Definitions

Inverse of a Matrix

- $(A^{-1})^{-1} = A$

- $(AB)^{-1} = B^{-1}A^{-1}$

- $AA^{-1} = A^{-1}A = I$

LDU Decomposition

- For a symmetric matrix

$$A : A = LDL^T$$

- L: lower triangular with unit diagonal

- D: diagonal matrix

Vector Space Axioms

- Addition: commutativity, associativity, identity, inverses

- Scalar Multiplication: distributivity, compatibility, identity

Subspaces

- Closed under addition and scalar multiplication

Linear Dependence and Independence

- Dependent: $\exists$ scalars, not all zero, s.t. $a_1 v_1 + \ldots + a_n v_n = 0$

- Independent: only solution is $a_1 = \ldots = a_n = 0$

Basis and Dimension

- Basis: linearly independent spanning set

- Dimension: number of vectors in a basis

General Principles for Subspaces

- Closed under vector addition

- Closed under scalar multiplication

Linear Transformation

- Preserves vector addition and scalar multiplication

Image and Kernel

- im(A): span of column vectors of A

- ker(A): $\{x \in \mathbb{R}^n : Ax = 0\}$

Basis Transformation

- Unique representation of a vector in terms of basis vectors

Determining Linear Independence (Standard Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 1)$, and $v_3 = (2, 5, 7)$, determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix A:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$$

2. Perform row reduction on A:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Since there are no rows of all zeros in the reduced row echelon form, the vectors are linearly independent.

Determining Linear Independence (Linearly Dependent Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, and $v_3 = (3, 6, 9)$, determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

2. Perform row reduction on A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. The presence of rows of all zeros indicates that the vectors are linearly dependent.

Finding a Basis for a Subspace (Polynomial Space)

Find a basis for the subspace of P_3 consisting of polynomials $p(x) = ax^3 + bx^2 + cx + d$ such that $p(1) = 0$.

Solution:

1. The condition $p(1) = 0$ gives $a + b + c + d = 0$. To find a basis, express this condition in terms of the coefficients and set up a system.

2. Considering the standard basis $\{1, x, x^2, x^3\}$ for P_3, impose the condition for $p(1) = 0$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = 0$$

This implies $a = -b - c - d$.

3. A basis satisfying this condition is $\{x^3 - x^2, x^2 - x, x - 1\}$ as these polynomials nullify at $x = 1$ and are linearly independent.

Finding the Matrix Inverse

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

Solution:

1. Set up the augmented matrix for A and the identity matrix: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$.

2. Perform row operations to get the identity matrix on the left side of the augmented matrix. Subtract twice the first row from the second row to start: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.

3. Then, subtract 3 times the second row from the first row: $\begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.

4. The matrix on the right side is now $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Solution:

1. Find the characteristic polynomial: $\det(B - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{bmatrix} = (4 - \lambda)^2 - 1$.

2. Solve for λ: $(4 - \lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.

3. Find eigenvectors for each eigenvalue: - For $\lambda_1 = 3$: Solve $(B - 3I)x = 0$. This gives $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. - For $\lambda_2 = 5$: Solve $(B - 5I)x = 0$. This gives $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Diagonalization

Determine if the matrix $C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable. If it is, find a matrix P that diagonalizes C.

Solution:

1. Find the eigenvalues of C: The characteristic polynomial is $(2 - \lambda)^2 = 0$, so the only eigenvalue is $\lambda = 2$.

2. Since C is a 2×2 matrix with only one distinct eigenvalue, we need to check if there are two linearly independent eigenvectors corresponding to $\lambda = 2$.

3. Solve $(C - 2I)x = 0$: This leads to the system $x_2 = 0$, indicating that every eigenvector has the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, which does not provide two independent eigenvectors.

4. Since we cannot find two linearly independent eigenvectors, C is not diagonalizable.

Verifying Vector Space Axioms

Verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space over the real numbers.

