

# Matrix and Household Size Transition Across Lifecycle

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## 1 Demographics and Transition Matrix

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### 1.1 Initial Population, Kids Count Transition, and Fertility

A household can have  $k = 0$ ,  $k = 1$ , or  $k = 2$  kids residing in the household. The household (head) has three possible age-groupings:  $a = 1$  (perhaps 18 to 30),  $a = 2$  (perhaps 31 to 50), and  $a = 3$  (perhaps 51 and after).

We have two  $3 \times 3$  transition matrixes for households,  $T_1$  and  $T_2$ .  $T_1$  shows the transition probabilities of going from  $k$  kids at age ( $a = 1$ ) to  $k'$  kids at ( $a = 2$ ), each row of  $T_1$  corresponds to  $k$ , and each column corresponds to  $k'$ , each cell represents  $P(k'|k)$ , and the probabilities for each row sums up to 1. Similarly for  $T_2$ . For example,  $T_1$  might be equal to the following matrix, which shows substantial persistence.

$$T_1 = \begin{bmatrix} t_{11}^1 & t_{12}^1 & t_{13}^1 \\ t_{21}^1 & t_{22}^1 & t_{23}^1 \\ t_{31}^1 & t_{32}^1 & t_{33}^1 \end{bmatrix} = \begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.05 & 0.80 & 0.15 \\ 0.05 & 0.16 & 0.79 \end{bmatrix}$$

Additionally, not all households survive from across  $a$  over time.  $0 < \delta_2 < 1$  and  $0 < \delta_3 < 1$  represent the probabilities that those in  $a = 1$  and  $a = 2$  survive into  $a = 2$  and  $a = 3$ , respectively. And  $\delta_1 = 1$ .

Finally, at the initial age, we have initial population share vector  $K_1$ , which is a  $3 \times 1$  vector that sums up to 1 and that indicates the share of households at the start of ( $a = 1$ ) with 0, 1, and 2 kids. Subsequently, we have  $K_2$  and also  $K_3$  population vectors for the population shares across kids count groups at later ages. For example,  $K_1$  might be equal to the following vector, with a larger share of households having less children.

$$K_1 = \begin{bmatrix} 0.70 \\ 0.20 \\ 0.10 \end{bmatrix}$$

Given the  $\{T_a\}_{a=1}^{a_{\max}}$  matrixes and the  $K_{a_{\min}}$  vector, we want to know the number of children born (the fertility level) given this demographic structure above. Think of the transition matrixes and the initial vector

as a “reduced-form” way of capturing underlying demographic dynamics of a particular type of household belonging to a particular cohort. A more complete model of demographics might consider age of each child explicitly and also changes in the structure of the household-head(s).

### 1.1.1 Arrival of Children Probabilities

We interpret an increase in the number of kids in the household as a fertility event. This means, when the household transitions from 0 to 1 (2) kids, there is 1 (2) more children born. For the household with 2 children, no new children can be born for that period. But if the 2 children household transitions to a 1 child household, then a new child can be born to this household. When the number of children go down, we interpret this as children leaving a household (perhaps reaching age 18 or for any other reason). In another word, at any age, we have the current number of kids who reside in the household, not the total number of children the household has had.

Given this interpretation, we can create fertility matrixes  $F_1$ ,  $F_2$ , and  $F_3$ , based on  $T_1$ ,  $T_2$ , and  $T_3$ . For example, for  $F_1$ , we have:

$$\begin{aligned}
F_{11} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t_{11}^1 & t_{12}^1 & t_{13}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ t_{21}^1 + t_{22}^1 & t_{23}^1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{13} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_{31}^1 + t_{32}^1 + t_{33}^1 & 0 & 0 \end{bmatrix} \\
F_1 &= F_{11} + F_{12} + F_{13}
\end{aligned}$$

In the  $F_1$  matrix we constructed, the first column corresponds to the probability of not increasing the number of kids, the second column corresponds to the probability of increasing the number of kids by 1, and the third column corresponds to the probability of increasing the number of kids by 2.

We construct similar matrixes  $F_{a=2}$  and  $F_{a=3}$  at the other two ages. These fertility matrixes are a function of the the kids transition matrixes. We use them in the subsection below to computer fertility.

### 1.1.2 Fertility Level

Given the fertility matrixes, we can compute, at each  $a$ , the number of new kids born as a fraction of the households available at the start of ( $a = 1$ ), where mass = 1.

