

## Homework 8

### Problem 3.7.12

#### Question

A point is chosen at random from the interior of a circle whose equation is  $x^2 + y^2 \leq 4$ . Let the random variables  $X$  and  $Y$  denote the  $x$ - and  $y$ -coordinates of the sampled point. Find the joint pdf  $f_{X,Y}(x, y)$ .

#### Solution

The area  $A$  of the circle is calculated as:

$$A = \pi r^2 = \pi \times 2^2 = 4\pi$$

The joint pdf  $f_{X,Y}(x, y)$  is given by:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

The integral of the pdf over the circle should equal 1:

$$\int \int_{x^2+y^2 \leq 4} \frac{1}{4\pi} dx dy = \frac{1}{4\pi} \times 4\pi = 1$$

### Problem 3.7.14

#### Question

Suppose that five independent observations are drawn from the continuous pdf  $f_T(t) = 2t$ ,  $0 \leq t \leq 1$ . Let  $X$  denote the number of  $t$ 's that fall in the interval  $0 \leq t \leq \frac{1}{3}$  and let  $Y$  denote the number of  $t$ 's that fall in the interval  $\frac{1}{3} \leq t \leq \frac{2}{3}$ . Find  $P_{XY}(1, 2)$ .

#### Solution

$$\begin{aligned} P(0 \leq T \leq \tfrac{1}{3}) &= \int_0^{\frac{1}{3}} 2t dt = [t^2]_0^{\frac{1}{3}} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}, \\ P(\tfrac{1}{3} \leq T \leq \tfrac{2}{3}) &= \int_{\frac{1}{3}}^{\frac{2}{3}} 2t dt = [t^2]_{\frac{1}{3}}^{\frac{2}{3}} = \left(\frac{4}{9}\right) - \left(\frac{1}{9}\right) = \frac{3}{9} = \frac{1}{3}. \\ P_{XY}(1, 2) &= \binom{5}{1} \left(\frac{1}{9}\right) \binom{4}{2} \left(\frac{1}{3}\right)^2 \left(\frac{5}{9}\right)^2, \\ &= 5 \cdot \frac{1}{9} \cdot 6 \cdot \frac{1}{9} \cdot \left(\frac{25}{81}\right) = \frac{750}{6561}. \end{aligned}$$