New Results on the Computation-Communication Tradeoff for Heterogeneous Coded Distributed Computing

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***Abstract*—Coded distributed computing (CDC) can alleviate the communication bottleneck in distributed computing systems by leveraging coding opportunities via redundant computation. While the optimal computation-communication tradeoff has been well studied for homogeneous systems, it remains largely unknown for heterogeneous systems where workers have different computation capabilities. We obtain a new achievable upper bound of the optimal communication load for a general heterogeneous CDC system using the MapReduce framework. The proposed scheme uses a flexible coding strategy, rather than greedy algorithms, so that it can avoid unicasting in the data shuffling. A lower bound is also established by extending the cut-set bound and the method in [1]. The obtained upper and lower bounds coincide in some non-trivial cases, and degenerate to the existing optimal results in homogeneous systems.**

I. Introduction

Communication overhead has been a major bottleneck in large-scale distributed computing. Coded distributed computing (CDC), introduced in [2], offers a promising solution to reduce the communication load in the MapReduce framework. MapReduce is a well-established distributed framework that consists of three phases. First, the input data is split into multiple files, and passed to distributed workers to produce intermediate values (I.V.s) by Map functions (Map phase). These I.V.s are then shuffled among the workers via a shared link (Shuffle phase). Finally, each worker combines its associated I.V.s and reduces the target outputs (Reduce phase). In [2], Li *et. al.* show that the communication load in the Shuffle phase can be substantially reduced by exploiting coded multicasting opportunities, at the expense of assigning redundant computation tasks at each worker in the Map phase.

Many existing studies on the computation-communication tradeoff in CDC focus on homogeneous systems [2]–[7], where each worker has an even computation load in the Map phase and computes the same number of target functions in the Reduce phase. These results cannot be directly generalized to heterogeneous CDC (HetCDC) systems where each worker can be assigned with different computation load and different number of target functions. Recently, several attempts have been made to study the computation-communication tradeoff in HetCDC [8]–[13], as reviewed below, however the optimal tradeoff still remains largely unknown. It is difficult on both ends of this problem, either the achievable upper bound or the theoretical lower bound.

Prior Works: Achievable schemes of HetCDC have been studied in [8]–[13]. The authors in [8] obtain the optimal computation-communication tradeoff for the systems with $K = 3$ workers. The authors in [9] also obtain an achievable upper bound, which is optimal for $K \leq 3$ workers, but is only optimal in two extreme cases for $K > 3$ workers. These works [8], [9] reveal that the number of input files repetitively computed by workers in different worker sets should be different so as to efficiently exploit coded multicasting opportunities in the Shuffle phase. Note that [8], [9] only consider heterogeneous computation load of workers in the Map phase, and still assume homogeneous target function assignment in the Reduce phase. As for heterogeneous function assignment, the authors in [10], [11] obtain achievable upper bounds for cascaded and non-cascaded computing systems, respectively. These works [10], [11] reveal that the complexity of encoding and decoding in data shuffling can be largely reduced by using a combinatorial design in HetCDC. However, their achievable schemes are limited to specific function assignments, and their considered systems are composed of multiple worker groups where workers in the same group have the same computation load and the same target function assignment.

Establishing the optimal lower bound for the optimal computation-communication tradeoff is also a non-trivial problem. Intuitively, the method in [2] can be extended to HetCDC by iteratively calculating the communication load needed in the data shuffling of each worker subset. However, this method is generally not tight for heterogeneous systems. Another possible method is using cut-set bound, as in [9], [13]. However, just like in many other network coding problems, cut-set bound is not optimal in general, since it cannot subtly discribe the mutual information between nodes in a same set.

Contributions: Different to all the previous works on HetCDC, we consider a more general HetCDC system in the sense that not only the number of input files computed in each worker is different, but the number of target functions assigned to each worker is also different. Both a new achievable upper bound and a new theoretical lower bound for the optimal tradeoff are obtained in this work.

Though our achievable scheme also relies on an optimization problem like [8], the fundamental strategy which the

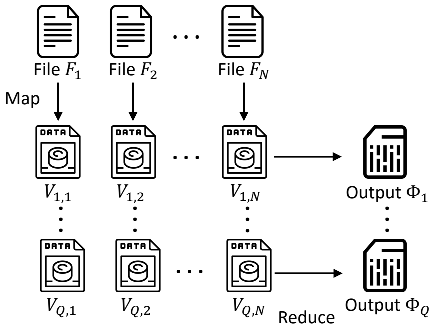

Fig. 1: MapReduce Framework.

optimization problem is based on is more delicately designed in our paper. As a natural idea, the more I.V.s are encoded in one multicast message, the higher communication efficiency this message will have. However in our data shuffling strategy, we choose to slightly reduce the number of multicast messages with the highest efficiency, so that the number of unicast messages, thus the messages with the lowest efficiency, can be largely decreased. Therefore, the overall communication load of our scheme is lower than those achieved by schemes which are only greedy on the number of messages with highest efficiency. Details are shown in Section III and IV.

As for the lower bound, two methods are proposed. The first method is improved from the conventional cut-set bound, while the second method is extended from the bounding strategy in [1]. The obtained lower bound characterizes the intersection of the two bounds obtained by these two methods. Numerical results show that our achievable upper bound coincides with the lower bound in some non-trivial cases, and both bounds degenerate to the optimal tradeoff in the homogeneous systems obtained by [2].

Notations: $[K]$ denotes the set $\{1, 2, \ldots, K\}$, for $K \in \mathbb{Z}^+$. $[a\!:\!b]$ denotes the set $\{a, a+1, \ldots, b-1, b\}$, for $a, b \in \mathbb{Z}, a < b$. $X_\mathcal{S} \triangleq \{X_k : k \in \mathcal{S}\}$, for any set $\mathcal{S}$.

II. System Model

A. MapReduce Model

We consider a HetCDC system consisted of K distributed workers, which aims to compute Q target functions from N input files, where $K, Q, N \in \mathbb{Z}^+$. The input files are denoted by $\{F_1, \ldots, F_N\}$. Each file F_n is an f-bit random variable which uniformly takes value from $[2^f]$, for all $n \in [N]$. The target functions are denoted by $\{\phi_1, \ldots, \phi_Q\}$. Each function ϕ_q maps all the input files into the output value Φ_q with length B bits, for all $q \in [Q]$.

Each worker k, for $k \in [K]$, can store $\lfloor m_k N \rfloor$ input files whose indices are denoted by $\mathcal{M}_k$, where $m_k \in [0, 1]$ and $m_k \in \mathbb{Q}$. We call m_k as the mapping load of worker k, which satisfies $\sum_{k\in[K]} m_k \geq 1$[1]. Each worker k, for $k \in [K]$, is also assigned to compute $\lfloor w_k Q \rfloor$ target functions whose indices are denoted by $\mathcal{W}_k$, where $w_k \in [0, 1]$ and $w_k \in \mathbb{Q}$. We call w_k as the reducing load of worker k. We assume that $\mathcal{W}_j \cap \mathcal{W}_k = \emptyset$ for any $j, k \in [K]$ and $j \neq k$, which implies that $\sum_{k\in[K]} w_k = 1$. The mapping load tuple and the reducing load tuple for all workers are defined as $\mathbf{m} \triangleq [m_1, \ldots, m_K]$ and $\mathbf{w} \triangleq [w_1, \ldots, w_K]$, respectively.

The HetCDC systems follow the MapReduce framework as in Fig. 1. In such systems, we have QN Map functions $g_{q,n}(F_n)$ for $q \in [Q]$ and $n \in [N]$. When an input file F_n is assigned to some worker k, this worker computes $g_{q,n}(F_n)$ for all $q \in [Q]$. We define I.V. $V_{q,n} \triangleq g_{q,n}(F_n) \in [2^T]$ as the result computed from $g_{q,n}(F_n)$, which has T bits. Meanwhile, we also have Q Reduce functions $h_q(\cdot)$ for $q \in [Q]$. Any individual worker can compute the target function ϕ_q, for all $q \in [Q]$, by applying the Reduce function as the following:

$$\phi_q(F_1, \ldots, F_N) = h_q(V_{q,1}, \ldots, V_{q,N}), \tag{1}$$

if all corresponding I.V.s $\{V_{q,n} : n \in [N]\}$ are collected. Following this decomposition, the considered HetCDC system consists of three phases: *Map*, *Shuffle* and *Reduce*.

Map phase: Each worker k, for $k \in [K]$, computes the Map functions of its stored input files, and obtains I.V.s $\{V_{q,n} : q \in [Q], n \in \mathcal{M}_k\}$.

Shuffle phase: Each worker k, for $k \in [K]$, uses its local I.V.s to generate an ℓ_k-bit message X_k, and broadcasts it to the other workers. After receiving messages from other workers, each worker k, for $k \in [K]$, uses its received messages and its local I.V.s to obtain its needed I.V.s $\{V_{q,n} : q \in \mathcal{W}_k, n \in [N]\}$.

Reduce phase: Each worker k, for $k \in [K]$, uses its obtained I.V.s $\{V_{q,n} : q \in \mathcal{W}_k, n \in [N]\}$ to compute the Reduce functions of its assigned target functions $\{\phi_q : q \in \mathcal{W}_k\}$ and obtain the output values $\{\Phi_q : q \in \mathcal{W}_k\}$.

B. Information Theoretic Definition

In the HetCDC system, we focus on the joint input file allocation in the Map phase and the design of encoding and decoding functions of workers in the Shuffle phase, similar to [2]–[13].

