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MATH 301GW
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March 10, 2024

Homework 6

Proposition 4.30. *For all $k, m \in \mathbb{N}$, where $m \geq 2$,*

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}.$$

Proof. **Base case:** et $k = 1$. We want to show that $f_{m+1} = f_{m-1}f_1 + f_m f_2$. Using the recursive definition of the sequence, $f_2 = f_1 + f_0$, we have:

$$\begin{aligned}
f_{m-1}f_1 + f_m f_2 &= f_{m-1}f_1 + f_m(f_1 + f_0) \\
&= f_{m-1}f_1 + f_m f_1 + f_m f_0 \\
&= (f_{m-1} + f_m)f_1 + f_m f_0 \\
&= f_{m+1}f_1 + f_m f_0 \\
&= f_{m+1}
\end{aligned}$$

Thus, the base case holds.

Inductive step: Assume the statement holds for some $k \in \mathbb{N}$

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$$

We want to show that the statement holds for $k + 1$

$$f_{m+(k+1)} = f_{m-1}f_{k+1} + f_m f_{k+2}$$

Using the recursive definition of the sequence, $f_{n+2} = f_{n+1} + f_n$, we have:

$$\begin{aligned}
f_{m+(k+1)} &= f_{(m+k)+1} \\
&= f_{m+k} + f_{(m+k)-1} \\
&= (f_{m-1}f_k + f_m f_{k+1}) + (f_{m-2}f_k + f_{m-1}f_{k+1}) \\
&= f_{m-1}(f_k + f_{k+1}) + f_m f_{k+1} + f_{m-2}f_k \\
&= f_{m-1}f_{k+2} + f_m f_{k+1} + f_{m-2}f_k \\
&= f_{m-1}f_{k+1} + (f_{m-1} + f_{m-2})f_k + f_m f_{k+1} \\
&= f_{m-1}f_{k+1} + f_m f_k + f_m f_{k+1} \\
&= f_{m-1}f_{k+1} + f_m(f_k + f_{k+1}) \\
&= f_{m-1}f_{k+1} + f_m f_{k+2}
\end{aligned}$$

Thus, the statement holds for $k + 1$. The $(m + k)$-th term of the sequence is equal to the product of the $(m - 1)$-th term and the k-th term, plus the product of the m-th term and the $(k + 1)$-th term. $\square$

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Proposition 4.31. *For all $k \in \mathbb{N}$, $f_{2k+1} = f_k^2 + f_{k+1}^2$.*

Proof. Let f_n be a sequence defined by $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, with initial values f_0 and f_1

Base case: Let $k = 1$. We want to show that $f_3 = f_1^2 + f_2^2$. Using the recursive definition of the sequence, $f_2 = f_1 + f_0$ and $f_3 = f_2 + f_1$, we have:

$$\begin{aligned}
f_1^2 + f_2^2 &= f_1^2 + (f_1 + f_0)^2 \\
&= f_1^2 + f_1^2 + 2f_1 f_0 + f_0^2 && \text{Axiom 1.1 (iii).} \\
&= 2f_1^2 + 2f_1 f_0 + f_0^2 && \text{Axiom 1.1 (i), (ii)} \\
&= (f_1 + f_0)(2f_1 + f_0) && \text{Axiom 1.1 (iii)} \\
&= f_2(f_2 + f_1) && \text{Recursive Definition} \\
&= f_2 f_3 && \text{Axiom 1.1 (iv)} \\
&= f_3
\end{aligned}$$

Thus, the base case holds.

Inductive step: Assume the statement holds for some $k \in \mathbb{N}$,

$$f_{2k+1} = f_k^2 + f_{k+1}^2$$

We want to show that the statement holds for $k + 1$,

$$f_{2(k+1)+1} = f_{k+1}^2 + f_{k+2}^2$$

Using the recursive definition of the sequence and the inductive hypothesis,

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we have:

$$
\begin{aligned}
f_{2(k+1)+1} &= f_{2k+3} \\
&= f_{2k+2} + f_{2k+1} && \text{Recursive definition} \\
&= (\mathbf{f_{2k+1}} + \mathbf{f_{2k}}) + (\mathbf{f_k^2} + \mathbf{f_{k+1}^2}) && \text{Inductive hypothesis} \\
&= \mathbf{f_{2k+1}} + \mathbf{f_{2k}} + \mathbf{f_k^2} + \mathbf{f_{k+1}^2} && \text{Axiom 1.1 (ii)} \\
&= \mathbf{f_k^2} + \mathbf{f_{k+1}^2} + \mathbf{f_{2k+1}} + \mathbf{f_{2k}} && \text{Axiom 1.1 (i)} \\
&= \mathbf{f_k^2} + \mathbf{f_{k+1}^2} + (f_{k+1} + f_k)^2 && \text{Recursive definition} \\
&= \mathbf{f_k^2} + \mathbf{f_{k+1}^2} + f_{k+1}^2 + 2f_{k+1}f_k + f_k^2 && \text{Axiom 1.1 (iii)} \\
&= \mathbf{f_k^2} + \mathbf{f_k^2} + \mathbf{f_{k+1}^2} + \mathbf{f_{k+1}^2} + 2f_{k+1}f_k && \text{Rearranging terms} \\
&= 2f_k^2 + 2f_{k+1}^2 + 2f_{k+1}f_k && \text{Combining like terms} \\
&= (f_k + f_{k+1})^2 + \mathbf{f_{k+1}^2} && \text{Axiom 1.1 (iii)} \\
&= f_{k+2}^2 + \mathbf{f_{k+1}^2} && \text{Recursive definition} \\
&= \mathbf{f_{k+1}^2} + f_{k+2}^2 && \text{Axiom 1.1 (i)}
\end{aligned}
$$

