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ASSIGNMENT - 3

Q1) On R^n, define two properties: $\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$ and $c\bar{\alpha} = -c\bar{\alpha}$. Which of the axioms for the vector space are satisfied by $(R^n, \oplus, \cdot)$?

A: A vector space (or linear space) consists of the following property:

1. a ~~set F~~ field F of scalars
2. a set V of objects, called vectors
3. a rule (or operation), called vector addition, which associates with each pair of vector α, β in V a vector $\alpha + \beta$ in V, called the sum of α and β, in such a way that
   (a) addition is commutative, $\alpha + \beta = \beta + \alpha$
   (b) addition is associative, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;
   (c) $\exists$ a unique vector 0 in V, called zero vector, such that $\alpha + 0 = \alpha$ ~~for~~ $\forall \alpha \in V$;
   (d) for each vector α in V, there is a unique vector $-\alpha$ in V such that $\alpha + (-\alpha) = 0$;
4. a rule (or operation), called scalar multiplication, which associates with each scalar c in F and vector α in V a vector $c\alpha$ in V, called product of c and α, such that:
   (a) $1\alpha = \alpha \ \forall \alpha \in V$;
   (b) $(c_1 c_2)\alpha = c_1(c_2\alpha)$;
   (c) $c(\alpha + \beta) = c\alpha + c\beta$;
   (d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$;

For ~~property~~ axiom ①,

$R^{\wedge}$ is a field of scalars
$\Rightarrow$ ~~property~~ axiom ① ~~holds~~ satisfied

For ~~property~~ axiom ②,

$\bar{\alpha}, \bar{\beta}$ are vectors
$\Rightarrow$ there exists a set of objects, called vectors in $(R^{\wedge}, \oplus, \cdot)$
$\Rightarrow$ ~~property~~ axiom ② ~~#~~ is satisfied

For axiom ③,

For condition ⓐ,

$$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$$
$$= -(\bar{\beta} - \bar{\alpha})$$
$$= -\bar{\beta} \oplus \bar{\alpha}$$

$\Rightarrow \bar{\alpha} \oplus \bar{\beta} \neq \bar{\beta} \oplus \bar{\alpha}$
$\Rightarrow$ commutativity does not hold
$\Rightarrow$ condition ⓐ is not satisfied

For condition ⓑ,

$$(\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{\gamma} = (\bar{\alpha} - \bar{\beta}) - \bar{\gamma}$$
$$= \bar{\alpha} - \bar{\beta} - \bar{\gamma}$$

$$\bar{\alpha} \oplus (\bar{\beta} \oplus \bar{\gamma}) = \bar{\alpha} - (\bar{\beta} - \bar{\gamma})$$
$$= \bar{\alpha} - \bar{\beta} + \bar{\gamma}$$

$$\Rightarrow \bar{\alpha} \oplus (\bar{\beta} \oplus \bar{\gamma}) \neq (\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{\gamma}$$

$\Rightarrow$ associativity does not hold

$\Rightarrow$ condition ② is not satisfied

For condition ⓒ,

$$\bar{\alpha} \oplus \bar{0} = \bar{\alpha} - \bar{0} = \bar{\alpha}$$

$$\bar{0} \oplus \bar{\alpha} = \bar{0} - \bar{\alpha} = -\bar{\alpha}$$

$\Rightarrow$ As $\bar{\alpha} \oplus \bar{0} \neq \bar{0} \oplus \bar{\alpha} \neq \bar{\alpha}$, additive identity '0' does not exist

$\Rightarrow$ condition ⓒ does not hold.

For condition ⓓ,

As the additive identity '0' does not exist, the additive inverse cannot exist.

$\Rightarrow$ condition ⓓ is not satisfied.

$\Rightarrow$ Axiom ② is not satisfied.

For axiom ③,

For condition ⓐ,

$$-1 . \alpha = -(-1(\alpha))$$
$$= \alpha$$

$\Rightarrow$ The multiplicative identity '1' exists and is equal to -1.

$\Rightarrow$ condition ⓐ is satisfied.