Solution

To verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space, we need to check that the following axioms hold:

- The set is closed under addition: The sum of any two polynomials of degree at most 2 is also a polynomial of degree at most 2.

- The set is closed under scalar multiplication: The scalar multiple of any polynomial of degree at most 2 is also a polynomial of degree at most 2.

- The set contains a zero vector, which is the zero polynomial.

- Each polynomial has an additive inverse within the set.

- Addition is associative and commutative.

- Scalar multiplication is distributive with respect to both scalar and vector addition.

- Scalar multiplication is compatible with field multiplication.

- The scalar 1 acts as a multiplicative identity.

Since all these properties are satisfied, the set is indeed a vector space.

Finding the Rank of a Matrix

Find the rank of the matrix

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Solution:

1. Perform row reduction on E:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. The rank is equal to the number of non-zero rows in the reduced row echelon form. Hence, rank$(E) = 1$.

Linear Dependence in $\mathbb{R}^2$

Given the vectors $v_1 = (3, -1)$ and $v_2 = (6, -2)$ in $\mathbb{R}^2$, determine if v_1 and v_2 are linearly dependent. **Solution**

1. Notice that $v_2 = 2v_1$, which means v_2 is a scalar multiple of v_1.

2. This implies the set $\{v_1, v_2\}$ is linearly dependent because v_2 can be expressed as a linear combination of v_1.

Identifying Subspaces

Determine if the set of all vectors in $\mathbb{R}^3$ of the form (a, a, b) is a subspace of $\mathbb{R}^3$.

Solution

To determine if the set of all vectors in $\mathbb{R}^3$ of the form (a, a, b) is a subspace, we check for the following properties:

- The zero vector $(0, 0, 0)$ is in the set, satisfying the non-emptiness requirement.

- The set is closed under addition: $(a_1, a_1, b_1) + (a_2, a_2, b_2) = (a_1 + a_2, a_1 + a_2, b_1 + b_2)$, which is of the form (a, a, b).

- The set is closed under scalar multiplication: For any real number c, $c(a, a, b) = (ca, ca, cb)$, which is still of the form (a, a, b).

Thus, the set is a subspace of $\mathbb{R}^3$.

Important Identities and Properties

- Rank-Nullity Theorem: rank(A) + nullity$(A) = n$, where n is the number of columns of A.

- Determinant properties: $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ for invertible A.

Determinant Calculation

Calculate the determinant of the matrix

$$D = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

Solution:

1. Apply the Laplace expansion using the first row:

$$\det(D) = 6 \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

2. Calculate each minor:

$$6 \cdot (3 \cdot 4 - 0 \cdot 1) - 1 \cdot (1 \cdot 4 - 1 \cdot 2) + 2 \cdot (1 \cdot 0 - 3 \cdot 2)$$
$$= 72 - 2 - 12$$
$$= 58 \tag{1}$$

3. Thus, $\det(D) = 58$.

Using Cramer's Rule

Solve the following system of equations using Cramer's Rule:

$$x + 2y - z = 4,$$
$$2x - y + 3z = -2, \tag{2}$$
$$x + 3y + z = 3.$$

Solution:

1. Write the coefficient matrix and calculate its determinant:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A) = -16.$$

2. For x, replace the first column of A with the constant terms and calculate its determinant:

$$A_x = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -1 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \quad \det(A_x) = -16.$$

3. For y, replace the second column of A:

$$A_y = \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A_y) = -32.$$

4. For z, replace the third column of A:

$$A_z = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 1 & 3 & 3 \end{bmatrix}, \quad \det(A_z) = -16.$$

5. Compute the solutions: $x = \frac{\det(A_x)}{\det(A)} = 1, y = \frac{\det(A_y)}{\det(A)} = 2, z = \frac{\det(A_z)}{\det(A)} = 1$.

LU Decomposition

Perform LU decomposition on the matrix $F = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$.

