Summary of Linear Algebra Concepts

Inverse of a Matrix

If the inverse of a matrix A, denoted as A^{-1}, exists, then it satisfies the following properties:

$$
\begin{aligned}
(A^{-1})^{-1} &= A, \\
(AB)^{-1} &= B^{-1}A^{-1}, \\
AA^{-1} &= A^{-1}A = I,
\end{aligned}
$$

where I is the identity matrix, and the inverse is unique.

LDU Decomposition

For a symmetric matrix A, the LDU decomposition is unique and satisfies:

$$A = LDL^T,$$

where L is a lower triangular matrix with unit diagonal, D is a diagonal matrix, and L^T is the transpose of L.

Vector Space Axioms

The axioms for vector space operations over the real numbers $\mathbb{R}$ include:

Addition

For all vectors $v, w, u \in V$:

- Commutativity: $v + w = w + v$.
- Associativity: $(v + w) + u = v + (w + u)$.
- Existence of additive identity: There exists a zero vector 0 such that $v + 0 = v$.
- Existence of additive inverses: For every v, there exists $-v$ such that $v + (-v) = 0$.

Scalar Multiplication

For all scalars $c, d \in \mathbb{R}$ and vectors $v \in V$:

- Distributivity over vector addition: $c(v + w) = cv + cw$.
- Distributivity over scalar addition: $(c + d)v = cv + dv$.
- Compatibility with field multiplication: $c(dv) = (cd)v$.
- Identity element of scalar multiplication: $1v = v$.

Subspaces

A subspace W of a vector space V is itself a vector space and must be closed under addition and scalar multiplication.

Linear Dependence and Independence

Vectors $\{v_1, v_2, \ldots, v_n\}$ are said to be linearly dependent if there exist scalars, not all zero, such that:

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0.$$

They are linearly independent if the only solution is $a_1 = a_2 = \ldots = a_n = 0$.

Basis and Dimension

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ forms a basis of vector space V if:

- They are linearly independent.
- They span V, meaning any vector in V can be expressed as a linear combination of the basis vectors.

The dimension of V, denoted $\dim(V)$, is the number of vectors in a basis for V.

General Principles

Given a vector space V, a subset W is a subspace of V if:

- For all $v, w \in W$, $v + w \in W$.
- For all $v \in W$ and $c \in \mathbb{R}$, $cv \in W$.

In other words, W must be closed under vector addition and scalar multiplication.

Further Definitions

Linear Transformation

A linear transformation is a mapping between two vector spaces that preserves the operations of vector addition and scalar multiplication.

Image and Kernel

For a matrix $A \in \mathbb{R}^{m \times n}$:

- The image of A, denoted $\mathrm{im}(A)$, is the span of the column vectors of A.
- The kernel of A, denoted $\ker(A)$, is the set of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

Basis Transformation

If $\{v_1, v_2, \ldots, v_n\}$ forms a basis for $\mathbb{R}^n$ and a vector $w \in \mathbb{R}^n$, then w can be expressed uniquely as:

$$w = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are unique scalars.