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Contents

Proof Methods

If then statements

Format:

Proof. **If A, then B:**

1. **Assume A.**
2. **Show that assuming A leads to B.**
3. **Therefore, B is concluded from A.**

$\square$

Example:

Proof. **If $m = 1$, then $m + 0 = 1$.**

1. **Assume $m = 1$.**
2. **Considering $m = 1$, we have $1 + 0 = 1$.**
3. **This simplifies to $1 = 1$, which is true.**

$\square$

If then types

Different types of implications and their meaning:

- $A \Rightarrow B$: "If it is Wednesday, Dr. Beck will get a cup of coffee from the student union."
- $B \Rightarrow A$ (Converse): "If Dr. Beck got a cup of coffee from the student union, then it is Wednesday."
- $A \Leftrightarrow B$ (Bi-conditional): "It is Wednesday if and only if Dr. Beck got a cup of coffee from the student union."
- $\neg B \Rightarrow \neg A$ (Contrapositive): "If Dr. Beck did not get a cup of coffee from the student union, then it is not Wednesday."

Induction Proof

Format:

Proof. **Prove that $F(x)$ is true for all $x \in A$:**

1. **Base case:** Show $F(a)$ is true, where a is the smallest element in set A.
2. **Induction step:** Assume $F(k)$ is true for an arbitrary $k \in A$. Show that $F(k) \Rightarrow F(k+1)$.
3. **Therefore, $F(x+1)$ is true for all $x \in A$.**

$\square$

Example:

Proof. **For all $n \in \mathbb{N}$, $n = n$:**

1. **Base case ($n = 1$):** $1 = 1$ is true.
2. **Induction step:** Assume $n = n$ is true for an arbitrary natural number n. Show that this implies $n + 1 = n + 1$.
3. By the induction hypothesis, $n = n$. Adding 1 to both sides, $n + 1 = n + 1$, which holds true.

$\square$

Proof by contradiction

Format:

Proof. **Prove that A is true by contradiction:**

1. Assume **not** A.
2. Show that this assumption leads to a contradiction (something that we know is false).
3. Therefore, A must be true.

$\square$

Different Negations

1. **AND $\Rightarrow$ OR:** If A and B, then **not** A or **not** B.

   Example: Dr. Beck is 5 ft tall and single $\Rightarrow$ Dr. Beck is **not** 5 ft tall or is **not** single.

2. **OR $\Rightarrow$ AND:** If A or B, then **not** A and **not** B.

   Example: Dr. Beck will drink a coffee or it is Wednesday $\Rightarrow$ Dr. Beck will **not** drink a coffee and it is **not** Wednesday.

3. **If, then $\Rightarrow$ AND:** If A, then B implies **not** A and **not** B.

   *If it is Monday, then Dr. Beck is on campus $\Rightarrow$ It is **not** Monday and Dr. Beck is **not** on campus.*

4. **For all $\Rightarrow$ There exists:** For all m, A is true implies there exists an m, A is **not** true.

   *For all $m \in \mathbb{Z}$, m is even $\Rightarrow$ There exists $m \in \mathbb{Z}$, m is **not** even.*

5. **There exists $\Rightarrow$ For all:** There exists an m, A is true implies for all m, A is **not** true.

   There exists an $m \in \mathbb{Z}$, $m + 1 = 0.5 \Rightarrow$ For all $m \in \mathbb{Z}$, $m + 1 \neq 0$.

Example:

Proof. **There is no $x \in \mathbb{N}$ that satisfies the equation $1 - x = 0 \cdot x$.**

1. Assume by way of contradiction that such an x exists in $\mathbb{N}$.
2. Since $x \neq 0$ for any $x \in \mathbb{N}$, cancelling x from both sides of the equation $1 - x = 0 \cdot x$ leads to $0 = 1$.
3. Since $0 \neq 1$ is a true mathematical contradiction, the initial statement is proven to be true by contradiction.

