1) Given a set of $n+1$ distinct ~~elements~~ numbers for $n > 1$, using pigeon hole principle, prove that there exists a pair of elements whose difference is divisible by n.

Let all distinct numbers be of the form $an+b$, $a, b \in N$, $0 \le b < n$.

Given the range of elements b can occupy, consider it as the number of pigeons. The maximum number of elements / pigeons is n. However, we are given there are $n+1$ elements/pigeons.

Via the pigeonhole principle, there exists at least two elements for which $b = b$ as there $n+1$ "holes" and ~~is~~ n "pigeons."

Let these two elements be represented as $a_1 n + b$ and $a_2 n + b$.

The difference of these two elements can be represented as:

$$
\begin{aligned}
D &= (a_1 n + b) - (a_2 n + b) \\
&= a_1 n - a_2 n \\
&= n(a_1 - a_2) \\
&= n p \qquad [\text{where } p \in N]
\end{aligned}
$$

$\Rightarrow$ There exist two elements whose difference is divisible by n.

Hence proved.

2. Given 51 points in a square of size 1 square unit, show that there exists a circl of radius 1/7 that would contain 3 of these points.

Let us divide the square of 1 unit into a 5 × 5 grid with 25 smaller squares of side length 0.2 units each

There are 51 points which need to be accommodated in 25 squares. Essentially, in terms of pigeon hole principle, there are 51 pigeons for 25 holes.

∴ At least one square contains 3 points

The radius of the ~~square~~ circle which circumscribes a smaller square (of 0.2 units) is given by:

$$\sqrt{\frac{1}{8^2}+\frac{1}{8^2}}$$

$$= \frac{\sqrt{2}}{16}$$

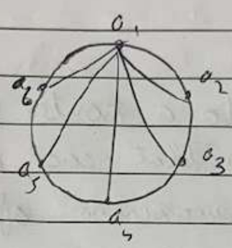
Given $\frac{\sqrt{2}}{16} < \frac{1}{7}$,

there exists a circle which has 3 holes

Hence proved.

3. Given a room of 6 people, show that there must be either 3 people who know each other, or 3 people who do not know each other.

Given 6 people in the room, focus on a person a_1. Assume this person can either know or not know the other 5 people in the room.



Assume a_1 knows at least 3 people in the room

Focusing on these 3 people, they can either know each other or not know each other. Thus, we are able to form 2 distinct cases.

Case 1: 3 people known by a_1 know each other

In this case, we have proved that 3 people know each other.

Case 2: 3 people ~~to~~ known by a_1 do not know each other

In this case, we have proved that 3 people do not know each other

In either case, ~~it has been proven that~~ either 3 people know each other or 3 people do not know each other in the group of 6. Hence proved.

9) What is the least number of computers required to connect 10 computers to 5 routers, in order to guarantee that 5 computers can directly access 5 routers

Connecting a computer to a router requires 1 wire. Direct connection means that each computer must be connected with a maximum of 1 wire with no intermediary nodes

$\Rightarrow$ for 1 computer to be directly connected to 5 routers, it takes 5 wires

$\Rightarrow$ for 5 computers, 25 wires will be required.

As there are 10 computers in total, the 5 other computers need to be connected to at least 1 router each, requiring 5 additional wires.

Therefore, the minimum amount of wires required to connect 10 computers to 5 routers in order to guarantee that 5 computers can directly access 5 routers is $25 + 5 = \boxed{30 \text{ wires.}}$

5) Show that for every integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion.

Assume $n+1$ integers as each a series of 1s such that the set of those integers is given as:

$$A = \{1, 11, \ldots, \underbrace{11\ldots1}_{n-2\text{ times; collectively } n+1 \text{ 1s}}\}$$

Expressing each element of set A in the form $a_i n + b_i$, for $a_i, b_i \in W$, $a_i \in N$, $i \in [1, n+1]$, $b \in [0, n]$, ~~we are b~~

Applying the pigeon hole principle for b_i as ~~pigeons~~ holes and the number of numbers as the number of ~~holes~~ pigeons, we realize that there ~~ar~~ must be at least 2 elements with the same b_i for some $i = p$ and $i = q$.

These numbers can be represented as $x = a_p n + r$ and $y = a_q n + r$

Expressing the difference of these terms,

$$x - y = (a_p n + r) - (a_q n + r)$$
$$= a_p n - a_q n$$
$$= n(a_p - a_q)$$

As both ~~a_p and a_q +~~ x and y are a series of 1s, their difference is also a string of 1s and 0s.

Hence proved.

6. Show that among any $n+1$ positive integers not exceeding $2n$, there must be an integer that divides one of the other integers.

Let $A = \{a_1, a_2, \ldots, a_3\}$ represent the set of $n+1$ positive integers such that ~~for~~ $\forall a \in A$, $a \in [1, 2n]$.

When any integer in A is divided by n, there are n possible remainders i.e. $0, 1, \ldots, n-1$.

~~If~~ If any integer, when divided by n, yields a remainder of 0, then it can be said that number is divisible by n.

If the remainder is not 0, the possible remainders are $1, 2, 3, \ldots, n-1$

$\Rightarrow$ There are n possible remainders.

Assume the number of remainders to be the number of holes ~~pigeons~~ and the ~~most~~ number of elements in A to be the number of ~~holes~~ pigeons.

