1) a) For all integers n, if n^2 is not divisible by 7, then n is not divisible by 7.

Proof:

Contraposition: For all integers n, if n is divisible by 7, n^2 is divisible by 7.

n is divisible by 7

$\Rightarrow n = 7K$ (K is an integer)

$\Rightarrow n^2 = (7K)^2$ (squaring on both sides)

$\Rightarrow n^2 = 49K^2$ (simplifying)

$\Rightarrow n^2 = 7(7K^2)$ (factoring 7)

$\Rightarrow n^2 = 7j$ (let $j = 7K^2$)

(j is an integer as mult. is closed on integers)

$\Rightarrow n^2$ is divisible by 7

Hence, the proposition is true.

1 b) If m and n are positive integers such that $mn = 100$, then $m \le 10$ or $n \le 10$.

Proof:

Contraposition: If $m > 10$ and $n > 10$, then $mn \neq 100$

Suppose $m > 10$ and $n > 10$

$\Rightarrow mn > 100$ (multiply the two inequalities)

$\Rightarrow mn \neq 100$

Hence, the proposition is true.

1c) If x is a real number such that $0 < x < 1$, then $x > x^2$.

Proof:

Contraposition: If $x^2 \geq x$, for any real number x, $x \leq 0$ or $x \geq 1$.

Suppose $x^2 \geq x$

$\Rightarrow \quad x^2 - x \geq 0$ (Subtracting x from both sides)

$\Rightarrow \quad x(x-1) \geq 0$ (Factoring x)

$\Rightarrow \quad x \geq 1$ or $x \leq 0$ (Solving the inequality)

Hence, the proposition is true.

2) 1) Let $x, y \in \mathbb{R}$. Prove that $|x+y| = |x| + |y|$ if and only if $xy \geq 0$.

Proof:

Part 1:

Statement: $|x+y| = |x| + |y| \Rightarrow xy \geq 0$

Suppose $|x+y| = |x| + |y|$

$\Rightarrow |x+y|^2 = (|x| + |y|)^2$ (Squaring)

$\Rightarrow (x+y)^2 = x^2 + y^2 + 2|x||y|$ (Expanding)

$\Rightarrow x^2 + y^2 + 2xy = x^2 + y^2 + 2|x||y|$ (Expanding)

$\Rightarrow 2xy = 2|x||y|$ (Subtracting x^2 and y^2)

$\Rightarrow xy = |x||y|$ (Dividing by 2)

As $|x| \geq 0$ and $|y| \geq 0$

$\Rightarrow |x||y| \geq 0$

$\Rightarrow xy \geq 0$ (LHS = RHS)

Part 2:

Statement: $xy \geq 0 \Rightarrow |x+y| = |x| + |y|$

If $xy \geq 0$, there are two cases:

~~1) x is x,y ∈ R~~

1) $x \geq 0$, $y \geq 0$

2) $x < 0$, $y < 0$

<u>Case 1</u>: $x \geq 0, y \geq 0$

Suppose $x \geq 0, y \geq 0$

$\Rightarrow |x| = x$ and $|y| = y$

Substituting in $|x+y| = |x| + |y|$,

$\Rightarrow |x+y| = x+y$

$\Rightarrow x+y = x+y$ $\quad$ ($x+y \geq 0 \Rightarrow |x+y| = x+y$)

<u>Case 2</u>: $x < 0, y < 0$

Suppose $x < 0, y \leq 0$

$\Rightarrow |x| = -x$ and $|y| = -y$

Furthermore, $|x+y| = -(x+y)$ (As $x+y < 0$)

Substituting in $|x+y| = |x| + |y|$:

$\Rightarrow -(x+y) = -x-y$

$\Rightarrow -x-y = -x-y$ $\quad$ (Distributing negative sign)

Thus, if $xy \geq 0$, $|x+y| = |x| + |y|$

Hence, $|x+y| = |x| + |y|$ if and only if $xy \geq 0$

2) 2) For all positive integers m and n, ~~[illegible]~~ $m \mid n$ and $n \mid m$ iff and only iff $m = n$.

Proof:

Part 1:

Statement: ~~[illegible]~~ $\forall m, n \in \mathbb{Z}^+$, $m \mid n$ and $n \mid m \Rightarrow m = n$.

Suppose $m \mid n$

$\Rightarrow m = an$ — ① (for some $a \in \mathbb{Z}^+$)

Suppose $n \mid m$

$\Rightarrow n = bm$ — ② (for some $b \in \mathbb{Z}^+$)

Substituting ② in ①:

$\Rightarrow m = a(bm)$

$\Rightarrow 1 = ab$ (dividing by m)

As $a, b \in \mathbb{Z}^+$, $a = 1$ and $b = 1$

Substituting into ①:

$m = (1)n$

$\Rightarrow m = n$

Part 2

Statement: $\forall m, n \in \mathbb{Z}^+$, $m = n \Rightarrow m \mid n$ and $n \mid m$

Suppose $m = n$

$\Rightarrow m = (1)(n)$

$\Rightarrow m = an$ (For $a = 1$)

$\Rightarrow m \mid n$

Also,

$n = (1)(m)$

$\Rightarrow n = bm$ (For $b = 1$)

$\Rightarrow n \mid m$

Hence, $\forall m, n \in \mathbb{Z}^+$, $m \mid n$ and $n \mid m$ if and only if $m = n$.