$\square$

Definitions

Equality $=$

The symbol $=$ means **equals**. To say $m = n$ means that m and n are the same number. Some properties are:

i. $m = m$ (**reflexivity**)

ii. If $m = n$ then $n = m$ (**symmetry**)

iii. If $m = n$ and $n = p$ then $m = p$ (**transitivity**)

iv. If $m = n$, then n can be substituted for m in any statement without changing the meaning (**replacement**)

Inequality $\neq$

The symbol $\neq$ means **is not equal to**. To say $m \neq n$ means that m and n are different numbers. Note that $\neq$ satisfies **symmetry**, but not **transitivity** and **reflexivity**.

In the set of $\in$

The symbol $\in$ means **is an element of**. For example, $0 \in \mathbb{Z}$ means ”0 is an element of the set $\mathbb{Z}$.”

Not in the set of $\notin$

The symbol $\notin$ means **is not an element of**. For example, $0.5 \notin \mathbb{Z}$ means ”0.5 is not an element of the set $\mathbb{Z}$.”

Divisibility

When m and n are integers, we say m is divisible by n (or alternatively, n divides m) if there exists $j \in \mathbb{Z}$ such that $m = jn$. We use the notation $n|m$.

2 and other integers

2 is defined as $2 = 1 + 1$ and **3** is $2 + 1$ and so on.

Even Integers

Even integers are defined to be those integers that are divisible by 2. That is, $x = 2j$, where $j \in \mathbb{Z}$.

Subtraction

Subtraction is defined as $m - n$ is defined to be $m + (-n)$.

Number Theory

Power

Let b be a fixed integer. We define b^k for all integers $k \geq 0$ by:

1. $\boldsymbol{b^0 := 1}$
2. Assuming b^n is defined, let $\boldsymbol{b^{n+1} := b^n \cdot b}$

Sum

Let $(x_j)_{j=1}^{\infty}$ be a sequence of integers. $(x_j)_{j=1}^{3} = \{1, 2, 3\}$. For each $k \in \mathbb{N}$, we want to define an integer called $\Sigma_{j=1}^{k} x_j$:

1. Define $\mathbf{\Sigma_{j=1}^{1} x_j}$ to be x_1
2. Assuming $\Sigma_{j=1}^{n} x_j$ is already defined, we define $\mathbf{\Sigma_{j=1}^{n+1} x_j}$ to be $\Sigma_{j=1}^{n} x_j + x_{n+1}$

Product

Let $(x_j)_{j=1}^{\infty}$ be a sequence of integers. $(x_j)_{j=1}^{3} = \{1, 2, 3\}$. For each $k \in \mathbb{N}$, we want to define an integer called $\Pi_{j=1}^{k} x_j$:

1. Define $\mathbf{\Pi_{j=1}^{1} x_j}$ to be x_1
2. Assuming $\Pi_{j=1}^{n} x_j$ is already defined, we define $\mathbf{\Pi_{j=1}^{n+1} x_j}$ to be $\Pi_{j=1}^{n} x_j \cdot x_{n+1}$

Non-negative integer ($\mathbb{Z}_{\geq 0}$)

$\mathbb{Z}_{\geq 0} := \{m \in \mathbb{Z} : m \geq 0\}$

Factorial

We define $k!$ ("k factorial") for all integers $k \geq 0$ by:

1. Define $\mathbf{0}! := \mathbf{1}$
2. Assuming $n!$ is defined (where $n \in \mathbb{Z}_{\geq 0}$), define $(\mathbf{n+1})! := (\mathbf{n}!) \cdot (\mathbf{n+1})$

Subset ($\subseteq$)

$A \subseteq B$ means that if $x \in A$, then $x \in B$

The Empty Set ($\emptyset$)

The empty set is defined as a set that contains no elements.

Equal Sets ($=$)

The set A is equal to B means that $A \subseteq B$ and $B \subseteq A$. In order to prove two sets are equal, you have to complete two proofs.

Functions

Informal Definition

A function consists of:

- a set A called the **domain** of the function
- a set B called the **codomain** of the function
- a rule f that assigns to each $a \in A$ an element $f(a) \in B$. Shorthand for this is $f : A \to B$

Abstract Definition

A function with domain A and codomain B is a subset of Γ of $A \times B$ such that for each $a \in A$, there is one and only one element of Γ whose first entry is a. If $(a, b) \in \Gamma$, we write $b = f(a)$.

