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Practice for the exam

Question 1

Suppose the random variables X and Y are jointly distributed according to the pdf:

$$f_{XY}(x,y) = 8xy, \quad 0 < y < x < 1$$

(a) Find $P(X < 2Y)$

(b) Find $P(Y < \frac{1}{4}|x = \frac{1}{2})$

Process

- X: Random variable representing the first dimension
- Y: Random variable representing the second dimension
- $f_{XY}(x,y)$: Joint probability density function of X and Y
- $f_X(x)$: Marginal probability density function of X
- $f_{Y|X}(y|x)$: Conditional probability density function of Y given X
- $P(X < 2Y)$: Probability that X is less than $2Y$
- $P(Y < \frac{1}{4}|X = \frac{1}{2})$: Conditional probability that Y is less than $\frac{1}{4}$ given $X = \frac{1}{2}$

Part (a) Find $P(X < 2Y)$

General Formula: For any joint PDF $f_{XY}(x,y)$, the probability $P(g(X,Y))$ for some condition g is given by:

$$P(g(X,Y)) = \int \int_{g(x,y)} f_{XY}(x,y)\, dx\, dy$$

Specific Calculation: This probability calculation involves integrating the joint PDF over the region defined by $X < 2Y$ and within the given bounds of $0 < y < x < 1$.

$$P(X < 2Y) = \int_0^1 \int_0^{X/2} 8xy\, dy\, dx$$

This integral setup reflects the condition $X < 2Y$ within the area bounded by $0 < y < x < 1$.

Integration Steps: Calculate the inner integral over y:

$$\int_0^{X/2} 8xy\, dy = 8x \left[\frac{y^2}{2}\right]_0^{X/2} = 8x \left[\frac{(X/2)^2}{2}\right] = X^3$$

Now integrate with respect to x:

$$\int_0^1 X^3\, dx = \left[\frac{X^4}{4}\right]_0^1 = \frac{1}{4}$$

Thus, $P(X < 2Y) = \frac{1}{4}$.

Part (b) Find $P(Y < \frac{1}{4}|x = \frac{1}{2})$

General Formula: For any joint PDF $f_{XY}(x, y)$, the conditional probability $P(A|B)$ is given by:

$$P(A|B) = \frac{\int\int_{A\cap B} f_{XY}(x, y)\, dx\, dy}{\int\int_B f_{XY}(x, y)\, dx\, dy}$$

Specific Calculation: Given $X = \frac{1}{2}$, we need to find the conditional probability $P(Y < \frac{1}{4}|X = \frac{1}{2})$. This involves determining the conditional PDF $f_{Y|X}(y|x)$ and integrating it over the desired range of Y.

Determine the Marginal Density of X, $f_X(x)$: The marginal density $f_X(x)$ is found by integrating out Y from the joint PDF:

$$f_X(x) = \int_0^x 8xy\, dy = 8x\left[\frac{y^2}{2}\right]_0^x = 4x^3$$

At $x = \frac{1}{2}$, the marginal density $f_X\left(\frac{1}{2}\right)$ is:

$$f_X\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

Calculate the Conditional PDF $f_{Y|X}(y|x)$: The conditional PDF $f_{Y|X}(y|\frac{1}{2})$ is:

$$f_{Y|X}(y|\frac{1}{2}) = \frac{8 \cdot \frac{1}{2} \cdot y}{\frac{1}{2}} = 8y$$

Compute the Conditional Probability $P(Y < \frac{1}{4}|X = \frac{1}{2})$:

$$P(Y < \frac{1}{4}|X = \frac{1}{2}) = \int_0^{1/4} 8y\, dy$$

Calculate the integral:

$$\int_0^{1/4} 8y\, dy = \left[4y^2\right]_0^{1/4} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$$

Conclusion: Therefore, $P(Y < \frac{1}{4}|X = \frac{1}{2}) = \frac{1}{4}$.