Solution:

1. Express F as the product of a lower triangular matrix L and an upper triangular matrix U.

2. Choose L with 1s on the diagonal: $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$.

3. Let $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$.

4. Since $F_{21} = l_{21} \cdot u_{11}$ and $F_{21} = 6$, $u_{11} = 4$, we get $l_{21} = \frac{6}{4} = \frac{3}{2}$.

5. Solve for U using the first row of F: $U = \begin{bmatrix} 4 & 3 \\ 0 & u_{22} \end{bmatrix}$. The second element of the second row gives $u_{22} = 3 - \frac{3}{2} \cdot 3 = -\frac{3}{2}$.

6. The LU decomposition is $L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 3 \\ 0 & -\frac{3}{2} \end{bmatrix}$

Orthogonal Diagonalization

Orthogonally diagonalize the matrix

$$F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Solution:

1. Find the eigenvalues by solving $\det(F - \lambda I) = 0$: $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 4$.

2. Find the eigenvectors: For $\lambda_1 = 2$, solve $(F - 2I)x = 0$ to get $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (after normalization).

For $\lambda_2 = 4$, solve $(F - 4I)x = 0$ to get $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (after normalization).

3. Construct $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ and verify

$$P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Orthogonal Diagonalization

Orthogonally diagonalize the matrix $F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

Solution:

1. Find the eigenvalues by solving $\det(F - \lambda I) = 0$: $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 4$.

2. Find the eigenvectors: For $\lambda_1 = 2$, solve $(F - 2I)x = 0$ to get $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (after normalization).

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$$P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Basis and Dimension of a Vector Space

Consider the vector space V of all vectors in $\mathbb{R}^4$ that satisfy the equation $x_1 - 2x_2 + x_3 - 2x_4 = 0$. Find a basis for V and state its dimension.

Solution

1. To find a basis, we need to solve for the vectors that satisfy the given equation. Let $x_4 = t$, then $x_3 = 2t$, $x_2 = s$, and $x_1 = 2s - t$.

2. Thus, any vector in V can be written as

$$\begin{bmatrix} 2s - t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

3. The vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent and span V, so they form a basis for V.

4. The dimension of V, denoted as $\dim(V)$, is the number of vectors in the basis, which is 2.

Span of Vectors

Determine if the vector $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ is in the span of the set of vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Solution:

1. We need to determine if there exist scalars c_1 and c_2 such that $\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$.

2. Setting up the equation, we get:

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

3. This leads to the system of equations:

$$c_1 = 4,$$
$$c_2 = -2,$$
$$-c_1 + 2c_2 = 1.$$

4. Solving this system, we find $c_1 = 4$ and $c_2 = -2$. However, these values do not satisfy the third equation $(-4 - 2 \neq 1)$. Therefore, $\mathbf{u}$ is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Change of Basis

Given a vector $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the standard basis of $\mathbb{R}^2$, find its coordinates in the new basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Solution

1. The coordinates of v in the basis B can be found by solving the equation $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

2. This equation translates into the system:

$$c_1 - c_2 = 3, c_1 + c_2 = 2.$$

3. Adding the two equations yields $2c_1 = 5$, so $c_1 = \frac{5}{2}$.

4. Substituting c_1 into the second equation gives $c_2 = \frac{2}{2} - \frac{5}{2} = -\frac{3}{2}$.

5. Therefore, the coordinates of v in the basis B are $\left(\frac{5}{2}, -\frac{3}{2} \right)$.

Perform LDU decomposition

Perform LDU decomposition on the matrix

$$H = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Solution:

1. First, we find the matrix L such that $H = LDU$ where L is a lower triangular matrix with unit diagonal, D is a diagonal matrix, and U is an upper triangular matrix.

2. Decompose H into LDU:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Verify the decomposition by calculating LDU and comparing it with H:

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}.$$

4. The result confirms the LDU decomposition of H.