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ASSIGNMENT - 5

1) a) Assume the matrix $A = \begin{pmatrix} 1 & h & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let us take $A - \lambda I$.

The characteristic equation is $\det A - \lambda I = 0$

$$\Rightarrow \det \begin{pmatrix} 1 & h & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \det \begin{pmatrix} 1 & h & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = 0$$

$$\Rightarrow \det \begin{pmatrix} 1-\lambda & h & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = \{1, 2\}$$

For $\lambda_1 = 1$,

$$(A - I \mid 0) = \left(\begin{array}{ccc|c} 0 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left[\because \begin{pmatrix} 1 & h & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right. - ①$$

For an ~~eigenvector~~ eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we look at equations

formed from ①:

$$x_2 = 0 \quad - (i)$$
$$2x_2 = 0 \quad - (ii)$$

Setting $x_1 = s$, $x_3 = t$

$$E_0 = \left\{ \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow E_0 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

For $\lambda_2 = 2$;

~~$(A - 2I \mid 0) =$~~

Take $A - 2I = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$(A - 2I \mid 0) = \left(\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right) \quad - ②$$

For ~~eigen~~vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we get (from the equations derived from ①):

$$-x_1 + k x_2 = 0$$
$$\Rightarrow x_1 = k x_2 \quad - ⓐ$$
and
$$x_3 = 0 \quad - ⓑ$$

Setting $x_2 = s$, $x_1 = ks$,

$$E_6 = \left\{ \begin{bmatrix} ks \\ s \\ 0 \end{bmatrix} \right\} = \left\{ s \cdot \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix} \right)$$

Hence, the eigenvectors are:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix}$$

$\forall k$, this forms a linearly independent set as

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = \begin{bmatrix} c_1 + k c_3 \\ c_3 \\ c_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ if and only if } c_1 = c_2 = c_3 = 0$$

Recalling the definition of linearly independent vectors as a set of vectors $\{v_1, v_2, \ldots\}$ such that:

$$c_1v_1 + \ldots + c_nv_n = 0 \text{ iff } c_1 = \ldots = c_n = 0,$$ we know that the above vectors are linearly independent.

Here, A is diagonalizable $\forall\, h \in R$

(c) Assume $A = \begin{pmatrix} 1 & 1 & h \\ 1 & 1 & h \\ 1 & 1 & h \end{pmatrix}$

The characteristic equation is:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{pmatrix} 1 & 1 & h \\ 1 & 1 & h \\ 1 & 1 & h \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$\Rightarrow \det\left(\begin{pmatrix} 1 & 1 & h \\ 1 & 1 & h \\ 1 & 1 & h \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$\Rightarrow \det\begin{pmatrix} 1-\lambda & 1 & h \\ 1 & 1-\lambda & h \\ 1 & 1 & h-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left((1-\lambda)(h-\lambda) - h\right) - (h-\lambda-h) + h(1-(1-\lambda)) = 0$$

$$\Rightarrow (1-\lambda)(h - h\lambda + \lambda^2 - \lambda - h) + \lambda + h\lambda = 0$$

$$\Rightarrow -\lambda^3 + (h+1)\lambda^2 + \lambda^2 - (h+1)\lambda + (h+1)\lambda = 0$$

$\Rightarrow \lambda^3 - (k+2)\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - k - 2) = 0$

$\Rightarrow \lambda = \{0, k+2\}$

For $\lambda_1 = 0$,

Take $A - 0I = \begin{pmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{pmatrix} - 0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & k \end{pmatrix}$

$\Rightarrow (A - 0I \mid 0) = \left(\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \end{array}\right) - ①$

For ~~eigen~~vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we take the equations

derived from ①:

$x_1 + x_2 + kx_3 = 0 - ①$

Setting $x_2 = s$, $x_3 = t$, $x_1 = -(s + kt)$

$E_0 = \left\{ \begin{bmatrix} -s-kt \\ s \\ t \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -k \\ 0 \\ 1 \end{bmatrix} \right\}$

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$$\Rightarrow E_0 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -h \\ 0 \\ 1 \end{bmatrix} \right)$$

For $\lambda_2 = h+2$,

Take $A - (h+2)I = \begin{pmatrix} 1 & 1 & h \\ 1 & 1 & h \\ 1 & 1 & h \end{pmatrix} - (h+2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & h \\ 1 & 1 & h \\ 1 & 1 & h \end{pmatrix} - \begin{pmatrix} h+2 & 0 & 0 \\ 0 & h+2 & 0 \\ 0 & 0 & h+2 \end{pmatrix}$$

$$= \begin{pmatrix} -1-h & 1 & h \\ 1 & -1-h & h \\ 1 & 1 & -2 \end{pmatrix}$$

$$\Rightarrow (A-(h+2)I \mid 0) = \left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ 1 & -1-h & h & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\Big\downarrow R_2 = R_1 + R_2$$

$$\left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ -h & -h & 2h & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\Big\downarrow R_2 = R_2 \times (-1/h)$$

$$\left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\Big\downarrow R_3 = R_3 - R_2$$

$$\left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow R_2 = R_2 - R_1$$

$$\left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ 2+h & 0 & -2-h & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow R_2 = R_2 \times \left(\frac{1}{2+h}\right)$$

$$\left(\begin{array}{ccc|c} -1-h & 1 & h & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) - ②$$

For vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we look at the conditions

derived from ②:

$x_1 - x_3 = 0$

$\Rightarrow x_1 = x_3 - ⓐ$

$(-1-h)x_1 + x_2 + hx_3 = 0$

$\Rightarrow -x_1 + (h-h)x_1 + x_2 = 0 \quad [\because ⓐ: x_1 = x_3]$

$\Rightarrow x_1 = x_2 - ⓑ$

Setting $x_1 = x_2 = x_3 = t$

$$E_0 = \left\{\begin{bmatrix} t \\ t \\ t \end{bmatrix}\right\} = \left\{t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

The given eigenvectors would be:

$$e_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -k \\ 0 \\ 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Once again, we check for the linear independence of e_1, e_2, e_3:

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = \begin{bmatrix} -c_1 - kc_2 + c_3 \\ c_1 + c_3 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow c_3 = -c_1 = -c_2$

and $kc_3 + 2c_3 = 0$

$\Rightarrow k = -2$ and $c_3 = -c_1 = -c_2$

or $k \neq -2$ and $c_1 = c_2 = c_3 = 0$

$\therefore$ if $k \neq -2$, $\{e_1, e_2, e_3\}$ is linearly independent as $c_1 = c_2 = c_3 = 0$.

Hence, for A to be diagonalizable,

$$k \in R - \{-2\}$$

2a) Consider a matrix $A_{n \times n}$ to be diagonalizable

Recalling that, for A to be diagonalizable, the eigenspace of A must consist of n linearly independent vectors

Consider v_0 to be an eigenvector with λ_0 as its eigenvalue. Then, by, definition,

$$A v_0 = \lambda_0 v_0$$

Multiplying both sides by A^{-1},

$$(A^{-1} . A) v_0 = \lambda_0 (A^{-1} v_0)$$
$$\Rightarrow I . v_0 = \lambda_0 (A^{-1} . v_0)$$
$$\Rightarrow \frac{1}{\lambda_0} . v_0 = A^{-1} . v_0 \qquad \{ \lambda_0 \neq 0 \}$$

Similarly, eigenvectors of A^{-1} are same as of A with corresponding $\lambda \neq 0$.

For $\lambda = 0$,

$$\det(A - 0) = 0$$
$$\Rightarrow \det A = 0$$

However, A is invertible and we know that if A is invertible, $\det A \neq 0$.

Hence, $\lambda \neq 0$ for A.

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Now, since the eigenspace of A^{-1} has the same vectors as A, they are linearly independent to each other.

$\therefore A^{-1}$ is diagonalizable,

Hence proved.

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Now, $A^{2015} = A^{2014} \cdot A$

$= [A^2]^{1007} \cdot A$

$= I^{1007} \cdot A = I \cdot A = A$

$$\therefore A^{2015} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3a) Given that the inner product $\langle x, y \rangle$ is defined as the dot product $x \cdot \bar{y}$.

~~Assume~~ Given $a = \begin{bmatrix} 1-i \\ 1+2i \end{bmatrix}$ and

$b = \begin{bmatrix} 2+i \\ 2 \end{bmatrix}$

To find the inner product,

$$(1-i)(\overline{2+i}) + (1+2i)(\bar{z}) = 0$$

$$\Rightarrow (1-i)(2-i) + (1+2i)(\bar{z}) = 0$$

$$\Rightarrow 2-i-2i+i^2 + (1+2i)(\bar{z}) = 0$$

$$\Rightarrow i^2 - 3i + 2 + (1+2i)(\bar{z}) = 0$$

$$\Rightarrow -1-3i+2+(1+2i)(\bar{z}) = 0$$

$$\Rightarrow -3i+1+(1+2i)(\bar{z}) = 0$$

$$\bar{z} = \frac{3i-1}{2i+1}$$

Rationalizing,

$$\bar{z} = \frac{3i-1}{2i+1} \times \frac{2i-1}{2i-1}$$

$$= \frac{6i^2-5+1}{4i^2-1}$$

$$= \frac{6i^2-5i+1}{-5}$$

$$= \frac{-5-5i}{-5}$$

$\Rightarrow \bar{z} = i + 1$

$\Rightarrow z = i - 1$

Normalizing the vectors :

