MATH 440 Marty Martin
2:00 PM - 3:20 PM May 2024

Homework 9

Problem 3.8.2

Question

Let $f_Y(y) = \frac{3(1+y^2)}{14}$, $0 \leq y \leq 2$. Define the random variable W by $W = 3Y + 2$. Find $f_W(w)$. Be sure to specify the values of w for which $f_W(w) \neq 0$.

Solution

$$Y = \frac{W-2}{3}, \quad f_W(w) = f_Y\left(\frac{w-2}{3}\right)\left|\frac{d}{dw}\left(\frac{w-2}{3}\right)\right|$$

$$f_W(w) = \frac{3\left(1 + \left(\frac{w-2}{3}\right)^2\right)}{14} \cdot \frac{1}{3}$$

$$f_W(w) = \frac{1 + \left(\frac{w-2}{3}\right)^2}{14}$$

$$w : 2 \leq w \leq 8.$$

Problem 3.8.6

Question

If a random variable V is independent of two dependent random variables X and Y, prove that V is independent of $X + Y$.

Proof

Given V is independent of X and Y,

$$P(V \leq v, X + Y \leq x) = P(V \leq v)P(X + Y \leq x).$$

By the definition of independence and the law of total probability,

$$\begin{aligned}
P(V \leq v, X + Y \leq x) &= \int P(V \leq v, X + Y \leq x \mid X = t) f_X(t)\, dt \\
&= \int P(V \leq v) P(X + Y \leq x \mid X = t) f_X(t)\, dt \\
&= P(V \leq v) \int P(X + Y \leq x \mid X = t) f_X(t)\, dt.
\end{aligned}$$

Since $P(X + Y \leq x \mid X = t)$ simplifies to $P(Y \leq x - t)$ by the definition of conditional probability,

$$\begin{aligned}
\int P(X + Y \leq x \mid X = t) f_X(t)\, dt &= \int P(Y \leq x - t) f_X(t)\, dt \\
&= P(X + Y \leq x).
\end{aligned}$$

Thus,

$$P(V \leq v, X + Y \leq x) = P(V \leq v)P(X + Y \leq x).$$

This confirms V is independent of $X + Y$.

Problem 3.8.8

Question

Let Y be a uniform random variable over the interval $[0, 1]$. Find the pdf of $W = Y^2$.

Solution

$$f_Y(y) = 1 \text{ for } 0 \leq y \leq 1, \quad W = Y^2 \Rightarrow Y = \sqrt{W}$$

$$f_W(w) = f_Y(\sqrt{w}) \left| \frac{d}{dw}(\sqrt{w}) \right| = 1 \cdot \frac{1}{2\sqrt{w}}$$

$$f_W(w) = \frac{1}{2\sqrt{w}} \text{ for } 0 \leq w \leq 1.$$

Problem 3.9.4

Question

Marksmanship competition at a certain level requires each contestant to take ten shots with each of two different handguns. Final scores are computed by taking a weighted average of four times the number of bull's-eyes made with the first gun plus six times the number gotten with the second. If Cathie has a 30% chance of hitting the bull's-eye with each shot from the first gun and a 40% chance with each shot from the second gun, what is her expected score?

Solution

We denote the number of bull's-eyes made with the first gun as X_1 and with the second gun as X_2. The variable X_1 follows a binomial distribution with parameters $n = 10$ (shots) and $p = 0.3$ (probability of success), and similarly for X_2 with $p = 0.4$.

$$E(X_1) = n \times p_1 = 10 \times 0.3 = 3, \quad E(X_2) = 10 \times 0.4 = 4$$

The competition score is calculated using the formula:

$$\text{Score} = 4 \times X_1 + 6 \times X_2$$

$$E(\text{Score}) = E(4 \times X_1 + 6 \times X_2)$$

$$E(\text{Score}) = 4 \times E(X_1) + 6 \times E(X_2) = 4 \times 3 + 6 \times 4$$

$$= 12 + 24 = 36$$

Thus, Cathie's expected score in the competition is 36 points.

Problem 3.9.10

Question

Suppose that X and Y are both uniformly distributed over the interval $[0, 1]$. Calculate the expected value of the square of the distance of the random point (X, Y) from the origin; that is, find $E(X^2 + Y^2)$.

Solution

$$E(X^2 + Y^2) = E(X^2) + E(Y^2)$$
$$E(X^2) = \int_0^1 x^2\, dx = \frac{1}{3}, \quad E(Y^2) = \int_0^1 y^2\, dy = \frac{1}{3}$$
$$E(X^2 + Y^2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$