Hence, there are n remainders and $n+1$ elements. Therefore, at least 2 integers will have the same remainder when divided by n

$$\text{i.e. } a_p \equiv r \pmod{n} \quad a_q = r \pmod{n}$$

$$a_p - a_q \equiv 0 \pmod{n}$$

Hence, $a_p - a_q$ is divisible by n.

Therefore, there must be at least 1 integer $(a_p - a_q)$ in set A, which divides another integer a_p or a_q.

Hence proved.

7) Let A be a non empty set. Show that the following are equivalent:

a) A is countable
b) There exists a surjection $f: N \to A$
c) There exists an injection $g: A \to N$

To prove the above, we must prove:

i) $a \Rightarrow b$
ii) $b \Rightarrow c$
iii) $c \Rightarrow a$

Part 1: $a \Rightarrow b$

For a set to be countable, it must be one-one (injective) and onto (surjective.

A bijective function refers to a function which is both injective and surjective.

Therefore, by definition, if set A is countable, it is bijective, and thus, ~~& the~~ there must exist a surjection $f: N \to A$, hence proved.

Part 2: $b \Rightarrow c$

Since there exists a surjection from $f N \to A$, we know for every element in A, there exists a pre-image in N.

To form an injection from $A \to N$, we simply need to form a relation from ~~one~~ any subset of A to any subset of N. Thus the question only demands an injection.

As we know, every element from N ~~be~~ matches to an element in A, therefore, we can create a simple injection i.e. $\{(a_1, 1), (a_2, 2)\}$.

Thus, there exists an injection $g: A \to N$. Hence proved

Part 3 : $c \Rightarrow a$

If there exists an injection $g: A \to N$, we know that ~~two~~ any element in A can only map to a maximum of 1 element in N.

Let $B \subset N$, $B = \{g(a); a \in A\}$.
Let $h(x): A \to N$.
Mapping each element of A to $g(a) \in N$, we check for bijectivity of h

For injectivity,
As g is injective, h is injective

For surjectivity,
g is in B by definition if and only if there exists an a in A such that $h(a) = y$

Thus, h is bijective
~~$\Rightarrow$ h is countable.~~
~~$\Rightarrow c \Rightarrow a$. Hen~~ $\Rightarrow$ A is countable, hence proved.

As $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow a$, all three statements are equivalent.

8) Prove that the set of all positive rational numbers is countable.

The set of all positive integers $\mathbb{N}$ has the same cardinality as the set of all positive integers ($\because$ cardinality is reflexive). Thus, the set of all positive integers is countably infinite.

~~Cre~~

Create a grid where $\mathbb{N}$ occupies the first row and first column. The numbers on the first row represent the ~~denomin~~ denominator while the first column numbers represent the numerator, as follows:

	1	2	3	4	5
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\cancel{\frac{5}{5}}$

Define a function F from Z^+ to Q^+ by starting to count at $\frac{1}{1}$ and following the arrows such that ~~numbers~~ numbers which have already been counted are skipped.

i.e. $F(1) = \frac{1}{1}$

$F(2) = \frac{1}{2}$

$F(3) = \frac{2}{1}$

$F(4) = \frac{3}{1}$

$\frac{2}{2}$ is skipped as it simplifies to $\frac{1}{1}$

$\Rightarrow F(5) = \frac{1}{3}$

$\vdots$

Continuing in this way, $F(n)$ is defined for every integer n.

Every positive rational number appears somewhere in the grid and the counting procedure ensures every point in the grid is reached eventually
$\Rightarrow$ F is onto.

Skipping of numbers ensures no number is counted twice.
$\Rightarrow$ F is one-one.

Consequently, F is a function from Z^+ to Q^+ that is one-to-one and onto, and so Q^+ is countably infinite and hence countable.

9) If A and B are countable sets, then show that $A \times B$ is countable.

Assume set A contains n elements such that
$A = \{a_1, a_2, a_3, \dots, a_n\}$

Assume set B contains m elements such that
$B = \{b_1, b_2, b_3, \dots, b_m\}$

Create a grid with set A as the first row and set B as the first column. The constituents of the grid will be ordered pairs (a_x, b_y), where x refers to x^{th} column and y refers to the y^{th} row.

i.e.

$$\begin{array}{ccccc}
 & a_1 & a_2 & a_3 & \cdots \quad a_n \\
b_1 & (a_1,b_1) & (a_2,b_1) \rightarrow & (a_3,b_1) & \\
b_2 & (a_1,b_2) & (a_2,b_2) & (a_3,b_2) & \\
b_3 & (a_1,b_3) \rightarrow & (a_2,b_3) & (a_3,b_3) & \\
\vdots & & & & \\
b_n & & & &
\end{array}$$

Define a function F from the grid $A \times B$ to Z^+ (N) by starting at (a_1, b_1), and following arrows such that every element of the grid is included.

i.e. $F(1) = (a_1, b_1)$

$F(2) = (a_1, b_2)$

$F(3) = (a_2, b_1)$

⋮

Counting in this manner, $F(n)$ is defined for every positive integer n:

$A \times B$	n
$(a_1, b_1) \to$	1
$(a_1, b_2) \to$	2
$(a_2, b_1) \to$	3
$(a_2, b_2) \to$	4
$(a_1, b_3) \to$	5

Every possible ordered pair present in $A \times B$ is present somewhere in the grid and the counting procedure ensures every point in the grid is reached eventually