$$x = \frac{\vec{a}}{||\vec{a}||}, \quad y = \frac{\vec{b}}{||\vec{b}||}$$

$$||\vec{a}|| = \sqrt{\begin{bmatrix} 1 - i \\ 1 + 2i \end{bmatrix} \begin{bmatrix} 1 + i \\ 1 - 2i \end{bmatrix}}$$

$$= \sqrt{(1 - i)(1 + i) + (1 + 2i)(1 - 2i)}$$

$$= \sqrt{1 - i^2 + 1 - 4i^2}$$

$$= \sqrt{2 - 5i^2} = \sqrt{2 + 5} = \sqrt{7}$$

$$||\vec{b}|| = \sqrt{\begin{bmatrix} 2 + i \\ 1 - i \end{bmatrix} \begin{bmatrix} 2 - i \\ 1 + i \end{bmatrix}}$$

$$= \sqrt{(2 + i)(2 - i) + (1 - i)(1 + i)}$$

$$= \sqrt{4 - i^2 + 1 - i^2}$$

$$= \sqrt{5 - i^2}$$

$$= \sqrt{5 + 1}$$

$$= \sqrt{6}$$

$$q_1 = \frac{1}{\sqrt{7}}(1-i,\ 1+2i)$$
$$= \left(\frac{1-i}{\sqrt{7}},\ \frac{1+2i}{\sqrt{7}}\right)$$

$$q_2 = \frac{1}{\sqrt{6}}(2+i,\ 1-i)$$
$$= \left(\frac{2+i}{\sqrt{6}},\ \frac{1-i}{\sqrt{6}}\right)$$

b) The Gram-Schmidt process, when applied to a set of three vectors $\{a_1, a_2, a_3\}$ ~~yield~~, yields the same basis as the Gram-Schmidt applied to the vectors $\{a_3, a_2, a_1\}$.

Using the following vectors a_1, a_2, a_3:

Let $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix}$

Starting building the basis from a_1:

$$v_1 = a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v_2 = a_2 - \left(\frac{v_1 \cdot a_2}{v_1 \cdot v_1}\right) v_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 \\ 5/7 \\ -3/7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$$

$$v_3 = a_3 - \left(\frac{v_2 \cdot a_3}{v_2 \cdot v_2} \right) v_2 - \left(\frac{v_1 \cdot a_3}{v_1 \cdot v_1} \right) v_1$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{-3}{35} \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} - \frac{3}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3/35 - 3/14 \\ 15/35 - 3/7 \\ 1 - 9/35 - 9/14 \end{bmatrix}$$

$$= \begin{bmatrix} -21/70 \\ 0 \\ 7/70 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

From the above calculations, it can be concluded that our basis is:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} - ①$$

Building the basis starting from a_3 instead of a_1:

$$v_1 = a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = a_2 - \left(\frac{v_1 \cdot a_2}{v_1 \cdot v_1} \right) v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = a_1 - \left(\frac{v_2 \cdot a_1}{v_2 \cdot v_2} \right) v_2 - \left(\frac{v_1 \cdot a_1}{v_1 \cdot v_1} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From the above calculations, our basis turns out to be:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} - ②$$

As ① and ② are different, the statement is false.

Hence proved

(via counterexample).

4(b) Let D be a diagonal matrix such that

$$P^{-1}.M.P = D \qquad \text{(P is an arbitrary matrix)}$$

Squaring both sides,

$$(P^{-1}.M.P).(P^{-1}.M.P) = D^2$$
$$\Rightarrow P^{-1}.M.(P.P^{-1}).P = D^2 \qquad \left[\because \text{Using associativity of matrix multiplication}\right]$$
$$\Rightarrow P^{-1}.M.I.MP = D^2$$
$$\Rightarrow P^{-1}.M.M.P = D^2$$
$$\Rightarrow P^{-1}.M^2.P = D^2$$

Also, if D is a diagonal matrix,

$$\begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 \cdots & \alpha_3 \end{bmatrix},$$

D^2 is also a diagonal matrix

$\therefore M^2$ is also diagonalizable.

4b) If M^2 is diagonalizable, so is M.

Proving by counterexample,
we take $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can take any invertible matrix P,

$$P^{-1}M^2P = P^{-1}OP = 0$$

as $[0]$ is a diagonalizable matrix,
M^2 is diagonalizable

Finding P for M:

$$M - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

$$\det(M - \lambda I) = \lambda^2$$

$$\lambda_1 = 0, \quad \lambda_2 = 0$$

Taking $\lambda = 0$:

$$(M - 0I \mid 0) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_2 = 0$$
let $x_1 = t$

E_0 has basis $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

As the algebraic and geometric multiplicity of $\lambda = 0$ are not equal, i.e. there is only one corresponding eigenvector the matrix M is not diagonalizable although M^2 is.

Hence, the statement B is false.

Hence proved.