Practice for the exam

Question 1

Suppose the random variables X and Y are jointly distributed according to the pdf:

$$f_{XY}(x,y) = 8xy, \quad 0 < y < x < 1$$

(a) Find $P(X < 2Y)$

(b) Find $P(Y < \frac{1}{4} | x = \frac{1}{2})$

Solution

- X: Random variable representing the first dimension
- Y: Random variable representing the second dimension
- $f_{XY}(x,y)$: Joint probability density function of X and Y
- $f_X(x)$: Marginal probability density function of X
- $f_{Y|X}(y|x)$: Conditional probability density function of Y given X
- $P(X < 2Y)$: Probability that X is less than $2Y$
- $P(Y < \frac{1}{4} | X = \frac{1}{2})$: Conditional probability that Y is less than $\frac{1}{4}$ given $X = \frac{1}{2}$

Part (a) Find $P(X < 2Y)$

General Formula: For any joint PDF $f_{XY}(x,y)$, the probability $P(g(X,Y))$ for some condition g is given by:

$$P(g(X,Y)) = \int \int_{g(x,y)} f_{XY}(x,y)\, dx\, dy$$

Specific Calculation: This probability calculation involves integrating the joint PDF over the region defined by $X < 2Y$ and within the given bounds of $0 < y < x < 1$.

$$P(X < 2Y) = \int_0^1 \int_0^{X/2} 8xy\, dy\, dx$$

This integral setup reflects the condition $X < 2Y$ within the area bounded by $0 < y < x < 1$.

Integration Steps: Calculate the inner integral over y:

$$\int_0^{X/2} 8xy\, dy = 8x \left[\frac{y^2}{2}\right]_0^{X/2} = 8x \left[\frac{(X/2)^2}{2}\right] = X^3$$

Now integrate with respect to x:

$$\int_0^1 X^3\,dx = \left[\frac{X^4}{4}\right]_0^1 = \frac{1}{4}$$

Thus, $P(X < 2Y) = \frac{1}{4}$.

Part (b) Find $P(Y < \frac{1}{4}|x = \frac{1}{2})$

General Formula: For any joint PDF $f_{XY}(x, y)$, the conditional probability $P(A|B)$ is given by:

$$P(A|B) = \frac{\int\int_{A\cap B} f_{XY}(x, y)\,dx\,dy}{\int\int_B f_{XY}(x, y)\,dx\,dy}$$

Specific Calculation: Given $X = \frac{1}{2}$, we need to find the conditional probability $P(Y < \frac{1}{4}|X = \frac{1}{2})$. This involves determining the conditional PDF $f_{Y|X}(y|x)$ and integrating it over the desired range of Y.

Determine the Marginal Density of X, $f_X(x)$: The marginal density $f_X(x)$ is found by integrating out Y from the joint PDF:

$$f_X(x) = \int_0^x 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_0^x = 4x^3$$

At $x = \frac{1}{2}$, the marginal density $f_X\left(\frac{1}{2}\right)$ is:

$$f_X\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

Calculate the Conditional PDF $f_{Y|X}(y|x)$: The conditional PDF $f_{Y|X}(y|\frac{1}{2})$ is:

$$f_{Y|X}(y|\frac{1}{2}) = \frac{8\cdot\frac{1}{2}\cdot y}{\frac{1}{2}} = 8y$$

Compute the Conditional Probability $P(Y < \frac{1}{4}|X = \frac{1}{2})$:

$$P(Y < \frac{1}{4}|X = \frac{1}{2}) = \int_0^{1/4} 8y\,dy$$

Calculate the integral:

$$\int_0^{1/4} 8y\,dy = \left[4y^2\right]_0^{1/4} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$$

Conclusion: Therefore, $P(Y < \frac{1}{4}|X = \frac{1}{2}) = \frac{1}{4}$.

Question 2

A random variable X has the moment generating function $M_X(t) = \left(\frac{2+e^t}{3}\right)^9$. Find $\text{Var}(X)$.