$\Rightarrow$ ~~$A \times B$ is onto~~ F is onto

~~$\Rightarrow A \times B$ is onto~~

The counting procedure ensures no number is repeated

$\Rightarrow$ F is one-one

~~$\Rightarrow A \times B$ is~~

$\Rightarrow$ F is ~~countably infinite, and hence~~ countable

$\Rightarrow$ $A \times B$ is ~~countably infinite~~ countable

10) Prove or disprove (for $f: X \to Y$ and $A, B \subseteq X$)

a) $f(A \cup B) = f(A) \cup f(B)$

<u>To prove</u> $f(A \cup B) = f(A) \cup f(B)$, we must prove two <u>statements</u>:

i) $f(A \cup B) \subseteq f(A) \cup f(B)$

ii) $f(A) \cup f(B) \subseteq f(A \cup B)$

Part 1: prove $f(A \cup B) \subseteq f(A) \cup f(B)$

~~Assume $f(x) = y$~~

~~Assume there exists an element y such that $f(x) = y$ and $y \in f(A, B)$. Thus, there exists an element $x \in (A, B)$~~

Assume $y = f(x)$ and $y \in f(A \cup B)$

$\Rightarrow$ there exists an element $x \in (A \cup B)$

$\Rightarrow$ $x \in A$ or $x \in B$

Case 1: $x \in A$

$\Rightarrow f(x) \in f(A)$

$\Rightarrow f(x) \in f(A) \cup f(B)$ $\quad [\because f(x) \in f(A)]$

Case 2: $x \in B$

$\Rightarrow f(x) \in f(B)$

$\Rightarrow f(x) \in f(A) \cup f(B)$ $\quad [\because f(x) \in f(B)]$

In both cases, $f(x) \in f(A) \cup f(B)$

$\Rightarrow f(A \cup B) \subseteq f(A) \cup f(B)$

Part 2: prove $f(A) \cup f(B) \subseteq f(A \cup B)$

Assume $y = f(x)$ such that $y \in f(A) \cup f(B)$

$\Rightarrow \quad y \in f(A) \quad$ or $\quad y \in f(B)$

$\Rightarrow \quad x \in A \quad$ or $\quad x \in B$

Case 1: $x \in A$

$\Rightarrow x \in A \cup B \qquad (\because x \in A)$

$\Rightarrow f(x) \in f(A \cup B)$

Case 2: $x \in B$

$\Rightarrow x \in A \cup B \qquad (\because x \in B)$

$\Rightarrow f(x) \in f(A \cup B)$

In both cases, $f(x) \in f(A \cup B)$

$\Rightarrow \cancel{f(A \cup B) \subseteq f}\ f(A) \cup f(B) \subseteq f(A \cup B)$

$\Rightarrow f(A \cup B) = f(A) \cup f(B)$,

Hence proved

10b) $f(A \cap B) \subseteq f(A) \cap f(B)$

Assume $y = f(x)$ and ~~f~~ $y \in f(A \cap B)$

$\Rightarrow f(x) \in f(A \cap B)$
$\Rightarrow x \in A \cap B$
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow f(x) \in f(A)$ and $f(x) \in f(B)$
$\Rightarrow f(x) \in f(A) \cap f(B)$

[$\because$ If $x \in A$ and $x \in B$, $x \in A \cap B$]

$\Rightarrow y \in f(A) \cap f(B)$

Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$

10c) $f(A - B) \subseteq f(A) - f(B)$, if f is injective

Assume $y \in f(A-B)$ and $y = f(x)$

$\Rightarrow f(x) \in f(A-B)$
$\Rightarrow x \in A - B$
$\Rightarrow x \in A \cap B'$ [$\because (A-B) = (A \cap B')$]
$\Rightarrow x \in A$ and $x \in B'$

The symmetric difference $f(A)-f(B)$ represents all elements of $f(A)$ which are not in $f(B)$. Thus, for ~~x to be~~ $f(x) \in f(A) - f(B')$, two conditions must be satisfied:

i) $f(x) \in f(A)$

ii) $f(x) \notin f(B)$

As $x \in A$

$\Rightarrow f(x) \in f(A)$

$\Rightarrow$ condition (i) is satisfied

As $x \in B'$

$\Rightarrow x \notin B$

$\Rightarrow f(x) \notin f(B)$

$\Rightarrow$ condition (ii) is satisfied

$\Rightarrow f(x) \in f(A) - f(B)$

$\Rightarrow f(A - B) \subseteq f(A) - f(B)$

Hence proved

11) For $f: X \to Y$, let $S \subseteq Y$. Define $f^{-1}(S) = x \in X \mid f(x) \in S$.
Let $A, B \subseteq Y$. Prove (or disprove):

a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Assume $\cancel{f^{-1}(A \cup B) = g}$ & $f^{-1}(x) = g(x)$

$\Rightarrow f^{-1}(A \cup B) = g(A \cup B)$

To prove $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, we need to prove:

i) $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

ii) $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Part 1: prove $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

Assume $y = g(x)$ ~~and g~~ and $y \in f^{-1}(A \cup B)$

$\cancel{\Rightarrow y = f^{-1}(A \cup B)}$

$\cancel{\Rightarrow y = f^{-1}(x)}$

$\cancel{\Rightarrow y \in f^{-1}(A \cup B)}$ $\cancel{[\because g(x) = f^{-1}(x)]}$

