ASSIGNMENT 3
ISHAAN ROMIL
2023116011

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Q1) If A, B, C are matrices over a field F such that the products BC and $A(BC)$ are well-defined, and so are the products AB and $(AB)C$, then prove $A(BC) = (AB)C$.

A: Source: Hoffman and Kunze (Linear Algebra - Second Edition) i.e. the textbook (for all questions)

Let B be an $n \times p$ matrix. As we are given that BC is well-defined, we know that C has p rows and ~~can be~~ is therefore a $p \times m$ matrix, where m is an arbitrary positive integer, representing the number of columns in C.

As BC is well-defined, we know that BC has the dimensions $n \times m$ ~~$n \times p$, where a is an arbitrary positive integer representing the number of rows in BC.~~

As the matrix ~~multiplication~~ product $A(BC)$ is well-defined, we know that it is the result of the matrix multiplication of A and BC (where A is assumed to have dimensions $a \times n$ so as to allow the matrix multiplication to be valid). The matrix $A(BC)$, therefore, has dimensions $a \times p$.

As AB and $(AB)C$ are also well-defined, and keeping the dimensions of all matrices constant, we can conclude that AB exists as a $a \times p$ matrix. When multiplied by C, the well-defined product

$(AB)C$ has the dimensions $a \times p$.

Thus, the dimensions of $A(BC)$ and $(AB)C$ are equal, and both matrices are well-defined, which serves as a ~~pre~~ prerequisite condition for the equality of $A(BC)$ and $(AB)C$.

To show $A(BC) = (AB)C$, it must be proved that

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

for all i, j.

Essentially, for equality to be proved between two matrices, it must be proved that every element in both matrices is equal to the corresponding element in the other matrix.

By definition of a matrix,

$$[A(BC)]_{ij} = \sum_r A_{ir} (BC)_{rj}$$

$$= \sum_r A_{ir} \sum_s B_{rs} C_{sj}$$

$$= \sum_r \sum_s A_{ir} B_{rs} C_{sj}$$

$$= \sum_s \sum_r A_{ir} B_{rs} C_{sj}$$

$$= \sum_s \left(\sum_r A_{ir} B_{rs} \right) C_{sj}$$

$$= \sum_{s} (AB)_{is} C_{sj}$$

$$= [(AB)C]_{ij}.$$

Hence, via the properties of a matrix and via the definition of matrix multiplication, it can be proved that the well-defined matrix $A(BC)$ is equivalent to the well-defined matrix $(AB)C$.

Mathematically,

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

Hence proved

Q2) Let e be an elementary row operation and let E be an elementary matrix of size $m \times m$ such that $E = e(I_{m\times m})$ then prove that $e(A) = EA$ holds $\forall$ matrices A of size $m \times n$.

A: To prove this ~~statement~~, we must prove that the entry in the i^{th} row and j^{th} column of product matrix EA is ~~taken~~ obtained from the i^{th} row and j^{th} column of A.

To start, we must distinguish amongst the three elementary row operations on an ~~m×n~~ $m \times n$ matrix, over the field F:

① Multiplication of one row of A by a non-zero scalar c

i.e. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = cA_{rj}$

② Replacement of the r^{th} row of A by row r plus c times row s, with c being any scalar and $r \neq s$.

i.e. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = A_{rj} + cA_{sj}$

③ Interchange of two rows of A

i.e. $e(A)_{ij} = A_{ij}$ if i is different from both r and s, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$

Now, we must consider each elementary row operation individually and prove the given statement.

