

Definitions

Inverse of a Matrix

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = A^{-1}A = I$

LDU Decomposition

- For a symmetric matrix
- $A : A = LDL^T$
- $L$ : lower triangular with unit diagonal
- $D$ : diagonal matrix

Vector Space Axioms

- Addition: commutativity, associativity, identity, inverses
- Scalar Multiplication: distributivity, compatibility, identity

Subspaces

- Closed under addition and scalar multiplication

Linear Dependence and Independence

- Dependent:  $\exists$  scalars, not all zero, s.t.  $a_1v_1 + \dots + a_nv_n = 0$
- Independent: only solution is  $a_1 = \dots = a_n = 0$

Basis and Dimension

- Basis: linearly independent spanning set
- Dimension: number of vectors in a basis

General Principles for Subspaces

- Closed under vector addition
- Closed under scalar multiplication

Linear Transformation

- Preserves vector addition and scalar multiplication

Image and Kernel

- $\text{im}(A)$ : span of column vectors of  $A$
- $\text{ker}(A)$ :  $\{x \in \mathbb{R}^n : Ax = 0\}$

Basis Transformation

- Unique representation of a vector in terms of basis vectors

Determining Linear Independence (Standard Case)

Given vectors  $v_1 = (1, 2, 3)$ ,  $v_2 = (0, 1, 1)$ , and  $v_3 = (2, 5, 7)$ , determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix  $A$ :

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$$

2. Perform row reduction on  $A$ :

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Since there are no rows of all zeros in the reduced row echelon form, the vectors are linearly independent.

Determining Linear Independence (Linearly Dependent Case)

Given vectors  $v_1 = (1, 2, 3)$ ,  $v_2 = (2, 4, 6)$ , and  $v_3 = (3, 6, 9)$ , determine if they are linearly independent.

Solution:

1. Arrange the vectors as columns in a matrix  $A$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

2. Perform row reduction on  $A$ :

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. The presence of rows of all zeros indicates that the vectors are linearly dependent.

Finding a Basis for a Subspace (Polynomial Space)

Find a basis for the subspace of  $P_3$  consisting of polynomials  $p(x) = ax^3 + bx^2 + cx + d$  such that  $p(1) = 0$ .

Solution:

1. The condition  $p(1) = 0$  gives  $a + b + c + d = 0$ . To find a basis, express this condition in terms of the coefficients and set up a system.

2. Considering the standard basis  $\{1, x, x^2, x^3\}$  for  $P_3$ , impose the condition for  $p(1) = 0$ :

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = 0$$

This implies  $a = -b - c - d$ .

3. A basis satisfying this condition is  $\{x^3 - x^2, x^2 - x, x - 1\}$  as these polynomials nullify at  $x = 1$  and are linearly independent.

Finding the Matrix Inverse

Find the inverse of the matrix  $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ .

Solution:

1. Set up the augmented matrix for  $A$  and the identity matrix:  $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$ .
2. Perform row operations to get the identity matrix on the left side of the augmented matrix. Subtract twice the first row from the second row to start:  $\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$ .
3. Then, subtract 3 times the second row from the first row:  $\begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$ .
4. The matrix on the right side is now  $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ .

Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors for the matrix  $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ .

Solution:

1. Find the characteristic polynomial:  $\det(B - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{bmatrix} = (4 - \lambda)^2 - 1$ .
2. Solve for  $\lambda$ :  $(4 - \lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = 0$ . The eigenvalues are  $\lambda_1 = 3$  and  $\lambda_2 = 5$ .
3. Find eigenvectors for each eigenvalue:
  - For  $\lambda_1 = 3$ : Solve  $(B - 3I)x = 0$ . This gives  $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .
  - For  $\lambda_2 = 5$ : Solve  $(B - 5I)x = 0$ . This gives  $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

Diagonalization

Determine if the matrix  $C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$  is diagonalizable. If it is, find a matrix  $P$  that diagonalizes  $C$ .

Solution:

1. Find the eigenvalues of  $C$ : The characteristic polynomial is  $(2 - \lambda)^2 = 0$ , so the only eigenvalue is  $\lambda = 2$ .
2. Since  $C$  is a  $2 \times 2$  matrix with only one distinct eigenvalue, we need to check if there are two linearly independent eigenvectors corresponding to  $\lambda = 2$ .
3. Solve  $(C - 2I)x = 0$ : This leads to the system  $x_2 = 0$ , indicating that every eigenvector has the form  $\begin{bmatrix} t \\ 0 \end{bmatrix}$ , which does not provide two independent eigenvectors.
4. Since we cannot find two linearly independent eigenvectors,  $C$  is not diagonalizable.

