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Proposition 2.18. *(ii) For all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.*

Proof. **Base Case:** For $k = 1$, we have:

$$1^4 - 6 \cdot 1^3 + 11 \cdot 1^2 - 6 \cdot 1 = 0$$

Since 0 is divisible by 4, the base case holds.

Inductive Step: We need to show that if the statement holds for $k = n$, then it must hold for $k = n + 1$. Consider:

$(n + 1)^4 - 6(n + 1)^3 + 11(n + 1)^2 - 6(n + 1)$
$n^4 + 4n^3 + 6n^2 + 4n + 1 - 6(n^3 + 3n^2 + 3n + 1) + 11(n^2 + 2n + 1) - 6n - 6$
$(n^4 - 6n^3 + 11n^2 - 6n) + 4(n^3 + 6n^2 + 9n + 1)$.

The equation $4(n^3 + 6n^2 + 9n + 1)$ is divisible by 4.

Thus the equation $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4 □

Proposition 2.18. *(iii) For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.*

Proof. **Base Case:** For $k = 1$:

$$1^3 + 5 \times 1 = 6$$

which is divisible by 6.

Inductive Step: Assume the statement holds for $k = n$, meaning $n^3 + 5n$ is divisible by 6. For $k = n + 1$, we have:

$$\begin{aligned}(n + 1)^3 + 5(n + 1) &= n^3 + 3n^2 + 3n + 1 + 5n + 5 \\ &= (n^3 + 5n) + 3n^2 + 3n + 6.\end{aligned}$$

By the inductive step, $n^3 + 5n$ is divisible by 6. Since $3n^2 + 3n + 6$ is also divisible by 6, the statement is true for $n + 1$. □

Proposition 2.21. *There exists no integer x such that $0 < x < 1$.*

Proof. Assume for contradiction there exists an integer x such that

$$0 < x < 1$$

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From Proposition 2.2, since x is not 0, x must be in $\mathbb{N}$ or $-x$ is in $\mathbb{N}$.
If $x \in \mathbb{N}$, then by Proposition 2.20, $x \geq 1$, which contradicts $x < 1$.
If $-x \in \mathbb{N}$, then x must be negative, which contradicts $x > 0$.
Hence, no such x exists. □

Proposition 2.24. *For all $k \in \mathbb{N}$, $k^2 + 1 > k$.*

Proof. **Base Case (k = 1):** For $k = 1$, the inequality becomes:

$$1^2 + 1 = 2 > 1,$$

which is clearly true.

Inductive Step: We need to show that the inequality holds for $k = k+1$:

$$(k+1)^2 + 1 > k + 1.$$

Expanding the left-hand side gives us:

$$k^2 + 2k + 1 + 1 > k + 1.$$

Simplifying the inequality, we get:

$$k^2 + 2k + 2 > k + 1.$$

Since by the inductive hypothesis we know $k^2 + 1 > k$, and clearly $2k + 1 > 1$ for $n \geq 1$, it follows that:

$$k^2 + 2k + 2 > k + 1.$$

Hence, the inequality $(k+1)^2 + 1 > k + 1$ is true, which completes the inductive step.
Therefore, by induction, the inequality $k^2 + 1 > k$ holds for all natural numbers k. □

Proposition 2.27. *For all integers $k > 2$, $2^k < k^3$.*

Proof. **Base Case (k = 3):**

$$2^3 = 8 < 27 = 3^3,$$

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which holds true.

Inductive Step (Prove for k = n + 1):

$$2^{n+1} = 2 \cdot 2^n < 2 \cdot n^3,$$

since $2^n < n^3$ by the inductive step and $n > 2$ implies $2 < n^2$
so $2 \cdot n^3 < n^2 \cdot n^3 = n^5$
$2 \cdot n^3 < (n+1)^3$. Since $n > 2$, we have:

$$(n+1)^3 - 2 \cdot n^3 = n^3 + 3n^2 + 3n + 1 - 2n^3 = n^3 - 3n^2 + 3n + 1,$$

and since $n > 2$, $n^2 - 3n = n(n-3) \geq 0$, which implies that $n^3 - 3n^2 + 3n + 1 > 0$, thus $2 \cdot n^3 < (n+1)^3$.
Therefore, by induction, $2^k < k^3$ for all integers $k > 2$. □

Proposition 2.28. *Determine for which natural numbers $k^2 - 3k \geq 4$ and prove your answer.*

Proof. We need to solve the inequality $k^2 - 3k \geq 4$ for natural numbers k.

First, we rearrange the inequality as follows:

$$k^2 - 3k - 4 \geq 0$$

Factoring the quadratic expression, we get:

$$(k-4)(k+1) \geq 0$$

This product is non-negative if both factors are non-negative or non-positive. Since k is a natural number, $k + 1 > 0$. Therefore, we only need to consider when $k - 4 \geq 0$, which simplifies to $k \geq 4$.

Thus, for all natural numbers $k \geq 4$, the inequality $k^2 - 3k \geq 4$ holds true. □