Matrix Inverse and Properties

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- If A^{-1} exists, it is unique.

LDU Decomposition

- The LDU decomposition is unique for symmetric matrices A.
- For a symmetric matrix A, $A = LDU$ and $A^T = (LDU)^T = U^T D^T L^T$.

Vector Space Axioms

Addition

- $v + w = w + v$
- $(v + w) + u = v + (w + u)$
- $v + 0 = v$
- $v + (-v) = 0$

Scalar Multiplication

- $c(v + w) = cv + cw$
- $(c + d)v = cv + dv$
- $1v = v$

Subspaces

- Must be closed under addition and scalar multiplication.

Linear Dependence and Independence

- A set of vectors including the zero vector is linearly dependent.
- Vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent if the only solution to $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ is $a_1 = a_2 = \ldots = a_n = 0$.

Basis and Dimension

- A basis is a set of linearly independent vectors that span the vector space V.
- The dimension of V, denoted $\dim(V)$, is the number of vectors in a basis of V.

Subspaces of $\mathbb{R}^n$

- For $A \in \mathbb{R}^{m\times n}$, $\mathrm{im}(A) = \mathrm{span}\{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^m$.
- $\ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$.

Linear Transformation

- If V and W are subspaces, then $V - W \subseteq \ker(A)$.

Basis Transformation

- Given basis B_1 of $\mathbb{R}^n$ and B_2 of $\mathbb{R}^m$, a vector v in $\mathbb{R}^n$ is transformed by $B_2(A(B_1^{-1}(v)))$.

Problems

2.2.27

(a) Show that the set of even functions, $f(-x) = f(x)$, is a subspace of the vector space of all functions $F(\mathbb{R})$.

(b) Show that the set of odd functions, $g(-x) = -g(x)$, forms a complementary subspace, as defined in Exercise 2.2.24.

(c) Explain why every function can be uniquely written as the sum of an even function and an odd function.

2.2.15 Determine which of the following conditions describe subspaces of the vector space C^1 consisting of all continuously differentiable scalar functions $f(x)$.

(a) $f(2) = f(3)$,

(b) $f'(2) = f'(3)$,

(c) $f'(x) + f(x) = 0$,

(d) $f(2) = -f(-2)$,

(e) $f(x) = f(-x) + 2f(2)$, for all x.

2.2.9 A square matrix is called *strictly lower triangular* if all entries on or above the main diagonal are 0. Prove that the space of strictly lower triangular matrices is a subspace of the vector space of all $n \times n$ matrices.

2.2.22 Which of the following are subspaces of $\mathbb{R}^3$? Justify your answers!

(a) The set of all vectors $(x, y, z)^T$ satisfying $x + y + z + 1 = 0$.

(b) The set of vectors of the form $(t, -t, 0)^T$ for $t \in \mathbb{R}$.

(c) The set of vectors $(x, y, z)^T$ with $z \geq x \geq y$.

2.1.6

(a) Let $x_1 = 0$, $x_2 = 1$. Find the unique linear function $f(x) = ax + b$ that has the sample vector $f = (3, -1)^T$.

(b) Let $x_1 = 0$, $x_2 = 1$, $x_3 = -1$. Find the unique quadratic function $f(x) = ax^2 + bx + c$ with sample vector $f = (1, -2, 0)^T$.

1.6.26 Find the LDL^T factorization of the matrices M_1, M_2, M_3, M_4.

1.6.8

(a) Prove that the inverse transpose operation respects matrix multiplication: $(AB)^T = B^T A^T$.

(b) Verify this identity for $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \end{pmatrix}$.

1.4.21 For each of the listed matrices A and vectors b, find a permuted LU factorization of the matrix, and use your factorization to solve the system $Ax = b$.

2.4.14

(a) Prove that the vector space of all 2×2 matrices is a four-dimensional vector space by exhibiting a basis.

(b) Generalize your result and prove that the vector space $M_{m \times n}$ has dimension mn.

2.4.11

(a) Show that $1, 1-t, (1-t)^2, (1-t)^3$ is a basis for $\mathcal{P}_3$.

(b) Write $p(t) = 1 + \frac{1}{1+t^3}$ in terms of the basis elements.

2.4.8 Find a basis for and the dimension of the following subspaces:

(a) The space of solutions to the linear system $Ax = 0$, where $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$.

(b) The set of all quadratic polynomials $p(r) = a^2 + b^2 + c$ that satisfy $p(1) = 0$.

