Homework 10

Problem 3.10.4

Question

A random sample of size 5 is drawn from the pdf $f_Y(y) = 2y$, $0 \leq y \leq 1$. Calculate $P(Y_1' < 0.6 < Y_2')$.

Solution

$$F_Y(y) = \int_0^y 2t\,dt = y^2$$

$$F_Y(0.6) = 0.6^2 = 0.36$$

$$P(\text{exactly one } Y_i < 0.6) = \binom{5}{1} \times 0.36 \times (1 - 0.36)^4$$

$$P(\text{no } Y_i < 0.6) = (1 - 0.36)^5$$

$$P(Y_1' < 0.6 < Y_2') = 1 - \left[\binom{5}{1} \times 0.36 \times (1 - 0.36)^4 + (1 - 0.36)^5\right]$$

Problem 3.10.10

Question

Suppose that n observations are chosen at random from a continuous pdf $f_Y(y)$. What is the probability that the last observation recorded will be the smallest number in the entire sample?

Solution

$$\text{Probability that the last observation is the smallest} = \frac{1}{n}$$

Problem 3.10.12

Question

Consider a system containing n components, where the lifetimes of the components are independent random variables and each has pdf $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$. Show that the average time elapsing before the first component failure occurs is $\frac{1}{n\lambda}$.

Solution

$$\text{CDF of } T = 1 - e^{-n\lambda t}$$
$$\mathbb{E}[T] = \frac{1}{n\lambda}$$

Problem 3.11.4

Question

Five cards are dealt from a standard poker deck. Let X be the number of aces received, and Y the number of kings. Compute $P(X = 2|Y = 2)$.

Solution

$$\text{Total ways to choose 5 cards from a 52-card deck} = \binom{52}{5}$$

$$\text{Ways to choose 2 kings out of 4} = \binom{4}{2}$$

$$\text{Ways to choose 2 aces out of 4} = \binom{4}{2}$$

$$\text{Ways to choose 1 remaining card from the remaining 46 cards} = \binom{46}{1}$$

$$P(X = 2|Y = 2) = \frac{\binom{4}{2}\binom{4}{2}\binom{46}{1}}{\binom{52}{5}}$$

$$P(X = 2|Y = 2) = \frac{6 \times 6 \times 46}{2598960}$$

Problem 3.11.8

Question

Define the random variable W to be the "majority" of x, y, and z. For example, $W(2,2,1) = 2$ and $W(1,1,1) = 1$. Given the joint PDF $p_{X,Y,Z}(x,y,z) = \frac{xy+xz+yz}{54}$, find the PDF of W given X.

Majority Calculations for W given $X = 1$

Probability $W = 1$ given $X = 1$

$$P(W = 1|X = 1) = P(1,1,1) + P(1,1,2) + P(1,2,1) + P(2,1,1)$$

$$P(1,1,1) = \frac{1*1+1*1+1*1}{54} = \frac{3}{54}, \quad P(1,1,2) = \frac{1*1+1*2+1*2}{54} = \frac{5}{54}$$

$$P(1,2,1) = \frac{1*2+1*1+2*1}{54} = \frac{5}{54}, \quad P(2,1,1) = \frac{2*1+1*1+1*2}{54} = \frac{5}{54}$$

$$P(W = 1|X = 1) = \frac{3}{54} + 3 \times \frac{5}{54} = \frac{18}{54} = \frac{1}{3}$$

Probability $W = 2$ given $X = 1$

$$P(W = 2|X = 1) = P(1,2,2) + P(2,1,2) + P(2,2,1) + P(2,2,2)$$

$$P(1,2,2) = \frac{1*2+1*2+2*2}{54} = \frac{9}{54}, \quad P(2,1,2) = \frac{2*1+1*2+2*2}{54} = \frac{9}{54}$$

$$P(2,2,1) = \frac{2*2+1*2+2*1}{54} = \frac{9}{54}, \quad P(2,2,2) = \frac{2*2+2*2+2*2}{54} = \frac{12}{54}$$

$$P(W = 2|X = 1) = 3 \times \frac{9}{54} + \frac{12}{54} = \frac{39}{54} = \frac{13}{18}$$

Majority Calculations for W given $X = 2$

Probability $W = 1$ given $X = 2$

$$P(W = 1|X = 2) = P(2,1,1)$$

$$P(2,1,1) = \frac{2*1+1*1+1*2}{54} = \frac{5}{54}$$

$$P(W = 1|X = 2) = \frac{5}{54}$$

Probability $W = 2$ given $X = 2$

$$P(W = 2|X = 2) = P(2, 2, 2) + P(2, 2, 1) + P(2, 1, 2) + P(1, 2, 2)$$

$$P(2, 2, 2) = \frac{2 * 2 + 2 * 2 + 2 * 2}{54} = \frac{12}{54}, \quad P(2, 2, 1) = \frac{2 * 2 + 1 * 2 + 2 * 1}{54} = \frac{9}{54}$$

$$P(2, 1, 2) = \frac{2 * 1 + 1 * 2 + 2 * 2}{54} = \frac{9}{54}, \quad P(1, 2, 2) = \frac{1 * 2 + 2 * 2 + 2 * 2}{54} = \frac{9}{54}$$

$$P(W = 2|X = 2) = \frac{12}{54} + 3 \times \frac{9}{54} = \frac{39}{54} = \frac{13}{18}$$