$\Rightarrow g(x) \in f^{-1}(A \cup B)$

$\Rightarrow x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$

Case 1: $x \in A$

$\Rightarrow g(x) \in g(A)$

$\Rightarrow g(x) \in f^{-1}(A)$

$\Rightarrow g(x) \in f^{-1}(A) \cup f^{-1}(B)$ $[\because g(x) \in f^{-1}(A)]$

Case 2: $x \in B$

$\Rightarrow g(x) \in g(B)$

$\Rightarrow g(x) \in f^{-1}(B)$

$\Rightarrow g(x) \in f^{-1}(A) \cup f^{-1}(B)$ $[\because g(x) \in f^{-1}(B)]$

~~In both ca~~

In both cases, $g(x) \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

Part 2: prove $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Assume $y = g(x)$ and $y \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow g(x) \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow \cancel{x}\, g(x) \in f^{-1}(A)$ or $g(x) \in f^{-1}(B)$

Case 1: $g(x) \in f^{-1}(A)$

$\Rightarrow x \in A$

$\Rightarrow x \in A \cup B$ $\quad [\because x \in A]$

$\Rightarrow g(x) \in g(A \cup B)$

$\Rightarrow f^{-1}(x) \in f^{-1}(A \cup B)$ $\quad [\because g(x) = f^{-1}(x)]$

Case 2: $g(x) \in f^{-1}(B)$

$\Rightarrow x \in B$

$\Rightarrow x \in A \cup B$ $\quad [\because x \in B]$

$\Rightarrow g(x) \in g(A \cup B)$

$\Rightarrow f^{-1}(x) \in f^{-1}(A \cup B)$ $\quad [\because g(x) = f^{-1}(x)]$

In both cases, $f^{-1}(x) \in f^{-1}(A \cup B)$

~~$\Rightarrow f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$~~

$\Rightarrow f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

$\Rightarrow f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

Hence proved

1 b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

To prove $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, we need to prove:

i) $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$

ii) $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$

~~Part 1: $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$~~

Assume $g(x) = f^{-1}(x)$

$\Rightarrow f^{-1}(A \cup B) = g(A \cup B)$

Part 1: $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$

Assume $y = g(x)$ and $y \in f^{-1}(A \cap B)$

$\Rightarrow g(x) \in f^{-1}(A \cap B)$

$\Rightarrow x \in A \cap B$ $\quad [\because g(x) = f^{-1}(x)]$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow g(x) \in g(A)$ and $g(x) \in g(B)$

$\Rightarrow g(x) \in g(A) \cap g(B)$

$[\because$ if $x \in A$ and $x \in B$, $x \in A \cap B]$

$\Rightarrow f^{-1}(x) \in f^{-1}(A) \cap f^{-1}(B)$

$\Rightarrow f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$

Part 2: $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$

Assume $y = g(x)$ and $y \in f^{-1}(A) \cap f^{-1}(B)$

$\Rightarrow \quad g(x) \in f^{-1}(A) \cap f^{-1}(B)$

$\Rightarrow \quad g(x) \in f^{-1}(A)$ and $g(x) \in f^{-1}(B)$

$\Rightarrow \quad x \in A$ and $x \in B$ $[\because g(x) = f^{-1}(x)]$

$\Rightarrow \quad x \in A \cap B$ $[\because$ if $x \in A$ and $x \in B$, $x \in A \cap B]$

$\Rightarrow \quad g(x) \in g(A \cap B)$

$\Rightarrow \quad f^{-1}(x) \in f^{-1}(A \cap B)$

$\Rightarrow \quad f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$

$\Rightarrow \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Hence proved

11c) $f^{-1}(A-B) = f^{-1}(A) \cdot f^{-1}(B)$

Assume $g(x) = f^{-1}(x)$
$\Rightarrow g(A-B) = f^{-1}(A-B)$

To prove $f^{-1}(A-B) = f^{-1}(A) - f^{-1}(B)$, we must prove:
i) $f^{-1}(A-B) \subseteq f^{-1}(A) - f^{-1}(B)$
ii) $f^{-1}(A) - f^{-1}(B) \subseteq f^{-1}(A-B)$

Part 1: prove $f^{-1}(A-B) \subseteq f^{-1}(A) - f^{-1}(B)$

Assume $y = g(x)$ and $y \in f^{-1}(A-B)$
$\Rightarrow g(x) \in f^{-1}(A-B)$
$\Rightarrow x \in A-B$ $\quad$ $(\because f^{-1}(x) = g(x))$
$\Rightarrow x \in A \cap B'$ $\quad$ $(\because (A-B) = (A \cap B'))$
$\Rightarrow x \in A$ and $x \in B'$
$\Rightarrow x \in A$ and $x \notin B$
$\Rightarrow g(x) \in g(A)$ and $g(x) \notin g(B)$
$\Rightarrow f^{-1}(x) \in f^{-1}(A)$ and $f^{-1}(x) \notin f^{-1}(B)$

$f^{-1}(A) - f^{-1}(B) = g(A) - g(B)$

For an element ~~x to be~~ $g(a) \in g(A) - g(B)$, it must be in A and not in B.

