Matrix and Household Size Transition Across Lifecycle

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1 Demographics and Transition Matrix

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1.1 Initial Population, Kids Count Transition, and Fertility

A household can have $k = 0$, $k = 1$, or $k = 2$ kids. The household head has three possible age-groupings: $a = 1$ (perhaps 18 to 30), $a = 2$ (perhaps 31 to 50), and $a = 3$ (perhaps 51 and after).

We have two 3×3 transition matrixes, T_1 and T_2. T_1 shows the transition probabilities of going from k kids at age $a = 1$ to k' kids at $a = 2$, each row of T_1 corresponds to k, and each column corresponds to k', each cell represents $P(k'|k)$, and the probabilities for each row sums up to 1. Similarly for T_2. For example, T_1 might be equal to the following matrix, which shows substantial persistence.

$$T_1 = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ t^1_{21} & t^1_{22} & t^1_{23} \\ t^1_{31} & t^1_{32} & t^1_{33} \end{bmatrix} = \begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.05 & 0.80 & 0.15 \\ 0.05 & 0.16 & 0.79 \end{bmatrix}$$

Additionally, not all individuals survive from across a over time. $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$ represent the probabilities that those in $a = 1$ and $a = 2$ survive into $a = 2$ and $a = 3$, respectively.

Finally, at the initial age, we have initial population share vector K_1, which is a 3×1 vector that sums up to 1 and that indicates the share of households at the start of $a = 1$ with 0, 1, and 2 kids. Subsequently, we have K_2 and also K_2. For example, K_1 might be equal to the following matrix, with more households having less children.

$$K_1 = \begin{bmatrix} 0.70 \\ 0.20 \\ 0.10 \end{bmatrix}$$

We want to know the number of children born (the fertility level) given this demographic structure above.

1.1.1 Arrival of Children Probabilities

We interpret an increase in the number of kids in the household as a fertility event. This means, when the household transitions from 0 to 1 (2) kids, there is 1 (2) more children born. For the household with 2 children, no new children can be born for that period. But if the 2 children household transitions to a 1 child household, then a new child can be born to this household. When the number of children go down, we interpret this as children leaving a household.

Given this interpretation, we can create fertility matrixes F_1, F_2, and F_3, based on T_1, T_2, and T_3. For example, for F_1, we have:

$$
\begin{aligned}
F_{11} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ t^1_{21} + t^1_{22} & t^1_{23} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{13} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^1_{31} + t^1_{32} + t^1_{33} & 0 & 0 \end{bmatrix} \\
F_1 &= F_{11} + F_{12} + F_{13}
\end{aligned}
$$

The construction of this is similar for F_2 and F_3.

1.1.2 Fertility Level

Given the fertility matrixes, we can compute, at each a, the number of new kids born as a fraction of the households available at the start of $a = 1$, where mass $= 1$. Specifically, the number of kids born during $a = 1$, meaning "by the end of" $a = 1$, is f_1:

$$
f_1 = \overbrace{K_1'}^{\text{1 by 3}} \cdot \underbrace{F_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1} \\ \text{Fertility} \\ \text{by current kids}}}
$$

Proceeding to $a = 2$, we have a similar formula for f_2, but have to make three adjustments: we use F_2 rather than F_1, we need to compute K_2 based on K_1 and T_1, and we need to account for δ_1.

$$
f_2 = \overbrace{(T_1 \cdot K_1)'}^{\substack{\text{1 by 3} \\ =K_2'}} \cdot \underbrace{F_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1} \\ \text{Fertility} \\ \text{by current kids}}} \cdot \delta_1
$$

Proceeding to $a = 3$, following what we just did, we have a similar formula for f_3.

$$f_3 = \overbrace{(T_2 \cdot T_1 \cdot K_1)}^{\substack{\text{1 by 3}\\ =K_3'}}{}' \cdot \underbrace{F_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1}\\ \text{Fertility}\\ \text{by current kids}}} \cdot (\delta_1 \cdot \delta_2)$$

The total number of new kids born to our population by the end of $a = 3$ is therefore:

$$f_{\text{total}} = f_1 + f_2 + f_3$$

If we are dealing with households that have a household head and a spouse, then the number of individuals born to each household member is

$$f_{\text{per-indi}} = \frac{f_{\text{total}}}{2}$$

When we have different types of households, we compute $f_{\text{total}}\,(\text{EDU}, \text{MARITAL})$, and compute the weighted average across households types to obtain the total national fertility level.

1.1.3 Overall Formula

Given what we have developed above, we have the following fertility formula. Let N be the maximum number of kids

$$f_{\text{total}} = \sum_{a=a_{min}}^{a_{max}} \left(\Pi_{i=1}^{a-1} T_i \cdot K_1\right)' \cdot \left(\sum_{j=1}^{N+1} \Psi_j \cdot T_a \cdot \Omega_j\right) \begin{bmatrix} 0 \\ 1 \\ .. \\ N-1 \\ N \end{bmatrix} \cdot \Pi_i^{a-1} \delta_i = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$