Solution

- X: Random variable
- $M_X(t)$: Moment generating function of X
- $E(X)$: Expected value (mean) of X
- $E(X^2)$: Second moment of X
- $\text{Var}(X)$: Variance of X

General Formula: The moment generating function (MGF) $M_X(t)$ of a random variable X is defined as $M_X(t) = E(e^{tX})$. The n-th moment of X is given by the n-th derivative of $M_X(t)$ evaluated at $t = 0$:

$$E(X^n) = M_X^{(n)}(0)$$

First, find the first derivative of $M_X(t)$:

$$M_X(t) = \left(\frac{2+e^t}{3}\right)^9$$

$$M_X'(t) = 9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}$$

Evaluate the first derivative at $t = 0$:

$$M_X'(0) = 9\left(\frac{2+e^0}{3}\right)^8 \cdot \frac{e^0}{3} = 9\left(\frac{3}{3}\right)^8 \cdot \frac{1}{3} = 9 \cdot 1 \cdot \frac{1}{3} = 3$$

Thus, the mean μ of X is:

$$\mu = E(X) = M_X'(0) = 3$$

Next, find the second derivative of $M_X(t)$:

$$M_X''(t) = \frac{d}{dt}\left[9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}\right]$$

Using the product rule:

$$M_X''(t) = 9\left[8\left(\frac{2+e^t}{3}\right)^7 \cdot \frac{e^t}{3} \cdot \frac{e^t}{3} + \left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}\right]$$

Evaluate the second derivative at $t = 0$:

$$M_X''(0) = 9\left[8\left(\frac{3}{3}\right)^7 \cdot \frac{1}{3} \cdot \frac{1}{3} + \left(\frac{3}{3}\right)^8 \cdot \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[8 \cdot 1 \cdot \frac{1}{9} + 1 \cdot \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[\frac{8}{9} + \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[\frac{8}{9} + \frac{3}{9}\right]$$

$$M_X''(0) = 9 \cdot \frac{11}{9} = 11$$

Thus, the second moment $E(X^2)$ is:

$$E(X^2) = M_X''(0) = 11$$

The variance of X is given by:

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$\text{Var}(X) = 11 - 3^2 = 11 - 9 = 2$$

Therefore, the variance of X is:

$$\text{Var}(X) = 2$$

Question 3

The driver of a truck loaded with 900 boxes of books will be fined if the total weight of the boxes exceeds 36450 pounds. If the distribution of the weight of a box has a mean of 40 pounds and a variance of 36, find the approximate probability that the driver will be fined.

Solution

Givens:

- $n = 900$
- $\mu = 40$ pounds
- $\sigma^2 = 36$ pounds^2
- $T = 36450$ pounds

Legend:

- n: Number of boxes
- μ: Mean weight of a single box
- σ^2: Variance of the weight of a single box
- T: Total weight threshold for the fine
- W: Total weight of the 900 boxes

Symbols to Find:

- $E(W)$: Expected total weight
- $\text{Var}(W)$: Variance of the total weight
- σ_W: Standard deviation of the total weight
- $P(W > 36450)$: Probability of exceeding the weight threshold

Step 1: Determine the Distribution of the Total Weight

Let X_i be the weight of the i-th box. The total weight W is:

$$W = \sum_{i=1}^{900} X_i$$

Since the weights X_i are independently and identically distributed, W follows a normal distribution by the Central Limit Theorem (CLT):

$$W \sim N(n\mu, n\sigma^2)$$

Calculate the mean and variance of W:

$$E(W) = n\mu = 900 \times 40 = 36000$$

$$\text{Var}(W) = n\sigma^2 = 900 \times 36 = 32400$$

$$\sigma_W = \sqrt{32400} = 180$$

Step 2: Standardize the Problem

We need to find $P(W > 36450)$.

Convert this to the standard normal variable Z:

$$Z = \frac{W - E(W)}{\sigma_W} = \frac{W - 36000}{180}$$

Thus, we need to find:

$$P\left(Z > \frac{36450 - 36000}{180}\right) = P(Z > 2.5)$$

Step 3: Find the Probability Using the Standard Normal Distribution Table

From the standard normal distribution table, $P(Z > 2.5)$ is the area to the right of $Z = 2.5$.

$$P(Z > 2.5) = 1 - P(Z \leq 2.5)$$

From the Z-table, $P(Z \leq 2.5) \approx 0.9938$.

Therefore:

$$P(Z > 2.5) = 1 - 0.9938 = 0.0062$$

Thus, the approximate probability that the driver will be fined is:

$$P(W > 36450) \approx 0.0062$$

Question 4

Suppose that the number of calls per hour to an answering service follows a Poisson distribution with rate $\lambda = 4$.

(a) What is the probability that fewer than 2 calls came in the first hour?

(b) What is the probability that there will be no calls in the next two hours?

Solution

Part (a)

What is the probability that fewer than 2 calls came in the first hour?

Step 1: Define the Poisson Distribution

The number of calls per hour follows a Poisson distribution with parameter $\lambda = 4$.