Verifying Vector Space Axioms

Verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space over the real numbers.

Solution

To verify that the set of all polynomials of degree at most 2 with real coefficients forms a vector space, we need to check that the following axioms hold:

- The set is closed under addition: The sum of any two polynomials of degree at most 2 is also a polynomial of degree at most 2.
- The set is closed under scalar multiplication: The scalar multiple of any polynomial of degree at most 2 is also a polynomial of degree at most 2.
- The set contains a zero vector, which is the zero polynomial.
- Each polynomial has an additive inverse within the set.
- Addition is associative and commutative.
- Scalar multiplication is distributive with respect to both scalar and vector addition.
- Scalar multiplication is compatible with field multiplication.

- The scalar 1 acts as a multiplicative identity.

Since all these properties are satisfied, the set is indeed a vector space.

Finding the Rank of a Matrix

Find the rank of the matrix  $E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Solution:

1. Perform row reduction on  $E$ :  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .
2. The rank is equal to the number of non-zero rows in the reduced row echelon form. Hence,  $\text{rank}(E) = 1$ .

Linear Dependence in  $\mathbb{R}^2$

Given the vectors  $v_1 = (3, -1)$  and  $v_2 = (6, -2)$  in  $\mathbb{R}^2$ , determine if  $v_1$  and  $v_2$  are linearly dependent. **Solution**

1. Notice that  $v_2 = 2v_1$ , which means  $v_2$  is a scalar multiple of  $v_1$ .
2. This implies the set  $\{v_1, v_2\}$  is linearly dependent because  $v_2$  can be expressed as a linear combination of  $v_1$ .

Identifying Subspaces

Determine if the set of all vectors in  $\mathbb{R}^3$  of the form  $(a, a, b)$  is a subspace of  $\mathbb{R}^3$ .

Solution

To determine if the set of all vectors in  $\mathbb{R}^3$  of the form  $(a, a, b)$  is a subspace, we check for the following properties:

- The zero vector  $(0, 0, 0)$  is in the set, satisfying the non-emptiness requirement.
- The set is closed under addition:  $(a_1, a_1, b_1) + (a_2, a_2, b_2) = (a_1 + a_2, a_1 + a_2, b_1 + b_2)$ , which is of the form  $(a, a, b)$ .
- The set is closed under scalar multiplication: For any real number  $c$ ,  $c(a, a, b) = (ca, ca, cb)$ , which is still of the form  $(a, a, b)$ .

Thus, the set is a subspace of  $\mathbb{R}^3$ . **Important Identities and Properties**

- Rank-Nullity Theorem:  $\text{rank}(A) + \text{nullity}(A) = n$ , where  $n$  is the number of columns of  $A$ .
- Determinant properties:  $\det(AB) = \det(A)\det(B)$  and  $\det(A^{-1}) = \frac{1}{\det(A)}$  for invertible  $A$ .

### Determinant Calculation

Calculate the determinant of the matrix

$$D = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

#### Solution:

1. Apply the Laplace expansion using the first row:

$$\det(D) = 6 \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

2. Calculate each minor:

$$\begin{aligned} & 6 \cdot (3 \cdot 4 - 0 \cdot 1) - 1 \cdot (1 \cdot 4 - 1 \cdot 2) + 2 \cdot (1 \cdot 0 - 3 \cdot 2) \\ &= 72 - 2 - 12 \\ &= 58 \end{aligned}$$

(1)

3. Thus,  $\det(D) = 58$ .

### Using Cramer's Rule

Solve the following system of equations using Cramer's Rule:

$$\begin{aligned} x + 2y - z &= 4, \\ 2x - y + 3z &= -2, \\ x + 3y + z &= 3. \end{aligned}$$

(2)

#### Solution:

1. Write the coefficient matrix and calculate its determinant:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A) = -16.$$

2. For  $x$ , replace the first column of  $A$  with the constant terms and calculate its determinant:

$$A_x = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -1 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \quad \det(A_x) = -16.$$

3. For  $y$ , replace the second column of  $A$ :

$$A_y = \begin{bmatrix} 1 & 4 & -1 \\ 2 & -2 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \quad \det(A_y) = -32.$$

4. For  $z$ , replace the third column of  $A$ :

$$A_z = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & -2 \\ 1 & 3 & 3 \end{bmatrix}, \quad \det(A_z) = -16.$$

5. Compute the solutions:  $x = \frac{\det(A_x)}{\det(A)} = 1, y = \frac{\det(A_y)}{\det(A)} = 2, z = \frac{\det(A_z)}{\det(A)} = 1$ .

