5.4.1 Find the least squares solution to the linear system $Ax = b$

(a) Given:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We need to find the least squares solution to the equation $Ax = b$. The normal equation for this problem is:

$$A^T A x = A^T b$$

First, calculate $A^T A$:

$$A^T = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$$

Then, calculate $A^T b$:

$$A^T b = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$$

Now, solve the normal equation:

$$6x = 3$$

$$x = \frac{3}{6} = 0.5$$

Thus, the least squares solution x is:

$$x = 0.5$$

(b) Given:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}$$

We need to find the least squares solution to the equation $Ax = b$. The normal equation for this problem is:

$$A^T A x = A^T b$$

First, calculate $A^T A$:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 26 \end{bmatrix}$$

Then, calculate $A^T b$:

$$A^T b = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 38 \\ 28 \end{bmatrix}$$

Now, solve the normal equation:

$$\begin{bmatrix} 14 & 13 \\ 13 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 38 \\ 28 \end{bmatrix}$$

Thus, the least squares solution x is:

$$x = \begin{bmatrix} 3.2 \\ -0.52307692 \end{bmatrix}$$

(c) Given:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We need to find the least squares solution to the equation $Ax = b$. The normal equation for this problem is:

$$A^T Ax = A^T b$$

First, calculate $A^T A$:

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 14 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 2 \end{bmatrix}$$

Then, calculate $A^T b$:

$$A^T b = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

Now, solve the normal equation:

$$\begin{bmatrix} 14 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the least squares solution x is:

$$x = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.0 \end{bmatrix}$$

Exercise 5.4.6

Find the least squares solution to the linear systems in Exercise 5.4.1 under the weighted norm

$$\|x\|^2 = x_1^2 + 2x_2^2 + 3x_3^2$$

We use the same systems as in 5.4.1 for each part (a), (b), and (c), and solve them considering the weighted norm.

Part (a)

Given the same A and b as in 5.4.1(a):

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

First, we adjust the normal equation considering the weighted norm:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

$$\tilde{A} = WA, \quad \tilde{b} = Wb$$

Calculate $\tilde{A}^T\tilde{A}$ and $\tilde{A}^T\tilde{b}$:

$$\tilde{A} = \begin{bmatrix} 1 \\ 2\sqrt{2} \\ \sqrt{3} \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$\tilde{A}^T\tilde{A} = \begin{bmatrix} 1 & 2\sqrt{2} & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2\sqrt{2} \\ \sqrt{3} \end{bmatrix}, \quad \tilde{A}^T\tilde{b} = \begin{bmatrix} 1 & 2\sqrt{2} & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

Now, solve the weighted normal equation for x:

$$\tilde{A}^T\tilde{A}x = \tilde{A}^T\tilde{b}$$

Part (b)

Given the same A and b as in 5.4.1(b):

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}$$

As in part (a), first define the weight matrix W and adjust A and b:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

Adjust A and b to reflect the weights:

$$\tilde{A} = WA = \begin{bmatrix} 1 & 0 \\ 2\sqrt{2} & -\sqrt{2} \\ 3\sqrt{3} & 5\sqrt{3} \end{bmatrix}, \quad \tilde{b} = Wb = \begin{bmatrix} 3 \\ 7\sqrt{2} \\ 7\sqrt{3} \end{bmatrix}$$

Calculate $\tilde{A}^T\tilde{A}$ and $\tilde{A}^T\tilde{b}$:

$$\tilde{A}^T = \begin{bmatrix} 1 & 2\sqrt{2} & 3\sqrt{3} \\ 0 & -\sqrt{2} & 5\sqrt{3} \end{bmatrix}$$

$$\tilde{A}^T\tilde{A} = \tilde{A}^T\tilde{A} = \begin{bmatrix} 1+8+27 & 0-2\sqrt{2}+15\sqrt{6} \\ 0-2\sqrt{2}+15\sqrt{6} & 2+75 \end{bmatrix}$$

$$\tilde{A}^T\tilde{b} = \tilde{A}^T\tilde{b} = \begin{bmatrix} 1\cdot 3 + 2\sqrt{2}\cdot 7\sqrt{2} + 3\sqrt{3}\cdot 7\sqrt{3} \\ 0\cdot 3 - \sqrt{2}\cdot 7\sqrt{2} + 5\sqrt{3}\cdot 7\sqrt{3} \end{bmatrix}$$

Now solve the weighted normal equation:

$$\tilde{A}^T\tilde{A}x = \tilde{A}^T\tilde{b}$$

Part (c)

