Matrix and Household Size Transition Across Lifecycle

Fan Wang

2022-07-26

Contents

1 Demographics and Transition Matrix

Go to the **RMD**, **R**, **PDF**, or **HTML** version of this file. Go back to fan's REconTools research support package, R4Econ examples page, PkgTestR packaging guide, or Stat4Econ course page.

1.1 Initial Population, Kids Count Transition, and Fertility

A household can have $k = 0$, $k = 1$, or $k = 2$ kids residing in the household. The household (head) has three possible age-groupings: $a = 1$ (perhaps 18 to 30), $a = 2$ (perhaps 31 to 50), and $a = 3$ (perhaps 51 and after).

We have two 3×3 transition matrixes for households, T_1 and T_2. T_1 shows the transition probabilities of going from k kids at age $(a = 1)$ to k' kids at $(a = 2)$, each row of T_1 corresponds to k, and each column corresponds to k', each cell represents $P(k'|k)$, and the probabilities for each row sums up to 1. Similarly for T_2. For example, T_1 might be equal to the following matrix, which shows substantial persistence.

$$T_1 = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ t^1_{21} & t^1_{22} & t^1_{23} \\ t^1_{31} & t^1_{32} & t^1_{33} \end{bmatrix} = \begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.05 & 0.80 & 0.15 \\ 0.05 & 0.16 & 0.79 \end{bmatrix}$$

Additionally, not all households survive from across a over time. $0 < \delta_2 < 1$ and $0 < \delta_3 < 1$ represent the probabilities that those in $a = 1$ and $a = 2$ survive into $a = 2$ and $a = 3$, respectively. And $\delta_1 = 1$.

Finally, at the initial age, we have initial population share vector K_1, which is a 3×1 vector that sums up to 1 and that indicates the share of households at the start of $(a = 1)$ with 0, 1, and 2 kids. Subsequently, we have K_2 and also K_3 population vectors for the population shares across kids count groups at later ages. For example, K_1 might be equal to the following vector, with a larger share of households having less children.

$$K_1 = \begin{bmatrix} 0.70 \\ 0.20 \\ 0.10 \end{bmatrix}$$

Given the $\{T_a\}_{a=1}^{a_{\max}}$ matrixes and the $K_{a_{\min}}$ vector, we want to know the number of children born (the fertility level) given this demographic structure above. Think of the transition matrixes and the initial vector as a “reduced-form” way of capturing underlying demographic dynamics of a particular type of household belonging to a particular cohort. A more complete model of demographics might consider age of each child explicitly and also changes in the structure of the household-head(s).

1.1.1 Arrival of Children Probabilities

We interpret an increase in the number of kids in the household as a fertility event. This means, when the household transitions from 0 to 1 (2) kids, there is 1 (2) more children born. For the household with 2 children, no new children can be born for that period. But if the 2 children household transitions to a 1 child household, then a new child can be born to this household. When the number of children go down, we interpret this as children leaving a household (perhaps reaching age 18 or for any other reason). In another word, at any age, we have the current number of kids who reside in the household, not the total number of children the household has had.

Given this interpretation, we can create fertility matrixes F_1, F_2, and F_3, based on T_1, T_2, and T_3. For example, for F_1, we have:

$$
\begin{aligned}
F_{11} =& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{12} =& \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ t^1_{21} + t^1_{22} & t^1_{23} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
F_{13} =& \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T_1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^1_{31} + t^1_{32} + t^1_{33} & 0 & 0 \end{bmatrix} \\
F_1 =& F_{11} + F_{12} + F_{13}
\end{aligned}
$$

In the F_1 matrix we constructed, the first column corresponds to the probability of not increasing the number of kids, the second column corresponds to the probability of increasing the number of kids by 1, and the third column corresponds to the probability of increasing the number of kids by 2.

We construct similar matrixes $F_{a=2}$ and $F_{a=3}$ at the other two ages. These fertility matrixes are a function of the the kids transition matrixes. We use them in the subsection below to computer fertility.

1.1.2 Fertility Level

Given the fertility matrixes, we can compute, at each a, the number of new kids born as a fraction of the households available at the start of ($a = 1$), where mass = 1.