(c) The space of all solutions to the homogeneous ordinary differential equation $u'''' - u''' + 4u' - 4u = 0$.

2.4.6

(a) Show that $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are two different bases for the plane $x - 2y - 4z = 0$.

(b) Show how to write both elements of the second basis as linear combinations of the first.

(c) Can you find a third basis?

2.4.9

(a) Prove that $1 + t^2, t + t^2, 1 + 2t + t^2$ is a basis for the space of quadratic polynomials $\mathcal{P}_2$.

(b) Find the coordinates of $p(t) = 1 + 4t + 7t^2$ in this basis.

2.4.5 Find a basis for:

(a) the plane given by the equation $4x - 2y = 0$ in $\mathbb{R}^3$;

(b) the plane given by the equation $4x + 3y - z = 0$ in $\mathbb{R}^3$;

(c) the hyperplane $x + 2y + z - w = 0$ in $\mathbb{R}^4$.

2.4.3 Let $v_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $v_4 = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$. Do v_1, v_2, v_3, v_4 span $\mathbb{R}^3$?

1.3.21 Find the LU factorization of the following matrices:

(a) $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$

1.5.31 Solve the following systems of linear equations by computing the inverses of their coefficient matrices.

1.5.25 Find the inverse of each of the following matrices, if possible, by applying the Gauss-Jordan Method.

(a) $\begin{pmatrix} 1 & -2 \\ -3 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

1.5.16 Prove that a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ is invertible if and only if all its diagonal entries are nonzero, in which case $D^{-1} = \operatorname{diag}(1/d_1, \ldots, 1/d_n)$.

1.5.7

(a) Find the inverse of the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$.

(b) Use your result to solve the system $x = a\cos\theta - b\sin\theta$, $y = a\sin\theta + b\cos\theta$, for a and b in terms of x and y.

(c) Prove that, for all $a \in \mathbb{R}$ and $0 < \theta < \pi$, the matrix $R_\theta - aI$ has an inverse.

7.1.7 Find a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $L\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

7.1.3 Which of the following functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ are linear?

(a) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ x+y \end{pmatrix}$

(b) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2y \end{pmatrix}$

1.8.7 Determine the rank of the following matrices:

(a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 3 \\ -2 & -1 & -3 \end{pmatrix}$

1.8.4 Let $A = \begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 2 & b \\ b & 2 & a & 0 \\ 2 & b & 0 & a \end{pmatrix}$ be the augmented matrix for a linear system. For which values of a and b does the system have (i) a unique solution? (ii) infinitely many solutions? (iii) no solution?

1.8.1 Which of the following systems has (i) a unique solution? (ii) infinitely many solutions? (iii) no solution? In each case, find all solutions.

1.2.14 Find all matrices B that commute (under matrix multiplication) with $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

1.2.7 Consider the matrices $A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 4 & -2 \\ 3 & 5 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \\ 0 & 3 & -4 \end{pmatrix}$, $C = \begin{pmatrix} -2 & 3 & 2 \\ 4 & -1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$.

1.2.8 Which of the following pairs of matrices commute under matrix multiplication?

1.4.3 Find the equation $z = ax + by + c$ for the plane passing through the three points $P_1 = (0, 2, -1)$, $P_2 = (-2, 4, 3)$, $P_3 = (2, -1, -3)$.

1.3.7 Find the equation of the parabola $y = ax^2 + bx + c$ that goes through the points $(1, 6)$, $(2, 4)$, and $(3, 0)$.

1.3.1 Solve the following linear systems by Gaussian Elimination.

1. $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

1.1.1 Solve the following systems of linear equations by reducing to triangular form and then using Back Substitution.

1. $$\begin{aligned} 6x - y + 7u &= 5, \\ 2x + 2y &= 3, \\ x - y + z &= 1; \end{aligned}$$

1.1.2 How should the coefficients a, b, and c be chosen so that the system $ax + by + cz = 3$, $ax - y + cz = 1$, $a + by - cz = 2$ has the solution $x = 1, y = 2$ and $z = -1$?

1.3.2 Write out the augmented matrix for the following linear systems. Then solve the system by first applying elementary row operations of type 1 to place the augmented matrix in upper triangular form, followed by Back Substitution.

1. $$\begin{aligned} x - 2y + z &= 0, \\ a - 2z_1 - 9z_2 &= 2; \end{aligned}$$

1.3.1 (continued) Solve the following linear systems by Gaussian Elimination. (continued)

1. $\begin{pmatrix} 6 & 1 & u \\ 3 & -2 & v \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$