Axiom 1.1. *If m, n, and p are integers, then*

(i) $m + n = n + m$. *(commutativity of addition)*

(ii) $(m + n) + p = m + (n + p)$. *(associativity of addition)*

(iii) $m \cdot (n + p) = m \cdot n + m \cdot p$. *(distributivity)*

(iv) $m \cdot n = n \cdot m$. *(commutativity of multiplication)*

(v) $(m \cdot n) \cdot p = m \cdot (n \cdot p)$. *(associativity of multiplication)*

Axiom 1.2. *There exists an integer* 0 *such that whenever* $m \in \mathbb{Z}, m+0 = m$. *(identity element for addition)*

Axiom 1.3. *There exists an integer* 1 *such that* $1 \neq 0$ *and whenever* $m \in \mathbb{Z}, 1 \cdot m = m$. *(identity element for multiplication)*

Axiom 1.4. *For each* $m \in \mathbb{Z}$, *there exists an integer, denoted by* $-m$, *such that* $m + (-m) = 0$. *(additive inverse)*

Axiom 1.5. *Let m, n, and p be integers. If* $m \cdot n = m \cdot p$ *and* $m \neq 0$, *then* $n = p$. *(cancellation)*

Axiom 2.1. *There exists a subset* $\mathbb{N} \subseteq \mathbb{Z}$ *with the following properties:*

(i) If $m, n \in \mathbb{N}$ *then* $m + n \in \mathbb{N}$.

(ii) If $m, n \in \mathbb{N}$ *then* $m \cdot n \in \mathbb{N}$.

(iii) $0 \notin \mathbb{N}$.

(iv) For every $m \in \mathbb{Z}$, *we have* $m \in \mathbb{N}$ *or* $m = 0$ *or* $-m \in \mathbb{N}$.

Axiom 2.15 (Induction Axiom). *If a subset* $A \subseteq \mathbb{Z}$ *satisfies*

(i) $1 \in A$ *and*

(ii) if $n \in A$ *then* $n + 1 \in A$,

then $\mathbb{N} \subseteq A$.

Proposition 1.18. *Let* $x \in \mathbb{Z}$. *If x has the property that for all* $m \in \mathbb{Z}$, $mx = m$, *then* $x = 1$.

Proposition 1.19. *Let $x \in \mathbb{Z}$. If x has the property that for some nonzero $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.*

Proposition 1.20. *For all $m, n \in \mathbb{Z}$, $(-m)(-n) = mn$.*

Proposition 1.21. $(-1)(-1) = 1$.

Proposition 1.22.

(i) *For all $m \in \mathbb{Z}$, $-(-m) = m$.*

(ii) $-0 = 0$.

Proposition 1.23. *Given $m, n \in \mathbb{Z}$ there exists one and only one $x \in \mathbb{Z}$ such that $m + x = n$.*

Proposition 1.24. *Let $x \in \mathbb{Z}$. If $x \cdot x = x$ then $x = 0$ or 1.*

Proposition 1.25. *For all $m, n \in \mathbb{Z}$:*

(i) $-(m + n) = (-m) + (-n)$.

(ii) $-m = (-1)m$.

(iii) $(-m)n = m(-n) = -(mn)$.

Proposition 1.26. *Let $m, n \in \mathbb{Z}$. If $mn = 0$, then $m = 0$ or $n = 0$.*

Proposition 1.27. *For all $m, n, p, q \in \mathbb{Z}$:*

(i) $(m - n) + (p - q) = (m + p) - (n + q)$.

(ii) $(m - n) - (p - q) = (m + q) - (n + p)$.

(iii) $(m - n)(p - q) = (mp + nq) - (mq + np)$.

(iv) $m = n = p = q$ *if and only if* $m + q = n + p$.

(v) $(m - n)p = mp - np$.

Proposition 2.2.

For $m \in \mathbb{Z}$, one and only one of the following is true: $m \in \mathbb{N}$, $-m \in \mathbb{N}$, $m = 0$.

Proposition 2.3. $1 \in \mathbb{N}$.

Proposition 2.4. *Let $m, n, p \in \mathbb{Z}$. If $m < n$ and $n < p$ then $m < p$.*

Proposition 2.5. *For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m > n$.*

Proposition 2.6. *Let $m, n \in \mathbb{Z}$. If $m \leq n$ and $n \leq m$ then $m = n$.*

Proposition 2.7. *Let $m, n, p, q \in \mathbb{Z}$.*

(i) *If $m < n$ then $m + p < n + p$.*

(ii) *If $m < n$ and $p < q$ then $m + p < n + q$.*

(iii) *If $0 < m$ and $0 < p$ then $mp < mq$.*

(iv) *If $m < n$ and $p < 0$ then $np < mp$.*

Proposition 2.8. *Let $m, n \in \mathbb{Z}$. Exactly one of the following is true: $m < n$, $m = n$, $m > n$.*

Proposition 2.9. *Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.*

Proposition 2.10. *The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.*

Proposition 2.11. *Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. If $mn \in \mathbb{N}$, then $n \in \mathbb{N}$.*

Proposition 2.12. *For all $m, n, p \in \mathbb{Z}$:*

(i) *$-m < -n$ if and only if $m > n$.*

(ii) *If $p > 0$ and $mp < np$ then $m < n$.*

(iii) *If $p < 0$ and $mp < np$ then $m > n$.*

(iv) *If $m \leq n$ and $0 < p$ then $mp \leq np$.*

Proposition 2.13. $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$.

Proposition 2.14.

(i) $1 \in \mathbb{N}$.

(ii) *If $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$.*

Proposition 2.16. *Let $B \subseteq \mathbb{N}$ be such that:*

(i) $1 \in B$

(ii) *If $n \in B$ then $n + 1 \in B$. Then $B = \mathbb{N}$.*

Proposition 2.18.

(i) *For all $k \in \mathbb{N}$, $k^3 + 2k$ is divisible by 3.*

(ii) For all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.

(iii) m For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.

Proposition 2.20. *For all $k \in \mathbb{N}$, $k \geq 1$.*

Proposition 2.21. *There exists no integer x such that $0 < x < 1$.*

Proposition 2.24. *For all $k \in \mathbb{N}$, $k^2 + 1 > k$.*

Proposition 2.27. *For all integers $k > 2$, $2^k < k^3$.*

Proposition 2.28. *Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.*