(1) $\begin{bmatrix} 1 & -2 & 0 & -1 & y_4-y_3 \\ 0 & 0 & 1 & 1 & y_3 \\ 0 & 0 & 0 & 0 & y_2+2y_4-3y_3 \\ 0 & 0 & 0 & 0 & y_1-3y_4+y_3 \end{bmatrix}$

cond^n for solution of $AX=Y$ are:

$y_1 + 0\cdot y_2 + y_3 - 3\cdot y_4 = 0$

$0\cdot y_1 + y_2 - 3y_3 + 2y_4 = 0$

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Already RREF, so assigning y_3 & y_4 arbitrary values y_3^c & y_4^c

$y_1 + y_3^c - 3y_4^c = 0 \Rightarrow y_1 = -y_3^c + 3y_4^c$

and

$y_2 - 3y_3^c + 2y_4^c = 0 \Rightarrow y_2 = 3y_3^c - 2y_4^c$

system has sol^n when:

$Y = (-y_3^c + 3y_4^c, 3y_3^c - 2y_4^c, y_3^c, y_4^c)$

where y_3^c & y_4^c are arbitrary

Q3) $\Rightarrow$ Assume $b \in F$ and

$T = \{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$

where T is a subspace of F^4

$\because$ T is a subspace by assumption that means $0 \in F$

$\Rightarrow (x_1, x_2, x_3, x_4) = (0,0,0,0)$

$\Rightarrow x_3$ & x_4 both 0

$\Rightarrow 0 = 5\cdot(0) + b$

$\Rightarrow b = 0$

$\Leftarrow$ Given $b=0$

$x_3 = 5x_4$

$A = (a_1, a_2, a_3, a_4) \in F^4$

$B = (b_1\ b_2\ b_3\ b_4) \in F^4$

$cA + B \in F^4 \ \forall c \in F$

Also, $a_3 = 5a_4$

$b_3 = 5b_4$

$cA + B = (\underbrace{ca_1+b_1}_{\in F}, \underbrace{ca_2+b_2}_{\in F}, c5a_4+5b_4, \underbrace{ca_4+b_4}_{\in 4})$

$5(\underbrace{ca_4+b_4}_{\in F})$

Q2)

c) Each element v in a vector space has an additive inverse.

For unique:

Let w & w' be inverses of v

$w = w + 0 = w + (w' + v)$

$= (w+v) + w'$

$= 0 + w'$

$= w'$

$w = w' \rightarrow$ unique

d) Let $\overset{w}{0}$ & $\overset{w'}{0'}$ be additive identities

$0' = \underbrace{0' + 0}_{0 \text{ is additive identity}} = \underbrace{0 + 0'}_{\text{commutativity}} = 0 \rightarrow$ since $0'$ is additive identity

$\Rightarrow 0 = 0'$

b) By def^n,

$(-v) + (-(-v)) = 0$ and

$v + (-v) = 0$

$\Rightarrow v$ & $(-(-v))$ are additive inverses of $-v$

& by uniqueness $v = (-(-v))$

a) $a = 0$, then done

if $a \neq 0$, then a^{-1} exists s.t $a^{-1}a = 1$

$v = 1\cdot v = (a^{-1}a)v = a^{-1}(av) \rightarrow$ associativity

$= a^{-1}\cdot 0 = 0$

Q4) $\Rightarrow$ Given: A is invertible.

$\rightarrow$ A should not have a zero column, if A

BA would also have a zero column & then $BA \neq I$

has a zero column.

$\Rightarrow a$ or c is not zero

Let $a \neq 0$ $\quad R_2 \leftarrow R_2 - \frac{c}{a}R_1$

$\begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \rightarrow$ should be I to be invertible

$\Rightarrow d - \frac{bc}{a} \neq 0$

$\Leftarrow$ given $ad - bc \neq 0$

A is invertible only if it can be transformed to I by elementary row op^n

$\begin{pmatrix} 1 & b/a \\ 0 & \frac{d-bc}{a} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$

$\frac{d-bc}{a} \neq 0$

$\Downarrow \ R_1 \leftarrow R_1 + \left(\frac{-b}{a}\right)R_2$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$