For condition (b),

$$c_1 \cdot (c_2 \cdot \alpha) = c_1 \cdot (-c_2 \alpha)$$
$$= -c_1(-c_2 \alpha)$$
$$= c_1 c_2 \alpha$$

$$(c_1 \cdot c_2) \cdot \alpha = (c_1 c_2) \cdot \alpha$$
$$= -c_1 c_2 \alpha$$

$\Rightarrow \quad c_1 \cdot (c_2 \cdot \alpha) \neq (c_1 \cdot c_2) \cdot \alpha$

$\Rightarrow$ commutativity does not hold

$\Rightarrow$ condition (b) is not satisfied

For condition (c),

$$c \cdot (\alpha \oplus \beta) = c \cdot (\alpha - \beta)$$
$$= -c(\alpha - \beta)$$
$$= -c\alpha + c\beta$$

$$c \cdot \alpha \oplus c \cdot \beta = -c\alpha \oplus (-c\beta)$$
$$= -c\alpha - (-c\beta)$$
$$= -c\alpha + c\beta$$

$\Rightarrow \quad c \cdot (\alpha \oplus \beta) = c \cdot \alpha \oplus c \cdot \beta$

$\Rightarrow$ condition (c) holds.

For condition ②,

$$(c_1 + c_2) \cdot \bar{\alpha} = -(c_1 + c_2)\bar{\alpha}$$
$$= -c_1 \alpha - c_2 \bar{\alpha} \quad [\because \text{distributive prop.}]$$

$$c_1 \cdot \bar{\alpha} \oplus c_2 \cdot \bar{\alpha} = -c_1 \bar{\alpha} \oplus (-c_2 \bar{\alpha})$$
$$= -c_1 \bar{\alpha} - (-c_2 \alpha)$$
$$= -c_1 \bar{\alpha} + c_2 \bar{\alpha}$$

$\Rightarrow (c_1 + c_2) \cdot \bar{\alpha} \neq c_1 \cdot \bar{\alpha} \oplus c_2 \cdot \bar{\alpha}$

$\Rightarrow$ condition ② is not satisfied.

$\Rightarrow$ Axiom ③ is not completely satisfied.

Q2) Let V be the set of all complex-valued functions f on real line such that $\forall t \in R$, $f(-t) = f(t))^* = f^*(t)$, where $f^*(t)$ denotes the complex conjugation of $f(t)$.

(a) Show that V with operations $(f+g)(t) = f(t) + g(t)$ and $(cf)(t) = cf(t)$ is a vector space over the field R.

(b) Give an example of a function f_n in V which is not real-valued.

A(a) Using the axioms as defined in question 1's answer,

For axiom ① (existence of field of scalars),

R is a field of scalars
$\Rightarrow$ axiom ① holds

For axiom ② (existence of set of vectors),

Set of all complex-valued functions f is a set of vectors
$\Rightarrow$ axiom ② holds

For axiom ③ (vector addition),

For condition ⓐ (commutativity),

$$(f+g)t = f(t) + g(t) = g(t) + f(t) = (g+f)(t)$$

$\Rightarrow$ commutativity holds
$\Rightarrow$ condition ⓐ is satisfied

For condition ⓑ (associativity),

$$(f+(g+h))(t) = f(t)+(g+h)(t)$$
$$= f(t)+g(t)+h(t)$$
$$= [f(t)+g(t)]+h(t)$$
$$= (f+g)(t)+h(t)$$
$$= ((f+g)+h)(t)$$

$\Rightarrow (f+(g+h))(t) = ((f+g)+h)(t)$

$\Rightarrow$ associativity holds

$\Rightarrow$ condition ⓑ is satisfied

For condition ⓒ (additive identity),

Consider a function $g \in V$, such that $g(t)=0$, $\forall t \in R$.