In the following, we give the formal information theoretic model for the achievable communication load in the Shuffle phase. Since each I.V. are of same bits, permuting target functions' indices does not change the communication load. Thus, without loss of generality, we can assume that the indices of the target functions assigned to worker k, for $k \in [K]$, are given by $\mathcal{W}_k = \left[\sum_{i=1}^{k-1} \lfloor w_i Q \rfloor + 1 : \sum_{i=1}^{k} \lfloor w_i Q \rfloor\right]$. We claim an overall communication load L is achievable in a K-worker HetCDC system with mapping load $\mathbf{m}$ and reducing load $\mathbf{w}$, if we can find following functions:

1) An input file allocation function
$$c : [K] \rightarrow [2]^N$$
that associates each worker k to a set of indices
$$\mathcal{M}_k = c(k) \subseteq [N],$$
where $|\mathcal{M}_k| \leq \lfloor N m_k \rfloor$, for each $k \in [K]$.
2) K Shuffle phase encoding functions
$$e_k : [2^T]^{\lfloor m_k N \rfloor Q} \rightarrow [2^{\ell_k}],$$
for $k \in [K]$. Each e_k maps the I.V.s computed in worker k to an ℓ_k-bit message
$$X_k = e_k(\{V_{q,n} : q \in [Q], n \in \mathcal{M}_k\})$$

[1]Since this work considers the heterogeneous computation load of workers in both the Map phase and the Reduce phase, we call m_k as the mapping load, rather than the computation load, of worker k, so as to distinguish it from the computation load in the Reduce phase.

for each $k \in [K]$. The overall communication load L in the Shuffle phase is defined as

$$L \triangleq \frac{\sum_{k=1}^{K} \ell_k}{QNT},$$

which is normalized by the total number of bits of all the QN I.V.s computed in the Map phase.

3) K Shuffle phase decoding functions $d_k : [2^{\ell_1}] \times \cdots \times [2^{\ell_K}] \times [2^T]^{\lfloor m_k N \rfloor Q} \to [2^T]^{N \lfloor w_k Q \rfloor}$, for $k \in [K]$. Each d_k maps all the messages and the I.V.s computed in worker k to decode the needed I.V.s

$$\{V_{q,n} : q \in \mathcal{W}_k, n \in [N]\} = d_k(X_1, \ldots, X_k, \{V_{q,n} : q \in [Q], n \in \mathcal{M}_k\})$$

of worker k, for each $k \in [K]$.

The optimal communication load is defined by

$$L^*(\mathbf{m}, \mathbf{w}) \triangleq \inf\{L : (L, \mathbf{m}, \mathbf{w}) \text{ is achievable.}\}.$$

In this work, our goal is to characterize the optimal communication load for any given mapping load $\mathbf{m}$ and any given reducing load $\mathbf{w}$.

III. Main Results

In this section, we first present our achievable upper bound and the theoretical lower bound for the optimal communication load, and then compare our results with existing works.

A. Upper Bound

We start by illustrating the main ideas of our achievable scheme in the general HetCDC system. In the Map phase, we exclusively allocate $a_{\mathcal{S}} N$ input files to worker set $\mathcal{S} \subseteq [K]$. In the Shuffle phase, we consider the data shuffling within all possible worker sets. Specifically, each worker in an arbitrary worker set $\mathcal{S}$ multicasts a coded message to the rest workers in this set. Due to heterogeneity, some workers may need more I.V.s than other workers. These unmatched I.V.s will be sent in the data shuffling within smaller worker sets.

In the data shuffling within an arbitrary worker set $\mathcal{S}$, we use $L_{k,\mathcal{S}}$ to denote the number of multicast messages sent from worker $k \in \mathcal{S}$ to workers $\mathcal{S} \setminus \{k\}$, and use $d_{k,\mathcal{S}}^{\mathcal{S}\setminus\{i\}}$ to denote the number of I.V.s needed by worker $k \in \mathcal{S}$ and will be sent in the data shuffling within a smaller worker set $\mathcal{S} \setminus \{i\}$. By jointly optimizing the file allocation strategy $\{a_{\mathcal{S}}\}$ and the data shuffling strategy $\{L_{k,\mathcal{S}}, d_{k,\mathcal{S}}^{\mathcal{S}\setminus\{i\}}\}$, we can obtain our achievable communication load, which is given in the following theorem. The proof is in Section IV.

Theorem 1. *(Achievable load) For a heterogeneous MapReduce computing system with K workers, mapping load $\mathbf{m} = [m_1, \ldots, m_K]$, and reducing load $\mathbf{w} = [w_1, \ldots, w_K]$, the optimal communication load $L^*(\mathbf{m}, \mathbf{w})$ is upper bounded by $L^*(\mathbf{m}, \mathbf{w}) \leq L_{UB}(\mathbf{m}, \mathbf{w})$ where $L_{UB}(\mathbf{m}, \mathbf{w})$ is defined by the following linear optimization problem:*

$$\mathcal{P}_{UB} : L_{UB}(\mathbf{m}, \mathbf{w}) \triangleq$$

$$\min_{\{a_{\mathcal{S}}, L_{k,\mathcal{S}}, d_{k,\mathcal{S}}^{\mathcal{S}\setminus\{i\}}\}} \sum_{\substack{\mathcal{S},k:\mathcal{S}\subseteq[K],\\ |\mathcal{S}|\geq 2, k\in\mathcal{S}}} L_{k,\mathcal{S}} \tag{2a}$$

$$\text{s.t.} \sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\ |\mathcal{S}|\geq 1}} a_{\mathcal{S}} = 1, \tag{2b}$$

$$\sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\ |\mathcal{S}|\geq 1, \mathcal{S}\ni k}} a_{\mathcal{S}} \leq m_k, \forall k \in [K] \tag{2c}$$

$$w_k a_{\mathcal{S}\setminus\{k\}} + \sum_{i\in[K]\setminus\mathcal{S}} d_{k,\mathcal{S}\cup\{i\}}^{\mathcal{S}} = \sum_{i\in\mathcal{S}\setminus\{k\}} L_{i,\mathcal{S}} + \sum_{i\in\mathcal{S}\setminus\{k\}} d_{k,\mathcal{S}}^{\mathcal{S}\setminus\{i\}},$$

$$\forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 2, k \in \mathcal{S} \tag{2d}$$

$$a_{\mathcal{S}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 1 \tag{2e}$$

$$L_{k,\mathcal{S}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 2, k \in \mathcal{S} \tag{2f}$$

$$d_{k,\mathcal{S}}^{\mathcal{S}\setminus\{i\}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 3, k \in \mathcal{S} \tag{2g}$$

B. Lower Bound

Theorem 2. *(Lower bound) For a heterogeneous MapReduce computing system with K workers, mapping load $\mathbf{m} = [m_1, \ldots, m_K]$, and reducing load $\mathbf{w} = [w_1, \ldots, w_K]$, the optimal communication load $L^*(\mathbf{m}, \mathbf{w})$ is lower bounded by $L^*(\mathbf{m}, \mathbf{w}) \geq L_{LB}(\mathbf{m}, \mathbf{w})$, where $L_{LB}(\mathbf{m}, \mathbf{w})$ is defined by the following optimization problem:*

$$\mathcal{P}_{LB} : L_{LB}(\mathbf{m}, \mathbf{w}) \triangleq$$

$$\min_{\{a_{\mathcal{S}}\}} \max_{\substack{\mathcal{S},p:\\ \mathcal{S}\subseteq[K], p\in\mathcal{P}_{\mathcal{S}}}} \{L_{\mathcal{S},p}, L_{\mathcal{S}}\} \tag{3a}$$

$$\text{s.t.} \sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\ |\mathcal{S}|\geq 1}} a_{\mathcal{S}} = 1, \tag{3b}$$

$$\sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\ |\mathcal{S}|\geq 1, \mathcal{S}\ni k}} a_{\mathcal{S}} \leq m_k, \forall k \in [K] \tag{3c}$$

$$a_{\mathcal{S}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 1 \tag{3d}$$

where

$$L_{\mathcal{S},p} \triangleq \sum_{k\in|\mathcal{S}|} \sum_{\mathcal{T}\subseteq[K]\setminus\mathcal{S}_p([k])} w_{\mathcal{S}_p(k)} a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|}, \tag{4}$$

$$L_{\mathcal{S}} \triangleq \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}^c} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|}, \tag{5}$$

and $\mathcal{S}^c \triangleq [K] \setminus \mathcal{S}$, $\mathcal{P}_{\mathcal{S}}$ denotes all the permutations of workers in $\mathcal{S}$, $\mathcal{S}_p(k)$ and $\mathcal{S}_p([k])$ denote the k-th worker and the first k workers in $\mathcal{S}$ under permutation $p \in \mathcal{P}_{\mathcal{S}}$, respectively.

Sketch of the proof: The complete proof is in Appendix A. We establish the lower bound in two different ways. Our first method is by improving the traditional cut-set bound. We create virtual workers on the network graph so that we can characterize more supporting hyperplanes of the achievable region. Our second method is by extending Yu *et. al.*'s bounding technique [1] to the HetCDC system. We manage

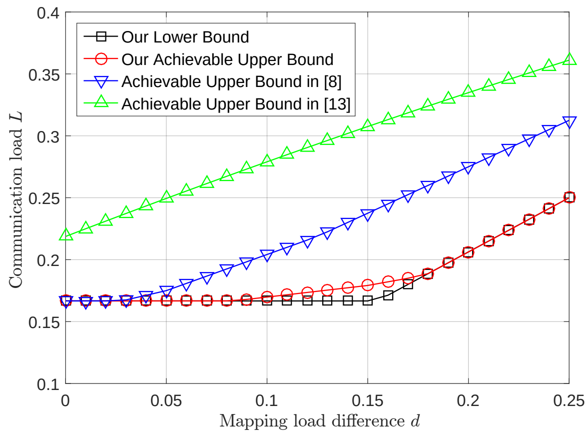

Fig. 2: Communication load in a 4-worker system with $\mathbf{m} = [\frac{5}{8} - \frac{3}{2}d, \frac{5}{8} - \frac{1}{2}d, \frac{5}{8} + \frac{1}{2}d, \frac{5}{8} + \frac{3}{2}d]$ and $\mathbf{w} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$.

to unify these two methods and describe the intersection of two lower bounds in one optimization problem.

