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Professor Matthias Beck March 19, 2024

Homework 5

User Project 3.7. Negate the following statements.

(i) There exists a cubic polynomial that does not have a real root.

(ii) G is not normal or H is not regular.

(iii) There does not exist exactly one element 0 such that for all x, $x+0 = x$.

(iv) The newspaper article was accurate or it was entertaining.

(v) There exist m and n such that gcd(m, n) is odd and both m and n are even.

(vi) H/N is a normal subgroup of G/N and H is not a normal subgroup of G, or H/N is not a normal subgroup of G/N and H is a normal subgroup of G.

(vii) There exists some $\varepsilon > 0$ for which, for every $N \in \mathbb{N}$, there is some $n \geq N$ such that $|a_n L| \geq \varepsilon$.

Negation of Statements

(i) All cubic polynomials have at least one real root.

(ii) G is normal and H is regular.

(iii) There exists exactly one element 0 such that for all x, $x + 0 = x$.

(iv) The newspaper article was neither accurate and entertaining.

(v) For all m and n, if gcd(m, n) is odd, then m or n is odd.

(vi) H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G.

(vii) For each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n L| < \varepsilon$.

1. **Not** every cubic polynomial has a real root. (There exists at least one cubic polynomial that does **not** have a real root.)

2. G is **not** normal **or** H is **not** regular.

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3. There does **not** exist exactly one zero such that for all x, $x + 0 = x$. (Either no such zero exists, **or** there exists more than one such zero.)

4. The newspaper article was accurate **or** entertaining. (It was either accurate, entertaining, or both.)

5. There exist m and n such that $\gcd(m, n)$ is odd, and **neither** m **nor** n is odd.

6. H/N is a normal subgroup of G/N and H is **not** a normal subgroup of G, **or** H/N is **not** a normal subgroup of G/N and H is a normal subgroup of G.

7. There exists some $\varepsilon > 0$ such that for every $N \in \mathbb{N}$, there exists some $n \geq N$ such that $|a_n - L| \geq \varepsilon$.

Proposition 4.7

(i) $5^{2k} - 1$ **is divisible by 24**

Base Case:

Let's check the base case where $k = 1$:

$$5^{2 \cdot 1} - 1 = 25 - 1 = 24$$

Since 24 is divisible by 24, the base case holds.

Inductive Step:

Suppose for some $k \in \mathbb{N}$, the statement is true; that is, $5^{2k} - 1$ is divisible by 24. We need to show that $5^{2(k+1)} - 1$ is also divisible by 24.

Consider $5^{2(k+1)} - 1$:

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^{2k} \cdot 5^2 - 1 \\ &= 5^{2k} \cdot 25 - 1 \\ &= (5^{2k} - 1) + 24 \cdot 5^{2k} \end{aligned}$$

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Thus Since by the inductive hypothesis $5^{2k}-1$ is divisible by 24, and $24 \cdot 5^{2k}$ is divisible by 24, their sum $5^{2(k+1)}-1$ is also divisible by 24.

Proof of Proposition 4.7(ii)

We want to prove that for all $k \in \mathbb{N}$, $2^{2k+1}+1$ is divisible by 3. We will proceed by induction on k.

Base Case ($k = 1$):

Let's start by checking the base case where $k = 1$:

$$2^{2\cdot 1+1}+1 = 2^3+1 = 8+1 = 9$$

Since 9 is divisible by 3, the base case holds.

Inductive Step:

Assume that the statement holds for some $k \in \mathbb{N}$, that is $2^{2k+1}+1$ is divisible by 3. We need to show that $2^{2(k+1)+1}+1$ is also divisible by 3.

Consider $2^{2(k+1)+1}+1$:

$$\begin{aligned} 2^{2(k+1)+1}+1 &= 2^{2k+2+1}+1 \\ &= 2^{2k+1}\cdot 2^2+1 \\ &= 2^{2k+1}\cdot 4+1 \\ &= (2^{2k+1}+1)+2^{2k+1}\cdot 3 \end{aligned}$$

By the inductive hypothesis, $2^{2k+1}+1$ is divisible by 3, and since $2^{2k+1}\cdot 3$ is clearly divisible by 3, their sum $2^{2(k+1)+1}+1$ is also divisible by 3.

Conclusion:

By the principle of mathematical induction, the statement that $2^{2k+1}+1$ is divisible by 3 holds for all $k \in \mathbb{N}$.

