Homework 11

Problem 3.12.4

Question

Find the moment-generating function for the discrete random variable X whose probability function is given by:

$$p_X(k) = \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right), \quad k = 0, 1, 2, \ldots$$

Solution

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)$$

$$M_X(t) = \frac{1}{4} \sum_{k=0}^{\infty} \left(e^t \frac{3}{4}\right)^k$$

$$M_X(t) = \frac{1}{4} \cdot \frac{1}{1 - \left(e^t \frac{3}{4}\right)} = \frac{1}{4 - 3e^t}$$

Problem 3.12.8

Question

Let Y be a continuous random variable with $f_Y(y) = ye^{-y}$ for $0 \leq y \leq \infty$. Show that $M_Y(t) = \frac{1}{(1-t)^2}$ for $t < 1$.

Solution

$$M_Y(t) = \int_0^\infty e^{ty} y e^{-y}\, dy$$

$$M_Y(t) = \int_0^\infty y e^{(t-1)y}\, dy$$

$$u = y, \quad dv = e^{(t-1)y} dy$$

$$du = dy, \quad v = \frac{e^{(t-1)y}}{t-1}$$

$$M_Y(t) = y\frac{e^{(t-1)y}}{t-1}\bigg|_0^\infty - \int_0^\infty \frac{e^{(t-1)y}}{t-1} dy$$

$$M_Y(t) = 0 - \left(-\frac{1}{(t-1)^2}\right) = \frac{1}{(1-t)^2}$$

Problem 4.2.4

Question

A chromosome mutation linked with colorblindness is known to occur, on average, once in every ten thousand births.

(a) Approximate the probability that exactly three of the next twenty thousand babies born will have the mutation.

(b) How many babies out of the next twenty thousand would have to be born with the mutation to convince you that the "one in ten thousand" estimate is too low?

Solution

$$\text{(a)} \quad \lambda = \frac{20,000}{10,000} = 2$$

$$P(X=3) = e^{-\lambda}\frac{\lambda^3}{3!} = e^{-2}\frac{2^3}{3!} = e^{-2}\frac{8}{6} = \frac{4}{3}e^{-2}$$

(b) Let k be the number where $P(X > k)$ indicates an underestimation.

$$P(X>5) = 1 - (P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5))$$

$$P(X=0) = e^{-2}\frac{2^0}{0!}, \quad P(X=1) = e^{-2}\frac{2^1}{1!}, \quad P(X=2) = e^{-2}\frac{2^2}{2!}$$

$$P(X=3) = \frac{4}{3}e^{-2}, \quad P(X=4) = e^{-2}\frac{2^4}{4!}, \quad P(X=5) = e^{-2}\frac{2^5}{5!}$$

$$P(X>5) = 1 - \left(e^{-2}(1+2+2+\frac{4}{3}+\frac{8}{3}+\frac{16}{15})\right)$$

Problem 4.2.6

Question

A newly formed life insurance company has underwritten term policies on 120 women between the ages of forty and forty-four. Suppose that each woman has a 1/150 probability of dying during the next calendar year, and that each death requires the company to pay out \$50,000 in benefits. Approximate the probability that the company will have to pay at least \$150,000 in benefits next year.

Solution

$$\lambda = \frac{120}{150} = 0.8 \quad \text{(expected number of deaths)}$$

$$\text{Minimum number of deaths} = \frac{150,000}{50,000} = 3$$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - (P(X = 0) + P(X = 1) + P(X = 2))$$

$$P(X = 0) = e^{-0.8}\frac{0.8^0}{0!}, \quad P(X = 1) = e^{-0.8}\frac{0.8^1}{1!}, \quad P(X = 2) = e^{-0.8}\frac{0.8^2}{2!}$$

$$P(X \geq 3) = 1 - (e^{-0.8} + 0.8e^{-0.8} + 0.32e^{-0.8}) = 1 - 2.12e^{-0.8}$$