Question 2

A random variable X has the moment generating function $M_X(t) = \left(\frac{2+e^t}{3}\right)^9$. Find $\text{Var}(X)$.

Process

- X: Random variable
- $M_X(t)$: Moment generating function of X
- $E(X)$: Expected value (mean) of X
- $E(X^2)$: Second moment of X
- $\text{Var}(X)$: Variance of X

General Formula: The moment generating function (MGF) $M_X(t)$ of a random variable X is defined as $M_X(t) = E(e^{tX})$. The n-th moment of X is given by the n-th derivative of $M_X(t)$ evaluated at $t = 0$:

$$E(X^n) = M_X^{(n)}(0)$$

First, find the first derivative of $M_X(t)$:

$$M_X(t) = \left(\frac{2+e^t}{3}\right)^9$$

$$M_X'(t) = 9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}$$

Evaluate the first derivative at $t = 0$:

$$M_X'(0) = 9\left(\frac{2+e^0}{3}\right)^8 \cdot \frac{e^0}{3} = 9\left(\frac{3}{3}\right)^8 \cdot \frac{1}{3} = 9 \cdot 1 \cdot \frac{1}{3} = 3$$

Thus, the mean μ of X is:

$$\mu = E(X) = M_X'(0) = 3$$

Next, find the second derivative of $M_X(t)$:

$$M_X''(t) = \frac{d}{dt}\left[9\left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}\right]$$

Using the product rule:

$$M_X''(t) = 9\left[8\left(\frac{2+e^t}{3}\right)^7 \cdot \frac{e^t}{3} \cdot \frac{e^t}{3} + \left(\frac{2+e^t}{3}\right)^8 \cdot \frac{e^t}{3}\right]$$

Evaluate the second derivative at $t = 0$:

$$M_X''(0) = 9\left[8\left(\frac{3}{3}\right)^7 \cdot \frac{1}{3} \cdot \frac{1}{3} + \left(\frac{3}{3}\right)^8 \cdot \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[8 \cdot 1 \cdot \frac{1}{9} + 1 \cdot \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[\frac{8}{9} + \frac{1}{3}\right]$$

$$M_X''(0) = 9\left[\frac{8}{9} + \frac{3}{9}\right]$$

$$M_X''(0) = 9 \cdot \frac{11}{9} = 11$$

Thus, the second moment $E(X^2)$ is:

$$E(X^2) = M_X''(0) = 11$$

The variance of X is given by:

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$\text{Var}(X) = 11 - 3^2 = 11 - 9 = 2$$

Therefore, the variance of X is:

$$\text{Var}(X) = 2$$

Question 3

The driver of a truck loaded with 900 boxes of books will be fined if the total weight of the boxes exceeds 36450 pounds. If the distribution of the weight of a box has a mean of 40 pounds and a variance of 36, find the approximate probability that the driver will be fined.

Process

Givens:

- $n = 900$
- $\mu = 40$ pounds
- $\sigma^2 = 36$ pounds^2
- $T = 36450$ pounds

Legend:

- n: Number of boxes
- μ: Mean weight of a single box
- σ^2: Variance of the weight of a single box
- T: Total weight threshold for the fine
- W: Total weight of the 900 boxes

Symbols to Find:

- $E(W)$: Expected total weight
- $\text{Var}(W)$: Variance of the total weight
- σ_W: Standard deviation of the total weight
- $P(W > 36450)$: Probability of exceeding the weight threshold

Step 1: Determine the Distribution of the Total Weight

Let X_i be the weight of the i-th box. The total weight W is:

$$W = \sum_{i=1}^{900} X_i$$

Since the weights X_i are independently and identically distributed, W follows a normal distribution by the Central Limit Theorem (CLT):

$$W \sim N(n\mu, n\sigma^2)$$

Calculate the mean and variance of W:

$$E(W) = n\mu = 900 \times 40 = 36000$$

$$\text{Var}(W) = n\sigma^2 = 900 \times 36 = 32400$$

$$\sigma_W = \sqrt{32400} = 180$$

Step 2: Standardize the Problem

We need to find $P(W > 36450)$.