~~$\Rightarrow f(t)+g(t)$~~ -

$\Rightarrow (f+g)(t) = f(t)+g(t) = f(t), \; \forall t \in R, f \in V.$

$\Rightarrow$ additive identity exists

$\Rightarrow$ condition ⓒ is satisfied

For condition ⓓ (additive inverse),

Consider a function $-f \in V$

$$\Rightarrow (f+(-f))(t) = f(t)+(-f(t))$$
$$= 0$$

$\Rightarrow$ additive inverse exists

$\Rightarrow$ condition d is satisfied

$\Rightarrow$ Axiom ③ is satisfied

~~For~~

For axiom (b) (scalar multiplication),

For condition (a) ($1\alpha = \alpha$),

$1 \in R$,

$\forall f \in V$,

$1 \cdot f(t) = f(t)$; $\forall t \in R$

$\Rightarrow$ Multiplicative identity '1' exists $\forall f \in V$

$\Rightarrow$ condition (a) is satisfied

For condition (b) ($(c_1 c_2)\alpha = c_1(c_2\alpha)$),

Consider ~~f_1, f~~ $c_1, c_2 \in R$,

$(c_1 c_2)(f(t)) = c_1 c_2 f(t)$, $\forall f \in V$, $t \in R$

$c_1(c_2(f(t))) = c_1 c_2 f(t)$, $\forall f \in V$, $t \in R$

$\Rightarrow (c_1 c_2) f(t) = c_1(c_2 f(t))$ $\forall f \in V$, $t \in R$

$\Rightarrow$ Condition (b) is satisfied

For condition (c) ($c(\alpha+\beta) = c\alpha + c\beta$),

Consider $c \in R$, $f, g \in V$, $t \in R$

~~$c(f(t) + g(t)) = cf(t) + cg(t)$~~

$c(f(t) + g(t)) = c[(f+g)(t)]$

$= cf(t) + cg(t)$

$\Rightarrow$ Condition (c) is satisfied.

For condition ② $((c_1+c_2)\alpha = c_1\alpha + c_2\alpha)$,

$\forall f \in V,\ c, d \in R,$

$((c+d)f)(t) = ((c+d)f(t)) = cf(t) + df(t)$

$= (cf + df)(t)$

$\Rightarrow$ Condition ② is satisfied.

$\Rightarrow$ Axiom ⑤ is satisfied

$\Rightarrow$ V is a vector space over field R.

1(b) $f(t) = it$ is a function such that $f \in V$ which is not real-valued. Here, i refers to $\sqrt{-1}$.

Q3) Prove the following theorem:

A non-empty subset W of vector space V is a subspace of V iff, for each pair of vectors $\bar{\alpha}, \bar{\beta} \in W$ and each scalar $c \in F$, the vector $c\bar{\alpha} + \bar{\beta} \in W$.

A To prove the theorem, we must prove:

i) the vector $c\alpha + \beta \in W$ if non-empty subset W of vector space V is a subspace of V.

ii) Non-empty subset W of vector space V is a subspace of V if $c\bar{\alpha} + \bar{\beta} \in W$

For ①,

If W is a subspace of V, $\alpha, \beta \in W$, $c \in F$, we know that vector addition exists in V and is closed generating vector sum:

$$\therefore \alpha + \beta \in W, \forall \alpha, \beta \in W$$

We also known that scalar multiplication exists in V generating a product vector:

$$c\alpha \in W, \forall \alpha \in W, c \in F.$$

$$\Rightarrow c\alpha + \beta \in W, \forall \alpha, \beta \in W, c \in F.$$

For (ii),

If W is a non-empty subset of V such that $c\alpha+\beta \in W$, we can conclude that, since the operations of addition and multiplication are the same for V and W, the properties of the operations are followed (i.e. closure under addition and multiplication, commutativity under vector addition, associativity under vector addition, vector addn. distributivity of scalars in ~~scalar multiplication~~, distributivity of vectors in scalar addition, $(c_1c_2)\alpha = c_1(c_2\alpha)$).

We also know that, since W is non-empty, it contains at least one vector $\gamma \in W$.

As $c\alpha+\beta \in W$, assuming $c=-1$, $\alpha=\gamma$, $\beta=\gamma$,

$$(-1)\gamma + \gamma \in W$$
$$\Rightarrow 0 \in W$$

$\Rightarrow$ the additive identity exists.

Assuming $c \in F$, ~~$\alpha \in W$~~, $\alpha=\gamma$, $\beta=0$,

$$c\gamma + 0 \in W$$
$$\Rightarrow c\gamma \in W$$

Further, assuming $c=-1$,

$$-1(\gamma) \in W, \ \gamma \in W.$$
$$\Rightarrow -\gamma \in W$$

$\Rightarrow$ the additive inverse is in W.

Finally, if $\alpha, \beta \in W$,

$$\alpha + \beta = 1\alpha + \beta \in W$$

$\Rightarrow$ Multiplicative identity '1' is in W.

$\Rightarrow$ W is a subspace of V.

Hence, as both (i) and (ii) are true, non-empty subset W of V is a subspace of iff for each pair of vectors $\alpha, \beta \in W$, $c \in F$, vector $c\alpha + \beta \in W$.

Hence Proved.