$g(x) \in g(A)$ and $g(x) \notin g(B)$
$\Rightarrow g(x) \in g(A) - g(B)$
$\Rightarrow f^{-1}(x) \in f^{-1}(A) \cdot f^{-1}(B)$

$\Rightarrow f^{-1}(A-B) \subseteq f^{-1}(A) - f^{-1}(B)$

Part 2: $f^{-1}(A) - f^{-1}(B) \subseteq f^{-1}(A-B)$

Assume $y = g(x)$ and $y \in f^{-1}(A) - f^{-1}(B)$

$\Rightarrow g(x) \in f^{-1}(A) - f^{-1}(B)$

$\cancel{\Rightarrow g(x) \in}$

$\Rightarrow g(x) \in f^{-1}(A)$ and $g(x) \notin f^{-1}(B)$

$\Rightarrow g(x) \in g(A)$ and $g(x) \notin g(B)$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow f^{-1}(x) \in f^{-1}(A)$ and $f^{-1}(x) \notin f^{-1}(B)$

For an element $a \in f^{-1}(A) - f^{-1}(B)$, $f^{-1}(a) \in f^{-1}(A)$ and $f^{-1}(a) \notin f^{-1}(B)$

$\cancel{\Rightarrow}$

As $f^{-1}(x) \in f^{-1}(A)$ and $f^{-1}(x) \notin f^{-1}(B)$,

$f^{-1}(x) \in f^{-1}(A) - f^{-1}(B)$

$\Rightarrow f^{-1}(A) - f^{-1}(B) \subseteq f^{-1}(A-B)$

$\Rightarrow f^{-1}(A-B) = f^{-1}(A) - f^{-1}(B)$

Hence proved.

12) Show that a strictly increasing or strictly decreasing function is injective.

Case 1: Function is strictly increasing

Assume a function $F: R \to R$ is strictly increasing

$\Rightarrow \forall x_1, x_2 \in R : x_1 < x_2,$
$F(x_1) < F(x_2)$

~~For contradiction,~~ Assume there exists Two elements x and y such that $f(x) = f(y)$

If $x < y$, $f(x) < f(y)$ (def. of F)

If $x > y$, $f(x) > f(y)$ (def. of F)

$\Rightarrow x = y$ if $f(x) = f(y)$

$\Rightarrow$ F is injective.

Case 2: Function is strictly decreasing

Assume a function $F: R \to R$ is strictly decreasing

$\Rightarrow \forall x_1, x_2 \in R : x_1 > x_2,$
$F(x_1) < F(x_2)$

Assume there exist two elements x and y such that
$f(x) = f(y)$

If $x < y$, $f(x) > f(y)$ (def. of F)

If $x > y$, $f(x) < f(y)$ (def. of F)

$\Rightarrow x = y$ if $f(x) = f(y)$

$\Rightarrow$ F is injective

Whether the function is strictly increasing or strictly decreasing, we come to the conclusion that the function is injective.

13) Define the range and domain of the following functions

~~Dr~~

a) $f(x) = \frac{x}{1+x^2}$

Domain:

$1+x^2 \neq 0 \ \forall \ x \in R$

$\Rightarrow f(x)$ is defined $\forall \ x \in R$

$\Rightarrow$ Domain $= R$

Range:

~~Switch y and x~~:

$y = \frac{x}{1+x^2}$

Switch y and x:

$x = \frac{y}{1+y^2}$

$\Rightarrow \ x(1+y^2) = y$

$\Rightarrow \ y^2 x + x - y = 0$

Once again, switching x and y:

$$x^2 y \overset{-x+y}{\cancel{\quad}} = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$$ (Applying quadratic formula)

$\Rightarrow 1 - 4y^2 \geq 0, \cancel{y \neq 0}$

$\Rightarrow$ Range: $\left[-\frac{1}{2}, \frac{1}{2}\right]$

b) $f(x) = 2 - |x - 5|$

Domain:

The domain of an absolute value function and of a linear function are both $\mathbb{R}$. There are no ways for $f(x)$ to be undefined $\forall x \in \mathbb{R}$

$\Rightarrow$ Domain $= \mathbb{R}$

Range:

$f(x)$ is maximum when $|x - 5| = 0$ as $|x - 5| > 0$ $\forall x \in \mathbb{R}$

$\Rightarrow \forall x > 5, f(x) < 2$ and

$\forall x < 5, f(x) < 2$

$\Rightarrow$ Range : $[5, -\infty)$

c) $f(x) = \frac{x^2}{1+x^2}$

Domain:

$1 + x^2 \neq 0 \ \forall x \in R$

$\Rightarrow f(x)$ is defined $\forall x \in R$

$\Rightarrow$ Domain: R

Range:

$y = \frac{x^2}{1+x^2}$

Switch y and x:

$x = \frac{y^2}{1+y^2}$

$\Rightarrow x(1+y^2) = y^2$

$\Rightarrow x + y^2 x = y^2$

$\Rightarrow y^2 - y^2 x - x = 0$

$\Rightarrow y^2(1-x) - x = 0$

Switching y and x:

$x^2(1-y) - y = 0$

$$x = \frac{\pm\sqrt{-(-y)(1-y)}}{2(1-y)}$$

$$= \frac{\pm\sqrt{y(1-y)}}{2(1-y)}$$

$\Rightarrow y \neq 1, y \geq 0$

$\Rightarrow$ Range : $[0, 1)$

d) $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$

Simplifying,

$$f(x) = \frac{(x-2)(x-1)}{x-2}$$

$\Rightarrow f(x) = x - 1$

Domain :

The domain of all linear functions is R
$\Rightarrow$ Domain = R

Range :

The range of all linear functions is R
$\Rightarrow$ Range = R

e) $f(x) = \log_2 (x - x^2)$

Domain:

Argument of a logarithm has to greater than 0

$\Rightarrow \quad x - x^2 > 0$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad x > 1$ or $x < 0$

$\Rightarrow$ Domain : $(-\infty, 0] \cup (1, \infty)$

Range:

There are no limitations on the range of a logarithmic function

$\Rightarrow$ Range : R

14) a) $f\left(\frac{2}{3}\right) = \left|\frac{2}{3} - 1\right|$

$= \left|\frac{1}{3}\right|$

$= \frac{1}{3}$

b) $g \circ h\left(\frac{1}{2}\right) = g\left(h\left(\frac{1}{2}\right)\right)$

$= g\left(2\left(\frac{1}{2}\right) + 1\right)$

$= g(1+1)$

$= g(2)$

$= \frac{2(2)}{2^2 - 3}$

$= \frac{4}{4-3}$

$= 4$

c) $f \circ f(-2) = f(f(-2))$

$= f(|(-2) - 1|)$

$= f(|-3|)$

$= f(3)$

$= |3-1|$

$= |2| = 2$

d) $f \circ h\,(x) = f(h(x))$

The function is defined as the codomain of h = domain of f

$= f(2x+1)$

$= |(2x+1)-1|$

$= |2x|$

e) $f \circ g\,(x) = f(g(x))$

$g : \mathbb{Z} \rightarrow \mathbb{R}$

$f : \mathbb{R} \rightarrow \mathbb{R}$

The codomain of g is $\mathbb{R}$ and the domain of f is $\mathbb{R}$, Therefore the function is defined

$$\Rightarrow f(g(x)) = f\left(\frac{2x}{x^2-3}\right)$$

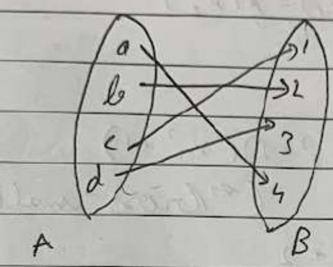
$$= \left| \frac{2x}{x^2-3} - 1 \right|$$

$$= \left| \frac{2x}{x^2-3} - \frac{x^2-3}{x^2-3} \right|$$

$$= \left| \frac{-(x^2 - 2x - 3)}{x^2-3} \right|$$

1.5) Determine which functions are injective, surjective, or both

a) $f: \{a, b, c, d\} \to \{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$



Assume $\{a, b, c, d\} = A$ and $\{1, 2, 3, 4\} = B$.

Each element of A matches to only one element of B

$\Rightarrow$ f is one-to-one

$\Rightarrow$ f is injective

Each element of B has a pre-image in A

$\Rightarrow$ f is onto

$\Rightarrow$ f is surjective

$\Rightarrow$ f is both injective and surjective

b) $f: R \to R, f(x) = \dfrac{x^2+1}{x^2+2}$

For injectivity,

Assume $x_1, x_2 \in R$

Assume ~~$x_1, x_2 \in R$~~ $f(x_1) = f(x_2)$

$\Rightarrow \dfrac{x_1^2+1}{x_1^2+2} = \dfrac{x_2^2+1}{x_2^2+2}$

$\Rightarrow (x_1^2+1)(x_2^2+2) = (x_1^2+2)(x_2^2+1)$

(Cross-multiplying)

$\Rightarrow x_1^2 x_2^2 + x_2^2 + 2x_1^2 + 2$

$= x_1^2 x_2^2 + x_1^2 + 2x_2^2 + 2$

(Expanding)

$\Rightarrow x_1^2 = x_2^2$ (Simplifying)

$\Rightarrow x_1 = x_2$ or $x_1 = -x_2$

$\Rightarrow$ f is not injective

For surjectivity,

$y = \dfrac{x^2+1}{x^2+2}$

Switch x and y

$x = \dfrac{y^2+1}{y^2+2}$

$\Rightarrow x(y^2+2) = y^2+1$

$\Rightarrow xy^2 + 2x = y^2 + 1$

Switch back x and y

$\Rightarrow \quad x^2y + 2y - x^2 - 1 = 0$

$\Rightarrow \quad x^2(y-1) + 2y - 1 = 0$

$\Rightarrow \quad x = \dfrac{\pm\sqrt{-(2y-1)(y-1)}}{2(y-1)}$

$\Rightarrow \quad x = \dfrac{\pm\sqrt{(2y-1)(1-y)}}{2(y-1)}$

Range: $[\frac{1}{2}, 1)$

$[\frac{1}{2}, 1) \neq \mathbb{R}$

$\Rightarrow$ R is not surjective.

c) $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = \left[\frac{n}{2}\right]$

For injectivity,

~~Assume $n_1, n_2 \in \mathbb{Z}$~~

~~Assume $f(n_1) = f(n_2)$~~

~~$\Rightarrow \left[\frac{n_1}{2}\right] = \left[\frac{n_2}{2}\right]$~~

$8, 9 \in \mathbb{Z}$

$f(8) = \left[\frac{8}{2}\right] = [4]$

$= 4$

$f(9) = \left[\frac{9}{2}\right] = [4.5]$

$= 4$

$8 \neq 9$

$\Rightarrow f(n)$ is not injective

(proof by counterexample)

For surjectivity,

Case 1: $f(n)$ is even

$f(n) = 2k$ (for arbitrary integer k)

$\Rightarrow y = 2K$

To find preimage, replace y with n

$\Rightarrow n = 2k$

$$\left[\frac{n}{2}\right] = \left[\frac{2k}{2}\right]$$

$$= [k]$$

$$= y$$

$\Rightarrow$ For every integer y, there exists an n such That $\left[\frac{n}{2}\right] = y$