- λ: Rate parameter of the Poisson distribution
- X: Number of calls in the first hour
- $P(X = k)$: Probability of getting exactly k calls in a given time period
- $P(X < k)$: Probability of getting fewer than k calls in a given time period

Step 2: Probability Mass Function

General Formula: The probability mass function of a Poisson random variable X with parameter λ is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Step 3: Calculate the Probability

We need to find the probability that fewer than 2 calls came in the first hour, i.e., $P(X < 2)$.

This can be calculated as:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

Using the Poisson pmf:

$$P(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4}$$

$$P(X = 1) = \frac{4^1 e^{-4}}{1!} = 4e^{-4}$$

Step 4: Sum the Probabilities

Therefore:

$$P(X < 2) = e^{-4} + 4e^{-4} = 5e^{-4}$$

Step 5: Numerical Value

Calculating the numerical value:

$$P(X < 2) \approx 5 \times 0.0183 = 0.0915$$

Part (b)

What is the probability that there will be no calls in the next two hours?

Step 1: Define the Poisson Distribution for Two Hours

The number of calls in two hours also follows a Poisson distribution, but with parameter $\lambda = 2 \times 4 = 8$.

- λ: Rate parameter of the Poisson distribution
- Y: Number of calls in the next two hours
- $P(Y = k)$: Probability of getting exactly k calls in a given time period
- $P(Y < k)$: Probability of getting fewer than k calls in a given time period

Step 2: Probability Mass Function

We need to find the probability of no calls in the next two hours, i.e., $P(Y = 0)$, where $Y \sim \text{Poisson}(\lambda = 8)$.

Step 3: Calculate the Probability

Using the Poisson pmf:

$$P(Y = 0) = \frac{8^0 e^{-8}}{0!} = e^{-8}$$

Step 4: Numerical Value

Calculating the numerical value:

$$P(Y = 0) \approx 0.00034$$

Question 5

Using moment generating functions (MGFs), show that if:

$$X \sim N(\mu_1, \sigma_1^2) \quad \text{and} \quad Y \sim N(\mu_2, \sigma_2^2),$$

then the expectation and variance of $X + Y$ are given by:

$$E(X+Y) = \mu_1 + \mu_2 \quad \text{and} \quad \text{Var}(X+Y) = \sigma_1^2 + \sigma_2^2,$$

and that:

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Note: The moment generating function of X, if X is normally distributed, is given by:

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}.$$

Solution

- μ_1: Mean of the normal distribution for X
- σ_1^2: Variance of the normal distribution for X
- μ_2: Mean of the normal distribution for Y
- σ_2^2: Variance of the normal distribution for Y
- $M_X(t)$: Moment generating function of X
- $M_Y(t)$: Moment generating function of Y
- $M_{X+Y}(t)$: Moment generating function of $X + Y$
- $E(X+Y)$: Expected value of $X + Y$
- $\text{Var}(X+Y)$: Variance of $X + Y$

Step 1: Moment Generating Function (MGF) of a Normal Distribution

The moment generating function (MGF) of a normally distributed random variable $X \sim N(\mu, \sigma^2)$ is given by:

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

Step 2: MGF of the Sum of Two Independent Normal Variables

If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, the MGF of $X + Y$ can be found using the property that the MGF of the sum of independent random variables is the product of their MGFs:

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Given:

$$M_X(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$$

$$M_Y(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

The MGF of $X + Y$ is:

$$M_{X+Y}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}} \cdot e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

Using properties of exponents:

$$M_{X+Y}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2} + t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

$$M_{X+Y}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$$

Step 3: Identifying the Distribution

The MGF of $X + Y$:

$$M_{X+Y}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$$

This is the MGF of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore:

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Step 4: Expectation and Variance of $X + Y$

Expectation

The expectation of $X + Y$ is:

$$E(X + Y) = \mu_1 + \mu_2$$

Variance

The variance of $X + Y$ is:

$$\mathrm{Var}(X + Y) = \sigma_1^2 + \sigma_2^2$$

Conclusion

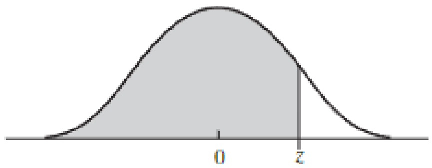
We have shown that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then:

$$E(X + Y) = \mu_1 + \mu_2$$

$$\mathrm{Var}(X + Y) = \sigma_1^2 + \sigma_2^2$$

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Table A.1 Cumulative Areas under the Standard Normal Distribution



z	0	1	2	3	4	5	6	7	8	9
−3.	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
−1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Figure 1: Cumulative Areas under the Standard Normal Distribution (Part 1)

Table A.1 Cumulative Areas under the Standard Normal Distribution (*cont.*)

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Figure 2: Cumulative Areas under the Standard Normal Distribution (Part 2)