We start with  $a = 0$ , which will provide us with the initial condition prior to the start of the first period. The number of kids at  $a = 0$  is  $f_0$ , which is the **dot product** of the distribution of households with 0, 1, and 2 kids and the vector  $[0, 1, 2]$ .

$$f_0 = \overbrace{K_1'}^{1 \text{ by } 3} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The number of kids born during  $a = 1$ , meaning “by the end of”  $a = 1$ , is  $f_1$ :

$$f_1 = \overbrace{K'_1}^{1 \text{ by } 3} \cdot \underbrace{F_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{3 \text{ by } 1 \\ \text{Fertility} \\ \text{by current kids}}}$$

Proceeding to  $a = 2$ , we have a similar formula for  $f_2$ , but have to make three adjustments: we use  $F_2$  rather than  $F_1$ , we need to compute  $K_2$  based on  $K_1$  and  $T_1$ , and we need to account for  $\delta_2$ .

$$f_2 = \overbrace{(T_1 \cdot K_1)'}^{\substack{1 \text{ by } 3 \\ =K'_2}} \cdot \underbrace{F_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{3 \text{ by } 1 \\ \text{Fertility} \\ \text{by current kids}}} \cdot \delta_2$$

Proceeding to  $a = 3$ , following what we just did, we have a similar formula for  $f_3$ .

$$f_3 = \overbrace{(T_2 \cdot T_1 \cdot K_1)'}^{\substack{1 \text{ by } 3 \\ =K'_3}} \cdot \underbrace{F_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{3 \text{ by } 1 \\ \text{Fertility} \\ \text{by current kids}}} \cdot (\delta_2 \cdot \delta_3)$$

The sum of the number of kids at the beginning of the first period and total number of new kids born to our population by the end of  $a = 3$  is therefore:

$$f_{\text{total}} = f_0 + \underbrace{(f_1 + f_2 + f_3)}_{\text{New born}}$$

If we are dealing with households that have a household head and a spouse, then the number of individuals born to each household member is

$$f_{\text{per-indi}} = \frac{f_{\text{total}}}{2}.$$

When we have different types of households, we compute  $f_{\text{total}}(\text{EDU}, \text{MARITAL})$ , and compute the weighted average across households types to obtain the total national fertility level.

$$f_{\text{total, all-groups}} = \sum_{\text{EDU}} \sum_{\text{MARITAL}} P(\text{EDU}, \text{MARITAL}) \cdot f_{\text{total}}(\text{EDU}, \text{MARITAL})$$

### 1.1.3 Overall Fertility Formula

Given what we have developed above, we have the following fertility formula. Let  $N$  be the maximum number of kids and  $a_{\text{max}}$  be the maximum age group index, and remember  $\delta_1 = 1$ , we have:

$$f_{\text{total}} = f_0(N, K_1) + \sum_{a=1}^{a_{\text{max}}} \left( \underbrace{\left( \prod_{i=1}^a T_i \cdot K_1 \right)}_{\substack{1 \text{ by } (N+1) \\ = K'_a}} \cdot \underbrace{\left( \sum_{j=1}^{N+1} \Psi_j \cdot T_a \cdot \Omega_j \right)}_{\substack{(N+1) \text{ by } 1 \\ \text{Fertility} \\ \text{by current kids}}} \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \\ \dots \\ N-1 \\ N \end{bmatrix}}_{\substack{(N+1) \text{ by } (N+1) \\ F_a, \text{ Fertility matrix}}} \cdot \prod_{i=1}^a \delta_i \right)$$

The notations from the equation above follow what we have written. We introduce above also  $\Psi_j$  and  $\Omega_j$  which are helper matrixes of 0s and 1s to help compute the fertility matrixes from the kids transition matrixes. We saw examples of these matrixes in the 3 by 3 example earlier. More generally and specifically, these matrixes are:

- $\Psi_j$  are [standard basis](#) matrixes with  $\psi_{jj}^j = 1$  and  $\psi_{j,j'}^j = 0$  for all  $j \neq j'$
- $\Omega_j$  are  $((N+1) + 1 - j)$  by  $((N+1) + 1 - j)$  identity matrixes with  $(j-1)$  zero columns appended to the right and with  $(j-1)$  standard basis with leading 1 appended to the top. In another word, it is a partially sequentially-[lower-shifted matrixes](#).

Given the formulation above, we can write  $f_{\text{total}}$  as an function of the transition matrixes, initial  $K_1$  vector, and the mortality rates. Let's call this the *FERT* function.

$$f_{\text{total}} = \text{FERT}(\{T_a\}_{a=1}^{a_{\text{max}}}, K_1, \{\delta_i\}_{a=1}^{a_{\text{max}}})$$

#### 1.1.4 Overall Fertility and Binomial Chance of Success Rescaler

Following the [binomial fitting procedure](#), the transition matrixes and initial  $K_1$  vector can be fitted as functions of varying  $\theta$  parameters as well as the shared  $M$  parameter that define binomial distributions. In this section, we develop a way to rescale these  $\theta$  parameters to increase and decrease fertility.

