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ASSIGNMENT 6

Q1) Let m and n be positive integers and F be a field. Suppose W is a subspace of F^n and $\dim W \le m$. Then, there is precisely one $m \times n$ row reduced echelon matrix over F which has W as its row space. Prove it.

Source: Hoffman and Kunze

A: There is at least one $m \times n$ row-reduced echelon matrix with row space W.

As dimension of W, denoted $\dim W$, is less than or equal to m: $\dim W \le m$, there are at least m-vectors $\alpha_1, \alpha_2, \alpha_3, \ldots$ which can be selected in W such that the vectors span W.

Let A be $m \times n$ matrix with row vectors $\alpha_1, \alpha_2, \ldots \alpha_m$ in W, which is equivalent to a row-reduced echelon matrix which is row-equivalent to A, denoted as R. Then, the row space of R is W.

Assume R be any row-reduced echelon matrix which has W as its row space.

Let $\rho_1, \rho_2, \rho_3, \ldots, \rho_r$ be non-zero vectors of R and suppose the leading non-zero entry of ρ_i occurs in column k_i, $i = 1, \ldots, r$. The vectors $\rho_1, \ldots, \rho_r$ form a basis for W.

It has been observed that if $\beta = (b_1, \ldots b_2)$ is in W, then

$$\beta = c_1 \rho_1 + \ldots + c_r \rho_r$$

and

$$c_i = b_s$$

Thus, the unique expression for β as a linear combination of $\rho_1, \ldots, \rho_r$ is

$$\beta = \sum_{i=1}^{r} b_{si} \rho_i \quad \left[\because \beta = \sum_{i=1}^{r} c_i \rho_i \right]$$

We are given $R_{ij} = 0$ if $i > s$, $j \leq k$. Thus

$$\beta = (0, \ldots, b_{k_s}, \ldots b_n) \qquad [\text{given } b_{k_s} \neq 0]$$

This also demonstrates that the leading non-zero element of β occurs in column k_s. Note also that for each k_s, $s = 1, \ldots, r$, there exists a vector in W which has a non-zero k_s^{th} coordinate, namely ρ_s.

From these assertions, it can be derived that R is uniquely determined by W.

To describe R in terms of W,

~~Consider~~

Consider all vectors $\beta = (b_1, b_2, \ldots, b_n)$ in W.

If $\beta \neq 0$, the first non-zero occurence of β must come in column t.

i.e. $\beta = (0, 0, 0, \ldots, 0, b_t, \ldots, b_n)$, $b_t \neq 0$.

Let $k_1, k_2, \dots, k_r$ be positive integers t such that there exists some $\beta \neq 0$ in W the first non-zero occurence of which occurs in column t.

Arrange the aforementioned k_x, $1 \le x \le r$ in an ascending order, $k_1 < k_2 < \dots < k_r$.

For each positive integer k_x, there will be one and only one vector ρ_s in W such that the k_s^{th} coordinate of ρ_s is 1 and the k_i^{th} coordinate of $\rho_s = 0$ $(i \neq s)$.
Then, R is $r \times n$ matrix which has row vectors

$$\rho_1, \rho_2, \dots, \rho_r, 0, 0, \dots, 0.$$

Hence proved.

(Q2) Consider the vector space P_4 which consists of all polynomials of degree 4 or less with real number coefficients. W be the subspace of P_4 given by

$$W = \{ p(x) \in P_4 \mid p(1) + p(-1) = 0 \text{ and } p(2) + p(-2) = 0 \}$$

a) Find a basis for subspace W.
b) Determine dimension of W

A:(a) As P_4 consists of all polynomials of degree 4, let $p \in P$ be defined as:

$$p = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

To belong to subspace W, $p \in P$ must satisfy two conditions:

$$p(1) + p(-1) = 0 \quad - \text{①}$$
$$p(2) + p(-2) = 0 \quad - \text{②}$$

For ①,

$$p(1) + p(-1) = 0$$
$$\Rightarrow (c_0 + c_1 + c_2 + c_3 + c_4) + (c_0 - c_1 + c_2 - c_3 + c_4) = 0$$
$$\Rightarrow 2(c_0 + c_2 + c_4) = 0$$
$$\Rightarrow c_0 + c_2 + c_4 = 0 \quad - \text{(i)}$$