C. Comparison with Existing Results

Remark 1. When the system degenerates to a homogeneous system, i.e., $m_k = m$ and $w_k = \frac{1}{K}$ for all $k \in [K]$, Theorem 1 and Theorem 2 reduce to the optimal communication load in [2, Theorem 1], i.e., $L_{UB} = L_{LB} = L^*$, where L^* is given by the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$.

Remark 2. In general HetCDC systems, our upper bound is tighter than those achieved by [8], [13], and coincides with the lower bound in some non-trivial cases.

Remark 1 is proved in Appendix B. Now we focus on the numerical proof of Remark 2. In Fig. 2, we consider a 4-worker HetCDC system with heterogeneous mapping load $\mathbf{m} = [\frac{5}{8} - \frac{3}{2}d, \frac{5}{8} - \frac{1}{2}d, \frac{5}{8} + \frac{1}{2}d, \frac{5}{8} + \frac{3}{2}d]$ and homogeneous reducing load $w_k = \frac{1}{4}$ for $k \in [4]$, where $d \in [0, 0.25]$. The mapping load in our system forms an arithmetic progression, and this system degenerates to the homogeneous CDC system when $d = 0$.

We can see that when the difference between each worker's mapping load increases, thus when $d \in [0.03, 0.25]$, our scheme (the red circle line) can achieve a strictly lower communication load than what the scheme in [8] achieves (the blue triangle line). Moreover, when $d \in [0 : 0.08] \cup [0.18 : 0.25]$, our scheme matches with our lower bound (the black rectangle line), which shows the partial optimality of our Theorem 1 and 2. The scheme in our previous work [13] (the green triangle line) sacrifices its communication load to make the scheme feasible for general HetCDC systems. In this paper we have made a significant improvement in the communication load under the system generality requirement.

IV. Achievable Scheme (Proof of Theorem 1)

In this section, we first provide the general scheme which achieves Theorem 1 in Section III-A. Then, we show the performance of our proposed scheme in a 4-worker HetCDC system as an example.

A. General Scheme

Let $\{a^*_{\mathcal{S}}, L^*_{k,\mathcal{S}}, d^{\mathcal{S}\setminus\{i\}*}_{k,\mathcal{S}}\}$ denote the optimal solution of Problem $\mathcal{P}_{UB}$.

1) Map phase: We consider the minsets consisting of the canonical forms of $\mathcal{M}_k$ for all $k \in [K]$. We assign the indices of $a^*_{\mathcal{S}}N$ input files to minset $\bigcap_{k\in\mathcal{S}} \mathcal{M}_k \cap \bigcap_{k\in[K]\setminus\mathcal{S}} \mathcal{M}^c_k$, for $\mathcal{S} \subseteq [K], |\mathcal{S}| \geq 1$, where $\mathcal{M}^c_k \triangleq [N] \setminus \mathcal{M}_k$[2]. Thus, each worker $k \in [K]$ computes $\sum_{\mathcal{S}\ni k} a^*_{\mathcal{S}}N$ input files. Given (2b) and (2c), this strategy allocates all the input files to workers, and satisfies the constraint of the mapping load of each worker.

2) Shuffle phase: Our main idea is to go through all possible worker sets in the descending order of the worker set sizes, and iteratively design the data shuffling within each worker set. Specifically, in an arbitrary worker set $\mathcal{S}$ with $|\mathcal{S}| = s$, we let each worker $k \in \mathcal{S}$ multicast a common message $X^k_{\mathcal{S}}$ to the rest workers $\mathcal{S}\setminus\{k\}$, and call the transmission of these s messages $\{X^k_{\mathcal{S}} : k \in \mathcal{S}\}$ as the data shuffling within worker set $\mathcal{S}$. Due to the heterogeneity on both the mapping load and the reducing load, some workers may need more I.V.s than the rest in some worker set $\mathcal{S}$ with $|\mathcal{S}| = s$. Instead of being unicast, these extra I.V.s will be multicast in the data shuffling within smaller worker sets $\{\mathcal{S}' : \mathcal{S}' \subset \mathcal{S}, |\mathcal{S}'| = s-1\}$.

We iterate over the worker set size $s \in [2 : K]$ in the descending order to design our data shuffling strategy. For an arbitrary $s \in [2 : K]$, We illustrate the data shuffling within an arbitrary worker set $\mathcal{S} \subseteq [K]$ with $|\mathcal{S}| = s$. In set $\mathcal{S}$, each worker $k \in \mathcal{S}$ needs I.V.s which are exclusively computed by the rest workers $\mathcal{S} \setminus \{k\}$. We let $\mathcal{V}^k_{\mathcal{S}\setminus\{k\}}$ denote these needed I.V.s for each worker $k \in \mathcal{S}$, with $|\mathcal{V}^k_{\mathcal{S}\setminus\{k\}}| = w_k a^*_{\mathcal{S}\setminus\{k\}}QN$. Note that there are some I.V.s which are not sent in the data shuffling within worker sets $\{\mathcal{S}' : \mathcal{S}' \supset \mathcal{S}, |\mathcal{S}'| = s+1\}$. Some of them will be sent in the data shuffling within this set $\mathcal{S}$. Specifically, for each worker set $\mathcal{S} \cup \{i\}$ with $i \in [K] \setminus \mathcal{S}$, there are $d^{\mathcal{S}*}_{k,\mathcal{S}\cup\{i\}}QN$ I.V.s, denoted by $\mathcal{D}^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}}$, which are needed by worker k and assigned to be sent in the data shuffling within set $\mathcal{S}$[3]. Note that these I.V.s are also available at workers $\mathcal{S} \setminus \{k\}$.

Thus, each worker k, for $k \in \mathcal{S}$, totally needs $(w_k a^*_{\mathcal{S}\setminus\{k\}} + \sum_{i\in[K]\setminus\mathcal{S}} d^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}}*)QN$ I.V.s, which are available at workers $\mathcal{S} \setminus \{k\}$. Let $\mathcal{I}_{k,\mathcal{S}} \triangleq \mathcal{V}^k_{\mathcal{S}\setminus\{k\}} \cup \bigcup_{i\in[K]\setminus\mathcal{S}} \mathcal{D}^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}}$ denote the needed I.V.s of worker k, for $k \in \mathcal{S}$. Then, for each $k \in \mathcal{S}$, we partition $\mathcal{I}_{k,\mathcal{S}}$ into $2s - 2$ disjoint subsets

$$\mathcal{I}_{k,\mathcal{S}} = \bigcup_{i\in\mathcal{S}\setminus\{k\}} \mathcal{I}^i_{k,\mathcal{S}} \cup \bigcup_{j\in\mathcal{S}\setminus\{k\}} \mathcal{D}^{\mathcal{S}\setminus\{j\}}_{k,\mathcal{S}}. \tag{6}$$

In (6), $\mathcal{I}^i_{k,\mathcal{S}}$ contains $L^*_{i,\mathcal{S}}QN$ I.V.s, and will be sent by worker i, for $i \in \mathcal{S} \setminus \{k\}$, while $\mathcal{D}^{\mathcal{S}\setminus\{j\}}_{k,\mathcal{S}}$ contains $d^{\mathcal{S}\setminus\{j\}*}_{k,\mathcal{S}}QN$ I.V.s, and will be sent in the data shuffling within set $\mathcal{S}\setminus\{j\}$, for $j \in \mathcal{S} \setminus \{k\}$[4]. Note that, given (2d), this partition is feasible.

After the partition of I.V.s $\{\mathcal{I}_{k,\mathcal{S}} : k \in \mathcal{S}\}$, each worker k, for $k \in \mathcal{S}$, creates a coded message $X^k_{\mathcal{S}} \triangleq \bigoplus_{i\in\mathcal{S}\setminus\{k\}} \mathcal{I}^k_{i,\mathcal{S}}$ via

[2] N is assumed to be big enough so that our scheme avoids rounding down.

[3] If $\mathcal{S} = [K]$, then $\mathcal{D}^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}} = \emptyset$ and $d^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}} = 0$.

[4] If $|\mathcal{S}| = 2$, then $\mathcal{D}^{\mathcal{S}\setminus\{j\}}_{k,\mathcal{S}} = \emptyset$ and $d^{\mathcal{S}\setminus\{j\}}_{k,\mathcal{S}} = 0$, since data shuffling does not exist in sets which contain only one worker.

TABLE I: Data shuffling of our scheme

Send \ Need	Worker 1	Worker 2	Worker 3	Worker 4	Size (normalized by QNT)
Worker 2	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,2,4\}} \oplus$	$\mathcal{V}^4_{\{1,2,3\}}$	0.014
	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,2\}}$		0.014
	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$		$\mathcal{V}^4_{\{1,2\}}$	0.014
			$\mathcal{V}^3_{\{2,4\}} \oplus$	$\mathcal{V}^4_{\{2,3\}}$	0.003
Worker 3	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,3,4\}}$	$\oplus$	$\mathcal{V}^4_{\{1,2,3\}}$	0.014
	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,3\}}$			0.007
	$\mathcal{V}^1_{\{2,3\}} \oplus$	$\mathcal{V}^2_{\{1,3\}}$			0.007
	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$		$\mathcal{V}^4_{\{1,3\}}$	0.014
		$\mathcal{V}^2_{\{3,4\}}$	$\oplus$	$\mathcal{V}^4_{\{2,3\}}$	0.003
Worker 4	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,3,4\}} \oplus$	$\mathcal{V}^3_{\{1,2,4\}}$		0.014
	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,4\}}$			0.007
	$\mathcal{V}^1_{\{2,4\}} \oplus$	$\mathcal{V}^2_{\{1,4\}}$			0.007
	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,4\}}$		0.007
	$\mathcal{V}^1_{\{3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,4\}}$		0.007
		$\mathcal{V}^2_{\{3,4\}} \oplus$	$\mathcal{V}^3_{\{2,4\}}$		0.003

bit-wise XOR, and multicasts it to the rest workers $\mathcal{S} \setminus \{k\}$. The communication load in worker set $\mathcal{S}$ is thus given by $L_{\mathcal{S}} = \sum_{k \in \mathcal{S}} L^*_{k,\mathcal{S}}$. Since each worker k, for $k \in \mathcal{S}$, already has I.V.s $\{\mathcal{I}_{i,\mathcal{S}} : i \in \mathcal{S} \setminus \{k\}\}$, it can successfully obtain its needed I.V.s $\bigcup_{i \in \mathcal{S} \setminus \{k\}} \mathcal{I}^i_{k,\mathcal{S}}$ from messages $\{X^i_{\mathcal{S}} : i \in \mathcal{S} \setminus \{k\}\}$.

