Your Document Title

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Contents

Chapter 1

Proof Methods

If then statements

Format: Proof. If A, then B.

1. Assume A.
2. Show that assuming A leads to B.
3. Therefore, B is concluded from A.

Example:

Proof. Proof. If $m = 1$, then $m + 0 = 1$.

1. Assume $m = 1$.
2. Considering $m = 1$, we have $1 + 0 = 1$.
3. This simplifies to $1 = 1$, which is true.

□

If then types

Various types of implications and their representations:

- $A \Rightarrow B$: "If it is Wednesday, Dr. Beck will get a cup of coffee from the student union."
- $B \Rightarrow A$ (Converse of the first): "If Dr. Beck got a cup of coffee from the student union, it is Wednesday."
- $A \Leftrightarrow B$ (Bi-conditional, if and only if): "It is Wednesday if and only if Dr. Beck got a cup of coffee from the student union."
- (not B) $\Rightarrow$ (not A) (Contrapositive): "If Dr. Beck did not get a cup of coffee from the student union, then it is not Wednesday."

Induction Proof

Format:

Proof. Proof that $F(x)$ is true for all $x \in A$:

1. **Base case:** Show $F(a)$ is true, where a is the smallest element in A.
2. **Induction step:** Assume $F(k)$ is true for an arbitrary $k \in A$. Show that $F(k) \Rightarrow F(k+1)$.
3. Therefore, $F(x)$ is true for all $x \in A$.

$\square$

Example:

Proof. For all $n \in \mathbb{N}$, $n = n$.

1. Base case ($n = 1$): $1 = 1$ is true.
2. Induction step: Assume $n = n$ is true for an arbitrary natural number n. Show that this implies $n + 1 = n + 1$.
3. By the induction hypothesis, $n = n$. Adding 1 to both sides, $n+1 = n+1$, which holds true.

$\square$

Proof by contradiction

Format:

Proof. Prove A is true by Contradiction:

1. Assume A is false.
2. Show that this assumption leads to a contradiction.
3. Therefore, A must be true.

$\square$

Example:

Proof. Prove there is no smallest negative integer.

1. Assume, by way of contradiction, that there is a smallest negative integer, call it n.
2. Consider $n-1$. $n-1$ is also an integer and is smaller than n, contradicting the assumption that n is the smallest negative integer.
3. Therefore, there cannot be a smallest negative integer.

$\square$

Chapter 2

Axioms

Axioms of Integers

The axioms of integers describe the basic properties that define the structure of the set of integers ($\mathbb{Z}$).

Axiom 1.1 (Commutativity and Associativity)

- For any integers m, n, the operation of addition is commutative: $m + n = n + m$.
- For any integers m, n, p, the operation of addition is associative: $(m+n) + p = m + (n + p)$.
- For any integers m, n, p, the distributive property connects the operations of multiplication and addition: $m \cdot (n + p) = m \cdot n + m \cdot p$.
- For any integers m, n, the operation of multiplication is commutative: $m \cdot n = n \cdot m$.
- For any integers m, n, p, the operation of multiplication is associative: $(m \cdot n) \cdot p = m \cdot (n \cdot p)$.

Axiom 1.2 (Identity Elements)

- There exists an integer 0 such that for any integer m, adding 0 to m leaves it unchanged: $m + 0 = m$.
- There exists an integer 1 ($1 \neq 0$) such that for any integer m, multiplying m by 1 leaves it unchanged: $m \cdot 1 = m$.

Axiom 1.3 (Additive Inverse)

For each integer m, there exists an integer denoted by $-m$ such that their sum is 0: $m + (-m) = 0$.

Axiom 1.4 (Cancellation Law)

For any integers m, n, p, if $m \neq 0$ and $m \cdot n = m \cdot p$, then $n = p$.

Proof Example

Proof. If m is an integer and $m \cdot 0 = 0$, then $m = m$.

- Consider an integer m.
- Multiplying by 0 gives $m \cdot 0 = 0$.
- Since $m \cdot 0 = 0$, by the property of zero in multiplication, we have $m = m$.
- Thus, the statement is proven. □