Case 2: $f(n)$ is odd

$f(n) = 2k + 1$ (for arbitrary integer k)

$\Rightarrow \quad y = 2k + 1$

To find preimage, replace y with n

$$\Rightarrow \left[\frac{n}{2}\right] = \left[\frac{2k+1}{2}\right]$$

$$= [k + 0.5]$$

$$= k = y$$

$\Rightarrow$ For every odd integer y, ~~we can find a~~ there exists an n such that $\left[\frac{n}{2}\right] = y$

$\Rightarrow f(n)$ is surjective

d) $f: R \to R, f(x) = e^x$

For injectivity,

Assume $x_1, x_2 \in R$

Assume $f(x_1) = f(x_2)$

$\Rightarrow e^{x_1} = e^{x_2}$

$\Rightarrow \cancel{\ln e^{x_2}_{x_1} = \ln x_2}\; x_1 \quad \ln e^{x_1} = \ln e^{x_2}$

$\Rightarrow x_1 = x_2$

$\Rightarrow f(x_2)$ is ~~so~~ injective.

For surjectivity,

Assume $x \in R$

$f(x) = e^x$

$\Rightarrow y = e^x$

To ~~find preimage~~, range, replace x with y

$\Rightarrow x = e^y$

$\Rightarrow \ln x = y$

$\Rightarrow$

$\Rightarrow \ln y = x$

As y is the argument of the logarithmic function y must be greater than 0.

$\Rightarrow$ Range $= (0, \infty)$

$(0, \infty) \neq R$

$\Rightarrow f(x)$ is not surjective

i) $f: R \to R^+$, $f(x) = e^x$

For injectivity,

Assume $x_1, x_2 \in R$

Assume $f(x_1) = f(x_2)$

$\Rightarrow e^{x_1} = e^{x_2}$

$\Rightarrow \cancel{\ln x_1} = \ln e^{x_1} = \cancel{x_2} \ln e^{x_2}$

$\Rightarrow x_1 = x_2$

$\Rightarrow f(x)$ is injective

For surjectivity,

Assume $x \in R$

$f(x) = e^x$

$\Rightarrow y = e^x$

To find ~~the~~ range, replace x with y

$\Rightarrow x = e^y$

$\Rightarrow \ln x = y$

Replacing y with x,

$\ln y = x$

As y is the argument of the logarithmic function, y must be greater than 0.

$\Rightarrow$ Range $= (0, \infty)$

$(0, \infty) = R^+$

$\Rightarrow f(x)$ is surjective.

16) Are the following functions invertible? If yes, find the inverse of the function. If not, check if they are left invertible or right-invertible.

For a function $f(x)$ to be left invertible, there should exist a function $g(x)$ such that $g \circ f = id_x$

$g \circ f = id_x$
$\Rightarrow g(f(x)) = id_x$
$\Rightarrow g(f(x)) = x$

For such a function $g(x)$ to exist, $f(x)$ must be one-one as otherwise, the function $g(x)$ may output multiple values for the same input.

$\Rightarrow$ $\boxed{\text{for a function } f(x) \text{ to be left invertible, it must be injective.}}$

For a function $f(x)$ to be right invertible, there should exist a function $g(x)$ such that $f \circ g = id_Y$ and $g \circ f = id_x$

$f \circ g = id_Y$
$\Rightarrow f(g(x)) = id_Y$
$\Rightarrow f(g(x)) = x$

For such a function $g(x)$ to exist, $f(x)$ must be onto as otherwise, certain elements would not have a defined preimage and the above condition would no longer be satisfied

$\Rightarrow$ $\boxed{\text{for a function } f(x) \text{ to be right-invertible, it must be surjective}}$

16(a) $f: N \to N$, $f(x) = x^2$

~~For inject~~
For invertibility, $f(x)$ must be both injective and surjective

For injectivity,

Assume $x_1, x_2 \in N$

Assume $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 = x_2^2$ $\quad (\because f(x) = x^2)$

$\Rightarrow x_1 = x_2$ $\quad (\because x_1 = -x_2$ is extraneous as $x \in N)$

$\Rightarrow f(x)$ is injective

For surjectivity,

$8 \in N$

Assume $f(x) = 8$ for some $x \in N$

$\Rightarrow x^2 = 8$ $\quad (\because f(x) = x^2)$

$\Rightarrow x = 2\sqrt{2}$

$2\sqrt{2} \notin N$

$\Rightarrow f(x)$ is not surjective

As $f(x)$ is not surjective, it is not invertible.

As $f(x)$ is injective, it is ~~left-inverti~~ left-invertible.

1(b) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = \begin{cases} n+4 & \text{if } n \equiv 0 \pmod 3 \\ -n-3 & \text{if } n \equiv 1 \pmod 3 \\ n+1 & \text{if } n \equiv 2 \pmod 3 \end{cases}$

