Definitions and General Principles

Inverse of a Matrix

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = A^{-1}A = I$

LDU Decomposition

- For a symmetric matrix A: $A = LDL^T$
- L: lower triangular with unit diagonal
- D: diagonal matrix

Vector Space Axioms

- Addition: commutativity, associativity, identity, inverses
- Scalar Multiplication: distributivity, compatibility, identity

Subspaces

- Closed under addition and scalar multiplication

Linear Dependence and Independence

- Dependent: $\exists$ scalars, not all zero, s.t. $a_1v_1 + \ldots + a_nv_n = 0$
- Independent: only solution is $a_1 = \ldots = a_n = 0$

Basis and Dimension

- Basis: linearly independent spanning set
- Dimension: number of vectors in a basis

General Principles for Subspaces

- Closed under vector addition
- Closed under scalar multiplication

Linear Transformation

- Preserves vector addition and scalar multiplication

Image and Kernel

- $\text{im}(A)$: span of column vectors of A
- $\ker(A)$: $\{x \in \mathbb{R}^n : Ax = 0\}$

Basis Transformation

- Unique representation of a vector in terms of basis vectors

Determining Linear Independence (Standard Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 1)$, and $v_3 = (2, 5, 7)$, determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix A:
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$$
2. Perform row reduction on A:
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Since there are no rows of all zeros in the reduced row echelon form, the vectors are linearly independent.

Determining Linear Independence (Linearly Dependent Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, and $v_3 = (3, 6, 9)$, determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix A:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$
2. Perform row reduction on A:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
3. The presence of rows of all zeros indicates that the vectors are linearly dependent.

Finding a Basis for a Subspace (Polynomial Space)

Find a basis for the subspace of P_3 consisting of polynomials $p(x) = ax^3 + bx^2 + cx + d$ such that $p(1) = 0$.

Solution:

1. The condition $p(1) = 0$ gives $a + b + c + d = 0$. To find a basis, express this condition in terms of the coefficients and set up a system.
2. Considering the standard basis $\{1, x, x^2, x^3\}$ for P_3, impose the condition for $p(1) = 0$:
$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = 0$$
This implies $a = -b - c - d$.
3. A basis satisfying this condition is $\{x^3 - x^2, x^2 - x, x - 1\}$ as these polynomials nullify at $x = 1$ and are linearly independent.

Finding the Matrix Inverse

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

Solution:

1. Set up the augmented matrix for A and the identity matrix: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$.
2. Perform row operations to get the identity matrix on the left side of the augmented matrix. Subtract twice the first row from the second row to start: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.
3. Then, subtract 3 times the second row from the first row: $\begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.
4. The matrix on the right side is now $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Solution:

1. Find the characteristic polynomial: $\det(B - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4 - \lambda)^2 - 1$.
2. Solve for λ: $(4 - \lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.
3. Find eigenvectors for each eigenvalue: - For $\lambda_1 = 3$: Solve $(B - 3I)x = 0$. This gives $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. - For $\lambda_2 = 5$: Solve $(B - 5I)x = 0$. This gives $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Diagonalization

Determine if the matrix $C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable. If it is, find a matrix P that diagonalizes C.

Solution:

1. Find the eigenvalues of C: The characteristic polynomial is $(2 - \lambda)^2 = 0$, so the only eigenvalue is $\lambda = 2$.
2. Since C is a 2×2 matrix with only one distinct eigenvalue, we need to check if there are two linearly independent eigenvectors corresponding to $\lambda = 2$.
3. Solve $(C - 2I)x = 0$: This leads to the system $x_2 = 0$, indicating that every eigenvector has the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, which does not provide two independent eigenvectors.
4. Since we cannot find two linearly independent eigenvectors, C is not diagonalizable.

Verifying Vector Space Axioms

Verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space over the real numbers.