3) Overall Communication Load: By considering all possible worker sets, the overall communication load is given by

$$L_{overall} = \sum_{\substack{\mathcal{S},k:\mathcal{S}\subseteq[K] \\ |\mathcal{S}|\geq 2, k \in \mathcal{S}}} L^*_{k,\mathcal{S}} = L_{UB}(\mathbf{m}, \mathbf{w}). \tag{7}$$

Thus, Theorem 1 is proved.

Remark 3. Compared with the scheme in [8], our scheme improves the communication efficiency in two ways. Firstly, our scheme is more flexible on the coding strategy. Specifically, our scheme allows to create multicast messages by combining I.V.s computed by different numbers of workers together, while the scheme in [8] only creates multicast messages for I.V.s computed by the same number of workers. Secondly, our scheme can combine those unmatched I.V.s together by introducing parameters $\{d^{\mathcal{S}\setminus\{i\}}_{k,\mathcal{S}}\}$ so as to exploit multicasting opportunities for them, while the scheme in [8] only unicasts these unmatched I.V.s.

B. An example

Now we consider a 4-worker HetCDC system with $\mathbf{m} = [\frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}]$ and $\mathbf{w} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$ to show the difference between our scheme and the scheme in [8].

Table I shows our data shuffling strategy by solving Problem $\mathcal{P}_{UB}$[5]. In our scheme, after receiving messages $\{\bigoplus_{i \in [4]\setminus\{k\}} \mathcal{V}^i_{[4]\setminus\{i\}} : k \in \{2,3,4\}\}$, each worker k, for $k \in \{2,3,4\}$, obtains its needed $0.028QN$ I.V.s of $\mathcal{V}^k_{[4]\setminus\{k\}}$, but worker 1 still needs $0.063QN$ I.V.s of $\mathcal{V}^1_{\{2,3,4\}}$. These I.V.s will be multicast with I.V.s computed by less number of workers, e.g, $0.014QN$ I.V.s of $\mathcal{V}^1_{\{2,3,4\}}$ will be multicast with $\mathcal{V}^3_{\{1,2\}}$ from worker 2 to workers 1 and 3. As a result,

[5]In Table I and II, if symbol "$\mathcal{V}^k_{\mathcal{S}}$" appears in multiple messages, then I.V.s $\mathcal{V}^k_{\mathcal{S}}$ will be partitioned into disjoint parts, each sent in a distinct message among these messages.

TABLE II: Data shuffling of the scheme in [8]

Send \ Need	Worker 1	Worker 2	Worker 3	Worker 4	Size (normalized by QNT)
Worker 1		$\mathcal{V}^2_{\{1,3\}} \oplus$	$\mathcal{V}^3_{\{1,2\}}$		0.005
		$\mathcal{V}^2_{\{1,4\}}$	$\oplus$	$\mathcal{V}^4_{\{1,2\}}$	0.005
			$\mathcal{V}^3_{\{1,4\}} \oplus$	$\mathcal{V}^4_{\{1,3\}}$	0.005
Worker 2	$\mathcal{V}^1_{\{2,3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,2,4\}} \oplus$	$\mathcal{V}^4_{\{1,2,3\}}$	0.016
	$\mathcal{V}^1_{\{2,3\}}$	$\oplus$	$\mathcal{V}^3_{\{1,2\}}$		0.005
	$\mathcal{V}^1_{\{2,4\}}$	$\oplus$		$\mathcal{V}^4_{\{1,2\}}$	0.005
			$\mathcal{V}^3_{\{2,4\}} \oplus$	$\mathcal{V}^4_{\{2,3\}}$	0.005
Worker 3	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,3,4\}}$	$\oplus$	$\mathcal{V}^4_{\{1,2,3\}}$	0.016
	$\mathcal{V}^1_{\{2,3\}} \oplus$	$\mathcal{V}^2_{\{1,3\}}$			0.005
	$\mathcal{V}^1_{\{3,4\}}$	$\oplus$		$\mathcal{V}^4_{\{1,3\}}$	0.005
		$\mathcal{V}^2_{\{3,4\}}$	$\oplus$	$\mathcal{V}^4_{\{2,3\}}$	0.005
Worker 4	$\mathcal{V}^1_{\{2,3,4\}} \oplus$	$\mathcal{V}^2_{\{1,3,4\}} \oplus$	$\mathcal{V}^3_{\{1,2,4\}}$		0.016
	$\mathcal{V}^1_{\{2,4\}} \oplus$	$\mathcal{V}^2_{\{1,4\}}$			0.005
	$\mathcal{V}^1_{\{3,4\}}$	$\oplus$	$\mathcal{V}^3_{\{1,4\}}$		0.005
		$\mathcal{V}^2_{\{3,4\}} \oplus$	$\mathcal{V}^3_{\{2,4\}}$		0.005
	$\mathcal{V}^1_{\{2,3,4\}}$				0.047

TABLE III: Communication load of Unicasting and Multicasting

Scheme \ Shuffling	Unicasting	Multicasting to 2 workers	Multicasting to 3 workers	Overall
Our Scheme	0	0.094	0.042	0.136
The scheme in [8]	0.047	0.063	0.047	0.157
The scheme in [13]	0.132	0.056	0.023	0.211

unicasting is avoided in our scheme. On the other hand, as shown in Table II, the scheme in [8] only exploits coded multicasting opportunities for I.V.s computed by the same number of workers. Therefore, worker 4 in [8] still needs to unicast $0.047QN$ I.V.s of $\mathcal{V}^1_{\{2,3,4\}}$ to worker 1.

Table III shows the communication load used for unicasting and multicasting in [8], [13] and in our scheme, respectively. The overall communication load in [13] is the highest since its scheme is extended from decentralized cache placement and content delivery [14]. Compared with the scheme in [8], though our scheme multicasts slightly less messages to three workers, it multicasts more messages to two workers, and totally avoids unicasting. Thus, it achieves the lowest overall communication load among the three considered schemes.

V. Conclusion

In this paper, we considered a HetCDC system with mapping load $\mathbf{m}$ and reducing load $\mathbf{w}$. We obtain both the achievable upper bound and the theoretical lower bound of the optimal communication load. If the system degenerates to the homogeneous case, our result recovers the result in [2]. When $K = 4$, $\mathbf{m} = [\frac{5}{8} - \frac{3}{2}d, \frac{5}{8} - \frac{1}{2}d, \frac{5}{8} + \frac{1}{2}d, \frac{5}{8} + \frac{3}{2}d]$, and $\mathbf{w} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, our result is strictly tighter than those achieved by [8], [13] for $d \in [0.03, 0.25]$, and optimal for $d \in [0, 0.08] \cup [0.18, 0.25]$. While this work only optimizes the communication load for a given reducing load $\mathbf{w}$, future works could further reduce the communication load by additionally optimizing the reducing load as in [13].

Appendix A: Proof of Theorem 2

Our main idea is to first derive the lower bound for any given file allocation, and then optimize the file allocation to minimize the lower bound. For an arbitrary file allocation, let $\mathcal{A}_S$ denote the files exclusively mapped by worker set $\mathcal{S}$, with $|\mathcal{A}_S| = a_{\mathcal{S}}N$. Let $\mathcal{V}_{\mathcal{S}_1;\mathcal{S}_2}$ denote the I.V.s needed by workers in set $\mathcal{S}_1$ and computed by at least one worker in set $\mathcal{S}_2$, i.e.,

$$\mathcal{V}_{\mathcal{S}_1;\mathcal{S}_2} \triangleq \{v_{q,n} : q \in \mathcal{W}_{k_1}, k_1 \in \mathcal{S}_1, n \in \mathcal{M}_{k_2}, k_2 \in \mathcal{S}_2\}.$$

We adopt two methods to obtain the lower bounds for the given file allocation. The first is improved from the conventional cut-set bound, and the second is called the peeling method, which is inspired and generalized from [1].