First, let  $\theta_n^a$  be the “chance of success” parameter fitted/estimated for the  $n$ th row of the transition matrix  $T_a$ , and let  $\theta_{K_1}$  be the “chance of success” parameter fitted/estimated for  $K_1$ . We have, given shared  $M$  (see [binomial fitting procedure](#)) :

$$f_{\text{total}} = \text{FERT} \left( \left\{ \left\{ \theta_n^a \right\}_{n=0}^{N-1} \right\}_{a=1}^{a_{\text{max}}}, \theta_{K_1}, M, \{\delta_i\}_{a=1}^{a_{\text{max}}} \right)$$

Second, to adjust the overall fertility rate, we can adjust the  $\lambda \in \mathbb{R}$  parameter, which provides a common value to rescale all  $\theta$  parameters by. Specifically, the adjustment follows this functional form:

$$\hat{\theta}(\theta, \lambda) = \theta \cdot \left( \frac{\exp(\lambda)}{1 + \theta(\exp(\lambda) - 1)} \right)$$

Given this function form, when  $\lambda = 0$ , we have  $\hat{\theta}(\theta, \lambda = 0) = \theta$ . Additionally,  $0 \leq \hat{\theta}(\lambda) \leq 1$  for all  $\lambda \in \mathbb{R}$ , which allows  $\hat{\theta}$  to be used as the “chance of success” parameter as we freely vary  $\lambda$ . See more information about this scaling method and visualization in [Rescale a Fraction Between 0 and 1](#).

Third, the fertility formula, allowing for  $\lambda$  adjustment becomes:

$$\begin{aligned} f_{\text{total}}(\lambda) &= \text{FERT} \left( \left\{ \left\{ \hat{\theta}(\theta_n^a, \lambda) \right\}_{n=0}^{N-1} \right\}_{a=1}^{a_{\text{max}}}, \hat{\theta}(\theta_{K_1}, \lambda), M, \{\delta_i\}_{a=1}^{a_{\text{max}}} \right) \\ &= \text{FERT}(\lambda) \end{aligned}$$

Fourth, we know that fertility is strictly increasing in  $\lambda$ .

$$\frac{df_{\text{total}}(\lambda)}{d\lambda} > 0$$

Fifth, we know that fertility is bounded above and below. The bounds on fertility are the extreme cases where all individuals either have 0 children, or where all individuals have the maximum number of children, in each period and initially.

$$0 \leq f_{\text{total}}(\lambda) \leq N + \sum_{a=1}^{a_{\max}} N \cdot \prod_{i=1}^a \delta_i = N \left( 1 + \sum_{a=1}^{a_{\max}} \prod_{i=1}^a \delta_i \right)$$

In another word, we have these limiting results:

$$\begin{aligned} \lim_{\lambda \rightarrow -\infty} f_{\text{total}}(\lambda) &= 0 \\ \lim_{\lambda \rightarrow \infty} f_{\text{total}}(\lambda) &= N \left( 1 + \sum_{a=1}^{a_{\max}} \prod_{i=1}^a \delta_i \right) \end{aligned}$$

### 1.1.5 Life-expectancy Formula

Related to fertility is the issue of life-expectancy, which is “the average number of years of life a person who has attained a given age can expect to live.” In our situation here, life expectancy is determined by the life-cycle of  $\{\delta_a\}_{a=1}^{a_{\max}}$  values, and the ages corresponding to each  $a$ .

Define  $\text{AGE}_1$  as the age at the start and during  $a = 1$ . Subsequently, define  $\text{AGE}_a$  as the age at the start and during age  $a$ . We have a unit mass of individuals, with different shares surviving until different ages.

During  $a = 1$ , all survived from  $a = 0$ , since  $\delta_1 = 1$ . However, only  $\delta_2$  fraction survives into  $a = 2$ . This means  $1 - \delta_2$  fraction of individuals pass away between  $a = 1$  and  $a = 2$ , giving these individuals a terminal age of  $a = 1$ . Of those that survived into  $a = 2$ , only  $\delta_3$  fraction last into  $a = 3$ , this means  $(\delta_2 \cdot (1 - \delta_3))$  fraction of individuals pass away between  $a = 2$  and  $a = 3$ , giving these individuals a terminal age of  $a = 2$ . Of those that survived into  $a = 3$ , only  $\delta_4$  fraction last into  $a = 4$ , this means  $(\delta_2 \cdot \delta_3 \cdot (1 - \delta_4))$  fraction of individuals pass away between  $a = 3$  and  $a = 4$ , giving these individuals a terminal age of  $a = 3$ .

The fraction of individuals passing away between  $a$  and  $a + 1$ , giving them a terminal age of  $a$ , is determined by the product of the fraction of individuals who survive until age  $a$ , multiplied by the fraction of individuals who pass away between age  $a$  and  $a + 1$ , conditional on surviving until age  $a$ .

$$P(\text{terminal age is } a) = (1 - \delta_{a+1}) \prod_{i=1}^a \delta_i$$

Let  $\delta_{(a_{\max}+1)} = 0$ , the life expectancy for the individual with the  $\{\delta_a\}_{a=1}^{a_{\max}}$  trajectory of mortality across the lifecycle, is:

$$\begin{aligned} \text{life expectancy} &= \sum_{a=1}^{a_{\max}} P(\text{terminal age is } a) \cdot \text{AGE}_a \\ &= \sum_{a=1}^{a_{\max}} \left( (1 - \delta_{a+1}) \prod_{i=1}^a \delta_i \right) \cdot \text{AGE}_a \end{aligned}$$