Solution

To verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space, we need to check that the following axioms hold:

- The set is closed under addition: The sum of any two polynomials of degree at most 2 is also a polynomial of degree at most 2.
- The set is closed under scalar multiplication: The scalar multiple of any polynomial of degree at most 2 is also a polynomial of degree at most 2.
- The set contains a zero vector, which is the zero polynomial.
- Each polynomial has an additive inverse within the set.
- Addition is associative and commutative.
- Scalar multiplication is distributive with respect to both scalar and vector addition.
- Scalar multiplication is compatible with field multiplication.
- The scalar 1 acts as a multiplicative identity.

Since all these properties are satisfied, the set is indeed a vector space.

Finding the Rank of a Matrix

Find the rank of the matrix
$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Solution:

1. Perform row reduction on E:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
2. The rank is equal to the number of non-zero rows in the reduced row echelon form. Hence, $\text{rank}(E) = 1$.

Linear Dependence in $\mathbb{R}^2$

Given the vectors $v_1 = (3, -1)$ and $v_2 = (6, -2)$ in $\mathbb{R}^2$, determine if v_1 and v_2 are linearly dependent. **Solution**

1. Notice that $v_2 = 2v_1$, which means v_2 is a scalar multiple of v_1.
2. This implies the set $\{v_1, v_2\}$ is linearly dependent because v_2 can be expressed as a linear combination of v_1.

Problem 16: Identifying Subspaces

Determine if the set of all vectors in $\mathbb{R}^3$ of the form (a, a, b) is a subspace of $\mathbb{R}^3$.

Solution

To determine if the set of all vectors in $\mathbb{R}^3$ of the form (a, a, b) is a subspace, we check for the following properties:

- The zero vector $(0, 0, 0)$ is in the set, satisfying the non-emptiness requirement.
- The set is closed under addition: $(a_1, a_1, b_1) + (a_2, a_2, b_2) = (a_1 + a_2, a_1 + a_2, b_1 + b_2)$, which is of the form (a, a, b).
- The set is closed under scalar multiplication: For any real number c, $c(a, a, b) = (ca, ca, cb)$, which is still of the form (a, a, b).

Thus, the set is a subspace of $\mathbb{R}^3$.

Important Identities and Properties

- Trace property: $\text{tr}(AB) = \text{tr}(BA)$
- Rank-Nullity Theorem: $\text{rank}(A) + \text{nullity}(A) = n$, where n is the number of columns of A.
- Determinant properties: $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ for invertible A.

Determinant Calculation
Calculate the determinant of the matrix

$$D = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution:

1. Apply the Laplace expansion using the first row:
$$\det(D) = 6\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$
2. Calculate each minor:
$$\begin{aligned} & 6 \cdot (3 \cdot 4 - 0 \cdot 1) - 1 \cdot (1 \cdot 4 - 1 \cdot 2) + 2 \cdot (1 \cdot 0 - 3 \cdot 2) \\ &= 72 - 2 - 12 \\ &= 58 \end{aligned} \tag{1}$$
3. Thus, $\det(D) = 58$.

Using Cramer's Rule
Solve the following system of equations using Cramer's Rule:

$$\begin{aligned} x + 2y - z &= 4, \\ 2x - y + 3z &= -2, \\ x + 3y + z &= 3. \end{aligned} \tag{2}$$

Solution:

1. Write the coefficient matrix and calculate its determinant:
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A) = -16.$$
2. For x, replace the first column of A with the constant terms and calculate its determinant:
$$A_x = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -1 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \quad \det(A_x) = -16.$$
3. For y, replace the second column of A:
$$A_y = \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A_y) = -32.$$
4. For z, replace the third column of A:
$$A_z = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 1 & 3 & 3 \end{bmatrix}, \quad \det(A_z) = -16.$$
5. Compute the solutions: $x = \frac{\det(A_x)}{\det(A)} = 1, y = \frac{\det(A_y)}{\det(A)} = 2, z = \frac{\det(A_z)}{\det(A)} = 1$.

LU Decomposition

Perform LU decomposition on the matrix $F = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$.