A. Improved Cut-set Bound

We partition the K workers into two sets: $\mathcal{S} \subseteq [K]$ and $\mathcal{S}^c \triangleq [K] \setminus \mathcal{S}$. *Given transmitted messages $X_{\mathcal{S}^c}$ (or $X_{\mathcal{S}}$) and I.V.s computed by workers $\mathcal{S}$ (or $\mathcal{S}^c$), I.V.s needed by workers $\mathcal{S}$ (or $\mathcal{S}^c$) can be obtained, i.e.,*

$$H(\mathcal{V}_{\mathcal{S};[K]}|\mathrm{X}_{\mathcal{S}^c}, \mathcal{V}_{[K];\mathcal{S}}) = H(\mathcal{V}_{\mathcal{S};[K]}|\mathrm{X}_{\mathcal{S}^c}, \mathcal{V}_{[K];\mathcal{S}}, \mathrm{X}_{\mathcal{S}}) = 0,$$
$$H(\mathcal{V}_{\mathcal{S}^c;[K]}|\mathrm{X}_{\mathcal{S}}, \mathcal{V}_{[K];\mathcal{S}^c}) = H(\mathcal{V}_{\mathcal{S}^c;[K]}|\mathrm{X}_{\mathcal{S}}, \mathcal{V}_{[K];\mathcal{S}^c}, \mathrm{X}_{\mathcal{S}^c}) = 0.$$

Thus, we have

$$\begin{aligned} &H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{[K];\mathcal{S}}) \\ =&H(\mathrm{X}_{\mathcal{S}^c}, \mathcal{V}_{\mathcal{S};[K]}|\mathcal{V}_{[K];\mathcal{S}}) \\ =&H(\mathcal{V}_{\mathcal{S};[K]}|\mathcal{V}_{[K];\mathcal{S}}) + H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}) \end{aligned} \tag{8}$$

and

$$\begin{aligned} &H(\mathrm{X}_{\mathcal{S}}|\mathcal{V}_{[K];\mathcal{S}^c}) \\ =&H(\mathrm{X}_{\mathcal{S}}, \mathcal{V}_{\mathcal{S}^c;[K]}|\mathcal{V}_{[K];\mathcal{S}^c}) \\ =&H(\mathcal{V}_{\mathcal{S}^c;[K]}|\mathcal{V}_{[K];\mathcal{S}^c}) + H(\mathrm{X}_{\mathcal{S}}|\mathcal{V}_{\mathcal{S}^c;[K]}, \mathcal{V}_{[K];\mathcal{S}^c}). \end{aligned} \tag{9}$$

In (8), $H(\mathcal{V}_{\mathcal{S};[K]}|\mathcal{V}_{[K];\mathcal{S}})$ is given by

$$H(\mathcal{V}_{\mathcal{S};[K]}|\mathcal{V}_{[K];\mathcal{S}}) = \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}^c} w_k a_{\mathcal{T}} QNT, \tag{10}$$

while $H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})$ is lower bounded by the following lemma, whose proof is in Appendix C. Note that the proof of Lemma 1 is extended from [2] to the heterogeneous case.

Lemma 1. *Given I.V.s needed by or computed by workers in an arbitrary set $\mathcal{S}$, the entropy of transmitted messages is lower bounded by*

$$H(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}) \geq \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|} QNT. \tag{11}$$

Thus, we have

$$\begin{aligned} &H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{[K];\mathcal{S}}) \\ =&H(\mathcal{V}_{\mathcal{S};[K]}|\mathcal{V}_{[K];\mathcal{S}}) + H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}) \\ \geq& \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}^c} w_k a_{\mathcal{T}} QNT + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|} QNT. \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned} &H(\mathrm{X}_{\mathcal{S}}|\mathcal{V}_{[K];\mathcal{S}^c}) \\ =&H(\mathcal{V}_{\mathcal{S}^c;[K]}|\mathcal{V}_{[K];\mathcal{S}^c}) + H(\mathrm{X}_{\mathcal{S}}|\mathcal{V}_{\mathcal{S}^c;[K]}, \mathcal{V}_{[K];\mathcal{S}^c}) \\ \geq& \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}} w_k a_{\mathcal{T}} QNT + \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}\setminus\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|} QNT. \end{aligned} \tag{13}$$

By combining (12) (13), the entropy of transmitted messages can be lower bounded by

$$\begin{aligned} &H(\mathrm{X}_{[K]}) \\ =&H(\mathrm{X}_{\mathcal{S}}) + H(\mathrm{X}_{\mathcal{S}^c}|\mathrm{X}_{\mathcal{S}}) \\ \geq&H(\mathrm{X}_{\mathcal{S}}|\mathcal{V}_{[K];\mathcal{S}^c}) + H(\mathrm{X}_{\mathcal{S}^c}|\mathcal{V}_{[K];\mathcal{S}}) \\ \geq& \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}} w_k a_{\mathcal{T}} QNT + \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} QNT \\ &+ \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}^c} w_k a_{\mathcal{T}} QNT + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} QNT \end{aligned} \tag{14}$$

Thus, by using the improved cut-set bound, the optimal communication load L^* is lower bounded by

$$\begin{aligned} L^* \geq& \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\mathcal{T}\subseteq\mathcal{S}^c} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|}, \end{aligned}$$

which is the same as $L_{\mathcal{S}}$ in (5).

B. Peeling Method

We still partition the workers into two sets: $\mathcal{S} \subseteq [K]$ and $\mathcal{S}^c = [K] \setminus \mathcal{S}$, and consider an arbitrary permutation $p \in \mathcal{P}_{\mathcal{S}}$ of workers in $\mathcal{S}$.

Lemma 2. *For a given permutation $p \in \mathcal{P}_{\mathcal{S}}$, given all the transmitted messages and I.V.s computed by worker $\mathcal{S}_p(k)$ and needed by workers $[K] \setminus \mathcal{S}_p([k-1])$ for each $k \in [|\mathcal{S}|]$, I.V.s needed by workers in $\mathcal{S}$ can be obtained, i.e.,*

$$H\Big(\mathcal{V}_{\mathcal{S};[K]}|X_{[K]}, \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \setminus \mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) = 0.$$

The proof of Lemma 2 is in Appendix D. Based on Lemma 2, we have

$$\begin{aligned} &H(X_{[K]}) \\ \geq&H\Big(X_{[K]}| \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \setminus \mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \\ =&H\Big(\mathcal{V}_{\mathcal{S};[K]}, X_{[K]}| \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}}\setminus\mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \\ =&H\Big(\mathcal{V}_{\mathcal{S};[K]}| \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \setminus \mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \\ &+H\Big(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}}\setminus\mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \end{aligned} \tag{15}$$

In (15), the first term is given by

$$\begin{aligned} &H\Big(\mathcal{V}_{\mathcal{S};[K]}| \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \setminus \mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \\ =& \sum_{k\in[|\mathcal{S}|]} H\Big(\mathcal{V}_{\mathcal{S}_p(k);[K]}| \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \setminus \mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big) \end{aligned} \tag{16a}$$

$$= \sum_{k\in[|\mathcal{S}|]} H\Big(\mathcal{V}_{\mathcal{S}_p(k);[K]}|\mathcal{V}_{\mathcal{S}_p(k);\mathcal{S}_p([k])}\Big) \tag{16b}$$

$$= \sum_{k\in[|\mathcal{S}|]} \sum_{\mathcal{T}\subseteq[K]\setminus\mathcal{S}_p([k])} w_{\mathcal{S}_p(k)} a_{\mathcal{T}} QNT, \tag{16c}$$

where (16a) and (16b) come from the fact that the I.V.s needed by different workers are independent with each other since there is no overlapping between the target functions computed by different workers. The second term in (15) can be lower bounded by

$$H\Big(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \bigcup_{k\in[|\mathcal{S}|]} \mathcal{V}_{[K];\{\mathcal{S}_p(k)\}} \backslash \mathcal{V}_{\mathcal{S}_p([k-1]);\mathcal{S}_p(k)}\Big)$$
$$=H\Big(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}\Big)$$
$$\geq \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\backslash\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|} QNT, \tag{17}$$

where the final inequality comes from Lemma 1. By combining (16c) and (17), the optimal communication load L^* can be lower bounded by

$$L^* \geq \sum_{k\in[|\mathcal{S}|]} \sum_{\mathcal{T}\subseteq[K]\backslash\mathcal{S}_p([k])} w_{\mathcal{S}_p(k)} a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\mathcal{T}\subseteq\mathcal{S}^c\backslash\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|},$$

which is the same as $L_{\mathcal{S},p}$ in (4).

Thus, by considering all possible worker partitions and all possible permutations of workers, we have

$$L^* \geq \max_{\mathcal{S}\subseteq[K],p\in\mathcal{P}_{\mathcal{S}}} \{L_{\mathcal{S}}, L_{\mathcal{S},p}\}$$

for an arbitrary file allocation $\{a_{\mathcal{S}}\}$. Now we need to minimize $\max_{\mathcal{S}\subseteq[K],p\in\mathcal{P}_{\mathcal{S}}}\{L_{\mathcal{S}}, L_{\mathcal{S},p}\}$ subject to file allocation $\{a_{\mathcal{S}}\}$, which is formulated as the optimization problem in Theorem 2. Thus, Theorem 2 is proved.

Appendix B: Proof of Remark 1

We first proof that Theorem 1 degenerates to the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \ldots, 1\}$, and then prove that Theorem 2 also degenerates to these points.

A. Proof of Theorem 1 in the Homogeneous Case

We first prove that $L_{UB}(\mathbf{m},\mathbf{w})$ obtained in Problem $\mathcal{P}_{UB}$ is lower bounded by the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \ldots, 1\}$ in the homogeneous case. Then, we prove that these points are achievable in Problem $\mathcal{P}_{UB}$.

For an arbitrary set $\mathcal{S}$ with $|\mathcal{S}| \geq 2$, we have

$$\sum_{k\in\mathcal{S}} w_k a_{\mathcal{S}\backslash\{k\}} + \sum_{k\in\mathcal{S}} \sum_{i\in[K]\backslash\mathcal{S}} d^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}}$$
$$=\sum_{k\in\mathcal{S}} (|\mathcal{S}|-1) L_{k,\mathcal{S}} + \sum_{k\in\mathcal{S}} \sum_{i\in\mathcal{S}\backslash\{k\}} d^{\mathcal{S}\backslash\{i\}}_{k,\mathcal{S}} \tag{18}$$

given (2d). We can rewrite (18) as

$$\frac{1}{|\mathcal{S}|-1} \sum_{k\in\mathcal{S}} w_k a_{\mathcal{S}\backslash\{k\}} + \frac{1}{|\mathcal{S}|-1} \sum_{k\in\mathcal{S}} \sum_{i\in[K]\backslash\mathcal{S}} d^{\mathcal{S}}_{k,\mathcal{S}\cup\{i\}}$$
$$=\sum_{k\in\mathcal{S}} L_{k,\mathcal{S}} + \frac{1}{|\mathcal{S}|-1} \sum_{k\in\mathcal{S}} \sum_{i\in\mathcal{S}\backslash\{k\}} d^{\mathcal{S}\backslash\{i\}}_{k,\mathcal{S}}. \tag{19}$$