Theorem 2.25 (Principle of Mathematical Induction — First Form Revisited):

Let $P(k)$ be a statement, depending on a variable $k \in \mathbb{Z}$, that makes sense for all $k \geq m$, where m is a fixed integer. In order to prove the statement "P(k) is true for all $k \geq m$," it is sufficient to prove:

1. $P(m)$ is true, and
2. For any given $n \geq m$, if $P(n)$ is true then $P(n+1)$ is true.

Axiom 1.1: Properties of Integers

If m, n, and p are integers, then:

(i) $m + n = n + m$ (**commutativity of addition**)

(ii) $(m + n) + p = m + (n + p)$ (**associativity of addition**)

(iii) $m \cdot (n + m) = m \cdot n + m \cdot p$ (**distributivity**)

(iv) $m \cdot n = n \cdot m$ (**commutativity of multiplication**)

(v) $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ (**associativity of multiplication**)

Axiom 1.2: Identity Element for Addition

There exists an integer 0 such that whenever $m \in \mathbb{Z}$, $m + 0 = m$ (**identity element for addition**).

Axiom 1.3: Identity Element for Multiplication

There exists an integer 1 such that $1 \neq 0$ and whenever $m \in \mathbb{Z}$, $m \cdot 1 = m$ (**identity element for multiplication**).

Axiom 1.4: Additive Inverse

For each $m \in \mathbb{Z}$, there exists an integer, denoted by $-m$, such that $m + (-m) = 0$ (**additive inverse**).

Axiom 1.5: Cancellation

Let m, n, and p be integers. If $m \cdot n = m \cdot p$ and $m \neq 0$, then $n = p$ (**cancellation**).

Proof Example

Proof. If m is an integer and $m \cdot 0 = 0$, then $m = m$.

- Consider an integer m.
- Multiplying by 0 gives $m \cdot 0 = 0$.
- Since $m \cdot 0 = 0$, by the property of zero in multiplication, we have $m = m$.
- Thus, the statement is proven. □

Axiom 2.1:

There exists a subset $\mathbb{N} \subseteq \mathbb{Z}$ with the following properties:

(i) If $m, n \in \mathbb{N}$ then $m + n \in \mathbb{N}$.

(ii) If $m, n \in \mathbb{N}$ then $mn \in \mathbb{N}$.

(iii) $0 \notin \mathbb{N}$.

(iv) For every $m \in \mathbb{Z}$, we have $m \in \mathbb{N}$ or $m = 0$ or $-m \in \mathbb{N}$.

Chapter 1: Propositions

Proposition 1.6

If m, n, and p are integers, then $(m+n) \cdot p = mp + np$.

Proposition 1.7

If m is an integer, then $0 + m = m$ and $1 \cdot m = m$.

Proposition 1.8

If m is an integer, then $(-m) + m = 0$.

Proposition 1.9

Let m, n, and p be integers. If $m + n = m + p$, then $n = p$.

Proposition 1.10

Let m, x_1, $x_2 \in \mathbb{Z}$. If m, x_1, x_2 satisfy the equation $m + x_1 = 0$ and $m + x_2 = 0$, then $x_1 = x_2$.

Proposition 1.11

If m, n, p, and q are integers, then:

(i) $(m+n)(p+q) = (mp+np) + (mq+nq)$.

(ii) $m + (n + (p+q)) = (m+n) + (p+q) = ((m+n)+p) + q$.

(iii) $m + (n+p) = (p+m) + n$.

(iv) $m(np) = p(mn)$.

(v) $m(n + (p+q)) = (mn + mp) + mq$.

(vi) $(m(n+p))q = (mn)q + m(pq)$.

Proposition 1.12

Let $x \in \mathbb{Z}$. If x has the property that for each integer m, $m + x = m$, then $x = 0$.

Proposition 1.13

Let $x \in \mathbb{Z}$. If x has the property that there exists an integer m such that $m + x = m$, then $x = 0$.

Proposition 1.14

For all $m \in \mathbb{Z}$, $m \cdot 0 = 0 = 0 \cdot m$.