Solution:

1. Express F as the product of a lower triangular matrix L and an upper triangular matrix U.
2. Choose L with 1s on the diagonal: $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$.
3. Let $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$.
4. Since $F_{21} = l_{21} \cdot u_{11}$ and $F_{21} = 6$, $u_{11} = 4$, we get $l_{21} = \frac{6}{4} = \frac{3}{2}$.
5. Solve for U using the first row of F: $U = \begin{bmatrix} 4 & 3 \\ 0 & u_{22} \end{bmatrix}$. The second element of the second row gives $u_{22} = 3 - \frac{3}{2} \cdot 3 = -\frac{3}{2}$.
6. The LU decomposition is $L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 3 \\ 0 & -\frac{3}{2} \end{bmatrix}$

Orthogonal Diagonalization
Orthogonally diagonalize the matrix

$$F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Solution:

1. Find the eigenvalues by solving $\det(F - \lambda I) = 0$: $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 4$.
2. Find the eigenvectors: For $\lambda_1 = 2$, solve $(F - 2I)x = 0$ to get $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (after normalization). For $\lambda_2 = 4$, solve $(F - 4I)x = 0$ to get $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (after normalization).
3. Construct $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ and verify $P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Orthogonal Diagonalization

Orthogonally diagonalize the matrix $F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

Solution:

1. Find the eigenvalues by solving $\det(F - \lambda I) = 0$: $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 4$.
2. Find the eigenvectors: For $\lambda_1 = 2$, solve $(F - 2I)x = 0$ to get $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (after normalization). For $\lambda_2 = 4$, solve $(F - 4I)x = 0$ to get $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (after normalization).
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Basis and Dimension of a Vector Space
Consider the vector space V of all vectors in $\mathbb{R}^4$ that satisfy the equation $x_1 - 2x_2 + x_3 - 2x_4 = 0$. Find a basis for V and state its dimension. **Solution**

1. To find a basis, we need to solve for the vectors that satisfy the given equation. Let $x_4 = t$, then $x_3 = 2t$, $x_2 = s$, and $x_1 = 2s - t$.
2. Thus, any vector in V can be written as $\begin{bmatrix} 2s - t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.
3. The vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent and span V, so they form a basis for V.
4. The dimension of V, denoted as $\dim(V)$, is the number of vectors in the basis, which is 2.

Image and Kernel of a Linear Transformation
Find the image and kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. **Solution**

1. To find the kernel of T, solve the homogeneous system $A\mathbf{x} = \mathbf{0}$.
2. Row reduce the matrix A to find the solution to the system:
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
3. The solutions to the system are of the form $\mathbf{x} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.
4. The kernel of T is therefore spanned by the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.
5. To find the image of T, we look at the column space of A, which is spanned by the pivot columns.
6. The only pivot column is the first column of A, so $\operatorname{im}(T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$.

Change of Basis
Given a vector $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the standard basis of $\mathbb{R}^2$, find its coordinates in the new basis $B = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.
Solution

1. The coordinates of v in the basis B can be found by solving the equation $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
2. This equation translates into the system:
$$c_1 - c_2 = 3, c_1 + c_2 = 2.$$
3. Adding the two equations yields $2c_1 = 5$, so $c_1 = \frac{5}{2}$.
4. Substituting c_1 into the second equation gives $c_2 = \frac{2}{2} - \frac{5}{2} = -\frac{3}{2}$.
5. Therefore, the coordinates of v in the basis B are $\left(\frac{5}{2}, -\frac{3}{2}\right)$.

Perform LDU decomposition
Perform LDU decomposition on the matrix

$$H = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Solution:

1. First, we find the matrix L such that $H = LDU$ where L is a lower triangular matrix with unit diagonal, D is a diagonal matrix, and U is an upper triangular matrix.
2. Decompose H into LDU:
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix},$$
$$U = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$
3. Verify the decomposition by calculating LDU and comparing it with H:
$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}.$$
4. The result confirms the LDU decomposition of H.