Given the same A and b as in 5.4.1(c):

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

As in previous parts, define the weight matrix W and adjust A and b:

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

Adjust A and b accordingly:

$$\tilde{A} = WA = \begin{bmatrix} 2 & 1 & -1 \\ \sqrt{2} & -2\sqrt{2} & 0 \\ 3\sqrt{3} & -\sqrt{3} & \sqrt{3} \end{bmatrix}, \quad \tilde{b} = Wb = \begin{bmatrix} 1 \\ 0 \\ \sqrt{3} \end{bmatrix}$$

Calculate $\tilde{A}^T\tilde{A}$ and $\tilde{A}^T\tilde{b}$:

$$\tilde{A}^T = \begin{bmatrix} 2 & \sqrt{2} & 3\sqrt{3} \\ 1 & -2\sqrt{2} & -\sqrt{3} \\ -1 & 0 & \sqrt{3} \end{bmatrix}$$

$$\tilde{A}^T\tilde{A} = \begin{bmatrix} 4+2+27 & 2-2\sqrt{2}-3\sqrt{3} & -2+0+3\sqrt{3} \\ 2-2\sqrt{2}-3\sqrt{3} & 1+8+1 & -1+0-1 \\ -2+0+3\sqrt{3} & -1+0-1 & 1+0+3 \end{bmatrix}$$

$$\tilde{A}^T\tilde{b} = \begin{bmatrix} 2\cdot 1+\sqrt{2}\cdot 0+3\sqrt{3}\cdot\sqrt{3} \\ 1\cdot 1-2\sqrt{2}\cdot 0-\sqrt{3}\cdot\sqrt{3} \\ -1\cdot 1+0\cdot 0+\sqrt{3}\cdot\sqrt{3} \end{bmatrix}$$

Now solve the weighted normal equation:

$$\tilde{A}^T\tilde{A}x = \tilde{A}^T\tilde{b}$$

Exercise 5.5.4

A 20-pound turkey that is at the room temperature of 72°F is placed in the oven at 1:00 PM. The temperature of the turkey is observed in 20-minute intervals to be 79°F, 88°F, and 96°F. A turkey is cooked when its temperature reaches 165°F. How much longer do you need to wait until the turkey is done?

Solution:

Given temperatures:

$$T_0 = 72, \quad T_{20} = 79, \quad T_{40} = 88, \quad T_{60} = 96 \quad \text{(all in degrees Fahrenheit)}$$

We need to find the constant k in the cooling equation. We can use the temperatures at different times to set up equations based on the model, and then solve for k using regression or a system of equations.

To find k, consider the equations:

$$\begin{aligned} 79 &= T_{\text{env}} + (72 - T_{\text{env}})e^{-20k}, \\ 88 &= T_{\text{env}} + (72 - T_{\text{env}})e^{-40k}, \\ 96 &= T_{\text{env}} + (72 - T_{\text{env}})e^{-60k}. \end{aligned}$$

Once k and T_{env} are determined, solve for t such that:

$$165 = T_{\text{env}} + (72 - T_{\text{env}})e^{-kt}.$$

Exercise 1.9.2

Verify the determinant product formula when matrices A and B are given as:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 1 \\ 4 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -3 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution:

To find $\det(A)$ and $\det(B)$:

$$\det(A) = 1\left((-1)(0) - (1)(-2)\right) - (-1)\left(2(0) - (1)(4)\right) + 3\left(2(-2) - (-1)(4)\right)$$

$$\det(B) = 0\left((-3)(1) - (-2)(0)\right) + 1\left((1)(1) - (-1)(2)\right) + (-1)\left(1(-2) - (-3)(2)\right)$$

First, compute the product AB:

$$AB = A \times B = \begin{bmatrix} 1\cdot 0 + (-1)\cdot 1 + 3\cdot 2 & 1\cdot 1 + (-1)(-3) + 3\cdot 0 & 1\cdot(-1) + (-1)(-2) + 3\cdot 1 \\ 2\cdot 0 + (-1)\cdot 1 + 1\cdot 2 & 2\cdot 1 + (-1)(-3) + 1\cdot 0 & 2\cdot(-1) + (-1)(-2) + 1\cdot 1 \\ 4\cdot 0 + (-2)\cdot 1 + 0\cdot 2 & 4\cdot 1 + (-2)(-3) + 0\cdot 0 & 4\cdot(-1) + (-2)(-2) + 0\cdot 1 \end{bmatrix}$$