Proposition 1.16

If m and n are even integers, then so are $m + n$ and mn.

Proposition 1.17

(i) 0 is divisible by every integer.

(ii) If m is an integer not equal to 0, then m is not divisible by 0.

Proposition 1.18

Let $x \in \mathbb{Z}$. If x has the property that for all $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.

Proposition 1.19

Let $x \in \mathbb{Z}$. If x has the property that for some nonzero $m \in \mathbb{Z}$, $mx = m$, then $x = 1$.

Proposition 1.20

For all $m, n \in \mathbb{Z}$, $(-m)(-n) = mn$.

Corollary 1.21

$(-1)(-1) = 1$.

Proposition 1.22

(i) For all $m \in \mathbb{Z}$, $-(m) = m$.

(ii) $-0 = 0$.

Proposition 1.23

Given $m, n \in \mathbb{Z}$, there exists one and only one $x \in \mathbb{Z}$ such that $m + x = n$.

Proposition 1.24

Let $x \in \mathbb{Z}$. If $x \cdot x = x$ then $x = 0$ or 1.

Proposition 1.25

(i) $-(m + n) = (-m) + (-n)$.

(ii) $-m = (-1)m$.

(iii) $(-m)n = m(-n) = -(mn)$.

Proposition 1.26

Let $m, n \in \mathbb{Z}$. If $mn = 0$, then $m = 0$ or $n = 0$.

Proposition 1.27

(i) $(m - n) + (p - q) = (m + p) - (n + q)$.

(ii) $(m - n) - (p - q) = (m + q) - (n + p)$.

(iii) $(m - n)(p - q) = (mp + nq) - (mq + np)$.

(iv) $m - n = p - q$ if and only if $m + q = n + p$.

(v) $(m-n)p = mp - np$.

Chapter 2: Propositions

Proposition 2.2:

For every $m \in \mathbb{Z}$, one and only one of the following is true: $m \in \mathbb{N}$, $-m \in \mathbb{N}$, or $m = 0$.

Proposition 2.3:

$1 \in \mathbb{N}$.

Proposition 2.4:

Let $m, n, p \in \mathbb{Z}$. If $m < n$ and $n < p$, then $m < p$.

Proposition 2.5:

For each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m > n$.

Proposition 2.6:

Let $m, n \in \mathbb{Z}$. If $m \leq n \leq m$, then $m = n$.

Proposition 2.7:

(i) If $m < n$, then $m + p < n + p$.

(ii) If $m < n$ and $p < q$, then $m + p < n + q$.

(iii) If $0 < m < n$ and $0 < p \leq q$, then $mp < nq$.

(iv) If $m < n$ and $p < 0$, then $np < mp$.

Proposition 2.8:

Let $m, n \in \mathbb{Z}$. Exactly one of the following is true: $m < n$, $m = n$, $m > n$.

Proposition 2.9:

Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

Proposition 2.10:

The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.

Proposition 2.11:

Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. If $mn \in \mathbb{N}$, then $n \in \mathbb{N}$.

Proposition 2.12:

For all $m, n, p \in \mathbb{Z}$:

(i) $-m < -n$ if and only if $m > n$.

(ii) If $p > 0$ and $mp < np$ then $m < n$.

(iii) If $p < 0$ and $mp < np$ then $n < m$.

(iv) If $m \leq m$ and $0 \leq p$ then $mp \leq np$.

Proposition 2.13:

$\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$.

Proposition 2.14:

(i) $1 \in \mathbb{N}$.

(ii) If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Axiom 2.15 (Induction Axiom):

If a subset $A \subseteq \mathbb{Z}$ satisfies:

1. $1 \in A$, and
2. If $n \in A$, then $n + 1 \in A$,

then $\mathbb{N} \subseteq A$.

Proposition 2.16:

Let $B \subseteq \mathbb{Z}$ be such that:

1. $1 \in B$, and
2. If $n \in B$, then $n + 1 \in B$,

then $B = \mathbb{N}$.