Convert this to the standard normal variable Z:

$$Z = \frac{W - E(W)}{\sigma_W} = \frac{W - 36000}{180}$$

Thus, we need to find:

$$P\left(Z > \frac{36450 - 36000}{180}\right) = P(Z > 2.5)$$

Step 3: Find the Probability Using the Standard Normal Distribution Table

From the standard normal distribution table, $P(Z > 2.5)$ is the area to the right of $Z = 2.5$.

$$P(Z > 2.5) = 1 - P(Z \leq 2.5)$$

From the Z-table, $P(Z \leq 2.5) \approx 0.9938$.

Therefore:

$$P(Z > 2.5) = 1 - 0.9938 = 0.0062$$

Thus, the approximate probability that the driver will be fined is:

$$P(W > 36450) \approx 0.0062$$

Question 4

Suppose that the number of calls per hour to an answering service follows a Poisson distribution with rate $\lambda = 4$.

(a) What is the probability that fewer than 2 calls came in the first hour?

(b) What is the probability that there will be no calls in the next two hours?

Process

Part (a)

What is the probability that fewer than 2 calls came in the first hour?

Step 1: Define the Poisson Distribution

The number of calls per hour follows a Poisson distribution with parameter $\lambda = 4$.

- λ: Rate parameter of the Poisson distribution
- X: Number of calls in the first hour
- $P(X = k)$: Probability of getting exactly k calls in a given time period
- $P(X < k)$: Probability of getting fewer than k calls in a given time period

Step 2: Probability Mass Function

General Formula: The probability mass function of a Poisson random variable X with parameter λ is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Step 3: Calculate the Probability

We need to find the probability that fewer than 2 calls came in the first hour, i.e., $P(X < 2)$.

This can be calculated as:

$$P(X < 2) = P(X = 0) + P(X = 1)$$

Using the Poisson pmf:

$$P(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4}$$

$$P(X = 1) = \frac{4^1 e^{-4}}{1!} = 4e^{-4}$$

Step 4: Sum the Probabilities

Therefore:

$$P(X < 2) = e^{-4} + 4e^{-4} = 5e^{-4}$$

Step 5: Numerical Value

Calculating the numerical value:

$$P(X < 2) \approx 5 \times 0.0183 = 0.0915$$

Part (b)

What is the probability that there will be no calls in the next two hours?

Step 1: Define the Poisson Distribution for Two Hours

The number of calls in two hours also follows a Poisson distribution, but with parameter $\lambda = 2 \times 4 = 8$.

- λ: Rate parameter of the Poisson distribution
- Y: Number of calls in the next two hours
- $P(Y = k)$: Probability of getting exactly k calls in a given time period
- $P(Y < k)$: Probability of getting fewer than k calls in a given time period

Step 2: Probability Mass Function

We need to find the probability of no calls in the next two hours, i.e., $P(Y = 0)$, where $Y \sim \text{Poisson}(\lambda = 8)$.

Step 3: Calculate the Probability

Using the Poisson pmf:

$$P(Y = 0) = \frac{8^0 e^{-8}}{0!} = e^{-8}$$

Step 4: Numerical Value

Calculating the numerical value:

$$P(Y = 0) \approx 0.00034$$

Question 5

Using moment generating functions (MGFs), show that if:

$$X \sim N(\mu_1, \sigma_1^2) \quad \text{and} \quad Y \sim N(\mu_2, \sigma_2^2),$$

then the expectation and variance of $X + Y$ are given by:

$$E(X+Y) = \mu_1 + \mu_2 \quad \text{and} \quad \mathrm{Var}(X+Y) = \sigma_1^2 + \sigma_2^2,$$

and that:

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Note: The moment generating function of X, if X is normally distributed, is given by:

