

# MA2.101: Linear Algebra (Spring 2022)

## Exam

Wednesday, 28 March 2024

Course outcomes: CO1, CO3, CO6.

1. ([4 marks]) Solve one of the following.

- The system of equations

$$x + y + z = 6$$

$$x + 4y + 6z = 20$$

$$x + 4y + \lambda z = \phi.$$

Find the values of  $\lambda$  and  $\phi$  for which this system of equations has no solutions.

- If  $Ax = b$  always has at least one solution, show that the only solution to  $A^T y = 0$  is  $y = 0$ . Here  $A^T$  denotes the transposition of matrix  $A$ .
2. ([3 marks])  $V$  is a finite-dimensional vector space and let  $T : V \rightarrow V$  be a linear operator on  $V$ . Suppose that  $\text{rank}(T^2) = \text{rank}(T)$ . Prove that the range and nullspace of  $T$  have only the zero vector  $0$  in common.
3. ([4 marks]) Two vector spaces are called *isomorphic* if there exists an invertible linear transformation from one vector space onto the other one. Prove that two finite-dimensional vector spaces over  $\mathbb{F}$  are isomorphic if and only if they have the same dimension.
4. ([4 marks]) Solve one of the following.
- (a) Prove both of the following statements.
- The image or the range of a linear transformation  $T : V \rightarrow W$  is a subspace of  $W$ .
  - A linear transformation  $T : V \rightarrow W$  is one-to-one if and only if the nullspace of  $T$  only contains  $0 \in V$ .

(b) Consider the ordered bases  $\mathcal{A} = \{(1, 2), (-2, -3)\}$  and  $\mathcal{B} = \{(2, 1), (1, 3)\}$  for a vector space  $V$ . Then find the following

- Matrix  $P$  that changes coordinates of any vector  $\vec{\alpha} \in V$  w.r.t. the ordered basis  $\mathcal{A}$  to coordinates w.r.t. the ordered basis  $\mathcal{B}$ .
- Matrix  $Q$  that changes coordinates of any vector  $\vec{\alpha} \in V$  w.r.t. the ordered basis  $\mathcal{B}$  to coordinates w.r.t. the ordered basis  $\mathcal{A}$ .