Nested Constant Elasticity of Substitution Production Function

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2021-06-26

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1 Nested CES Production Function

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1.1 Denesting the Nested CES Problem

We have the following production function with four inputs x_1, x_2, y_1 and y_2. There are three ρ parameters ρ_x, ρ_y and ρ_o that correspond to inner-nest and outter nest elasticity of substitution between inputs.

The firm's expenditure minimization problem has the following objective:

$$\min_{x_1,x_2,y_1,y_2} \left(x_1 \cdot p_{x_1} + x_2 \cdot p_{x_2} + y_1 \cdot p_{y_1} + y_2 \cdot p_{y_2}\right)$$

The production quantity constraint is, using a constant-returns doubly-nested production function:

$$Y = Z \cdot \left(\beta_{o_1}\left(\left(\beta_{x_1}x_1^{\rho_x} + \beta_{x_2}x_2^{\rho_x}\right)^{\frac{1}{\rho_x}}\right)^{\rho_o} + \beta_{o_2}\left(\left(\beta_{y_1}y_1^{\rho_y} + \beta_{y_2}y_2^{\rho_y}\right)^{\frac{1}{\rho_y}}\right)^{\rho_o}\right)^{\frac{1}{\rho_o}}$$

Note that we are assuming constant-returns to scale in a competitive setting, so firms do not make profits. We solve for expenditure minimization rather than profit maximization.

Rather than solving the problem above directly in an expenditure minimization, we can divide the problem above into three parts, the **X Problem**, the **Y Problem** and the **Z Problem**.

1.1.1 Marginal Product of Labor

A key object to consider is the marginal product of input (labor or capital). Taking derivative of output Y with respect to input x_1, we have:

$$\frac{\partial Y}{\partial x_1} = \left[\frac{1}{\rho_o}Z\left(\beta_{o_1}\left(\left(\beta_{x_1}x_1^{\rho_x} + \beta_{x_2}x_2^{\rho_x}\right)^{\frac{1}{\rho_x}}\right)^{\rho_o} + \beta_{o_2}\left(\left(\beta_{y_1}y_1^{\rho_y} + \beta_{y_2}y_2^{\rho_y}\right)^{\frac{1}{\rho_y}}\right)^{\rho_o}\right)^{\frac{1}{\rho_o}-1}\right] \cdot \left[\rho_o\beta_{o_1}\left(\left(\beta_{x_1}x_1^{\rho_x} + \beta_{x_2}x_2^{\rho_x}\right)^{\frac{1}{\rho_x}}\right)^{\rho_o-1}\right] \cdot \left[\frac{1}{\rho_x}\right.$$

What is the relationship between the marginal product of labor and the wage? Let λ be the lagrange multiplier for the overall problem:

$$p_{x_1} = \lambda \cdot \left(\frac{\partial Y}{\partial x_1}\right)$$

1.1.2 Three Denested Sub-problems, X, Y and O Problems

The X problem:

$$\min_{x_1, x_2} \left(x_1 \cdot p_{x_1} + x_2 \cdot p_{x_2}\right)$$

$$O_x = \left(\beta_{x_1} x_1^{\rho_x} + \beta_{x_2} x_2^{\rho_x}\right)^{\frac{1}{\rho_x}}$$

The Y problem:

$$\min_{y_1, y_2} \left(y_1 \cdot p_{y_1} + y_2 \cdot p_{y_2}\right)$$

$$O_y = \left(\beta_{y_1} y_1^{\rho_y} + \beta_{y_2} y_2^{\rho_y}\right)^{\frac{1}{\rho_y}}$$

The O problem:

$$\min_{o_1, o_2} \left(O_x \cdot p_{o_1} + O_y \cdot p_{o_2}\right)$$

$$Y = Z \cdot \left(\beta_{o_1} O_x^{\rho_o} + \beta_{o_2} O_y^{\rho_o}\right)^{\frac{1}{\rho_o}}$$

1.1.3 Marginal Product of Labor for De-nested Problem

We can also take the derivative of the output requirement for the X problem with respect to x_1, we have:

$$\frac{\partial O_x}{\partial x_1} = \left[\frac{1}{\rho_x}\left(\beta_{x_1} x_1^{\rho_x} + \beta_{x_2} x_2^{\rho_x}\right)^{\frac{1}{\rho_x}-1}\right] \cdot \left[\rho_x \beta_{x_1} x_1^{\rho_x - 1}\right]$$

Which simplifies a little bit to:

$$\frac{\partial O_x}{\partial x_1} = \left[\left(\beta_{x_1} x_1^{\rho_x} + \beta_{x_2} x_2^{\rho_x}\right)^{\frac{1}{\rho_x}-1}\right] \cdot \left[\beta_{x_1} x_1^{\rho_x - 1}\right]$$

What is the relationship between the marginal product of labor and the wage for the problem in the subnest? Let λ_x be the lagrange multiplier for the lagrange multiplier specific to the subnest:

$$p_{x_1} = \lambda_x \cdot \left(\frac{\partial O_x}{\partial x_1}\right)$$

This means that we have the following FOC from solving the expenditure minimization problem:

$$p_{x_1} = \lambda_x \cdot \left[\left(\beta_{x_1} x_1^{\rho_x} + \beta_{x_2} x_2^{\rho_x}\right)^{\frac{1}{\rho_x}-1}\right] \cdot \left[\beta_{x_1} x_1^{\rho_x - 1}\right]$$