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}.$$

Process

- μ_1: Mean of the normal distribution for X
- σ_1^2: Variance of the normal distribution for X
- μ_2: Mean of the normal distribution for Y
- σ_2^2: Variance of the normal distribution for Y
- $M_X(t)$: Moment generating function of X
- $M_Y(t)$: Moment generating function of Y
- $M_{X+Y}(t)$: Moment generating function of $X + Y$
- $E(X+Y)$: Expected value of $X + Y$
- $\mathrm{Var}(X+Y)$: Variance of $X + Y$

Step 1: Moment Generating Function (MGF) of a Normal Distribution

The moment generating function (MGF) of a normally distributed random variable $X \sim N(\mu, \sigma^2)$ is given by:

$$M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

Step 2: MGF of the Sum of Two Independent Normal Variables

If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, the MGF of $X + Y$ can be found using the property that the MGF of the sum of independent random variables is the product of their MGFs:

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Given:

$$M_X(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$$

$$M_Y(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

The MGF of $X + Y$ is:

$$M_{X+Y}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}} \cdot e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

Using properties of exponents:

$$M_{X+Y}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2} + t\mu_2 + \frac{t^2\sigma_2^2}{2}}$$

$$M_{X+Y}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$$

Step 3: Identifying the Distribution

The MGF of $X + Y$:

$$M_{X+Y}(t) = e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$$

This is the MGF of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Therefore:

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Step 4: Expectation and Variance of $X + Y$

Expectation

The expectation of $X + Y$ is:

$$E(X + Y) = \mu_1 + \mu_2$$

Variance

The variance of $X + Y$ is:

$$\mathrm{Var}(X + Y) = \sigma_1^2 + \sigma_2^2$$

Conclusion

We have shown that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then:

$$E(X + Y) = \mu_1 + \mu_2$$

$$\mathrm{Var}(X + Y) = \sigma_1^2 + \sigma_2^2$$

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Definitions, Formulas & Theorems

Joint Probability Density Function (PDF)

The joint PDF $f_{XY}(x, y)$ defines the probability density at each point (x, y) for the continuous random variables X and Y. It is used to compute the probability that (X, Y) falls within a specific region A.

$$P((X, Y) \in A) = \int \int_A f_{XY}(x, y)\, dx\, dy.$$

Marginal Probability Density Function

The marginal PDF of a random variable X, denoted as $f_X(x)$, is obtained by integrating the joint PDF $f_{XY}(x, y)$ over all possible values of the other variable Y.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy.$$

Conditional Probability Density Function

The conditional PDF $f_{Y|X}(y|x)$ gives the probability density of Y given that X takes a specific value x. It is derived from the joint PDF and the marginal PDF of X.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Expected Value and Variance of a Random Variable

The expected value $E(X)$ is the average value of a random variable X, and the variance $\mathrm{Var}(X)$ measures the spread of X around its mean.

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

$$\mathrm{Var}(X) = E((X - E(X))^2) = \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x)\, dx.$$

Moment Generating Function (MGF)

The moment generating function $M_X(t)$ of a random variable X is a function that summarizes all the moments of X. It is particularly useful for identifying the distribution of X based on its moments.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

Central Limit Theorem (CLT)

The Central Limit Theorem (CLT) states that the average of a large number of independent and identically distributed (i.i.d.) random variables, each with a finite mean and variance, tends toward a normal distribution regardless of the original distribution of the variables.

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2),$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Poisson Distribution

The Poisson distribution is used to model the number of events occurring within a fixed interval of time or space, assuming events occur independently at a constant rate λ.

Bayes' Theorem

Bayes' theorem is a fundamental result that describes how to update the probability of a hypothesis based on new evidence.