Theorem 2.17 (Principle of Mathematical Induction - First Form):

Let $P(k)$ be a statement depending on a variable $k \in \mathbb{N}$. In order to prove the statement "P(k) is true for all $k \in \mathbb{N}$," it is sufficient to prove:

1. $P(1)$ is true, and
2. For any given $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n + 1)$ is true.

Proposition 2.18:

(i) For all $k \in \mathbb{N}$, $k^3 + 2k$ is divisible by 3.

(ii) For all $k \in \mathbb{N}$, $k^4 - 6k^3 + 11k^2 - 6k$ is divisible by 4.

(iii) For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.

Proposition 2.20:

For all $k \in \mathbb{N}$, $k \geq 1$.

Proposition 2.21:

There exists no integer x such that $0 < x < 1$.

Corollary 2.22:

Let $n \in \mathbb{Z}$. There exists no integer x such that $n < x < n+1$.

Proposition 2.23:

Let $m, n \in \mathbb{N}$. If n is divisible by m, then $m \leq n$.

Proposition 2.24:

For all $k \in \mathbb{N}$, $k^2 + 1 > k$.

Proposition 2.26:

For all integers $k \geq -3$, $3k^2 + 21k + 37 \geq 0$.

Proposition 2.27:

For all integers $k \geq 2$, $k^2 < k^3$.

Theorem 2.32 (Well-Ordering Principle):

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Proposition 2.33:

Let A be a nonempty subset of $\mathbb{Z}$ and $b \in \mathbb{Z}$, such that for each $a \in A$, $b \leq a$. Then A has a smallest element.

Proposition 2.34:

If m and n are integers that are not both 0, then

$$S = \{k \in \mathbb{N} : k = mx + ny \text{ for some } x, y \in \mathbb{Z}\}$$

Theorem 4.4:

A legitimate method of describing a sequence $(y_j)_{j=m}^{\infty}$ is:

1. to name y_m, and
2. to state a formula describing y_{n+1} in terms of y_n, for each $n \geq m$.

Proposition 4.5:

For all $k \in \mathbb{Z}_{\geq 0}$, $k! \in \mathbb{N}$.

Proposition 4.6:

Let $b \in \mathbb{Z}$ and $k, m \in \mathbb{Z}_{\geq 0}$.

1. if $b \in \mathbb{N}$ then $b^k \in \mathbb{N}$
2. $b^m b^k = b^{m+k}$
3. $(b^m)^k = b^{mk}$

Proposition 4.7:

For all $k \in \mathbb{N}$:

1. 5^{2k-1} is divisible by 24
2. $2^{2k+1} + 1$ is divisible by 3
3. $10^{k+3} \cdot 4^{k+2} + 5$ is divisible by 9

Proposition 4.8:

For all $k \in \mathbb{N}$, $4k > k$.

Proposition 4.11:

Let $k \in \mathbb{N}$:

1. $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$
2. $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

Proposition 4.13:

For $x \neq 1$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\sum_{j=0}^{k} x^j = \frac{1 - x^{k+1}}{1 - x}$$

Proposition 4.15:

1. Let $m \in \mathbb{Z}$ and let $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then for all $k \in \mathbb{N}$:

$$m \cdot \sum_{j=1}^{k} x_j! = \sum_{j=1}^{k} (m x_j)$$

2. If $x_j = 1$ for all $j \in \mathbb{N}$, then for all $k \in \mathbb{N}$:

$$\sum_{j=1}^{k} x_j = k$$

3. If $x_j = n \in \mathbb{Z}$ for all $j \in \mathbb{N}$, then for all $k \in \mathbb{N}$:

$$\sum_{j=1}^{k} x_j = kn$$

Proposition 4.16:

Let $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$, and let $a, b, c \in \mathbb{Z}$ be such that $a \leq b < c$.

1. $\displaystyle\sum_{j=a}^{c} x_j = \sum_{j=a}^{b} x_j + \sum_{j=b+1}^{c} x_j$
2. $\displaystyle\sum_{j=a}^{b} (x_j + y_j) = \sum_{j=a}^{b} x_j! + \sum_{j=a}^{b} y_j!$

Proposition 4.17:

Let $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$, and let $a, b, r \in \mathbb{Z}$ be such that $a \leq b$. Then $\displaystyle\sum_{j=a}^{b} x_j = \sum_{j=a}^{b+r} x_j - r$

Proposition 4.18:

Let $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$ such that $x_j \leq y_j$ for all $j \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $\displaystyle\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$

Theorem 4.19:

Let $k, m \in \mathbb{Z}_{\geq 0}$, where $m \leq k$. Then $m!(k-m)!$ divides $k!$.