Case 1: Multiplication of one row of A by a non-zero scalar c

i.e. $r = cr$, $c \neq 0$

This operation has an elementary matrix, E_{ik}, defined by:

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ c\delta_{ik}, & i = r \end{cases}$$

for $1 \leq i \leq m$

Thus,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A_{ik}, & i \neq r \\ cA_{ik} & i = r \end{cases}$$

We also know that $e(A)$ of the same operation is defined as:

$$e(A)_{ij} = \begin{cases} A_{ij}, & i \neq r \\ cA_{ij}, & i = r \end{cases}$$

for $1 \leq i \leq m$

Thus, for this elementary row operation, $EA = e(A)$

Case 2: Replacement of the r^{th} row of A by row r plus c times row s, with c being any scalar and $r \neq s$
i.e. $r = r + cs$

This operation has elementary matrix, E_{ik}, defined as:

$$E_{ik} = \begin{cases} \delta_{ik} & , i \neq r \\ \delta_{rk} + c\,\delta_{sk} & , i = r \end{cases}$$

for $1 \leq i \leq m$

Thus,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A & , i \neq r \\ A_{rj} + cA_{sj} & , i = r \end{cases}$$

We also know that e(A) of the same operation is defined as:

$$e(A)_{ij} = \begin{cases} A_{ij} & , i \neq r \\ A_{rj} + cA_{sj} & , i = r \end{cases}$$

for $1 \leq i \leq m$

Thus, for this elementary row operation, ~~E(A) = eA.~~
$EA = e(A)$

Case 3: Interchange of two rows of A
i.e. $r \leftrightarrow s$

This operation has elementary matrix, E_{ik}, defined as:

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \neq s \\ \delta_{sk}, & i = r \\ \delta_{rk}, & i = s \end{cases}$$

for $1 \leq i \leq m$

Thus,

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj} = \begin{cases} A, & i \neq r \neq s \\ A_{rj}, & i = s \\ A_{sj}, & i = r \end{cases}$$

We also know $e(A)$ of the same operation is defined as:

$$\cancel{e(A)_{ij} = \begin{cases} A_{ri}, & i \\ A_{rj} + c A_{sj}, & i = r \end{cases}}$$

$$e(A)_{ij} = \begin{cases} A_{ij}, & i \neq r \neq s \\ A_{rj}, & i = s \\ A_{sj}, & i = r \end{cases}$$

Thus, for this elementary row operation, $EA = e(A)$.

Thus, for every $m \times n$ matrix A,

$$e(A) = EA$$

Hence proved.

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63) If A is an $n \times n$ matrix, prove that the following statements are equivalent:
a) A is invertible.
b) A is row-equivalent to the $n \times n$ identity matrix.
c) A is a product of elementary matrices.

A: To prove the above statement, we must prove that:
i) Statement ⓐ $\Rightarrow$ Statement ⓑ
ii) Statement ⓑ $\rightarrow$ Statement ⓒ
iii) Statement ⓒ $\rightarrow$ Statement ⓐ

A row-reduced echelon matrix is a $m \times n$ matrix R satisfying the following conditions:
i) R is row-reduced
ii) every row of R which has all its entries 0 occurs below every row which has a non-zero entry
iii) if rows $1, \ldots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in column k_i, $i = 1, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$

~~Let R~~ We also know that every matrix has a row-reduced echelon form, and the row-reduced echelon form is row-equivalent to the original matrix.

Let R be the row-reduced echelon form of ~~matri~~ an arbitrary matrix A.

We know that for a matrix to be row-equivalent to another matrix, it must be a product of $n \times n$ elementary matrices (or, alternatively, of elementary row operations).

$\Rightarrow$ R is row-equivalent to A iff $R = PA$, where P is a product matrix of $n \times n$ elementary matrices.

$$\Rightarrow R = E_k \ldots E_2 E_1 A,$$

where $E_1, E_2, \ldots, E_k$ represent k elementary matrices.

Furthermore, we know that for every elementary row operation e, there corresponds an elementary row operation e_1, which is its inverse such that:

$$e_1(e(A)) = e(e_1(A)) = A.$$

Therefore, we can also conclude that for every elementary matrix E_i, there exists an ~~matrix~~ elementary matrix E_i^{-1}, which serves as its inverse such that:

$$E(E^{-1}(A)) = E^{-1}(E(A)) = A$$

As every elementary matrix is invertible,

$$A = E_k^{-1} \ldots E_2^{-1} E_1^{-1} R$$

Since the product of invertible matrices is also an invertible matrix, it can be concluded that A is invertible iff R is invertible.