Summing up (19) for all worker sets $\mathcal{S}$ with $|\mathcal{S}| \geq 2$, we have

$$\sum_{\substack{\mathcal{S}\subseteq[K]\\|\mathcal{S}|\geq 2}} \sum_{k\in\mathcal{S}} \frac{1}{|\mathcal{S}|-1} w_k a_{\mathcal{S}\backslash\{k\}}$$
$$+ \sum_{\substack{\mathcal{S}\subseteq[K]\\|\mathcal{S}|\geq 3}} \sum_{k\in\mathcal{S}} \sum_{i\in\mathcal{S}\backslash\{k\}} (\frac{1}{|\mathcal{S}|-2} - \frac{1}{|\mathcal{S}|-1}) d^{\mathcal{S}\backslash\{i\}}_{k,\mathcal{S}}$$
$$= \sum_{\substack{\mathcal{S}\subseteq[K]\\|\mathcal{S}|\geq 2}} \sum_{k\in\mathcal{S}} L_{k,\mathcal{S}}. \tag{20}$$

Note that the right side in (20) is exactly the communication load in Theorem 1. Thus, for an arbitrary feasible solution $\{a_{\mathcal{S}}, L_{k,\mathcal{S}}, d^{\mathcal{S}\backslash\{i\}}_{k,\mathcal{S}}\}$, the achievable communication load L in Theorem 1 is lower bounded by

$$L \geq \sum_{\substack{\mathcal{S}\subseteq[K]\\|\mathcal{S}|\geq 2}} \sum_{k\in\mathcal{S}} \frac{1}{|\mathcal{S}|-1} w_k a_{\mathcal{S}\backslash\{k\}}$$
$$= \sum_{\mathcal{T}\subseteq[K]} \sum_{k\in[K]\backslash\mathcal{T}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|}$$
$$= \sum_{\mathcal{T}\subseteq[K]} \frac{K-|\mathcal{T}|}{K|\mathcal{T}|} a_{\mathcal{T}}. \tag{21}$$

Let $\{a^*_{\mathcal{S}}, L^*_{k,\mathcal{S}}, d^{\mathcal{S}\backslash\{i\}*}_{k,\mathcal{S}}\}$ denote the optimal solution in Problem $\mathcal{P}_{UB}$. Then, the optimal communication load L_{UB} obtained in Problem $\mathcal{P}_{UB}$ is lower bounded by

$$L_{UB} \geq \sum_{\mathcal{S}\subseteq[K]} \frac{K-|\mathcal{S}|}{K|\mathcal{S}|} a^*_{\mathcal{S}}. \tag{22}$$

Note that the right side in (22) is lower bounded by $L'_{UB}(\mathbf{m},\mathbf{w})$, where $L'_{UB}(\mathbf{m},\mathbf{w})$ is defined by the following optimization problem:

$$\mathcal{P}'_{UB}:\ L'_{UB}(\mathbf{m},\mathbf{w}) \triangleq \min_{\{a_{\mathcal{S}}\}} \sum_{\mathcal{S}\subseteq[K]} \frac{K-|\mathcal{S}|}{K|\mathcal{S}|} a_{\mathcal{S}} \tag{23a}$$
$$\text{s.t.} \sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\|\mathcal{S}|\geq 1}} a_{\mathcal{S}} = 1, \tag{23b}$$
$$\sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\|\mathcal{S}|\geq 1,\mathcal{S}\ni k}} a_{\mathcal{S}} \leq m, \forall k \in [K] \tag{23c}$$
$$a_{\mathcal{S}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 1 \tag{23d}$$

Define a^j as the number of files, normalized by N, exclusively allocated to j workers, i.e.,

$$a^j \triangleq \sum_{\mathcal{S}\subseteq[K],|\mathcal{S}|=j} a_{\mathcal{S}},$$

for $j \in [K]$. Then, the communication load in (23a) can be rewritten as $\sum_{j\in[K]} \frac{K-j}{Kj} a^j$, and Problem $\mathcal{P}'_{UB}$ can be degenerated to the following problem:

$$\mathcal{P}''_{UB}: L''_{UB}(\mathbf{m},\mathbf{w}) \triangleq \min_{\{j\}} \sum_{j\in[K]} \frac{K-j}{Kj} a^j \tag{24a}$$
$$\text{s.t.} \sum_{j\in[K]} a^j = 1, \tag{24b}$$
$$\sum_{j\in[K]} j a^j \leq mK, \tag{24c}$$
$$a_j \geq 0, \forall j \in [K]. \tag{24d}$$

Any feasible solution $\{a_{\mathcal{S}}\}$ of Problem $\mathcal{P}'_{UB}$ corresponds to an feasible solution $\{j\}$ of Problem $\mathcal{P}''_{UB}$, which implies that $L'_{UB}(\mathbf{m},\mathbf{w}) \geq L''_{UB}(\mathbf{m},\mathbf{w})$. Therefore, we have $L_{UB}(\mathbf{m},\mathbf{w}) \geq L''_{UB}(\mathbf{m},\mathbf{w})$.

Problem $\mathcal{P}''_{UB}$ is the optimization problem in [2, Eq.(24)-

(27)][6]. According to [2], $L''_{UB}(\mathbf{m},\mathbf{w})$ is given by the lower convex envelope of points $L''_{UB}(\mathbf{m},\mathbf{w}) = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$. Thus, we prove that $L_{UB}(\mathbf{m},\mathbf{w})$ is lower bounded by the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$.

Now, we prove that these points are achievable in Problem $\mathcal{P}_{UB}$. We first consider the case when $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$. By letting

$$\begin{cases} a_{\mathcal{S}} = \frac{1}{\binom{K}{Km}}, & \text{for all } \mathcal{S} \text{ with } |\mathcal{S}| = Km \\ L_{k,\mathcal{S}} = \frac{1}{\binom{K}{Km}KKm}, & \text{for all } \mathcal{S} \text{ with } |\mathcal{S}| = Km+1 \text{ and } k \in \mathcal{S} \end{cases}$$

and the rest parameters be 0 in Problem $\mathcal{P}_{UB}$, it is easy to prove that the achievable communication load is given by

$$L = \binom{K}{Km+1}(Km+1)\frac{1}{\binom{K}{Km}KKm} = \frac{1-m}{Km}.$$

As for the general m, we can use memory sharing between $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$ to obtain the file allocation strategy $\{a_{\mathcal{S}}\}$ and the data shuffling strategy $\{L_{k,\mathcal{S}}\}$, and prove that the achievable communication load also coincides with the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$. The detailed proof is omitted here.

Therefore, we prove that Theorem 1 degenerates to the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$ in the homogeneous case.

B. Proof of Theorem 2 in the Homogeneous Case

We first prove that $L_{LB}(\mathbf{m},\mathbf{w})$ obtained in Problem $\mathcal{P}_{LB}$ is lower bounded by the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$ in the homogeneous case. Then, we prove that these points are achievable in Problem $\mathcal{P}_{LB}$.

Letting $\mathcal{S} = [K]$ in (5), we have

$$\begin{aligned} L_{[K]} &= \sum_{k\in[K]} \sum_{\mathcal{T}\subseteq[K]\setminus\{k\}} w_k a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} \\ &= \sum_{k\in[K]} \sum_{\mathcal{T}\subseteq[K]\setminus\{k\}} \frac{1}{K} a_{\mathcal{T}} \frac{1}{|\mathcal{T}|} \\ &= \sum_{\mathcal{T}\subseteq[K]} \frac{K-|\mathcal{T}|}{K|\mathcal{T}|} a_{\mathcal{T}} \end{aligned} \tag{25}$$

and

$$\max_{\mathcal{S}\subseteq[K], p\in\mathcal{P}_{\mathcal{S}}} \{L_{\mathcal{S}}, L_{\mathcal{S},p}\} \geq L_{[K]}.$$

Therefore, $L_{LB}(\mathbf{m},\mathbf{w})$ is lower bounded by $L'_{LB}(\mathbf{m},\mathbf{w})$, where $L'_{LB}(\mathbf{m},\mathbf{w})$ is defined by the following optimization problem:

$$\mathcal{P}'_{LB} : L'_{LB}(\mathbf{m},\mathbf{w}) \triangleq$$

$$\min_{\{a_{\mathcal{S}}\}} \sum_{\mathcal{S}\subseteq[K]} \frac{K-|\mathcal{S}|}{K|\mathcal{S}|} a_{\mathcal{S}} \tag{26a}$$

$$\text{s.t.} \sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\|\mathcal{S}|\geq 1}} a_{\mathcal{S}} = 1, \tag{26b}$$

$$\sum_{\substack{\mathcal{S}:\mathcal{S}\subseteq[K],\\|\mathcal{S}|\geq 1, \mathcal{S}\ni k}} a_{\mathcal{S}} \leq m, \forall k \in [K] \tag{26c}$$

$$a_{\mathcal{S}} \geq 0, \forall \mathcal{S} \subseteq [K], |\mathcal{S}| \geq 1 \tag{26d}$$

Note that Problem $\mathcal{P}'_{LB}$ is the same as Problem $\mathcal{P}'_{UB}$ in (23). By using the same method in Appendix B-A, it is easy to prove that $L_{LB}(\mathbf{m},\mathbf{w})$ is lower bounded by the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$.

Now, we prove that these points are also achievable in Problem $\mathcal{P}_{LB}$. We first consider the case when $m \in \{\frac{1}{K}, \frac{2}{K}, \dots, 1\}$. Specifically, we aim to prove that when

$$a_{\mathcal{S}} = \frac{1}{\binom{K}{Km}}, \text{ for all } \mathcal{S} \text{ with } |\mathcal{S}| = Km$$

and the rest $a_{\mathcal{S}}$ being 0, we have

$$\max_{\mathcal{S}\subseteq[K], p\in\mathcal{P}_{\mathcal{S}}} \{L_{\mathcal{S}}, L_{\mathcal{S},p}\} = \frac{1-m}{Km}.$$

First, it is easy to prove that $L_{[K]} = \frac{1-m}{Km}$ in (25). Thus, we only need to prove that $L_{[K]}$ is the largest among $\{L_{\mathcal{S}} : \mathcal{S} \subseteq [K]\} \cup \{L_{\mathcal{S},p} : \mathcal{S} \subseteq [K], p \in \mathcal{P}_{\mathcal{S}}\}$.