Calculate $\det(AB)$:

$$\det(AB) = [\text{Cofactor expansion of the resultant matrix AB}]$$

Finally, verify if:

$$\det(AB) = \det(A) \times \det(B)$$

Exercise 8.2.1

Find the eigenvalues and eigenvectors of the following matrices:

Part (a)

Find the eigenvalues and eigenvectors for the matrix:

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = -1$$

$$(A - 3I)x = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} x = 0 \Rightarrow x_1 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A + I)x = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} x = 0 \Rightarrow x_2 = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Part (b)

Find the eigenvalues and eigenvectors for the matrix:

$$B = \begin{bmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$$

$$\det(B - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} - \lambda \end{bmatrix}\right) = (1-\lambda)\left(-\frac{1}{6} - \lambda\right) + \frac{1}{3}$$

$$\lambda^2 - \frac{5}{6}\lambda - \frac{1}{6} = 0$$

$$\begin{aligned} \lambda &= \frac{\frac{5}{6} \pm \sqrt{\left(\frac{5}{6}\right)^2 + \frac{4}{6}}}{2} \\ &= \frac{\frac{5}{6} \pm \sqrt{\frac{49}{36}}}{2} \\ &= \frac{\frac{5}{6} \pm \frac{7}{6}}{2} \\ &\Rightarrow \lambda_1 = 1, \quad \lambda_2 = -\frac{1}{6} \end{aligned}$$

$$(B - I)x = \begin{bmatrix} 0 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{7}{6} \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ (normalized)}$$

$$(B + \frac{1}{6}I)x = \begin{bmatrix} \frac{7}{6} & -\frac{2}{3} \\ \frac{1}{2} & 0 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ (normalized)}$$

Part (c)

Find the eigenvalues and eigenvectors for the matrix:

$$C = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\det(C - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4$$

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2 \text{ (doubly repeated eigenvalue)}$$

$$(C - 2I)x = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} x = 0 \Rightarrow x = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Part (d)

Find the eigenvalues and eigenvectors for the matrix:

$$D = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\det(D - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + 2$$

$$\lambda^2 - 2\lambda + 3 = 0$$

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4-12}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm 2i\sqrt{2}}{2} \\ \Rightarrow \lambda_1 &= 1 + i\sqrt{2}, \quad \lambda_2 = 1 - i\sqrt{2} \end{aligned}$$

$$(D - (1+i\sqrt{2})I)x = \begin{bmatrix} -i\sqrt{2} & 2 \\ -1 & -i\sqrt{2} \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} i\sqrt{2} \\ 1 \end{bmatrix}$$

$$(D - (1-i\sqrt{2})I)x = \begin{bmatrix} i\sqrt{2} & 2 \\ -1 & i\sqrt{2} \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} -i\sqrt{2} \\ 1 \end{bmatrix}$$

Part (e)

Find the eigenvalues and eigenvectors for the matrix:

$$E = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\det(E-\lambda I) = \det\left(\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)[(2-\lambda)(3-\lambda)-1]+1 = \lambda^3-8\lambda^2+21\lambda-18$$

$$\lambda^3 - 8\lambda^2 + 21\lambda - 18 = 0 \Rightarrow \lambda = 1, 2, 4$$

$$(E - I)x = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$(E - 2I)x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(E - 4I)x = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} x = 0 \Rightarrow x_3 = u \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Part (f)

Find the eigenvalues and eigenvectors for the matrix:

$$F = \begin{bmatrix} -1 & -1 & 4 \\ 1 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\det(F - \lambda I) = \det\left(\begin{bmatrix} -1-\lambda & -1 & 4 \\ 1 & 3-\lambda & -2 \\ 1 & 1 & -1-\lambda \end{bmatrix}\right) = -\lambda^3 + \lambda^2 + 5\lambda - 3$$

$$-\lambda^3 + \lambda^2 + 5\lambda - 3 = 0 \Rightarrow \lambda = 3, -1, 1$$

$$(F - 3I)x = \begin{bmatrix} -4 & -1 & 4 \\ 1 & 0 & -2 \\ 1 & 1 & -4 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$(F + I)x = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 1 & 0 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$(F - I)x = \begin{bmatrix} -2 & -1 & 4 \\ 1 & 2 & -2 \\ 1 & 1 & -2 \end{bmatrix} x = 0 \Rightarrow x_3 = u \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Part (g)

