ISHAAN ROMIL
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Q2) To each elementary row operation e, there corresponds an elementary row operation $e_1$ such that

$$e_1(e(A)) = e(e_1(A)) = A$$

for any matrix A,
Here $e_1$ is the same type of operation as e.

A: There exist three elementary row operations on an $m \times n$ matrix, A, over the field F:

① multiplication of one row of A by a non-zero scalar c
i.e. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = cA_{rj}$

② replacement of the $r^{th}$ row of A by row r plus c times ~~rows~~ row s, with c being any scalar and $r \neq s$
i.e. $e(A)_{ij} = A_{ij}$ if $i \neq r$, $e(A)_{rj} = A_{rj} + cA_{sj}$

③ interchange of two rows of A
i.e. $e(A)_{ij} = A_{ij}$ if i is different from both r and s, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$

Considering each operation a separate case,

Consider case ①,

If $e(A)_{ij} = A_{ij}$ for $i \neq r$ and $e(A)_{rj} = cA_{rj}$,
an operation exists such that scalar $c_1 = \frac{1}{c} = c^{-1}$
is ~~multiply~~ multiplied by a matrix

i.e. $e_1(A)_{ij} = A_{ij}$ for $i \neq r$, $e_1(A)_{rj} = \frac{1}{c} A_{rj}$

$e_1(e(A)_{ij}) = A_{ij}$ for $i \neq r$, $e_1(e(A)_{ij}) = \frac{1}{c}(c)(A_{rj})$

$= A_{rj}$

$e(e_1(A)_{ij}) = A_{ij}$ for $i \neq r$, $e(e_1(A)_{ij}) = c\left(\frac{1}{c}\right)(A_{rj})$

$= A_{rj}$

$\Rightarrow$ for case ①, there exists a rule $e_1$ such that $e(e_1(A)) = e_1(e(A)) = A$.

Consider case ②,

If $e(A)_{ij} = A_{ij}$ for $i \neq r$, $e(A)_{rj} = A_{rj} + cA_{sj}$, there exists an operation where $cA_{sj}$ is subtracted from $e(A)_{rj}$ instead of added.

i.e. $e_1(A)_{ij} = A_{ij}$ for $i \neq r$, $e_1(A)_{rj} = A_{rj} - cA_{sj}$

$e_1(e(A)_{ij}) = A_{ij}$ for $i \neq r$, $e_1(e(A)_{rj}) = A_{rj} + cA_{sj} - cA_{sj}$

$= A_{rj}$

$e(e_1(A)_{ij}) = A_{ij}$ for $i \neq r$, $e(e_1(A)_{rj}) = A_{rj} + cA_{sj} - cA_{sj}$

$= A_{rj}$

$\Rightarrow$ for case ②, there exists a rule $e_1$ such that $e(e_1(A)) = e_1(e(A)) = A$.

Consider case ③,

If $e(A)_{ij} = A_{ij}$, $i \neq r$; $i \neq s$, $e(A)_{rj} = A_{sj}$, $e(A)_{sj} = A_{rj}$
there exists an inverse operation $e_1$ such that
$e_1(A)_{sj} = A_{rj}$ and $e_1(A)_{rj} = A_{sj}$

ie. for $i \neq r, i \neq s$, $e_1(e(A)_{sj}) = A_{sj}$ and
$$e(e_1(A)_{rj}) = A_{rj}$$

for all other cases, ~~e(A~~ $e_1(e(A)_{ij}) = A_{ij}$ and
$$e(e_1(A)_{ij}) = A_{ij}$$

$\Rightarrow$ for case ③, there exists a rule $e_1$ such that
$e_1(e(A)) = A$ and $e(e_1(A)) = A$.