3)i) Prove that the set

$$A = \left\{ \frac{n-1}{n} : n \in \mathbb{Z}^+ \right\}$$

does not have a largest element

Proof:

Suppose for ~~an~~ a positive integer a,

$$\frac{a-1}{a} > \frac{n-1}{n}$$

for every integer n (Assume n is positive)

Let $b = a+1$,
b is a positive integer since addition is closed on integers

$$\Rightarrow \frac{b-1}{b} \in A$$

$$\Rightarrow \frac{(a+1)-1}{a+1} \in A \qquad \text{(Substituting } b = a+1\text{)}$$

Comparing $\frac{(a+1)-1}{a+1}$ and $\frac{a-1}{a}$,

$$\frac{(a+1)-1}{a+1} = \frac{a}{a+1} \qquad \text{(Simplifying)}$$

$$\Rightarrow 1 - \frac{1}{a+1}$$

Meanwhile,

$$\frac{a-1}{a} = 1 - \frac{1}{a} \quad \text{(Simplifying)}$$

As $a+1 > a$, $\frac{1}{a+1} < \frac{1}{a}$ (Taking multiplicative inverse)

$$\Rightarrow 1 - \frac{1}{a+1} > 1 - \frac{1}{a}$$

$$\Rightarrow \frac{b-1}{b} > \frac{a-1}{a}$$

So, $\frac{a-1}{a}$ is not the greatest element of set A

Contradiction! Hence, $A = \left\{ \frac{n-1}{n} ; n \in \mathbb{Z}^+ \right\}$

does not have a greatest element.

3)b) If the mean of four distinct numbers is $n \in \mathbb{Z}$, then at least one of the integers is greater than $n+1$.

Proof:

Average $= A = \frac{a_1 + a_2 + a_3 + a_4}{4}$

Negation: $a_1, a_2, a_3, a_4 \leq n+1$

Suppose $a_1, a_2, a_3, a_4 \leq n+1$.

As the four numbers are distinct integers, only one can be equal to $n+1$

$\Rightarrow a_1 \leq n+1$; $a_2, a_3, a_4 < n+1$

Supposing the greatest set of integers satisfying these conditions: $a_1 = n+1$, $a_2 = n$, $a_3 = n-1$, $a_4 = n-2$

$$\Rightarrow n = \frac{(n+1)+(n)+(n-1)+(n-2)}{4}$$

$$\Rightarrow n = \frac{4n-2}{4}$$

$$\Rightarrow 4n = 4n-2$$

Contradiction! Hence, at least one of a_1, a_2, a_3, a_4 is greater than $n+1$.

3) 3) Prove that there is no rational number r such that $2^r = 3$

Proof:

Negation: Suppose there is a rational number r such that $2^r = 3$

r is rational

$\Rightarrow$ r can be expressed as a ratio of integers with a non-zero denominator

$\Rightarrow$ $r = \frac{p}{q}$, where p and q are ~~no~~ integers, $q \neq 0$

$\Rightarrow$ $2^{p/q} = 3$ (Substituting $r = \frac{p}{q}$)

$\Rightarrow$ $\sqrt[q]{2^p} = 3$ (Simplifying)

$\Rightarrow$ $2^p = 3^q$ (Multiplying both sides to q power)

~~Powers of 2~~ Integral powers of 2 are even while integral powers of 3 are odd as integral powers of even numbers are even and integral powers of odd numbers are odd.

$\Rightarrow 2^p \neq 3^q$

Contradiction: Hence, there is no rational number such that $2^r = 3$

4) 1) A conference is being attended by 367 people. Prove that there exists at least two people born with same date of birth.

Proof:

Suppose each individual at the conference has a separate birthday.

Including leap year day, there are a maximum of 366 days in a year. Therefore, only 366 separate birthdays may exist.

Contradiction! Therefore, at least two people share the same date of birth as the 367^{th} person must share their birthday with another individual.

This can also be thought of as a version of the pigeonhole problem, insofar as there are only 366 possible birthdays and 367 individuals. As $367 > 366$, at least two people must share a birthday.

4)2) Let $\{b_1, b_2, \ldots, b_n\}$ be a set of integers such that $\sum b_k^2 < n$. Prove that at least one of the integers in the set is zero.

Proof:

Assume all integers in set $\{b_1, b_2, \ldots, b_n\} \neq 0$.

Thus, the lowest possible value of the absolute value of an integer in the set $\{b_1, b_2, \ldots, b_n\}$ is 1.

Assuming $b_1 = \pm 1,\ b_2 = \pm 1, \ldots, b_n = \pm 1$

$$\Rightarrow b_1^2 = 1,\ b_2^2 = 1, \ldots, b_n^2 = 1 \quad \text{(squaring)}$$

$$\Rightarrow \sum b_n^2 = 1 + 1 + \ldots + 1 \quad \text{(Summation of } b_1^2, b_2^2, \ldots, b_n^2\text{)}$$

$$= n$$

Contradiction! $\sum b_n^2$ cannot be greater than n and also equal to n.

Hence, at least one of the integers in the set has to be zero.