# Definitions and General Principles

### Inverse of a Matrix

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = A^{-1}A = I$

### LDU Decomposition

- For a symmetric matrix $A$: $A = LDL^T$
- $L$: lower triangular with unit diagonal
- $D$: diagonal matrix

### Vector Space Axioms

- Addition: commutativity, associativity, identity, inverses
- Scalar Multiplication: distributivity, compatibility, identity

### Subspaces

- Closed under addition and scalar multiplication

### Linear Dependence and Independence

- Dependent: $\exists$ scalars, not all zero, s.t. $a_1v_1 + \ldots + a_nv_n = 0$
- Independent: only solution is $a_1 = \ldots = a_n = 0$

### Basis and Dimension

- Basis: linearly independent spanning set
- Dimension: number of vectors in a basis

### General Principles for Subspaces

- Closed under vector addition
- Closed under scalar multiplication

### Linear Transformation

- Preserves vector addition and scalar multiplication

### Image and Kernel

- $\text{im}(A)$: span of column vectors of $A$
- $\ker(A)$: $\{x \in \mathbb{R}^n : Ax = 0\}$

### Basis Transformation

- Unique representation of a vector in terms of basis vectors

### Determining Linear Independence (Standard Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 1)$, and $v_3 = (2, 5, 7)$, determine if they are linearly independent.

**Solution:**

1. Arrange the vectors as columns in a matrix $A$:
$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$$
2. Perform row reduction on $A$:
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
3. Since there are no rows of all zeros in the reduced row echelon form, the vectors are linearly independent.

### Determining Linear Independence (Linearly Dependent Case)

Given vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, and $v_3 = (3, 6, 9)$, determine if they are linearly independent.

**Solution:**

1. Arrange the vectors as columns in a matrix $A$:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$
2. Perform row reduction on $A$:
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
3. The presence of rows of all zeros indicates that the vectors are linearly dependent.

### Finding a Basis for a Subspace (Polynomial Space)

Find a basis for the subspace of $P_3$ consisting of polynomials $p(x) = ax^3 + bx^2 + cx + d$ such that $p(1) = 0$.

**Solution:**

1. The condition $p(1) = 0$ gives $a + b + c + d = 0$. To find a basis, express this condition in terms of the coefficients and set up a system.
2. Considering the standard basis $\{1, x, x^2, x^3\}$ for $P_3$, impose the condition for $p(1) = 0$:
$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = 0$$
This implies $a = -b - c - d$.
3. A basis satisfying this condition is $\{x^3 - x^2, x^2 - x, x - 1\}$ as these polynomials nullify at $x = 1$ and are linearly independent.

### Finding the Matrix Inverse

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

**Solution:**

1. Set up the augmented matrix for $A$ and the identity matrix: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$.
2. Perform row operations to get the identity matrix on the left side of the augmented matrix. Subtract twice the first row from the second row to start: $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.
3. Then, subtract 3 times the second row from the first row: $\begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$.
4. The matrix on the right side is now $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

### Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

**Solution:**

1. Find the characteristic polynomial: $\det(B - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 - 1$.
2. Solve for $\lambda$: $(4-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$.
3. Find eigenvectors for each eigenvalue: - For $\lambda_1 = 3$: Solve $(B - 3I)x = 0$. This gives $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. - For $\lambda_2 = 5$: Solve $(B - 5I)x = 0$. This gives $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

### Diagonalization

Determine if the matrix $C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable. If it is, find a matrix $P$ that diagonalizes $C$.

**Solution:**

1. Find the eigenvalues of $C$: The characteristic polynomial is $(2-\lambda)^2 = 0$, so the only eigenvalue is $\lambda = 2$.
2. Since $C$ is a $2 \times 2$ matrix with only one distinct eigenvalue, we need to check if there are two linearly independent eigenvectors corresponding to $\lambda = 2$.
3. Solve $(C - 2I)x = 0$: This leads to the system $x_2 = 0$, indicating that every eigenvector has the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, which does not provide two independent eigenvectors.
4. Since we cannot find two linearly independent eigenvectors, $C$ is not diagonalizable.

### Perform LDU decomposition

Perform LDU decomposition on the matrix $H = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$.