Thus, we can conclude that for every elementary row operation $e$, there exists an elementary row operation $e_1$ such that
$$e_1(e(A)) = A = e(e_1(A))$$

Hence proved

Date

Q3. Find whether the given system of linear equations are equivalent or not.

a) $\cdot\ x + 2y = 18, \quad -x + 11y = 23$

$\cdot\ -2x + 9y = 5, \quad 8x - 10y = 62$

b) $\cdot\ x + y + z = 6, \quad y + 2z = 5, \quad x + 3z = 6$

$\cdot\ 3x + 2y - z = 12, \quad 3x + y - z = 10, \quad y + z = 3$

$\cdot\ x + y - z = 4, \quad 2x + y = 8, \quad x - z = 2$

A: Two $m \times n$ matrices A and B are said to be row-equivalent if they can be obtained from the other by a finite sequence of elementary row operations. Specifically, matrix A is said to be row-equivalent to matrix B if B can be obtained from A by a finite sequence of elementary row operations.

We know that row-equivalence is a equivalence relation. A equivalence relation is a binary relation which is reflexive, transitive, and symmetric.

$\Rightarrow$ row-equivalence is transitive.

Thus, if matrix A is row-equivalent to a row-reduced echelon matrix, and B is also row-equivalent to the same row-reduced echelon matrix, we can say that A is row equivalent to B.

We can verify this as we know that for every elementary row operation e, there corresponds an elementary row operation $e_1$ such that:

$$e(e_1(A)) = e_1(e(A)) = A$$

Thus if the row-reduced echelon matrix of A is the same as the row-reduced echelon matrix of B, B can be obtained from A via performing the inverse of the operations required to transform B to the row-reduced echelon matrix.

$$\xrightarrow{e_5'} \xrightarrow{e_4'} \xrightarrow{e_3'}$$

$$A \xrightarrow{e} \xrightarrow{e_1} \xrightarrow{e_2} \text{row-reduced echelon matrix} \xleftarrow{e_5} \xleftarrow{e_4} \xleftarrow{e_3} B$$

3(a) Solving via reduction to row-reduced echelon form,

Representing ~~they~~ the following system, denoted 1.1 ~~as~~ and 1.2, as matrices,

$$1.1 \begin{cases} x + 2y = 18 \\ -x + 11y = 23 \end{cases} \equiv \begin{bmatrix} 1 & 2 & 18 \\ -1 & 11 & 23 \end{bmatrix}$$

$$1.2 \begin{cases} -2x + 9y = 5 \\ 8x - 10y = 62 \end{cases} \equiv \begin{bmatrix} -2 & 9 & 5 \\ 8 & -10 & 62 \end{bmatrix}$$

Taking 1.1,

$$\begin{bmatrix} 1 & 2 & 18 \\ -1 & 11 & 23 \end{bmatrix}$$

$$\Big\downarrow \; ② = ② + ①$$

$$\begin{bmatrix} 1 & 2 & 18 \\ 0 & 13 & 41 \end{bmatrix}$$

$$\Big\downarrow \; ② = ② \times \frac{1}{13}$$

$$\begin{bmatrix} 1 & 2 & 18 \\ 0 & 1 & 41/13 \end{bmatrix}$$

$$\Big\downarrow \; ① = ① - 2 \times ②$$

$$\begin{bmatrix} 1 & 0 & 152/13 \\ 0 & 1 & 41/13 \end{bmatrix} = A^*$$

Taking (1.2),

$$\begin{bmatrix} -2 & 9 & 5 \\ 8 & -10 & 62 \end{bmatrix}$$

$$\Big\downarrow \; ① = ① \times \left(\frac{-1}{2}\right)$$

$$\begin{bmatrix} 1 & -9/2 & -5/2 \\ 8 & -10 & 62 \end{bmatrix}$$

$$\Big\downarrow \; ② = ② - 8 \times ①$$

$$\begin{bmatrix} 1 & -9/2 & -5/2 \\ 0 & 26 & 82 \end{bmatrix}$$

$$\Big\downarrow \; ② = ② \times \frac{1}{26}$$

$$\begin{bmatrix} 1 & -9/2 & -5/2 \\ 0 & 1 & 41/13 \end{bmatrix}$$

$$\Big\downarrow \; ① = ① + \frac{9}{2} \times ②$$

$$\begin{bmatrix} 1 & 0 & 152/13 \\ 0 & 1 & 41/13 \end{bmatrix} = B$$

As the row-reduced echelon matrix of ~~A~~ (1.1) and (1.2) are equal, and recognizing row-equivalence as a transitive relation, we can say that the given systems of linear equations.