Find the eigenvalues and eigenvectors for the matrix:

$$G = \begin{bmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{bmatrix}$$

$$\det(G - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -3 & 11 \\ 2 & -6-\lambda & 16 \\ 1 & -3 & 7-\lambda \end{bmatrix}\right) = -\lambda^3 + 2\lambda^2 - \lambda + 2$$

$$-\lambda^3 + 2\lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda = -1, 1, 2$$

$$(G + I)x = \begin{bmatrix} 2 & -3 & 11 \\ 2 & -5 & 16 \\ 1 & -3 & 8 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$(G - I)x = \begin{bmatrix} 0 & -3 & 11 \\ 2 & -7 & 16 \\ 1 & -3 & 6 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$(G - 2I)x = \begin{bmatrix} -1 & -3 & 11 \\ 2 & -8 & 16 \\ 1 & -3 & 5 \end{bmatrix} x = 0 \Rightarrow x_3 = u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Part (h)

Find the eigenvalues and eigenvectors for the matrix:

$$H = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(H - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 & -1 \\ -2 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0 \Rightarrow \lambda = 1, 1, 2$$

$$(H - I)x = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(H - 2I)x = \begin{bmatrix} 0 & -1 & -1 \\ -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Part (i)

Find the eigenvalues and eigenvectors for the matrix:

$$I = \begin{bmatrix} -4 & -4 & 2 \\ 3 & 4 & -1 \\ -3 & -2 & 3 \end{bmatrix}$$

$$\det(I - \lambda I) = \det\left(\begin{bmatrix} -4-\lambda & -4 & 2 \\ 3 & 4-\lambda & -1 \\ -3 & -2 & 3-\lambda \end{bmatrix}\right) = -\lambda^3 + 3\lambda^2 + 11\lambda - 27$$

$$-\lambda^3 + 3\lambda^2 + 11\lambda - 27 = 0 \Rightarrow \lambda = -3, 1, 9$$

$$(I + 3I)x = \begin{bmatrix} -1 & -4 & 2 \\ 3 & 7 & -1 \\ -3 & -2 & 6 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$(I - I)x = \begin{bmatrix} -5 & -4 & 2 \\ 3 & 3 & -1 \\ -3 & -2 & 2 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$$(I - 9I)x = \begin{bmatrix} -13 & -4 & 2 \\ 3 & -5 & -1 \\ -3 & -2 & -6 \end{bmatrix} x = 0 \Rightarrow x_3 = u \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Part (j)

Find the eigenvalues and eigenvectors for the matrix:

$$J = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

$$\det(J - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 4 & 0 & 0 \\ 4 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 3 \\ 0 & 0 & 4 & 5-\lambda \end{bmatrix}\right)$$
$$= ((3-\lambda)^2 - 16) \times ((1-\lambda)(5-\lambda) - 12)$$

$$= (\lambda^2 - 6\lambda + 25) \times (\lambda^2 - 6\lambda - 11)$$
$$= (\lambda - 7)(\lambda + 1)(\lambda - 1)(\lambda - 5)$$
$$\Rightarrow \lambda = 7, -1, 1, 5$$

$$(J - 7I)x = \begin{bmatrix} -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 \\ 0 & 0 & -6 & 3 \\ 0 & 0 & 4 & -2 \end{bmatrix} x = 0 \Rightarrow x_1 = t\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(J + I)x = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{bmatrix} x = 0 \Rightarrow x_2 = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(J - I)x = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 4 \end{bmatrix} x = 0 \Rightarrow x_3 = u\begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

$$(J - 5I)x = \begin{bmatrix} -2 & 4 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} x = 0 \Rightarrow x_4 = v\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Part (k)

Find the eigenvalues and eigenvectors for the matrix:

$$K = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\det(K - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 2-\lambda & 0 \\ 1 & -1 & 1 & 1-\lambda \end{bmatrix}\right)$$
$$= (4-\lambda) \times (3-\lambda) \times (2-\lambda) \times (1-\lambda)$$

$$\Rightarrow \lambda = 4, 3, 2, 1$$

$$(K - 4I)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & -2 & 0 \\ 1 & -1 & 1 & -3 \end{bmatrix} x = 0 \Rightarrow x_1 = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(K - 3I)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -2 \end{bmatrix} x = 0 \Rightarrow x_2 = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(K - 2I)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} x = 0 \Rightarrow x_3 = u \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(K - 1I)x = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} x = 0 \Rightarrow x_4 = v \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$