As R is a row-reduced echelon matrix, for R to be invertible, every row must have a non-zero entry

$\Rightarrow R = \mathbb{I}_{n \times n}$

$\therefore$ A is row-equivalent to $\mathbb{I}$ iff A is invertible

$\Rightarrow$ (a) $\Rightarrow$ (b)

If statement (b) is true, ~~every~~ A is row-equivalent to the $n \times n$ identity matrix

$\Rightarrow A = e_k(\dots e_2(e_1(\mathbb{I}))\dots)$ — (1)

We know that $e(A) = EA$ holds $\forall$ matrices A of size $m \times n$, where e is an elementary row operation and E is an elementary matrix of size $m \times m$.

Thus, (1) can be represented as a product of elementary matrices; ~~where~~

$$A = E_k \dots E_2 \cdot E_1 \cdot \mathbb{I}$$

$$\Rightarrow A = E_k \dots E_2 \cdot E_1 \qquad [\because M \cdot \mathbb{I} = M]$$

Where $e_k = \cdot E_k$.

Thus, A can be represented as a product of elementary matrices

$\Rightarrow$ (b) $\Rightarrow$ (c)

Assuming (c) to be true, A is a product of elementary matrices.

$\Rightarrow A = E_k \cdots E_2 E_1$

As we know that every elementary matrix has an inverse and can therefore be classified as invertible.

$$(A)\cdot(E_k^{-1}\cdot \ldots \cdot E_2^{-1}\cdot E_1^{-1}) = (E_k \ldots E_2 \cdot E_1)\cdot(E_k^{-1}\cdot \ldots \cdot E_2^{-1}\cdot E_1^{-1})$$

$$\Rightarrow A\cdot(E_k^{-1}\cdot \ldots \cdot E_2^{-1}\cdot E_1^{-1}) = \mathbb{1} \; (\text{~~where~~ as } E_i \cdot E_i^{-1} = \mathbb{1})$$

Grouping $E_k^{-1}\cdot \ldots \cdot E_2^{-1}\cdot E_1^{-1}$ as A^{-1}, as A^{-1} is the inverse of A,

$$A\cdot A^{-1} = \mathbb{1}$$

Thus, A has an inverse $A^{-1} = E_k^{-1}\cdot \ldots \cdot E_2^{-1}\cdot E_1^{-1}$.

Thus, A is invertible

$\Rightarrow$ (c) $\Rightarrow$ (a)

As we have proved (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a), we can say the three statements are equivalent,

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(5) Let A be a $n \times m$ matrix and B be a $n \times 1$ vector with real entries, suppose the equation $AX = B$ (here, X belongs to $\mathbb{R}^m$) admits a unique solution, then we can conclude that:

a) $m >= n$

✓ b) $n >= m$

c) $m = n$

d) $n > m$

A: The equation $AX = B$ can be represented as:

$$A_{n \times m} X = B_{n \times 1},$$

where $n \times m$ is the size of A and $n \times 1$ is the size of B.

Following the rules of matrix multiplication, X must have m rows and 1 column, making it a column matrix of size $m \times 1$.

For a unique solution to exist, the number of equations has to equal the number of variables.

The number of equations given by n (ie. the number of ~~equations~~ rows in A) while the number of variables is given by m (ie. the number of columns of A)

Thus, a unique solution exists when $m = n$.

ie. when $A_{n \times m} = A_{n \times n} = A_{m \times m}$

We are left with two other cases:

1. $n > m$
2. $m > n$

Case 1: $n > m$

For a unique solution to exist when there are more equations than variables, the number of independent equations (i.e. equations which provide unique and non-redundant information) must be equal to the number of unknowns.

The remaining equations can be dependant equations, or any linear combination of the independent equations. These can be identified via conversion to row-reduced form as the entire row ~~becomes equal~~ to consists solely of zero-value entries.

Thus, for $n > m$, a unique solution can exist.

Case 2: $m > n$

If the number of unknowns is greater than the number of equations, a unique solution cannot be found as there are insufficient constraints and a possibility of multiple solutions

i.e. $x + y = 5$
$z + x = 6$

Thus, for a unique solution to exist $n >= m$.