3b) Solving via reduction to row-reduced echelon form,

Representing the following systems, denoted 2.1, 2.2, and 2.3, as matrices,

$$(2.1)\begin{cases} x+y+z=6 \\ y+2z=5 \\ x+3z=6 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 1 & 0 & 3 & 6 \end{bmatrix}$$

$$(2.2)\begin{cases} 3x+2y-z=12 \\ 3x+y-z=10 \\ y+z=3 \end{cases} \Rightarrow \begin{bmatrix} 3 & 2 & -1 & 12 \\ 3 & 1 & -1 & 10 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$(2.3)\begin{cases} x+y-z=4 \\ 2x+y=8 \\ x-z=2 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 0 & 8 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

Taking (2.1),

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 1 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{③=③-①} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{③=③+②} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 5 \end{bmatrix} \xrightarrow{①=①-②} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

$\downarrow$

$$\xrightarrow{③ = ③ \times \frac{1}{4}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix} \xrightarrow{① = ① + ③} \begin{bmatrix} 1 & 0 & 0 & 9/4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix}$$

$$\Big\downarrow \; ② = ② - 2 \times ③$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 5/4 \end{bmatrix}$$

Taking 2.2,

$$\begin{bmatrix} 3 & 2 & -1 & 12 \\ 3 & 1 & -1 & 10 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{1 = 1 \times \frac{1}{3}} \begin{bmatrix} 1 & 2/3 & -1/3 & 4 \\ 3 & 1 & -1 & 10 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{2 = 2 - 3 \times 1} \begin{bmatrix} 1 & 2/3 & -1/3 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{2 = 2 \times (-1)} \begin{bmatrix} 1 & 2/3 & -1/3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{3 = 3 - 2} \begin{bmatrix} 1 & 2/3 & -1/3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{1 = 1 - \frac{2}{3} \times 2} \begin{bmatrix} 1 & 0 & -1/3 & 8/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{1 = 1 + 3 \times \frac{1}{3}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Taking (2.3),

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 0 & 8 \\ 1 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{(2) = (2) - 2\times(1)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{(3) = (3) - (1)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix} \xrightarrow{(2) = (2)\times(-1)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{(3) = (3) + (2)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \xrightarrow{(1) = (1) - (2)} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{(3)\times\left(-\frac{1}{2}\right) = (3)} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(1) = (1) - (3)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{(2) = (2) + 2\times(3)} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

As the row-reduced echelon matrix of (2.1) is not ~~the same~~ equal to as the row-reduced matrix of (2.2) and (2.3), we can say that the given systems of linear equations are not equivalent.

Q5) Prove that every matrix has a row-reduced form.

A: A $m \times n$ matrix denoted $R$ is called row-reduced if it satisfies the following two conditions:

(a) The 1st non-zero entry in each non-zero row of $R$ is equal to 1

(b) each column of $R$ which contains the leading nonzero entry of some row has all its other ~~elements~~ entries equal to 0 (zero).

Let there exist an $m \times n$ matrix $R$.

Consider the first row of $R$. There are two cases for the entries present in $R$:

<u>Case 1</u>: Every entry of the first row of $R$ is 0

<u>Case 2</u>: There exists a non-zero entry in the first row of $R$.

Consider Case 1:

If every entry of the first row of $R$ is zero, then condition (a) is met for the first row of $R$ and we can consider row 2.

Consider Case 2:

If row 1 contains a non-zero entry, consider $k$, where $k$ is the smallest positive integer $j$

for which $R_{ij} \neq 0$. In other words, $R_{ik}$ is the first element in the first row which has a nonzero value. Multiply row 1 by $A_{1k}^{-1}$ such that $R_{ik}$ is now equal to 1, thus once more satisfying condition ⓐ. Further, to satisfy condition ⓑ, for each row $i \geq 1$, add $(-R_{ik})$ times row 1 to row 1. This ensures that every other entry in column $k$ aside from $R_{ik}$ is 0, allowing us to consider ~~cond~~ the second row.

Considering row 2, we can split it into the same cases as row 1:

<u>Case 1</u>: row 2 consists only of entries which are zero.