We start with $a = 0$, which will provide us with the initial condition prior to the start of the first period. The number of kids at $a = 0$ is f_0, which is the dot product of the distribution of households with 0, 1, and 2 kids and the vector $[0, 1, 2]$.

$$
f_0 = \overbrace{K'_1}^{\text{1 by 3}} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}
$$

The number of kids born during $a = 1$, meaning “by the end of” $a = 1$, is f_1:

$$f_1 = \overbrace{K_1'}^{\text{1 by 3}} \cdot \underbrace{F_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1} \\ \text{Fertility} \\ \text{by current kids}}}$$

Proceeding to $a = 2$, we have a similar formula for f_2, but have to make three adjustments: we use F_2 rather than F_1, we need to compute K_2 based on K_1 and T_1, and we need to account for δ_2.

$$f_2 = \overbrace{(T_1 \cdot K_1)'}^{\substack{\text{1 by 3} \\ =K_2'}} \cdot \underbrace{F_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1} \\ \text{Fertility} \\ \text{by current kids}}} \cdot \delta_2$$

Proceeding to $a = 3$, following what we just did, we have a similar formula for f_3.

$$f_3 = \overbrace{(T_2 \cdot T_1 \cdot K_1)'}^{\substack{\text{1 by 3} \\ =K_3'}} \cdot \underbrace{F_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\substack{\text{3 by 1} \\ \text{Fertility} \\ \text{by current kids}}} \cdot (\delta_2 \cdot \delta_3)$$

The sum of the number of kids at the beginning of the first period and total number of new kids born to our population by the end of $a = 3$ is therefore:

$$f_{\text{total}} = f_0 + \underbrace{(f_1 + f_2 + f_3)}_{\text{New born}}$$

If we are dealing with households that have a household head and a spouse, then the number of individuals born to each household member is

$$f_{\text{per-indi}} = \frac{f_{\text{total}}}{2}.$$

When we have different types of households, we compute $f_{\text{total}}(\text{EDU}, \text{MARITAL})$, and compute the weighted average across households types to obtain the total national fertility level.

$$f_{\text{total, all-groups}} = \sum_{\text{EDU}} \sum_{\text{MARITAL}} P(\text{EDU}, \text{MARITAL}) \cdot f_{\text{total}}(\text{EDU}, \text{MARITAL})$$

1.1.3 Overall Fertility Formula

Given what we have developed above, we have the following fertility formula. Let N be the maximum number of kids and $a_{\max}$ be the maximum age group index, and remember $\delta_1 = 1$, we have:

$$
f_{\text{total}} = f_0(N, K_1) + \sum_{a=1}^{a_{\max}} \left(\overbrace{\left(\prod_{i=1}^{a} T_i \cdot K_1 \right)'}^{\substack{1 \text{ by } (N+1) \\ = K_a'}} \cdot \underbrace{\overbrace{\left(\sum_{j=1}^{N+1} \Psi_j \cdot T_a \cdot \Omega_j \right)}^{\substack{(N+1) \text{ by } (N+1) \\ F_a, \text{ Fertility matrix}}} \begin{bmatrix} 0 \\ 1 \\ .. \\ N-1 \\ N \end{bmatrix}}_{\substack{(N+1) \text{ by } 1 \\ \text{Fertility} \\ \text{by current kids}}} \cdot \prod_{i=1}^{a} \delta_i \right)
$$

The notations from the equation above follow what we have written. We introduce above also Ψ_j and Ω_j which are helper matrixes of 0s and 1s to help compute the fertility matrixes from the kids transition matrixes. We saw examples of these matrixes in the 3 by 3 example earlier. More generally and specifically, these matrixes are:

- Ψ_j are standard basis matrixes with $\psi^j_{jj} = 1$ and $\psi^j_{j,j'} = 0$ for all $j \neq j'$
- Ω_j are $((N+1)+1-j)$ by $((N+1)+1-j)$ identity matrixes with $(j-1)$ zero columns appended to the right and with $(j-1)$ standard basis with leading 1 appended to the top. In another word, it is a partially sequentially-lower-shifted matrixes.

Given the formulation above, we can write f_{total} as an function of the transition matrixes, initial K_1 vector, and the mortality rates. Let's call this the $FERT$ function.

$$
f_{\text{total}} = \text{FERT}(\{T_a\}_{a=1}^{a_{\max}}, K_1, \{\delta_i\}_{a=1}^{a_{\max}})
$$

1.1.4 Overall Fertility and Binomial Chance of Success Rescaler

Following the binomial fitting procedure, the transition matrixes and initial K_1 vector can be fitted as functions of varying θ parameters as well as the shared M parameter that define binomial distributions. In this section, we develop a way to rescale these θ parameters to increase and decrease fertility.

