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ASSIGNMENT-5 ISHAAN ROMIL

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Q1) Prove that the subspace spanned by a non-empty subset of a vector space V is the set of all linear combinations of vectors in S.

A: Recalling that a subspace of V is a subset W of V, where V is a vector space over field F, and where W is itself a vector space over F with operations of vector addition and scalar multiplication on V.

Further recalling that the subspace spanned by a set of vectors in vector space V, denoted S, is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_3\}$, we shall simply call W the subspace spanned by vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Let W be the subspace spanned by non-empty subset S of vector space V. Then, each linear combination,

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_m\alpha_m$$

of vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ in S is clearly in W. Therefore, W contains set L of all linear combinations of vectors in S. It further contains all elements of S and thus contains S. As S is non-empty, L is also non-empty.

If α, β belong to L, then α is a linear combination,

$$\alpha = x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_m \alpha_m$$

of vectors α_i in S, and β is a linear combination,

$$\beta = y_1 \beta_1 + y_2 \beta_2 + \dots + y_n \beta_n$$

of vectors β_j in S. For each scalar c,

$$c\alpha + \beta = \sum_{i=1}^{n} (c x_i) \alpha_i + \sum_{j=1}^{n} y_j \beta_j$$

Hence, $c\alpha + \beta$ belongs to L. Thus L is a subspace of V.

As it has been proven that L is a subspace of V which contains S, and also proven that any subspace which contains S also contains L, it follows that L is the intersection of all subspaces containing S. Thus, by definition of subspace spanning, L is the subspace spanned by set S.

Hence proved.

Q2. If W_1 and W_2 are finite dimensional subspaces of vector space V then prove the following:

a) $W_1 + W_2$ is finite dimensional

b) $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

A: Recall that if $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

of vectors α_i in S_i is called the sum of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by

$$S_1 + S_2 + \ldots + S_k$$

or by,

$$\sum_{i=1}^{k} S_i$$

Further recalling that in a vector space V, a basis for V is a linearly independent set of vectors in V which spans the space V, and by intension, the space V can be said to be finite-dimensional if it is has a finite basis.

~~Defining~~ Clarifying the definition of linearly dependent as there existing distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in subset S of ~~the~~ vector space V, and scalars $c_1, c_2, \ldots, c_n$ in F, not all of which are 0, such that

$$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$$

Further clarifying the definition of a linearly independent as not linearly dependent.

Let S be a linearly independent subset of a vector space V. Suppose $\alpha_1, \alpha_2, \dots, \alpha_m$ are distinct vectors in S, ~~and that~~ and β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is also linearly independent. ~~$\Rightarrow c_1\alpha_1 + c_2 \dots + c_m\alpha_m + b\beta = 0$~~

As a proof for the above lemma, assume $\alpha_1, \dots, \alpha_m$ are distinct vectors in S and that,

$$c_1\alpha_1 + \dots + c_m\alpha_m + b\beta = 0.$$

Then $b=0$, otherwise,

$$\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \dots + \left(-\frac{c_m}{b}\right)\alpha_m,$$

and β is in the subspace spanned by S. Thus, $c_1\alpha_1 + \dots + c_m\alpha_m = 0$, and since S is linearly independent, each $c_i = 0$

Using this, we can prove that if W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Suppose S_0 is a linearly independent subset of W. If S is a linearly independent subset of W containing S_0, then S is also a linearly independent subset of V; since V is finite-dimensional, S contains no more than $\dim V$ elements.

Extending S_0 to a basis for W, if ~~set~~ S_0 spans W, then S is a basis for W. Otherwise, if S_0 does not span W, we use the lemma to find vector β_1 such that $S_1 = S_0 \cup \{\beta_1\}$ is independent. If S_1 spans W, it proves the theorem. Otherwise repeat the previous step recursively to obtain S_m, which is defined by:

$$S_m = S_0 \cup \{\beta_1, \ldots, \beta_m\},$$

where S_m is a basis for W.

Based off of this theorem, we can say that in a finite-dimensional vector space V, every non-empty linearly independent set of vectors is part of a basis.

Using the theorem and above corollary, we know that $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \ldots, \alpha_k\}$ which is part of basis

$$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m\} \text{ for } W_1$$

and part of basis

$$\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\} \text{ for } W_2.$$

The subspace $W_1 + W_2$ is spanned by vectors

$$\alpha_1, \ldots, \alpha_k \quad \beta_1, \ldots, \beta_m, \quad \gamma_1, \ldots, \gamma_n$$

with these vectors forming an independent set. The reason for this is obvious. ~~Given~~ If

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$$

Then,

$$-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_j \beta_j$$

showing that $\sum z_r \gamma_r$ belongs to W_1. As $\sum z_r \gamma_r$ also belongs to W_2, it will follow that

$$\sum z_r \gamma_r = \sum c_i \alpha_i$$

for certain scalars $c_1, \ldots, c_k$. We know $z_r = 0$ for all of these scalars as

$$\{\alpha_1, \ldots, \alpha_k \quad \gamma_1, \ldots, \gamma_n\}$$

is independent. Thus,

$$\sum x_i \alpha_i + \sum y_j \beta_j = 0.$$

Since

$$\{\alpha_1, \ldots, \alpha_k \quad \beta_1, \ldots, \beta_m\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$.
Thus,

$$\{\alpha_1, \ldots, \alpha_k \quad \beta_1, \ldots, \beta_m \quad \gamma_1, \ldots, \gamma_n\}$$

is a basis for $W_1 + W_2$.

Finally, to prove (A),

$$\begin{aligned} \dim W_1 + \dim W_2 &= (k+m) + (k+n) \\ &= k + (m+k+n) \\ &= \dim(W_1 \cap W_2) + \dim(W_1 + W_2). \end{aligned}$$

Hence proved.

Q3) Let R be a non-zero row-reduced echelon matrix then prove that the non-zero row vectors of R form a basis for the row space of R.

A: Let r be the number of non-zero row vectors of R and $p_1, \ldots, p_r$ be the non-zero row vectors of R.

$$p_i = (R_{i1}, \ldots, R_{in}).$$

Obviously, these row vectors span the row space of R.

Recalling the definition of basis for a vector space V being a set of linearly independent vectors in V which spans the space V, we just need to prove that the row vectors are linearly independent.

As R is a row-reduced echelon matrix, there exist positive integers $k_1, \ldots, k_r$ such that, for $i \leq r$

$$① \begin{cases} (a) & R(i,j) = 0 \text{ if } j < k_i \\ (b) & R(i, k_j) = \delta_{ij} \\ (c) & k_1 < \ldots < k_r \end{cases}$$

by definition of row-reduced echelon matrix.

Suppose $\beta = (b_1, \ldots, b_n)$ is a vector in the row space of R:

$$\beta = c_1 \rho_1 + \ldots + c_r \rho_r \quad - \quad ①$$

Claiming $c_j = b_{k_j}$, via ~~(2.19)~~ ①,

$$b_{k_j} = \sum_{i=1}^{r} c_i R(i, k_j)$$

$$= \sum_{i=1}^{r} c_i \delta_{ij}$$

$$= c_j.$$

If $\beta = 0$, i.e. $c_1 \rho_1 + \ldots + c_r \rho_r = 0$, c_j must be the k_j th coordinate of zero vector in such a way that $c_j = 0$, $j = 1, \ldots, r$. Thus, $\rho_1, \ldots, \rho_r$ are linearly independent.

Thus, the non-zero row vectors of R form a basis for the row space of R.

Hence proved.