Thus, the statement holds for $k+1$. By the principle of mathematical induction, the statement holds for all $k \in \mathbb{N}$, $f_{2k+1} = f_k^2 + f_{k+1}^2$. □

Project 5.3. *Define the following sets:*

$$
\begin{aligned}
A &= \{3x : x \in \mathbb{N}\}, \\
B &= \{3x + 21 : x \in \mathbb{N}\}, \\
C &= \{x + 7 : x \in \mathbb{N}\}, \\
D &= \{3x : x \in \mathbb{N} \textit{ and } x > 7\}, \\
E &= \{x : x \in \mathbb{N}\}, \\
F &= \{3x - 21 : x \in \mathbb{N}\}, \\
G &= \{x : x \in \mathbb{N} \textit{ and } x > 7\}.
\end{aligned}
$$

Determine which of the following set equalities are true. If a statement is true, prove it. If it is false, explain why this set equality does not hold.

(i) $D = E$.

(ii) $C = G$.

(iii) $D = B$.

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Project 5.3. *Proof:*

(i) $D \neq E$
$D = 3x : x \in \mathbb{N}$ *and* $x > 7$ *and* $E = x : x \in \mathbb{N}$
The sets are not equal because D *only contains multiples of 3 greater than 21, while* E *contains all natural numbers. For example,* $1 \in E$ *but* $1 \notin D$.

(ii) $C = G$
To prove $C = G$*, we need to show that* $C \subseteq G$ *and* $G \subseteq C$.
Let $x \in C$*. Then* $x = y + 7$ *for some* $y \in \mathbb{N}$.
Since $y \in \mathbb{N}$, $y \geq 1$ *(by Proposition 2.20).*
So $x = y + 7 > 7$*, and* $x \in \mathbb{N}$ *(by closure of addition in* $\mathbb{N}$*, Axiom 2.1(i)).*
Thus, $x \in G$*. This proves* $C \subseteq G$.

Now let $x \in G$*. Then* $x \in \mathbb{N}$ *and* $x > 7$.
Let $y = x - 7$*. Since* $x > 7$, $y > 0$ *and* $y \in \mathbb{N}$ *(by Proposition 2.13).*
So $x = y + 7$ *for some* $y \in \mathbb{N}$*. Thus,* $x \in C$*. This proves* $G \subseteq C$.

Therefore, $C = G$.

(iii) $D = B$
To prove $D = B$*, we need to show that* $D \subseteq B$ *and* $B \subseteq D$.

Let $x \in D$*. Then* $x = 3y$ *for some* $y \in \mathbb{N}$ *with* $y > 7$.
Since $y > 7$, $y \geq 8$ *and* $y - 7 \in \mathbb{N}$ *(by Proposition 2.13).*
Let $z = y - 7$*. Then* $z \in \mathbb{N}$ *and* $x = 3y = 3(z + 7) = 3z + 21$.
Thus, $x \in B$*. This proves* $D \subseteq B$.

Now let $x \in B$*. Then* $x = 3y + 21$ *for some* $y \in \mathbb{N}$.
Let $z = y + 7$*. Since* $y \in \mathbb{N}$, $z > 7$ *and* $z \in \mathbb{N}$ *(by closure of addition in* $\mathbb{N}$*, Axiom 2.1(i)).*
So $x = 3y + 21 = 3(z - 7) + 21 = 3z - 21 + 21 = 3z$ *for some* $z \in \mathbb{N}$ *with* $z > 7$.
Thus, $x \in D$*. This proves* $B \subseteq D$.

Therefore, $D = B$.

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Proposition 5.4. *Let A, B, C be sets.*

(i) $A = A$.

(ii) *If* $A = B$ *then* $B = A$.

(iii) *If* $A = B$ *and* $B = C$ *then* $A = C$.

Proposition 5.4. *Proof.*

(i) $A = A$
Let $x \in A$. Then $x \in A$. This proves $A \subseteq A$.
Let $x \in A$. Then $x \in A$. This proves $A \subseteq A$.
Therefore, $A = A$.

(ii) If $A = B$ then $B = A$
Assume $A = B$. Let $x \in B$. Then $x \in A$ (since $A = B$). This proves $B \subseteq A$.
Let $x \in A$. Then $x \in B$ (since $A = B$). This proves $A \subseteq B$.
Therefore, $B = A$.

(iii) If $A = B$ and $B = C$ then $A = C$
Assume $A = B$ and $B = C$.
Let $x \in A$. Then $x \in B$ (since $A = B$). And $x \in C$ (since $B = C$).
This proves $A \subseteq C$.
Let $x \in C$. Then $x \in B$ (since $B = C$). And $x \in A$ (since $A = B$).
This proves $C \subseteq A$.
Therefore, $A = C$.

$\square$