1) $\{L_{\mathcal{S}} : \mathcal{S} \subseteq [K]\}$: Let us first consider $\{L_{\mathcal{S}} : \mathcal{S} \subseteq [K]\}$. We have

$$\begin{aligned} L_{\mathcal{S}} =& \sum_{k\in\mathcal{S}^c} \sum_{\substack{\mathcal{T}\subseteq\mathcal{S}\\|\mathcal{T}|=Km}} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}} \sum_{\substack{\mathcal{T}\subseteq\mathcal{S}\setminus\{k\}\\|\mathcal{T}|=Km}} w_k a_{\mathcal{T}} \frac{1}{Km} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\substack{\mathcal{T}\subseteq\mathcal{S}^c\\|\mathcal{T}|=Km}} w_k a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\substack{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}\\|\mathcal{T}|=Km}} w_k a_{\mathcal{T}} \frac{1}{Km} \\ =& \sum_{k\in\mathcal{S}^c} \sum_{\substack{\mathcal{T}_1\subseteq\mathcal{S}\\|\mathcal{T}_1|=Km}} w_k a_{\mathcal{T}_1} + \sum_{\substack{\mathcal{T}_1\subseteq\mathcal{S}\\|\mathcal{T}_1|=Km+1}} \sum_{k\in\mathcal{T}_1} w_k a_{\mathcal{T}_1\setminus\{k\}} \frac{1}{Km} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\substack{\mathcal{T}_2\subseteq\mathcal{S}^c\\|\mathcal{T}_2|=Km}} w_k a_{\mathcal{T}_2} + \sum_{\substack{\mathcal{T}_2\subseteq\mathcal{S}^c\\|\mathcal{T}_2|=Km+1}} \sum_{k\in\mathcal{T}_2} w_k a_{\mathcal{T}_2\setminus\{k\}} \frac{1}{Km} \\ =& \sum_{k\in\mathcal{S}^c} \sum_{\substack{\mathcal{T}_1\subseteq\mathcal{S}\\|\mathcal{T}_1|=Km}} w_k a_{\mathcal{T}_1} + \sum_{\substack{\mathcal{T}_1\subseteq\mathcal{S}\\|\mathcal{T}_1|=Km+1}} b_{\mathcal{T}_1} \\ &+ \sum_{k\in\mathcal{S}} \sum_{\substack{\mathcal{T}_2\subseteq\mathcal{S}^c\\|\mathcal{T}_2|=Km}} w_k a_{\mathcal{T}_2} + \sum_{\substack{\mathcal{T}_2\subseteq\mathcal{S}^c\\|\mathcal{T}_2|=Km+1}} b_{\mathcal{T}_2}, \end{aligned} \tag{27}$$

where

$$b_{\mathcal{T}_1} \triangleq \sum_{k\in\mathcal{T}_1} w_k a_{\mathcal{T}_1\setminus\{k\}} \frac{1}{Km},$$

$$b_{\mathcal{T}_2} \triangleq \sum_{k\in\mathcal{T}_2} w_k a_{\mathcal{T}_2\setminus\{k\}} \frac{1}{Km}.$$

If $m = \frac{1}{K}$, it is easy to prove that $L_{\mathcal{S}} = 1 - \frac{1}{K} = L_{[K]}$ for any $\mathcal{S} \subseteq [K]$. Thus, we only consider the case when $m \geq \frac{2}{K}$. We rewrite (25) as

$$\begin{aligned} L_{[K]} &= \sum_{\substack{\mathcal{S}_1\subseteq[K]\\|\mathcal{S}_1|=Km+1}} \sum_{k\in\mathcal{S}_1} w_k a_{\mathcal{T}\setminus\{k\}} \frac{1}{Km} \\ &= \sum_{\substack{\mathcal{S}_1\subseteq[K]\\|\mathcal{S}_1|=Km+1}} b_{\mathcal{S}_1} \end{aligned} \tag{28}$$

where

$$b_{\mathcal{S}_1} \triangleq \sum_{k\in\mathcal{S}_1} w_k a_{\mathcal{T}\setminus\{k\}} \frac{1}{Km}.$$

We prove that each term in (27) is no larger than a distinct

[6]Although inequality (24c) becomes equality [2, Eq.(27)] in [2], it is trivial to prove that the optimal solution must satisfy the equality.

$b_{\mathcal{S}_1}$ in (28). For the first term in (27) where $\mathcal{T}_1 \subseteq \mathcal{S}$, by letting $\mathcal{S}_1 = \mathcal{T}_1 \cup \{k\}$, we have

$$w_k a_{\mathcal{T}_1} = \frac{1}{K\binom{K}{Km}} < \frac{Km+1}{K\binom{K}{Km}Km} = \sum_{i\in\mathcal{S}_1} w_i a_{\mathcal{T}\setminus\{i\}} \frac{1}{Km} = b_{\mathcal{S}_1}.$$

Similarly, for the third term in (27) where $\mathcal{T}_2 \subseteq \mathcal{S}^c$, by letting $\mathcal{S}_1 = \mathcal{T}_2 \cup \{k\}$, we also have

$$w_k a_{\mathcal{T}_2} < b_{\mathcal{S}_1}.$$

Note that $\mathcal{T}_1 \cup \{k\} \neq \mathcal{T}_2 \cup \{k\}$ since $\mathcal{T}_1 \subseteq \mathcal{S}, \mathcal{T}_2 \subseteq \mathcal{S}^c$, and $|\mathcal{T}_1| = |\mathcal{T}_2| \geq 2$. As for the second term $b_{\mathcal{T}_1}$ and the fourth term $b_{\mathcal{T}_2}$ in (27), it is easy to see that they exactly appear once in (28), and do not coincide with each other or the terms in (28) corresponding to the first and the third terms in (27).

Since each term in (27) is no larger than a distinct $b_{\mathcal{S}_1}$ in (28), we proved that $L_{\mathcal{S}} \leq L_{[K]}$.

2) $\{L_{\mathcal{S},p} : \mathcal{S} \subseteq [K], p \in \mathcal{P}_{\mathcal{S}}\}$: Now let us consider $\{L_{\mathcal{S},p} : \mathcal{S} \subseteq [K], p \in \mathcal{P}_{\mathcal{S}}\}$. We have

$$L_{\mathcal{S},p} = \sum_{k\in[|\mathcal{S}|]} \sum_{\substack{\mathcal{T}\subseteq[K]\setminus\mathcal{S}_p([k])\\|\mathcal{T}|=Km}} w_{\mathcal{S}_p(k)} a_{\mathcal{T}} + \sum_{k\in\mathcal{S}^c} \sum_{\substack{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}\\|\mathcal{T}|=Km}} \frac{w_k a_{\mathcal{T}}}{Km}. \tag{29}$$

Similar to the proof for $\{L_{\mathcal{S}} : \mathcal{S} \subseteq [K]\}$, we also show that each term in (29) is no larger than a distinct $b_{\mathcal{S}_1}$ in (28) so as to prove $L_{\mathcal{S},p} \leq L_{[K]}$. Specifically, the proof for the first term in (29) is similar to the proof for the first and the third terms in (27), and the proof for the second term in (29) is similar to the proof for the second and the fourth terms in (27). The detailed proof is omitted here.

As a result, by using the file allocation $a_{\mathcal{S}} = \frac{1}{\binom{K}{Km}}$ for all $\mathcal{S}$ satisfying $|\mathcal{S}| = Km$ and the rest $a_{\mathcal{S}}$ being 0, we prove that $\frac{1-m}{Km}$ is achievable when $m \in \{\frac{1}{K}, \frac{2}{K}, \ldots, 1\}$. As for the general m, we can use memory sharing between $m \in \{\frac{1}{K}, \frac{2}{K}, \ldots, 1\}$ to construct the file allocation $\{a_{\mathcal{S}}\}$, and use the similar method as above to prove the achievability, which is omitted here.

Thus, we prove that Theorem 2 degenerates to the lower convex envelope of points $L^* = \frac{1-m}{Km}$ for $m \in \{\frac{1}{K}, \frac{2}{K}, \ldots, 1\}$ in the homogeneous case.

Appendix C: Proof of Lemma 1

We use induction over the size of $\mathcal{S}$ in the descending order to prove Lemma 1. If $|\mathcal{S}| = K$, i.e., $\mathcal{S} = [K]$, we have $H(X_{[K]}|\mathcal{V}_{[K];[K]}, \mathcal{V}_{[K];[K]}) \geq 0$ in (11). If $|\mathcal{S}| = K-1$, we assume that $\mathcal{S} = [K] \setminus \{k\}$ for an arbitrary $k \in [K]$ without loss of generality. Then, we also have

$$H(X_{[K]}|\mathcal{V}_{[K]\setminus\{k\};[K]}, \mathcal{V}_{[K];[K]\setminus\{k\}}) \geq 0 \tag{30}$$

in (11). Thus, Lemma 1 is true for sets $\mathcal{S}$ with $|\mathcal{S}| = K$ or $K-1$.