For (ii),

$$p(2) + p(-2) = 0$$

$$\Rightarrow (c_0 + 2c_1 + 2^2 c_2 + 2^3 c_3 + 2^4 c_4) + (c_0 - 2c_1 - 2^2 c_2 - 2^3 c_3 + 2^4 c_4) = 0$$

$$\Rightarrow 2(c_0 + 2^2 c_2 + 2^4 c_4) = 0$$

$$\Rightarrow c_0 + 2^2 c_2 + 2^4 c_4 = 0$$

$$\Rightarrow c_0 + 4c_2 + 16c_4 = 0. \quad - \text{(ii)}$$

We turn the given conditions into an augmented matrix and solve in the form $AX = 0$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix}$$

and $AX = 0$ is defined as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let us solve for A via row-reduction.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix}$$

$$\Big\downarrow \; (2) = (2) - 1 \times (1)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 15 \end{bmatrix}$$

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$$\xrightarrow{\textcircled{2} = \frac{1}{3} \times \textcircled{2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\Big\downarrow \textcircled{1} = \textcircled{1} - \textcircled{2}$$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix} = R$$

Using the row-reduced echelon form of A, denoted by R, we can substitute it back.

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_2 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad c_0 - 4c_4 = 0$
$\Rightarrow c_0 = 4c_4$

$\Rightarrow \quad c_2 + 5c_4 = 0$
$\Rightarrow c_2 = -5c_4$

The solution to the above system of equations is given by:

$$(c_0, c_2, c_4) = (4c, -5c, c)$$

Any $p \in W$ will therefore have the representation

$$p = c(x^4 - 5x^2 + 4) + a(x^3) + b(x)$$

As given by the above representation, coefficients of x^1 and x^3 will be independent.

Meanwhile, there will exist a relation between coefficients of x^4, x^2, x^0.

We also know that vectors $(x^4 - 5x^2 + 4)$, (x^3), (x) are linearly independent.

Thus, $\beta_W = \{(x^4 - 5x^2 + 4), (x^3), (x)\}$.

b) ~~Ba~~ Knowing that all basis have equal elements and that the dimension of W can be defined as the number of elements in the basis of W, and also noting that the basis has 3 elements, we can conclude the dimension of W is 3.

Q3) Suppose that a set of vectors $S_1 = \{v_1, v_2, v_3\}$ is spanning set of a subspace V in R^3. If v_4 is another vector in V, then is the set

$$S_2 = \{v_1, v_2, v_3, v_4\}$$

still a spanning set for V? If so, prove it. Otherwise, give a counterexample.

A: If set S_1 is a spanning set of subspace V to any vector $v \in V$, it can be represented as a linear combination of $\{v_1, v_2, v_3\}$.

If v is a linear combination of $\{v_1, v_2, v_3\}$, then by definition,

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 \quad - (1)$$

for some given values of c_1, c_2, c_3.

To prove, similarly, that ~~S_2~~ S_2 is also a spanning subset of V, we must show that any vector $v \in V$ can be expressed as a linear combination of the ~~se~~ vectors of S_2.

From (1), and given $v_4 \in V$, we know that v_4 can be expressed as a linear combination of v_1, v_2, v_3.

Consider a vector $v \in V$, where v is a linear combination of vectors in S_2,

ie. $v = d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4$

As coefficients of a vector in a linear combination can be zero,

Considering the case $d_4 = 0$ and thereby making a linear combination of v_1, v_2, v_3, we obtain the equation

$$v = d_1 v_1 + d_2 v_2 + d_3 v_3 + 0 v_4 \quad - ②$$

for any arbitrary d_1, d_2, d_3.

② can be used to ~~theref~~ claim that since coeff. of $v_4 = 0$, the vector v is essentially a linear combination of $\{v_1, v_2, v_3\}$.

Since $\{v_1, v_2, v_3\}$ has already been established as a spanning set of V, any $v \in V$ can therefore be expressed as v in ②.

Thus, every $v \in V$ is a linear combination of vectors of S_2, and by extension, S_2 is a spanning set of V.

Therefore, adding another vector to spanning set S_1, results in spanning set S_2 for same subspace V.

Hence proved.