**Solution:**

1. First, we find the matrix $L$ such that $H = LDU$ where $L$ is a lower triangular matrix with unit diagonal, $D$ is a diagonal matrix, and $U$ is an upper triangular matrix.
2. Decompose $H$ into $LDU$:
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix},$$
$$U = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$
3. Verify the decomposition by calculating $LDU$ and comparing it with $H$:
$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$
4. The result confirms the LDU decomposition of $H$.

## Section 3

**Problem 7: Determinant Calculation**

Calculate the determinant of the matrix $D = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$.

**Solution:**

1. Apply the Laplace expansion using the first row:

$$\det(D) = 6\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

2. Calculate each minor:

$$6(3 \cdot 4 - 0 \cdot 1) - 1(1 \cdot 4 - 1 \cdot 2) + 2(1 \cdot 0 - 3 \cdot 2) = 72 - 2 - 12 = 58$$

3. Thus, $\det(D) = 58$.

**Problem 8: Using Cramer's Rule**

Solve the following system of equations using Cramer's Rule:

$$\begin{aligned} x + 2y - z &= 4, \\ 2x - y + 3z &= -2, \\ x + 3y + z &= 3. \end{aligned} \tag{1}$$

**Solution:**

1. Write the coefficient matrix and calculate its determinant:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A) = -16.$$

2. For $x$, replace the first column of $A$ with the constant terms and calculate its determinant:

$$A_x = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -1 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \quad \det(A_x) = -16.$$

3. For $y$, replace the second column of $A$:

$$A_y = \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A_y) = -32.$$

4. For $z$, replace the third column of $A$:

$$A_z = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 1 & 3 & 3 \end{bmatrix}, \quad \det(A_z) = -16.$$

5. Compute the solutions: $x = \frac{\det(A_x)}{\det(A)} = 1, y = \frac{\det(A_y)}{\det(A)} = 2, z = \frac{\det(A_z)}{\det(A)} = 1$.

## Section 4

**Problem 9: Finding the Rank of a Matrix**

Find the rank of the matrix $E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

**Solution:**

1. Perform row reduction on $E$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. The rank is equal to the number of non-zero rows in the reduced row echelon form. Hence, $\text{rank}(E) = 1$.

**Problem 10: LU Decomposition**

Perform LU decomposition on the matrix $F = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$.

**Solution:**

1. Express $F$ as the product of a lower triangular matrix $L$ and an upper triangular matrix $U$.

2. Choose $L$ with 1s on the diagonal: $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$.

3. Let $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$.

4. Since $F_{21} = l_{21} \cdot u_{11}$ and $F_{21} = 6$, $u_{11} = 4$, we get $l_{21} = \frac{6}{4} = \frac{3}{2}$.

5. Solve for $U$ using the first row of $F$: $U = \begin{bmatrix} 4 & 3 \\ 0 & u_{22} \end{bmatrix}$. The second element of the second row gives $u_{22} = 3 - \frac{3}{2} \cdot 3 = -\frac{3}{2}$.

6. The LU decomposition is $L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 3 \\ 0 & -\frac{3}{2} \end{bmatrix}$.

## Section 5

**Problem 11: Orthogonal Diagonalization**

Orthogonally diagonalize the matrix $F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

**Solution:**

1. Find the eigenvalues by solving $\det(F - \lambda I) = 0$: $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda_1 = 2$ and $\lambda_2 = 4$.

2. Find the eigenvectors: For $\lambda_1 = 2$, solve $(F - 2I)x = 0$ to get $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (after normalization). For $\lambda_2 = 4$, solve $(F - 4I)x = 0$ to get $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (after normalization).

3. Construct $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ and verify $P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

**Problem 12: Singular Value Decomposition (SVD)**

Find the singular value decomposition of the matrix $G = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

**Solution:**

1. Compute the eigenvalues of $G^T G$: Since $G^T G = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, the eigenvalues are 9 and 4.

2. The singular values are the square roots of the eigenvalues: $\sigma_1 = 3$ and $\sigma_2 = 2$.

3. The right singular vectors are the eigenvectors of $G^T G$, which correspond to $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4. The left singular vectors are obtained by normalizing the columns of $GU$, where $U$ is the matrix of right singular vectors. In this case, they remain the same as $v_1$ and $v_2$.

5. Assemble the SVD: $G = U \Sigma V^T$ with $U = V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.