Math 301 Notes

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Sum

Let $(x_j)_{j=1}^{\infty}$ be a sequence of integers. For each $k \in \mathbb{N}$, we want to define a number called $\sum_{j=1}^{k} x_j$:

1. Define $\sum_{j=1}^{1} x_j$ to be x_1.
2. Assuming $\sum_{j=1}^{k} x_j$ already defined, we define $\sum_{j=1}^{k+1} x_j$ to be $\left(\sum_{j=1}^{k} x_j\right) + x_{k+1}$.

Product

Similarly, we define an integer called $\prod_{j=1}^{k} x_j$:

1. Define $\prod_{j=1}^{1} x_j$ to be x_1.
2. Assuming $\prod_{j=1}^{k} x_j$ already defined, we also find $\prod_{j=1}^{k+1} x_j$ to be $\left(\prod_{j=1}^{k} x_j\right) \cdot x_{k+1}$.

Factorial

As a third example, we define $k!$ ("k factorial") for all integers $k \geq 0$ by:

1. Define $0!$ to be 1.
2. Assuming $n!$ defined (where $n \in \mathbb{Z}_{\geq 0}$), define $(n+1)!$ to be $(n!) \cdot (n+1)$.

Power

Let b be a fixed integer. We define b^k for all integers $k \geq 0$ by:

1. Define b^0 to be 1.
2. Assuming b^n defined, let b^{n+1} be $b^n \cdot b$.

Proposition 4.13 for $x \neq 1$ and $k \in \mathbb{Z}_{\geq 0}$

$$\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x} \quad \text{if} \quad |x| < 1$$

$$\lim_{k\to\infty} \sum_{j=0}^{k} x^j = \frac{1}{1-x} \quad \text{if} \quad |x| < 1$$

Geometric series

Proof

Assume $x \neq 1$, $k \in \mathbb{Z}_{\geq 0}$

Base Case: $k = 0$

$$\sum_{j=0}^{0} x^j = x^0 = 1$$

$$\text{when} \quad j = 0, \quad x^j = 1$$

$$\frac{1-x^{0+1}}{1-x} = \frac{1-x}{1-x} = 1$$

For Induction Step: (Always If-Then Statement)

Assume Case k

$$\sum_{j=0}^{k} x^j = \frac{1-x^{k+1}}{1-x}$$

Show Case $k+1$

$$\sum_{j=0}^{k+1} x^j = \frac{1-x^{k+2}}{1-x}$$

Proof:

$$\begin{aligned}
\sum_{j=0}^{k+1} x^j &= \left(\sum_{j=0}^{k} x^j\right) + x^{k+1} \\
&= \frac{1-x^{k+1}}{1-x} + x^{k+1} \\
&= \frac{1-x^{k+1} + (1-x)x^{k+1}}{1-x} \\
&= \frac{1-x^{k+1} + x^{k+1} - x^{k+2}}{1-x} \\
&= \frac{1-x^{k+2}}{1-x}
\end{aligned}$$