If Lemma 1 is true for sets $\mathcal{S}'$ with $|\mathcal{S}'| = s$, then consider sets $\mathcal{S}$ with $|\mathcal{S}| = s-1$. Define $\mathcal{S}^c \triangleq [K] \setminus \mathcal{S}$. By applying [2, Eq.(43)-(52)], we have

$$\begin{aligned}
&H(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
=&H(X_{\mathcal{S}^c}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
\geq&\frac{1}{K-s}\sum_{k\in\mathcal{S}^c} H(X_{\mathcal{S}^c}|\mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
=&\frac{1}{K-s}\sum_{k\in\mathcal{S}^c} H(\mathcal{V}_{\{k\};[K]}|\mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
&+\frac{1}{K-s}\sum_{k\in\mathcal{S}^c} H(X_{\mathcal{S}^c}|\mathcal{V}_{\{k\};[K]}, \mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}).
\end{aligned} \tag{31}$$

In (31), $H(\mathcal{V}_{\{k\};[K]}|\mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})$ is given by

$$\begin{aligned}
&H(\mathcal{V}_{\{k\};[K]}|\mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
=&H(\mathcal{V}_{\{k\};[K]}|\mathcal{V}_{[K];\{k\}}, \mathcal{V}_{[K];\mathcal{S}}) &(32a)\\
=&\sum_{\mathcal{T}\subseteq[K]\setminus\{\mathcal{S}\cup\{k\}\}} w_k a_{\mathcal{T}} QNT, &(32b)
\end{aligned}$$

where (32a) comes from the fact that the I.V.s needed by different workers are independent with each other since there is no overlapping between the target functions computed by different workers. In (31), $H(X_{\mathcal{S}^c}|\mathcal{V}_{\{k\};[K]}, \mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})$ is lower bounded by

$$\begin{aligned}
&H(X_{\mathcal{S}^c}|\mathcal{V}_{\{k\};[K]}, \mathcal{V}_{[K];\{k\}}, \mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
=&H(X_{\mathcal{S}^c}|\mathcal{V}_{\mathcal{S}\cup\{k\};[K]}, \mathcal{V}_{[K];\mathcal{S}\cup\{k\}})\\
=&H(X_{\mathcal{S}^c\setminus\{k\}}|\mathcal{V}_{\mathcal{S}\cup\{k\};[K]}, \mathcal{V}_{[K];\mathcal{S}\cup\{k\}})\\
\geq&\sum_{i\in\mathcal{S}^c\setminus\{k\}}\sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k,i\}} \frac{w_i a_{\mathcal{T}}}{|\mathcal{T}|} QNT
\end{aligned} \tag{33}$$

where the final inequality comes from the induction assumption. By combining (31), (32b) and (33), we have

$$\begin{aligned}
&H(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}})\\
\geq&\frac{1}{K-s}\sum_{k\in\mathcal{S}^c}\sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} w_k a_{\mathcal{T}} QNT\\
&+\frac{1}{K-s}\sum_{k\in\mathcal{S}^c}\sum_{i\in\mathcal{S}^c\setminus\{k\}}\sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k,i\}} \frac{w_i a_{\mathcal{T}}}{|\mathcal{T}|} QNT
\end{aligned} \tag{34}$$

Now, we need to calculate the factor of $w_k a_{\mathcal{T}}$ in (34) for each $k \in \mathcal{S}^c, \mathcal{T} \subseteq \mathcal{S}^c\setminus\{k\}$. If $|\mathcal{T}| = K-s$, then $w_k a_{\mathcal{T}}$ only appears in the first term of (34). Thus, its factor is $\frac{QNT}{K-s} = \frac{QNT}{|\mathcal{T}|}$. If $|\mathcal{T}| < K-s$, then $w_k a_{\mathcal{T}}$ appears in both terms of (34). Specifically, in the second term of (34), $w_k a_{\mathcal{T}}$ appears once for any $i \in \mathcal{S}^c \setminus \{\{k\} \cup \mathcal{T}\}$. Thus, the factor of $w_k a_{\mathcal{T}}$ in (34) is $\frac{QNT}{K-s} + \frac{1}{K-s}(K-s-|\mathcal{T}|)\frac{QNT}{|\mathcal{T}|} = \frac{QNT}{|\mathcal{T}|}$. Thus, (34) can be rewritten as

$$H(X_{[K]}|\mathcal{V}_{\mathcal{S};[K]}, \mathcal{V}_{[K];\mathcal{S}}) \geq \sum_{k\in\mathcal{S}^c}\sum_{\mathcal{T}\subseteq\mathcal{S}^c\setminus\{k\}} \frac{w_k a_{\mathcal{T}}}{|\mathcal{T}|} QNT,$$

which implies that Lemma 2 is also true for sets $\mathcal{S}$ with $|\mathcal{S}| = s-1$. Thus, Lemma 2 is proved.

Appendix D: Proof of Lemma 2

To prove Lemma 2, we first prove

$$H\Big(\mathcal{V}_{\mathcal{S}_p([k]);[K]}|X_{[K]}, \bigcup_{i\in[k]} \mathcal{V}_{[K];\{\mathcal{S}_p(i)\}}\setminus\mathcal{V}_{\mathcal{S}_p([i-1]);\{\mathcal{S}_p(i)\}}\Big) = 0, \tag{35}$$

for any $k \in [|\mathcal{S}|]$. We use induction to prove (35). If $k = 1$, we have

$$H\Big(\mathcal{V}_{\mathcal{S}_p(1);[K]}|X_{[K]}, \mathcal{V}_{[K];\{\mathcal{S}_p(1)\}}\Big) = 0.$$

If (35) is true for $k = s \in [|\mathcal{S}|-1]$, then we have

$$H\Big(\mathcal{V}_{\mathcal{S}_p([s+1]);[K]}|X_{[K]}, \bigcup_{i\in[s+1]} \mathcal{V}_{[K];\{\mathcal{S}_p(i)\}}\setminus\mathcal{V}_{\mathcal{S}_p([i-1]);\{\mathcal{S}_p(i)\}}\Big)$$

$$
\begin{aligned}
=&H\Big(\mathcal{V}_{\mathcal{S}_p([s]);[K]}\big|X_{[K]},\bigcup_{i\in[s+1]}\mathcal{V}_{[K];\{\mathcal{S}_p(i)\}}\backslash\mathcal{V}_{\mathcal{S}_p([i-1]);\{\mathcal{S}_p(i)\}}\Big)\\
&+H\Big(\mathcal{V}_{\mathcal{S}_p(s+1);[K]}\big|\mathcal{V}_{\mathcal{S}_p([s]);[K]},X_{[K]},\\
&\qquad\bigcup_{i\in[s+1]}\mathcal{V}_{[K];\{\mathcal{S}_p(i)\}}\setminus\mathcal{V}_{\mathcal{S}_p([i-1]);\{\mathcal{S}_p(i)\}}\Big)\\
=&H\Big(\mathcal{V}_{\mathcal{S}_p(s+1);[K]}\big|\mathcal{V}_{\mathcal{S}_p([s]);[K]},X_{[K]},\mathcal{V}_{[K];\mathcal{S}_p([s+1])}\Big) \quad (36a)\\
=&0. \quad (36b)
\end{aligned}
$$

for $k = s+1$, where (36a) comes from the induction assumption. Thus, (35) is also true for $k=s+1$. Therefore, we prove that (35) holds for all $k\in[|\mathcal{S}|]$. By letting $k=|\mathcal{S}|$ in (35), we have

$$
H\Big(\mathcal{V}_{\mathcal{S};[K]}\big|X_{[K]},\bigcup_{k\in[|\mathcal{S}|]}\mathcal{V}_{[K];\{\mathcal{S}_p(k)\}}\setminus\mathcal{V}_{\mathcal{S}_p([k-1]);\{\mathcal{S}_p(k)\}}\Big)=0.
$$

Thus, Lemma 2 is proved.

References

[1] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, “Characterizing the rate-memory tradeoff in cache networks within a factor of 2,” *IEEE Trans. Inf. Theory*, vol. 65, no. 1, pp. 647–663, Jan 2019.

[2] S. Li, M. A. Maddah-Ali, Q. Yu, and A. S. Avestimehr, “A fundamental tradeoff between computation and communication in distributed computing,” *IEEE Trans. Inf. Theory*, vol. 64, no. 1, pp. 109–128, Jan 2018.

[3] N. Woolsey, R. Chen, and M. Ji, “A new combinatorial design of coded distributed computing,” in *IEEE ISIT*, June 2018.

[4] K. Konstantinidis and A. Ramamoorthy, “Leveraging coding techniques for speeding up distributed computing,” in *IEEE GLOBECOM*, Dec 2018.

[5] Q. Yan, S. Yang, and M. Wigger, “Storage, computation, and communication: A fundamental tradeoff in distributed computing,” in *IEEE ITW*, Nov 2018.

[6] M. Zhao, W. Wang, Y. Wang, and Z. Zhang, “Load scheduling for distributed edge computing: A communication-computation tradeoff,” *Peer-to-Peer Networking and Applications*, Oct 2018. [Online]. Available: https://doi.org/10.1007/s12083-018-0695-4

[7] F. Li, J. Chen, and Z. Wang, “Wireless MapReduce distributed computing,” in *IEEE ISIT*, June 2018.

[8] M. Kiamari, C. Wang, and A. S. Avestimehr, “On heterogeneous coded distributed computing,” in *IEEE GLOBECOM*, Dec 2017, pp. 1–7.

[9] N. Shakya, F. Li, and J. Chen, “On distributed computing with heterogeneous communication constraints,” in *52nd Asilomar Conf. Signals, Syst. Comput.*, Oct 2018, pp. 1795–1799.

[10] N. Woolsey, R. Chen, and M. Ji, “Cascaded coded distributed computing on heterogeneous networks,” in *IEEE ISIT*, July 2019.

[11] ——, “Coded distributed computing with heterogeneous function assignments,” 2019. [Online]. Available: http://arxiv.org/abs/1902.10738

[12] M. Kiamari, C. Wang, and A. S. Avestimehr, “Coding for edge-facilitated wireless distributed computing with heterogeneous users,” in *51st Asilomar Conf. Signals, Syst. Comput.*, Oct 2017, pp. 536–540.

[13] F. Xu and M. Tao, “Heterogeneous coded distribtued computing: joint design of file allocation and function assignment,” in *IEEE GLOBECOM*, Dec 2019.

[14] M. A. Maddah-Ali and U. Niesen, “Decentralized coded caching attains order-optimal memory-rate tradeoff,” *IEEE/ACM Trans. Networking*, vol. 23, no. 4, pp. 1029–1040, Aug 2015.