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k = 0, 1, 2, \ldots.$$

Properties of Normal Distribution

The normal distribution, characterized by its bell-shaped curve, is defined by its mean μ and variance σ^2. It is one of the most important distributions in statistics due to its occurrence in many natural phenomena.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Sum of Independent Normal Variables

When two independent normal variables X and Y are summed, the result is also a normal variable with mean equal to the sum of their means and variance equal to the sum of their variances.

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Union of Sets

The union of two sets A and B, denoted $A \cup B$, is the set of all elements that belong to either A, B, or both. In probability, $P(A \cup B)$ is the probability that at least one of the events A or B occurs.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Intersection of Sets

The intersection of two sets A and B, denoted $A \cap B$, is the set of all elements that belong to both A and B. In probability, $P(A \cap B)$ is the probability that both events A and B occur simultaneously.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$
$$P(A \cap B) = \text{If } A \text{ and } B \text{ are independent, } P(A \cap B) = P(A) \times P(B).$$

Complement of a Set

The complement of a set A, denoted A^c, is the set of all elements not in A. In probability, $P(A^c)$ is the probability that event A does not occur.

$$A^c = \{x : x \notin A\}.$$
$$P(A^c) = 1 - P(A).$$

Conditional Probability

Conditional probability, denoted $P(A|B)$, is the probability of an event A occurring given that another event B has already occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0.$$

Inclusion-Exclusion Principle

The inclusion-exclusion principle is used to compute the probability of the union of multiple events by correcting for overlaps.

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Basic Set Operations

Understand and utilize the fundamental operations in set theory. Here's a breakdown of basic operations and their properties.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A^c = \{x : x \notin A\}$$
$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Properties of Set Operations

Recognize the properties that allow the transformation and simplification of expressions involving sets.

$$\begin{aligned} A \cup B = B \cup A, &\quad A \cap B = B \cap A \\ (A \cup B) \cup C = A \cup (B \cup C), &\quad (A \cap B) \cap C = A \cap (B \cap C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C), &\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ (A \cup B)^c = A^c \cap B^c, &\quad (A \cap B)^c = A^c \cup B^c \end{aligned}$$

Breaking Down Complex Set Expressions

Use algebraic techniques to decompose complex set expressions into simpler, more manageable components.

Consider $A \cap (B \cup C)$. Using distributivity:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Consider $(A \cup B) \cap (A^c \cup C)$. Applying distributivity:

$$(A \cup B) \cap (A^c \cup C) = (A \cap A^c) \cup (A \cap C) \cup (B \cap A^c) \cup (B \cap C)$$

Simplifying $A \cap A^c = \emptyset$, the expression reduces to:

$$(A \cap C) \cup (B \cap A^c) \cup (B \cap C)$$

Basic Probability Concepts

Familiarize yourself with the core concepts of probability theory, which are foundational for analyzing random events.

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$
$$P(A^c) = 1 - P(A)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability and Independence

Understand how the occurrence of one event affects the probability of another and define when two events are independent.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Events A and B are independent if $P(A \cap B) = P(A) \times P(B)$.

Breaking Down Complex Probability Expressions

Apply probability rules and algebraic manipulation to decompose and understand complex probability expressions.

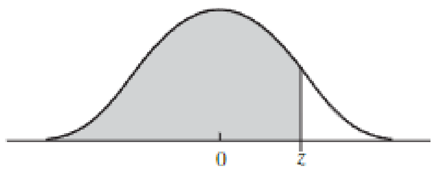
Consider $P(A \cup B \cup C)$. Using the inclusion-exclusion principle:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Consider $P(A \cap (B \cup C))$. Distributing the intersection:

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

Table A.1 Cumulative Areas under the Standard Normal Distribution



z	0	1	2	3	4	5	6	7	8	9
−3.	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0238	0.0233
−1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0570	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Figure 1: Cumulative Areas under the Standard Normal Distribution (Part 1)

Table A.1 Cumulative Areas under the Standard Normal Distribution (*cont.*)

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9430	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9648	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9762	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Figure 2: Cumulative Areas under the Standard Normal Distribution (Part 2)