### LU Decomposition

Perform LU decomposition on the matrix  $F = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$ .

#### Solution:

1. Express  $F$  as the product of a lower triangular matrix  $L$  and an upper triangular matrix  $U$ .

2. Choose  $L$  with 1s on the diagonal:  $L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$ .

3. Let  $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ .

4. Since  $F_{21} = l_{21} \cdot u_{11}$  and  $F_{21} = 6, u_{11} = 4$ , we get  $l_{21} = \frac{6}{4} = \frac{3}{2}$ .

5. Solve for  $U$  using the first row of  $F$ :  $U = \begin{bmatrix} 4 & 3 \\ 0 & u_{22} \end{bmatrix}$ . The second element of the second row gives  $u_{22} = 3 - \frac{3}{2} \cdot 3 = -\frac{3}{2}$ .

6. The LU decomposition is  $L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}, U =$

$$\begin{bmatrix} 4 & 3 \\ 0 & -\frac{3}{2} \end{bmatrix}$$

### Orthogonal Diagonalization

Orthogonally diagonalize the matrix

$$F = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

#### Solution:

1. Find the eigenvalues by solving  $\det(F - \lambda I) = 0$ :  $\lambda^2 - 6\lambda + 8 = 0$  gives  $\lambda_1 = 2$  and  $\lambda_2 = 4$ .

2. Find the eigenvectors: For  $\lambda_1 = 2$ , solve  $(F - 2I)x = 0$  to get  $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  (after normalization).

For  $\lambda_2 = 4$ , solve  $(F - 4I)x = 0$  to get  $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  (after normalization).

3. Construct  $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$  and verify

$$P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

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$$P^T F P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

### Basis and Dimension of a Vector Space

Consider the vector space  $V$  of all vectors in  $\mathbb{R}^4$  that satisfy the equation  $x_1 - 2x_2 + x_3 - 2x_4 = 0$ . Find a basis for  $V$  and state its dimension.

#### Solution

1. To find a basis, we need to solve for the vectors that satisfy the given equation. Let  $x_4 = t$ , then  $x_3 = 2t, x_2 = s$ , and  $x_1 = 2s - t$ .

2. Thus, any vector in  $V$  can be written as  $\begin{bmatrix} 2s-t \\ s \\ 2t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ .

3. The vectors  $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$  are linearly independent and span  $V$ , so they form a basis for  $V$ .

4. The dimension of  $V$ , denoted as  $\dim(V)$ , is the number of vectors in the basis, which is 2.

### Span of Vectors

Determine if the vector  $\mathbf{u} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$  is in the span of

the set of vectors  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ .

#### Solution:

1. We need to determine if there exist scalars  $c_1$  and  $c_2$  such that  $\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$ .

2. Setting up the equation, we get:

$$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

3. This leads to the system of equations:

$$\begin{aligned} c_1 &= 4, \\ c_2 &= -2, \\ -c_1 + 2c_2 &= 1. \end{aligned}$$

4. Solving this system, we find  $c_1 = 4$  and  $c_2 = -2$ . However, these values do not satisfy the third equation ( $-4 - 2 \neq 1$ ). Therefore,  $\mathbf{u}$  is not in the span of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

### Change of Basis

Given a vector  $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  in the standard basis of  $\mathbb{R}^2$ , find

its coordinates in the new basis  $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

#### Solution

1. The coordinates of  $v$  in the basis  $B$  can be found by solving the equation  $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ .

2. This equation translates into the system:

$$c_1 - c_2 = 3, c_1 + c_2 = 2.$$

3. Adding the two equations yields  $2c_1 = 5$ , so  $c_1 = \frac{5}{2}$ .

4. Substituting  $c_1$  into the second equation gives  $c_2 = \frac{5}{2} - \frac{5}{2} = -\frac{3}{2}$ .

5. Therefore, the coordinates of  $v$  in the basis  $B$  are  $\left(\frac{5}{2}, -\frac{3}{2}\right)$ .

### Perform LDU decomposition

Perform LDU decomposition on the matrix

$$H = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

#### Solution:

1. First, we find the matrix  $L$  such that  $H = LDU$  where  $L$  is a lower triangular matrix with unit diagonal,  $D$  is a diagonal matrix, and  $U$  is an upper triangular matrix.

2. Decompose  $H$  into  $LDU$ :

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Verify the decomposition by calculating  $LDU$  and comparing it with  $H$ :

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}.$$

4. The result confirms the LDU decomposition of  $H$ .