First, let θ^a_n be the "chance of success" parameter fitted/estimated for the nth row of the transition matrix T_a, and let θ_{K_1} be the "chance of success" parameter fitted/estimated for K_1. We have, given shared M (see binomial fitting procedure) :

$$
f_{\text{total}} = \text{FERT}\left(\left\{ \{\theta^a_n\}_{n=0}^{N-1} \right\}_{a=1}^{a_{\max}}, \theta_{K_1}, M, \{\delta_i\}_{a=1}^{a_{\max}} \right)
$$

Second, to adjust the overall fertility rate, we can adjust the $\lambda \in \mathbb{R}$ parameter, which provides a common value to rescale all θ parameters by. Specifically, the adjustment follows this functional form:

$$
\widehat{\theta}(\theta, \lambda) = \theta \cdot \left(\frac{\exp(\lambda)}{1 + \theta(\exp(\lambda) - 1)} \right)
$$

Given this function form, when $\lambda = 0$, we have $\widehat{\theta}(\theta, \lambda = 0) = \theta$. Additionally, $0 \leq \widehat{\theta}(\lambda) \leq 1$ for all $\lambda \in \mathbb{R}$, which allows $\widehat{\theta}$ to be used as the "chance of success" parameter as we freely vary λ. See more information about this scaling method and visualization in Rescale a Fraction Between 0 and 1.

Third, the fertility formula, allowing for λ adjustment becomes:

$$
\begin{aligned}
f_{\text{total}}(\lambda) &= \text{FERT}\left(\left\{ \left\{ \widehat{\theta}(\theta^a_n, \lambda) \right\}_{n=0}^{N-1} \right\}_{a=1}^{a_{\max}}, \widehat{\theta}(\theta_{K_1}, \lambda), M, \{\delta_i\}_{a=1}^{a_{\max}} \right) \\
&= \text{FERT}(\lambda)
\end{aligned}
$$

Fourth, we know that fertility is strictly increasing in λ.

$$\frac{df_{\text{total}}(\lambda)}{d\lambda} > 0$$

Fifth, we know that fertility is bounded above and below. The bounds on fertility are the extreme cases where all individuals either have 0 children, or where all individuals have the maximum number of children, in each period and initially.

$$0 \le f_{\text{total}}(\lambda) \le N + \sum_{a=1}^{a_{\max}} N \cdot \prod_{i=1}^{a} \delta_i = N \left(1 + \sum_{a=1}^{a_{\max}} \prod_{i=1}^{a} \delta_i \right)$$

In another word, we have these limiting results:

$$\begin{aligned}
\lim_{\lambda \to -\infty} f_{\text{total}}(\lambda) &= 0 \\
\lim_{\lambda \to \infty} f_{\text{total}}(\lambda) &= N \left(1 + \sum_{a=1}^{a_{\max}} \prod_{i=1}^{a} \delta_i \right)
\end{aligned}$$

1.1.5 Life-expectancy Formula

Related to fertility is the issue of life-expectancy, which is "the average number of years of life a person who has attained a given age can expect to live." In our situation here, life expectancy is determined by the life-cycle of $\{\delta_a\}_{a=1}^{a_{\max}}$ values, and the ages corresponding to each a.

Define AGE_1 as the age at the start and during $a = 1$. Subsequently, define AGE_a as the age at the start and during age a. We have a unit mass of individuals, with different shares surviving until different ages.

During $a = 1$, all survived from $a = 0$, since $\delta_1 = 1$. However, only δ_2 fraction survives into $a = 2$. This means $1 - \delta_2$ fraction of individuals pass away between $a = 1$ and $a = 2$, giving these individuals a terminal age of $a = 1$. Of those that survived into $a = 2$, only δ_3 fraction last into $a = 3$, this means $(\delta_2 \cdot (1 - \delta_3))$ fraction of individuals pass away between $a = 2$ and $a = 3$, giving these individuals a terminal age of $a = 2$. Of those that survived into $a = 3$, only δ_4 fraction last into $a = 4$, this means $(\delta_2 \cdot \delta_3 \cdot (1 - \delta_4))$ fraction of individuals pass away between $a = 3$ and $a = 4$, giving these individuals a terminal age of $a = 3$.

The fraction of individuals passing away between a and $a + 1$, giving them a terminal age of a, is determined by the product of the fraction of individuals who survive until age a, multiplied by the fraction of individuals who pass away between age a and $a + 1$, conditional on surviving until age a.

$$P(\text{terminal age is a}) = (1 - \delta_{a+1}) \prod_{i=1}^{a} \delta_i$$

Let $\delta_{(a_{\max}+1)} = 0$, the life expectancy for the individual with the $\{\delta_a\}_{a=1}^{a_{\max}}$ trajectory of mortality across the lifecycle, is:

$$\begin{aligned}
\text{life expectancy} &= \sum_{a=1}^{a_{\max}} P\left(\text{terminal age is a}\right) \cdot \text{AGE}_a \\
&= \sum_{a=1}^{a_{\max}} \left((1 - \delta_{a+1}) \prod_{i=1}^{a} \delta_i \right) \cdot \text{AGE}_a
\end{aligned}$$