ASSIGNMENT 1 2023115011

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1) a) $\left(\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right)$

$\Big\downarrow R_1 = \frac{1}{a} \times R_1$

$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/a & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & d & 0 & 0 & 0 & 1 \end{array}\right)$

$\Big\downarrow R_3 = \frac{1}{d} \times R_3$

$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/a & 0 & 0 \\ b & 0 & c & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right)$

$\Big\downarrow R_2 = \frac{1}{b} \times R_2$

$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/a & 0 & 0 \\ 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right)$

$\Big\downarrow R_1 \Leftrightarrow R_2$

$\left(\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right)$

$\Big\downarrow R_3 = R_3 - R_2$

$\left(\begin{array}{ccc|ccc} 1 & 0 & c/b & 0 & 1/b & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 0 & 1/a & 0 & 1/d \end{array}\right)$

The inverse of the given matrix does not exist as the RREF of the left part of the augmented matrix has a zero row, and thus cannot be equivalent to the identity matrix.

b) $$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \; R_1 = \frac{1}{a} \times R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \; R_2 = R_2 - R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & a & 0 & -1/a & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \; R_2 = \frac{1}{a} \times R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \; R_3 = R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 0 & a & 1/a^2 & -1/a & 1 \end{array}\right)$$

$$\Big\downarrow \; R_3 = \frac{1}{a} \times R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/a^2 & 1/a & 0 \\ 0 & 0 & 1 & 1/a^3 & -1/a^2 & 1/a \end{array}\right)$$

As the left part of the augmented matrix (the RREF of the original matrix) is row-equivalent to the identity matrix, the inverse of given matrix is

$$\begin{pmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{pmatrix}$$

2) Let there exist a non-singular matrix A, which is symmetric i.e.

$$A = A^T$$

Also, there exists an inverse to matrix A, denoted A^{-1} such that

$$AA^{-1} = A^{-1}A = I \quad - ①$$

We know that the identity matrix, denoted I, is also symmetric i.e.

$$I = I^T$$

From ①, we know that

$$AA^{-1} = (AA^{-1})^T$$

$$\Rightarrow \quad AA^{-1} = (A^{-1}A)^T$$

$$\Rightarrow \quad AA^{-1} = A^T (A^{-1})^T \qquad (\text{as } (AB)^T = B^T A^T)$$

$$\Rightarrow \quad AA^{-1} = A (A^{-1})^T \qquad (\text{as } A = A^T)$$

Left-multiply by A^{-1} on both sides,

$$A^{-1}AA^{-1} = A^{-1}A\,(A^{-1})^T$$

$$\Rightarrow \quad I \cdot A^{-1} = I \cdot (A^{-1})^T$$

$$\Rightarrow \quad A^{-1} = (A^{-1})^T$$

Thus, the inverse of a non-singular symmetric matrix is also symmetric.

Hence proved.

3. $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$A^4 = A^3 \cdot A = -I \cdot A = -A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$A^5 = A^3 \cdot A^2 = -I \cdot A^2 = -A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$A^6 = A^3 \cdot A^3 = -I \cdot (-I) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^7 = A^6 \cdot A = I \cdot A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$\begin{aligned} A^{2015} &= A^{2010+5} = A^{2010} A^5 = A^{6 \cdot 335} \cdot A^5 \\ &= (A^6)^{335} \cdot A^5 \\ &= (I)^{335} \cdot A^5 \\ &= A^5 \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = A^{2015} \end{aligned}$$

4. The trace of a matrix A, denoted $tr(A)$ is the sum of the diagonal elements of the matrix

i.e. $tr(A) = \sum_{i=1}^{n} a_{ii}$

As XY and YX are both defined, X and Y must both be $n \times n$ matrices, where n is a positive integer

$$XY = \begin{bmatrix} \sum_{i=1}^{n} x_{1i} y_{i1} & & \\ & \sum_{i=1}^{n} x_{2i} y_{i2} & \\ & & \sum_{i=1}^{n} x_{ni} y_{in} \end{bmatrix}$$

$$YX = \begin{bmatrix} \sum_{i=1}^{n} y_{1i} x_{i1} & & \\ & \sum_{i=1}^{n} y_{2i} x_{i2} & \\ & & \sum_{i=1}^{n} y_{ni} x_{in} \end{bmatrix}$$

$$tr(XY) = \sum_{i=1}^{n} x_{1i} y_{i1} + \sum_{i=1}^{n} x_{2i} y_{i2} + \dots + \sum_{i=1}^{n} x_{ni} y_{in}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} x_{ji} y_{ij}$$

$$tr(YX) = \sum_{i=1}^{n} y_{1i} x_{i1} + \dots + \sum_{i=1}^{n} y_{ni} x_{in}$$

$$\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} y_{ji} x_{ij} \Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} x_{ij} y_{ji}$$

$$\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{n} x_{ji} y_{ij}$$

$\therefore tr(XY) = tr(YX)$

$\Rightarrow tr(XY) - tr(YX) = 0$

As ~~XY-YX~~ $I \neq 0$,

$XY - YX \neq I$

Hence proved

5) For the vectors $\{v_1, v_2, v_3\}$ to form an orthogonal basis, $\langle v_1, v_2 \rangle = 0$, $\langle v_2, v_3 \rangle = 0$, $\langle v_3, v_1 \rangle = 0$.

$\langle v_1, v_2 \rangle = 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 0 = 0$

$\langle v_2, v_3 \rangle = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot (-2) = 0$

$\langle v_3, v_1 \rangle = 1 \cdot 1 + 1 \cdot 1 + (-2) \cdot (1) = 0$

$\therefore \{v_1, v_2, v_3\}$ are orthogonal to each other and hence, v_1, v_2, v_3 are linearly independent of each other, forming an orthogonal basis. ~~for~~

The formula to calculate the coordinate c_i is given by:

$$c_i = \frac{\langle w, v_i \rangle}{\langle v_i, v_i \rangle}, \quad i = \{1, 2, 3\}$$

The coordinates of ~~each~~ vector w with respect to the basis vectors c_1, c_2, c_3 are as follows:

$$c_1 = \frac{4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} = 5$$

$$c_2 = \frac{4 \cdot 1 + 5 \cdot (-1) + 6 \cdot 0}{1 + 1 + 0} = -\frac{1}{2}$$

$$c_3 = \frac{4 \cdot 1 + 5 \cdot 1 + 6 \cdot (-2)}{1 + 1 + 4} = -\frac{1}{2}$$

$\therefore w = 5 \cdot u - \frac{u_2}{2} - \frac{u_3}{2}$

6) Let A and B be upper triangular matrices.
i.e. all entries in A and B below main diagonal are 0.

$$a_{ij} = \begin{cases} a_{ij}, & i \le j \\ 0, & i > j \end{cases}, \text{ where } A = [a_{ij}]$$

$$b_{ij} = \begin{cases} b_{ij}, & i \le j \\ 0, & i > j \end{cases}, \text{ where } B = [b_{ij}]$$

Let C be defined as $C = AB$.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= 0 + \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

Thus, for $k > j$, $c_{ij} = 0$

i.e.

$$c_{ij} = \begin{cases} c_{ij}, & k \le j \\ 0, & k > j \end{cases}, \text{ where } C = [c_{ij}]$$

Thus, by definition of an upper triangular matrix, C is an upper triangular matrix.

Hence proved