Homework 8

Problem 3.7.12

Question

A point is chosen at random from the interior of a circle whose equation is $x^2 + y^2 \leq 4$. Let the random variables X and Y denote the x- and y-coordinates of the sampled point. Find the joint pdf $f_{X,Y}(x, y)$.

Solution

The area A of the circle is calculated as:

$$A = \pi r^2 = \pi \times 2^2 = 4\pi$$

The joint pdf $f_{X,Y}(x, y)$ is given by:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

The integral of the pdf over the circle should equal 1:

$$\int\int_{x^2+y^2\leq 4} \frac{1}{4\pi}\, dx\, dy = \frac{1}{4\pi} \times 4\pi = 1$$

Problem 3.7.14

Question

Suppose that five independent observations are drawn from the continuous pdf $f_T(t) = 2t$, $0 \leq t \leq 1$. Let X denote the number of t's that fall in the interval $0 \leq t \leq \frac{1}{3}$ and let Y denote the number of t's that fall in the interval $\frac{1}{3} \leq t \leq \frac{2}{3}$. Find $P_{XY}(1, 2)$.

Solution

$$P(0 \leq T \leq \frac{1}{3}) = \int_0^{\frac{1}{3}} 2t\, dt = \left[t^2\right]_0^{\frac{1}{3}} = \left(\frac{1}{3}\right)^2 = \frac{1}{9},$$

$$P(\frac{1}{3} \leq T \leq \frac{2}{3}) = \int_{\frac{1}{3}}^{\frac{2}{3}} 2t\, dt = \left[t^2\right]_{\frac{1}{3}}^{\frac{2}{3}} = \left(\frac{4}{9}\right) - \left(\frac{1}{9}\right) = \frac{3}{9} = \frac{1}{3}.$$

$$P_{XY}(1, 2) = \binom{5}{1}\left(\frac{1}{9}\right)\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{5}{9}\right)^2,$$

$$= 5 \cdot \frac{1}{9} \cdot 6 \cdot \frac{1}{9} \cdot \left(\frac{25}{81}\right) = \frac{750}{6561}.$$