Proof of Proposition 4.7(iii)

We aim to show that for all $k \in \mathbb{N}$, the expression $10^k+3\cdot 4^{k+2}+5$ is divisible by 9. We will use the method of mathematical induction.

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Base Case ($k = 1$):

First, we verify the base case where $k = 1$:

$$10^1 + 3 \cdot 4^{1+2} + 5 = 10 + 3 \cdot 4^3 + 5 = 10 + 3 \cdot 64 + 5 = 10 + 192 + 5 = 207$$

Since 207 is divisible by 9 ($207 = 23 \cdot 9$), the base case is satisfied.

Inductive Step:

Assume for the sake of induction that the statement is true for some $k \in \mathbb{N}$, i.e., $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9. We need to show that $10^{k+1} + 3 \cdot 4^{(k+1)+2} + 5$ is also divisible by 9.

Consider $10^{k+1} + 3 \cdot 4^{(k+1)+2} + 5$:

$$\begin{aligned} 10^{k+1} + 3 \cdot 4^{(k+1)+2} + 5 &= 10 \cdot 10^k + 3 \cdot 16 \cdot 4^{k+2} + 5 \\ &= 10 \cdot (10^k + 3 \cdot 4^{k+2} + 5) - 5 \cdot (10 - 1) \\ &= 10 \cdot (10^k + 3 \cdot 4^{k+2} + 5) - 45 \end{aligned}$$

By the inductive hypothesis, $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9, and 45 is obviously divisible by 9. Hence, since we are subtracting a multiple of 9 from a number that is a multiple of 9 when multiplied by 10, $10^{k+1} + 3 \cdot 4^{(k+1)+2} + 5$ must also be divisible by 9.

Conclusion:

Therefore, by the principle of mathematical induction, it follows that $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9 for all $k \in \mathbb{N}$.

Project 4.9

We wish to determine for which natural numbers k the inequality $k^2 < 2^k$ holds true. We will analyze the inequality for different values of k to establish a pattern.

Analysis:

- For $k = 1$: $1^2 = 1 < 2 = 2^1$, so the inequality holds.

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- For $k = 2$: $2^2 = 4 = 2^2$, the inequality does not hold as both sides are equal.
- For $k = 3$: $3^2 = 9 < 8 = 2^3$, the inequality does not hold.
- For $k = 4$: $4^2 = 16 < 16 = 2^4$, the inequality does not hold as both sides are equal.

To find the values of k for which the inequality $k^2 < 2^k$ strictly holds, we can use mathematical induction or direct observation for higher values of k. It is well known that exponential functions grow faster than polynomial functions, which implies that after a certain point, 2^k will always be greater than k^2.

By testing various values (which we can do by direct calculation or programming), we find that the inequality $k^2 < 2^k$ holds for $k = 1$ and for all $k \geq 5$.

Conclusion:

The natural numbers k for which the inequality $k^2 < 2^k$ holds are $k = 1$ and all $k \geq 5$.

Proof of Proposition 4.11(i)

We want to prove that for any $k \in \mathbb{N}$, the following equality holds:

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

We will proceed by induction on k.

Base Case ($k = 1$):

For $k = 1$, the sum on the left is simply 1, and the right-hand side is $\frac{1(1+1)}{2} = 1$, so the base case holds.

Inductive Step:

Suppose the statement is true for some k, i.e.,

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

We need to show that the statement holds for $k+1$, i.e.,

$$\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$$

Starting with the left-hand side,

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \sum_{j=1}^{k} j + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Thus, the statement holds for $k+1$.

Conclusion:

By the principle of mathematical induction, the formula for the sum of the first k natural numbers is proved for all $k \in \mathbb{N}$.

Proof of Proposition 4.11(ii)

We aim to prove that for any $k \in \mathbb{N}$, the following equality is true:

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Again, we apply the method of induction on k.

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Base Case ($k = 1$):

For $k = 1$, the sum of squares on the left is $1^2 = 1$, and the right-hand side is $\frac{1(1+1)(2\cdot1+1)}{6} = 1$, so the base case holds.

Inductive Step:

Assume the formula holds for some k, i.e.,

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

We need to prove it for $k+1$, i.e.,

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Starting with the left-hand side,

$$\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Thus, the statement holds for $k+1$.

Conclusion:

By the principle of mathematical induction, the formula for the sum of the squares of the first k natural numbers is proved for all $k \in \mathbb{N}$.