<u>Case 2</u>: row 2 consists of at least one non-zero entry.

Consider Case 1:

If every entry is zero, we leave row 2 unchanged as it satisfies condition ⓐ and thus, allows us to consider the next row.

Consider Case 2:

If the row contains a non-zero entry, we multiply row 2 by a scalar such that the leading non-zero element becomes 1 in a similar method to case 2 of the first row. ~~However~~ It is important to note that if row 1 contained a leading non-zero entry in column $k_1$, column $k_1$ cannot contain the leading nonzero entry of row 2 as all ~~the~~ entries of column $k_1$ except the non-zero entry in row 1 have been changed to 0 via elementary row transformations. Taking the column of the leading nonzero element of row 2, denoted $k_2$, we arrange all entries of column $k_2$ as 0 via the same technique applied in case 2 of the first row. We can now consider further rows.

For successive rows, the same technique is followed as was followed for row 2. It is important to note that the entries of row $i$ in column $k_i$ - that is, the leading non-zero elements of row $i$ - will not change because of successive elementary row operations. We will also leave every ~~the~~ entry in ~~columns $k_a$ unchanged, where $k$ represents every prior row~~. The aforementioned columns unchanged so as to meet condition (b) of the definition of a row-reduced matrix.

Following this method for all rows of matrix R, we can construct a row-reduced matrix of R.

Therefore, ~~for any~~ every matrix can be reduced to a row-reduced form and has a row-reduced form which can be reached by the above steps.

Hence proved.

~~Q5) Every $m \times n$ matrix~~

Q5) Prove that every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix.

A: Via Q4, we know that A is row-equivalent to a row-reduced matrix.

Taking the row-reduced matrix form of A, we can perform a finite number of row interchanges to achieve a row-reduced echelon matrix

An $m \times n$ matrix R is said to be ~~reduced if~~: a row-reduced echelon matrix if it satisfies the following conditions:

(a) R is row-reduced.

(b) every row of R which has all its entries 0 occurs ~~before~~ below every row which has a non-zero ~~entries~~ entry.

(c) if rows $1, \ldots, r$ are the non-zero rows of R and if the leading non-zero entry of row $i$ occurs in column $k_i$, $i = 1, \ldots, r$, then $k_1 < k_2 < \ldots < k_r$.

First, interchange the rows of the row-reduced matrix in such a way that they are located at the end. To do this assume there are $r$ rows containing a nonzero element and $k$ rows ~~of~~ consisting of only zeroes.

This leads to three cases:

1. $k = 0$
2. $r = 0$
3. $k \neq 0, r \neq 0$

Consider case 1,

Arrange the rows in such a way that ~~of the~~ the rows with leading nonzero elements towards the left are placed towards the top of matrix A. Mathematically,

Consider set $R = \{A_{ij}\}$ such that all elements of set R are leading nonzero elements of the matrix A. Via the elementary row operation of row interchange, place ~~column~~ row $i'$ as the first row where $i'$ is the row containing $A_{i'j'}$, where $j'$ represents the smallest possible value of $j$ in any element of set R.

Next, consider ~~the~~ $A_{i''j''}$, where $j''$ is the column number which is the second lowest in set R. Via row interchange, make $i''$ the second row.

Repeat this process for all $r$ rows until a row reduced echelon matrix is reached.

Consider case ②.

If all elements of matrix A are zero, A is already in row-reduced echelon form.

Consider case ③,

Using the elementary row operation of row interchange, interchange every row containing all zero elements such that they are located at the bottom of ©. To do this, ~~replace~~ exchange row $k_i$ with the lowest nonzero row, where i represents the topmost row containing all zero elements. Perform this interchange successively for every one of the k rows. ~~starting~~

Next, perform the same operations as listed in case 1 for the remaining unsorted rows - i.e. the row which contain nonzero elements.

Therefore, it is possible to reduce any arbitrary $m \times n$ matrix to a row-reduced echelon matrix, and Thus, ~~any~~ ~~or~~ every arbitrary $m \times n$ matrix A is equivalent to a row-reduced echelon matrix.

Hence proved.