ASSIGNMENT 7

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Q1) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite-dimensional. Then prove that:

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Source: Hoffman and Kunze

A: Recall that in a finite-dimensional vector space, the rank of a linear transformation can be defined as the dimension of the range of that linear transformation.

Further recall that the nullity of a linear transformation refers to the ~~nullity~~ dimension of the null space of that transformation.

Let $\{\alpha_1, \dots, \alpha_k\}$ be a basis for the null space of a linear transformation T, denoted N. Recall that the null space of a linear transformation is the set of all vectors α in V such that $T\alpha = 0$.

Further, let there be vectors $\{\alpha_{k+1}, \dots, \alpha_n\}$ in V, which form the basis for V.

To prove that $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ is a basis for the range of T, we must prove the two inherent properties of a basis:

i) the set of vectors must span the vector space V

ii) the set of vectors must be linearly independent

For ⓐ, it is obvious that $\{T\alpha_1, \dots, T\alpha_n\}$ span the range of T. Further, since $T\alpha_x = 0$, $x \le k$, $\{T\alpha_{k+1}, \dots, T_n\}$ must also span the range.

For ⓑ, suppose there exist scalars

$$\sum_{i=k+1}^{n} c_i(T\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

Accordingly, vector $\alpha = \sum_{i=k+1}^{n} c_i\alpha_i$ is in the null space of T. Thus, the vectors are also linearly independent, and therefore, $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ is the basis for the range of T.

Since $\{\alpha_1, \dots, \alpha_k\}$ form a basis for N, there must exist scalars $b_1, \dots, b_k$ such that

$$a = \sum_{i=1}^{k} b_i a_i.$$

Therefore,

$$\sum_{i=1}^{k} b_i a_i - \sum_{j=k+1}^{n} c_j a_j = 0$$

Since every vector $\alpha_x \in \{\alpha_1, \dots, \alpha_n\}$ is linearly independent, it can be claimed

$$b_1 = \dots = b_k = c_{k+1} = \dots = c_n = 0.$$

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Defining r to be the rank of T, and having defined n as the dimension of V and k as the nullity of T, we know that $r = n - k$, or

$$\cancel{\text{rank}(T) + \text{nullity}}$$

$$\text{rank}(T) = \text{nullity}(T) + \dim V$$

$$\Rightarrow \text{rank}(T) - \text{nullity}(T) = \dim V$$

Hence proved.

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62) Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vectorspace over F. Then the space $L(V,W)$ is finite-dimensional and has dimension mn. Formally, prove that:

$$\dim L(V,W) = \dim(V) \times \dim(W)$$

Source: Hoffman and Kunze

A: The space of linear transformations from V into W is denoted as $L(V,W)$.

Let $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_n\}$ be ordered bases for V and W, respectively. For each pair of integers, $1 \le x \le m$ and $1 \le y \le n$, let a linear transformation $E^{x,y}$ from V into W be defined by:

$$E^{x,y}(\alpha_i) = \begin{cases} 0, & \text{if } i \neq y \\ \beta_x, & \text{if } i = y \end{cases}$$
$$= \delta_{iy}\beta_x$$

We know that for a finite-dimensional vector space, V, and another finite dimensional vector space, W (both over field F), letting $\{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V and $\{\beta_1, \dots, \beta_n\}$ be an ordered basis for W, there is precisely one linear transformation T from V into W such that

$$T\alpha_j = \beta_j \quad , j = 1, \dots, n$$

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Therefore, we know there is a linear transformation from V into W such that $E^{x,y}(\alpha_i) = \delta_{iy} \beta_x$.

To prove that mn transformations $E^{x,y}$ form a basis for $L(V, W)$, let T be a linear transformation from V into W such that for each j, $1 \le j \le n$, let $A_{1j}, \dots, A_{mj}$ be the coordinates of the vector $T\alpha_j$ in the ordered basis B

$$\text{ie, } T\alpha_j = \sum_{p=1}^{m} A_{xj} \beta_x$$

and to complete the proof, it must be shown that

$$T = \sum_{x=1}^{m} \sum_{y=1}^{n} A_{xy} E^{x,y} \quad - (1)$$

Let U be the ~~right hand~~ linear transformation in the right-hand member of (1).

Then, for each j,

$$U\alpha_j = \sum_x \sum_y A_{xy} E^{x,y}(\alpha_j)$$

$$= \sum_x \sum_y A_{x,y} \delta_{jy} \beta_x$$

$$= \sum_{x=1}^{m} A_{xj} \beta_x$$

$$= T\alpha_j$$

Thus, consequently, $U = T$.

(i) shows that $E^{x,y}$ span $L(V,W)$. To prove that they are independent, we can simply refer to the above transformation. If the transformation

$$U = \sum_{x} \sum_{y} A_{xy} E^{x,y}$$

is the zero transformation, then $U\alpha_j = 0$ for each j, so

$$\sum_{x=1}^{m} A_{xj} \beta_x = 0$$

and the independence of β_x implies $A_{xj} = 0$ for every p and j.

Thus, the space $L(V,W)$ is finite-dimensional with dimensions mn

Hence proved.