For invertibility, $f(x)$ must be both injective and surjective

For injectivity,

Assume $x_1, x_2 \in \mathbb{Z}$

Assume $f(x_1) = f(x_2)$

Cases: ~~1) x_1, x_2 are divisible by~~

1) $x_1 = 3k$, $x_2 = 3b$

2) $x_1 = 3k+1$, $x_2 = 3b+1$

3) $x_2 = 3k+2$, $x_2 = 3b+2$

4) $x_1 = 3k$, $x_2 = 3b+1$

5) $x_1 = 3k$, $x_2 = 3b+2$

6) $x_1 = 3k+1$, $x_2 = 3b+2$

For Case 1: $x_1 = 3k$, $x_2 = 3b$

$\Rightarrow f(x_1) = 3k+4$ and $f(x_2) = 3b+4$

$\Rightarrow 3k+4 = 3b+4$

$\Rightarrow k = b$

$\Rightarrow x_1 = x_2$

Case 2: $x_1 = 3k+1$, $x_2 = 3b+1$

$\Rightarrow f(x_1) = -(3k+1)-3$ and $f(x_2) = -(3b+1)-3$

$\Rightarrow -3k-4 = -3b-4$

$\Rightarrow k = b$

$\Rightarrow x_1 = x_2$

Case 3 : $x_1 = 3k+2,\ x_2 = 3l+2$

$\Rightarrow f(x_1) = (3k+2)+1,\ f(x_2) = (3l+2)+1$

$\Rightarrow \cancel{f(x_1)}\ 3k+3 = 3l+3$

$\Rightarrow k = l$

$\Rightarrow x_1 = x_2$

Case 4 : $x_1 = 3k,\ x_2 = 3l+1$

$\Rightarrow f(x_1) = 3k+4,\ x_2 = -(3l+1)-3$

$\Rightarrow 3k+1 = -3l-4$

$\Rightarrow 3k = -3l-5$

$\Rightarrow x_1 = -x_2 - 4$

$\Rightarrow$ not possible for $f(x)$ to be equal

Case 5 : $x_1 = 3k,\ x_2 = 3l+1$

$\Rightarrow f(x_1) = 3k+4,\ x_2 = (3l+1)+1$

$\Rightarrow 3k+4 = 3l+2$

$\Rightarrow 3k = 3l-2$

$\Rightarrow x_1 = x_2 - 3$

$\Rightarrow$ not possible for $f(x)$ to be equal

Case 6 : $x_1 = 3k+1,\ x_2 = 3l+2$

~~$\Rightarrow f(x_1) = 3k+4,$~~

$\Rightarrow f(x_1) = -(3k+1)-3,\ f(x_2) = (3l+1)+1$

$\Rightarrow -3k-4 = 3l+2$

$\Rightarrow -3k = 3l+6$

$\Rightarrow k = -k-2$

$\Rightarrow$ not possible for $f(x)$ to be equal

$\Rightarrow f(x)$ is injective

For surjectivity,

$0 \in \mathbb{Z}$

Assume $f(x) = 0$

Cases: $x = 3K$, $x = 3k+1$, $x = 3k+2$, $k \in \mathbb{Z}$

Case 1: $x = 3k$

$\Rightarrow$ $\cancel{3k} = 3k + 4 = 0$

$\Rightarrow$ $k = -\frac{4}{3}$

$\Rightarrow$ not possible

Case 2: $x = 3k+1$

$\Rightarrow -(3k+1) - 3 = 0$

$\Rightarrow$ $k = \frac{4}{3}$

$\Rightarrow$ not possible

Case 3: $x = 3k+2$

$\Rightarrow$ $(3k+2)+1 = 0$

$\Rightarrow$ $k = -1$

$\Rightarrow$ ~~not a~~ $x = -3$

$\Rightarrow$ ~~so~~ The given value of x would not use this formula

$\Rightarrow$ $f(x)$ is not surjective.

As $f(x)$ is ~~neither injective nor~~ not surjective, it is not ~~neither~~ invertible.

As $f(x)$ is injective, it is left-invertible.

16) c $f : [0, \infty] \rightarrow (-\infty, \infty)$

For a function to be invertible, the function must be both injective or surjective

For injectivity,

$0 \in (-\infty, \infty)$

Also, $0 \in [0, \infty]$, $1 \in [0, \infty)$

Also, $f(0) = 0$

Also, $f(1) = \ln(1) = 0$

$\Rightarrow$ $f(0) = f(1)$

$\Rightarrow$ $f(x)$ is not injective

For surjectivity,

The range of the logarithmic function is $(-\infty, \infty)$

$\Rightarrow$ $f(x)$ is surjective as $(-\infty, \infty) =$ codomain of $f(x)$

As $f(x)$ is not injective, $f(x)$ is not invertible

As $f(x)$ is surjective, $f(x)$ is right-invertible.

17) $f: A \to B$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$

a) In how many functions, does the range not contain 1?

$2^5 = 32$

Explanation: number of functions $= n^m$ where $n =$ # of elements in codomain and $m =$ # of elements in domain.

b) In how many functions, does 1 in B have exactly 2 pre-images in A?

$\cancel{3 \times 5 \times 4} = {}^5C_2 \times 2^3 = 80$

Explanation: ${}^5C_2$ pre-images are possible for P. Excluding 1, codomain has 2 elements and excluding the two preimages, domain has 3 elements ⇒ number of functions $= n^m$ (from part a)

c) In how many functions, will $f(1) = 1$?

$3^4 = 81$

Explanation: Every element in A can only match to 1 element in B. Therefore, to find the number of functions, we ~~tempore~~ do not consider 1 in the domain. Applying n^m from part a, $3^4 = 81$.

d) How many functions would be one-one?

0

Explanation: In a function, every element in the domain has an image. Thus, a one-one function is not possible from ~~an element~~ a set with 5 elements to a set with 3 elements.

e) How many functions would be many-one?

$3^5 - 0 = 243$

Explanation: There are 3^5 total functions possible ($\because n^m$ as described in part a). There are 0 one-one functions. Therefore, there are 243 many-one functions.