Corollary 4.20:

For $1 \leq m \leq k$, $\binom{k+1}{m} = \binom{k}{m-1} + \binom{k}{m}$

Theorem 4.21 (Binomial theorem for integers):

If $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$ then $(a+b)^k = \displaystyle\sum_{m=0}^{k} \binom{k}{m} a^{k-m} b^m$

Corollary 4.22:

For $k \in \mathbb{Z}_{\geq 0}$, $\displaystyle\sum_{m=0}^{k} \binom{k}{m} = 2^k$

Theorem 4.24 (Principle of mathematical induction —second form):

Let $P(k)$ be a statement depending on a variable $k \in \mathbb{N}$. In order to prove the statement "$P(k)$ is true for all $k \in \mathbb{N}$" it is sufficient to prove:

1. $P(1)$ is true and
2. if $P(j)$ is true for all integers j such that $1 \leq j \leq n$, then $P(n+1)$ is true

Proposition 4.29:

The kth Fibonacci number is given directly by the formula $f_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right)$

Proposition 4.30:

For all $k, m \in \mathbb{N}$, where $m \geq 2$,

$$f_{m+k} = f_{m-1} f_k + f_m f_{k+1}$$

Proposition 4.31:

For all $k \in \mathbb{N}$,

$$f_{2k+1} = f_k^2 + f_{k+1}^2$$

Proposition 4.32:

For all $k, m \in \mathbb{N}$, f_{mk} is divisible by f_m.

Chapter 5

Proposition 5.1: Let A, B, C be sets:

1. $A \subseteq A$
2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proposition 5.2:

$$\{7m + 1 : m \in \mathbb{Z}\} = \{7n - 6 : n \in \mathbb{Z}\}$$

Proposition 5.4: Let A, B, C be sets:

1. $A = A$
2. if $A = B$ then $B = A$
3. If $A = B$ and $B = C$ then $A = C$

Proposition 5.6:

If the sets $\emptyset_1$ and $\emptyset_2$ have the property that $x \in \emptyset_1$ is never true and $x \in \emptyset_2$ is never true, then $\emptyset_1 = \emptyset_2$.

Proposition 5.7:

The empty set is a subset of every set, that is, for every set S, $\emptyset \subseteq S$.

Proposition 5.14: Let $A, B \subseteq X$:

$A \subseteq B$ if and only if $B^c \subseteq A^c$.

Theorem 5.15 (De Morgan's laws):

Given two subsets $A, B \subseteq X$,

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c$$

Proposition 5.20: Let A, B, C be sets:

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Chapter 3: Propositions

Chapter 4: Propositions

Theorem 4.4:

A legitimate method of describing a sequence $(y_j)_{j=m}^{\infty}$ is:

1. to name y_m, and
2. to state a formula describing y_{n+1} in terms of y_n, for each $n \geq m$.

Proposition 4.5:

For all $k \in \mathbb{Z}_{\geq 0}$, $k! \in \mathbb{N}$.

Proposition 4.6:

Let $b \in \mathbb{Z}$ and $k, m \in \mathbb{Z}_{\geq 0}$.

1. if $b \in \mathbb{N}$ then $b^k \in \mathbb{N}$
2. $b^m b^k = b^{m+k}$
3. $(b^m)^k = b^{mk}$

Proposition 4.7:

For all $k \in \mathbb{N}$:

1. 5^{2k-1} is divisible by 24
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Proposition 4.8:

For all $k \in \mathbb{N}$, $4k > k$.

