 0.21754

	<b>c1</b>	c2	<b>c</b> 3	c4
			<del></del>	
r1	0.13727	0.24893	0.065108	0.25809
r2	0.050551	0.21139	0.031132	0.0063057

xxx TABLE:mpl\_price\_c4 xxxxxxxxxxxxxxxxxx

	<b>c1</b>	c2	<b>c</b> 3	с4	<b>c</b> 5	с6	с7	c8
r1	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
r2	0.02507	0.099481	0.012272	0.0025507	0.027845	0.11203	0.018861	0.0038085

xxx	TABLE	:prho_c3 x <b>c1</b>	xxxxxxxxxx c2	xxxxxx					
	r1 r2	-1.3523 -0.41668	-0.3346 <sub>4</sub>						
xxx	TABLE	:prho_c4 x <b>c1</b> 	c2	c3	c4 				
	r1 r2	NaN -1.0512	NaN 0.58694	NaN 0.62089	NaN 0.16334				
xxx	TABLE	:pshare_c2 <b>c1</b> 	c2	xxxxxxxx					
	r1	0.28614	0.22685						
XXX	TABLE	:pshare_c3	c2	xxxxxxxx					
	r1 r2	0.55131 0.71947	0.42311 0.98076						
xxx	TABLE	:pshare_c4 <b>c1</b> 	c2	c3	c4				
	r1 r2	NaN 0.68483	NaN 0.48093	NaN 0.39212	NaN 0.34318				
xxx	TABLE	:yz_c1 xxx <b>c1</b>	c2	xxxx					
	r1	3.9225	4.1672						
XXX	TABLE	:yz_c2 xxx <b>c1</b>	c2	xxxx					
	r1 r2	4.0073 3.8468	3.8887 4.2727						
xxx	TABLE	:yz_c3 xxx <b>c1</b>	c2	c3	c4				
	r1 r2	3.6467 4.23	3.4605 4.2863	4.5876 3.0635	4.2488 3.7118				
xxx	TABLE	:yz_c4 xxx <b>c1</b>	c2	c3	c4	c5	c6 	c7	c8
	r1 r2	NaN 4.9508	NaN 4.5178	NaN 3.0212	NaN 3.0495	NaN 3.2221	NaN 4.0763	NaN 3.0909	NaN 4.1031

 xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

	i	idx	value
	-		
prho_c1 pshare_c1	1 2	5 9	0.50172 0.69647