**Axiom 1.1.** *If $m, n$, and $p$ are integers, then*

*(i)* $m + n = n + m$. *(commutativity of addition)*

*(ii)* $(m + n) + p = m + (n + p)$. *(associativity of addition)*

*(iii)* $m \cdot (n + p) = m \cdot n + m \cdot p$. *(distributivity)*

*(iv)* $m \cdot n = n \cdot m$. *(commutativity of multiplication)*

*(v)* $(m \cdot n) \cdot p = m \cdot (n \cdot p)$. *(associativity of multiplication)*

**Axiom 1.2.** *There exists an integer* $0$ *such that whenever* $m \in \mathbb{Z}, m+0 = m$. *(identity element for addition)*

**Axiom 1.3.** *There exists an integer* $1$ *such that* $1 \neq 0$ *and whenever* $m \in \mathbb{Z}, 1 \cdot m = m$. *(identity element for multiplication)*

**Axiom 1.4.** *For each* $m \in \mathbb{Z}$, *there exists an integer, denoted by* $-m$, *such that* $m + (-m) = 0$. *(additive inverse)*

**Axiom 1.5.** *Let* $m, n$, *and* $p$ *be integers. If* $m \cdot n = m \cdot p$ *and* $m \neq 0$, *then* $n = p$. *(cancellation)*

**Axiom 2.1.** *There exists a subset* $\mathbb{N} \subseteq \mathbb{Z}$ *with the following properties:*

*(i)* *If* $m, n \in \mathbb{N}$ *then* $m + n \in \mathbb{N}$.

*(ii)* *If* $m, n \in \mathbb{N}$ *then* $m \cdot n \in \mathbb{N}$.

*(iii)* $0 \notin \mathbb{N}$.

*(iv)* *For every* $m \in \mathbb{Z}$, *we have* $m \in \mathbb{N}$ *or* $m = 0$ *or* $-m \in \mathbb{N}$.

**Axiom 2.15 (Induction Axiom).** *If a subset* $A \subseteq \mathbb{Z}$ *satisfies*

*(i)* $1 \in A$ *and*

*(ii)* *if* $n \in A$ *then* $n + 1 \in A$,

*then* $\mathbb{N} \subseteq A$.

**Proposition 1.18.** *Let* $x \in \mathbb{Z}$. *If* $x$ *has the property that for all* $m \in \mathbb{Z}$, $mx = m$, *then* $x = 1$.

**Proposition 1.19.** *Let $x \in \mathbb{Z}$. If $x$ has the property that for some nonzero $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.*

**Proposition 1.20.** *For all $m, n \in \mathbb{Z}$, $(-m)(-n) = mn$.*

**Proposition 1.21.** $(-1)(-1) = 1$.

**Proposition 1.22.**

*(i)* *For all $m \in \mathbb{Z}$, $-(-m) = m$.*

*(ii)* $-0 = 0$.

**Proposition 1.23.** *Given $m, n \in \mathbb{Z}$ there exists one and only one $x \in \mathbb{Z}$ such that $m + x = n$.*

**Proposition 1.24.** *Let $x \in \mathbb{Z}$. If $x \cdot x = x$ then $x = 0$ or $1$.*

**Proposition 1.25.** *For all $m, n \in \mathbb{Z}$:*

*(i)* $-(m + n) = (-m) + (-n)$.

*(ii)* $-m = (-1)m$.

*(iii)* $(-m)n = m(-n) = -(mn)$.

**Proposition 1.26.** *Let $m, n \in \mathbb{Z}$. If $mn = 0$, then $m = 0$ or $n = 0$.*

**Proposition 1.27.** *For all $m, n, p, q \in \mathbb{Z}$:*

*(i)* $(m - n) + (p - q) = (m + p) - (n + q)$.

*(ii)* $(m - n) - (p - q) = (m + q) - (n + p)$.

*(iii)* $(m - n)(p - q) = (mp + nq) - (mq + np)$.

*(iv)* $m = n = p = q$ *if and only if* $m + q = n + p$.

*(v)* $(m - n)p = mp - np$.

**Proposition 2.2.**

*For $m \in \mathbb{Z}$, one and only one of the following is true: $m \in \mathbb{N}$, $-m \in \mathbb{N}$, $m = 0$.*

**Proposition 2.3.** $1 \in \mathbb{N}$.

**Proposition 2.4.** *Let $m, n, p \in \mathbb{Z}$. If $m < n$ and $n < p$ then $m < p$.*

**Proposition 2.5.** *For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m > n$.*

**Proposition 2.6.** *Let $m, n \in \mathbb{Z}$. If $m \leq n$ and $n \leq m$ then $m = n$.*

**Proposition 2.7.** *Let $m, n, p, q \in \mathbb{Z}$.*

*(i) If $m < n$ then $m + p < n + p$.*

*(ii) If $m < n$ and $p < q$ then $m + p < n + q$.*

*(iii) If $0 < m$ and $0 < p$ then $mp < mq$.*

*(iv) If $m < n$ and $p < 0$ then $np < mp$.*

**Proposition 2.8.** *Let $m, n \in \mathbb{Z}$. Exactly one of the following is true: $m < n$, $m = n$, $m > n$.*

**Proposition 2.9.** *Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.*

**Proposition 2.10.** *The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.*

**Proposition 2.11.** *Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. If $mn \in \mathbb{N}$, then $n \in \mathbb{N}$.*

**Proposition 2.12.** *For all $m, n, p \in \mathbb{Z}$:*

*(i) $-m < -n$ if and only if $m > n$.*

*(ii) If $p > 0$ and $mp < np$ then $m < n$.*

*(iii) If $p < 0$ and $mp < np$ then $m > n$.*

*(iv) If $m \leq n$ and $0 < p$ then $mp \leq np$.*

**Proposition 2.13.** $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$.

**Proposition 2.14.**

*(i) $1 \in \mathbb{N}$.*

*(ii) If $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$.*

**Proposition 2.16.** *Let $B \subseteq \mathbb{N}$ be such that:*

*(i) $1 \in B$*

*(ii) If $n \in B$ then $n + 1 \in B$. Then $B = \mathbb{N}$.*

**Proposition 2.18.**

*(i) For all $k \in \mathbb{N}$, $k^3 + 2k$ is divisible by 3.*

*(ii) For all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.*

*(iii) m For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.*

**Proposition 2.20.** *For all $k \in \mathbb{N}$, $k \geq 1$.*

**Proposition 2.21.** *There exists no integer $x$ such that $0 < x < 1$.*

**Proposition 2.24.** *For all $k \in \mathbb{N}$, $k^2 + 1 > k$.*

**Proposition 2.27.** *For all integers $k > 2$, $2^k < k^3$.*

**Proposition 2.28.** *Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.*