MATH 440 Marty Martin
2:00 PM - 3:20 PM May 2024

Practice for the exam

Question 1

Suppose the random variables X and Y are jointly distributed according to the pdf:

$$f_{XY}(x,y) = 8xy, \quad 0 < y < x < 1$$

(a) Find $P(X < 2Y)$

(b) Find $P(Y < \frac{1}{4} | x = \frac{1}{2})$

Part (a) Find $P(X < 2Y)$

General Formula: For any joint PDF $f_{XY}(x,y)$, the probability $P(g(X,Y))$ for some condition g is given by:

$$P(g(X,Y)) = \int\int_{g(x,y)} f_{XY}(x,y)\,dx\,dy$$

This probability calculation involves integrating the joint PDF over the region defined by $X < 2Y$ and within the given bounds of $0 < y < x < 1$.

$$P(X < 2Y) = \int_0^1 \int_0^{X/2} 8xy\,dy\,dx$$

This integral setup reflects the condition $X < 2Y$ within the area bounded by $0 < y < x < 1$.

Calculate the inner integral over y:

$$\int_0^{X/2} 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_0^{X/2} = 8x\left[\frac{(X/2)^2}{2}\right] = X^3$$

Now integrate with respect to x:

$$\int_0^1 X^3\,dx = \left[\frac{X^4}{4}\right]_0^1 = \frac{1}{4}$$

Thus, $P(X < 2Y) = \frac{1}{4}$.

Part (b) Find $P(Y < \frac{1}{4}|x = \frac{1}{2})$

For any joint PDF $f_{XY}(x, y)$, the conditional probability $P(A|B)$ is given by:

$$P(A|B) = \frac{\int\int_{A\cap B} f_{XY}(x, y)\,dx\,dy}{\int\int_B f_{XY}(x, y)\,dx\,dy}$$

Given $X = \frac{1}{2}$, we need to find the conditional probability $P(Y < \frac{1}{4}|X = \frac{1}{2})$. This involves determining the conditional PDF $f_{Y|X}(y|x)$ and integrating it over the desired range of Y.

The marginal density $f_X(x)$ is found by integrating out Y from the joint PDF:

$$f_X(x) = \int_0^x 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_0^x = 4x^3$$

At $x = \frac{1}{2}$, the marginal density $f_X\left(\frac{1}{2}\right)$ is:

$$f_X\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

The conditional PDF $f_{Y|X}(y|\frac{1}{2})$ is:

$$f_{Y|X}(y|\frac{1}{2}) = \frac{8 \cdot \frac{1}{2} \cdot y}{\frac{1}{2}} = 8y$$

$$P(Y < \frac{1}{4}|X = \frac{1}{2}) = \int_0^{1/4} 8y\,dy$$

Calculate the integral:

$$\int_0^{1/4} 8y\,dy = \left[4y^2\right]_0^{1/4} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$$

Therefore, $P(Y < \frac{1}{4}|X = \frac{1}{2}) = \frac{1}{4}$.

Question 2

A random variable has moment generating function $M_X(t) = \left(\frac{2+e^t}{3}\right)^9$. Find $\text{Var}(X)$.

Question 3

The driver of a truck loaded with 900 boxes of books will be fined if the total weight of the boxes exceeds 36450 pounds. If the distribution of the weight of a box has a mean of 40 pounds and a variance of 36, find the approximate probability that the driver will be fined.

Question 4

Suppose that the number of calls per hour to an answering service follows a Poisson distribution with rate $\lambda = 4$.

(a) What is the probability that fewer than 2 calls came in the first hour?

(b) What is the probability that there will be no calls in the next two hours?

Question 5

Using moment generating functions (MGFs), show that if:

$$X \sim N(\mu_1, \sigma_1^2) \quad \text{and} \quad Y \sim N(\mu_2, \sigma_2^2),$$

then the expectation and variance of $X + Y$ are given by:

$$E(X+Y) = \mu_1 + \mu_2 \quad \text{and} \quad \text{Var}(X+Y) = \sigma_1^2 + \sigma_2^2,$$

and that:

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Note: The moment generating function of X, if X is normally distributed, is given by:

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}.$$