Proposition 4.11:

Let $k \in \mathbb{N}$:

1. $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$
2. $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

Proposition 4.13:

For $x \neq 1$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\sum_{j=0}^{k} x^j = \frac{1 - x^{k+1}}{1 - x}$$

Proposition 4.15:

1. Let $m \in \mathbb{Z}$ and let $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then for all $k \in \mathbb{N}$:

$$m \cdot \sum_{j=1}^{k} x_j! = \sum_{j=1}^{k} (m x_j)$$

2. If $x_j = 1$ for all $j \in \mathbb{N}$, then for all $k \in \mathbb{N}$:
$$\sum_{j=1}^{k} x_j = k$$

3. If $x_j = n \in \mathbb{Z}$ for all $j \in \mathbb{N}$, then for all $k \in \mathbb{N}$:
$$\sum_{j=1}^{k} x_j = kn$$

Proposition 4.16:

Let $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$, and let $a, b, c \in \mathbb{Z}$ be such that $a \leq b < c$.

1. $\displaystyle\sum_{j=a}^{c} x_j = \sum_{j=a}^{b} x_j + \sum_{j=b+1}^{c} x_j$
2. $\displaystyle\sum_{j=a}^{b} (x_j + y_j) = \sum_{j=a}^{b} x_j! + \sum_{j=a}^{b} y_j!$

Proposition 4.17:

Let $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$, and let $a, b, r \in \mathbb{Z}$ be such that $a \leq b$. Then $\displaystyle\sum_{j=a}^{b} x_j = \sum_{j=a}^{b+r} x_j - r$

Proposition 4.18:

Let $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$ such that $x_j \leq y_j$ for all $j \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $\displaystyle\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$

Theorem 4.19:

Let $k, m \in \mathbb{Z}_{\geq 0}$, where $m \leq k$. Then $m!(k-m)!$ divides $k!$.

Corollary 4.20:

For $1 \leq m \leq k$, $\binom{k+1}{m} = \binom{k}{m-1} + \binom{k}{m}$

Theorem 4.21 (Binomial theorem for integers):

If $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$ then $\displaystyle(a+b)^k = \sum_{m=0}^{k} \binom{k}{m} a^{k-m} b^m$

Corollary 4.22:

For $k \in \mathbb{Z}_{\geq 0}$, $\displaystyle\sum_{m=0}^{k} \binom{k}{m} = 2^k$

Theorem 4.24 (Principle of mathematical induction —second form):

Let $P(k)$ be a statement depending on a variable $k \in \mathbb{N}$. In order to prove the statement "$P(k)$ is true for all $k \in \mathbb{N}$" it is sufficient to prove:

1. $P(1)$ is true and
2. if $P(j)$ is true for all integers j such that $1 \leq j \leq n$, then $P(n+1)$ is true

Proposition 4.29:

The kth Fibonacci number is given directly by the formula $f_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)$

Proposition 4.30:

For all $k, m \in \mathbb{N}$, where $m \geq 2$,

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$$

Proposition 4.31:

For all $k \in \mathbb{N}$,

$$f_{2k+1} = f_k^2 + f_{k+1}^2$$

Proposition 4.32:

For all $k, m \in \mathbb{N}$, f_{mk} is divisible by f_m.

Chapter 5: Propositions

Proposition 5.1: Let A, B, C be sets:

1. $A \subseteq A$
2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proposition 5.2:

$$\{7m+1 : m \in \mathbb{Z}\} = \{7n-6 : n \in \mathbb{Z}\}$$

Proposition 5.4: Let A, B, C be sets:

1. $A = A$
2. if $A = B$ then $B = A$
3. If $A = B$ and $B = C$ then $A = C$

Proposition 5.6:

If the sets $\emptyset_1$ and $\emptyset_2$ have the property that $x \in \emptyset_1$ is never true and $x \in \emptyset_2$ is never true, then $\emptyset_1 = \emptyset_2$.

Proposition 5.7:

The empty set is a subset of every set, that is, for every set S, $\emptyset \subseteq S$.

Proposition 5.14: Let $A, B \subseteq X$:

$A \subseteq B$ if and only if $B^c \subseteq A^c$.

Theorem 5.15 (De Morgan's laws):

Given two subsets $A, B \subseteq X$,

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c$$

Proposition 5.20: Let A, B, C be sets:

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$