Q5) Let V and W be subspaces of R^n such that $V \cap W = \{0\}$ and $\dim(V) + \dim(W) = n$.

a) If $v + w = 0$, then $v \in V$ and $w \in W$, then show that $v = 0$ and $w = 0$.

b) If B_1 is a basis for the subspace V and B_2 is a basis for the subspace W, then show that the union $B_1 \cup B_2$ is a basis for R^n.

c) If x is in R^n, show that x can be written in the form $x = v + w$, where $v \in V$ and $w \in W$.

d) Show that the representation obtained in part c) is unique

A.(a) Taking $v + w = 0$,

Adding the additive inverse of v, denoted $(-v)$ to both sides,

$$v + w + (-v) = -v$$
$$\Rightarrow w = -v \quad - (1)$$

Given the properties of fields and spaces, and given that $v \in V$, ~~$v \in W$~~ $-v \in V$
$$\Rightarrow w \in V \text{ (by (1))}$$

Given that $w \in W$ and $v \cap w = \{0\}$, it can therefore be concluded that $w = 0$.

Similarly, once more taking $v + w = 0$

Adding the additive inverse of w, denoted $(-w)$ to both sides,

$$v + w + (-w) = -w$$
$$\Rightarrow v = -w$$

Given the above discussed property (i.e., $-w \in W$ $\because w \in W$)

$$\Rightarrow v \in W$$
$$\Rightarrow v \in V \cap W$$

Given that $V \cap W = \{0\}$,

$$v = 0.$$

Hence proved.

b) From elementary set operations,

$$|B_1 \cup B_2| = |B_1| + |B_2| - |B_1 \cap B_2| \quad - (1)$$

However, the basis of a subspace doesn't contain 0 to ensure that the basis would become linearly independent.

Hence, $B_1 \cap B_2 = \emptyset$ $[\because B_1 \subseteq V,\ B_2 \subseteq W$ and $V \cap W = \{0\}]$

$$\Rightarrow |B_1 \cap B_2| = 0$$

Inserting into (1),

$$\Rightarrow |B_1 \cup B_2| = |B_1| + |B_2|$$
$$= \dim V + \dim W$$
$$= n.$$

Thus, $B_1 \cup B_2$ has n vectors.

To prove the theorem statement, it must be proved that B_1 and B_2 are linearly independent.

Considering, now, the elements of sets B_1 and B_2

$$B_1 = \{a_1, a_2, \ldots, a_m\}$$
$$B_2 = \{b_1, b_2, \ldots, b_r\}.$$

Knowing that the subspace of a basis contains the set of all linear combinations of the vectors of the basis.

Applying that in the context of V, we know that V contains all linear combinations of B_1 vectors but none of the combinations of B_2 vectors. Thus, the vectors of B_2 are not linear combinations of B_1 vectors.

To prove the linear ~~indepent~~ independent nature of B_1 and B_2, ~~append~~ adjoin b_1 to B_1

$$\Rightarrow B_1 \cup \{b_1\} = \{a_1, a_2, \ldots, a_r, b_1\}$$

This represents a linearly independent set as b_1 is not a linear combination of any element of B_1.

$$\Rightarrow B_1 \cup \{b_1\} \text{ is a linearly independent set.}$$

Next, adjoin b_2 to B_1

$$\Rightarrow B_1 \cup \{b_1, b_2\} = \{a_1, a_2, \ldots, a_n, b_1, b_2\}$$

Assume that the ~~above~~ linear combination of $B_1 \cup \{b_1, b_2\}$ belongs to B_2.

Then,

$$c_1 a_1 + c_2 a_2 + \ldots + c_n a_n + c_{n+1} b_n = b_2 \quad \text{—} \ ②$$

$$\Rightarrow c_1 a_1 + c_2 a_2 + \ldots + c_n a_n = b_2 - c_{n+1} b_n$$

Setting the right handside equal to W (via the closure property of vector spaces),

$$c_1 a_1 + c_2 a_2 + \ldots + c_n a_n \in W$$

However, by definition,

$$c_1 a_1 + c_2 a_2 + \ldots + c_n a_n \in V.$$

$$\Rightarrow c_1 a_1 + c_2 a_2 + \ldots + c_n a_n \in V \cap W = \{0\}$$

$$\Rightarrow c_1 a_1 + c_2 a_2 + \ldots + c_n a_n = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = \ldots = c_n = 0.$$

The ~~entire~~ equation ② then reduces to:

$$c_{n+1} B_1 = B_2$$

which is obviously false as B_1 and B_2 are linearly independent.

Therefore, by contradiction, $B_1 \cup \{b_1, b_2\}$ is also linearly independent.

Continuing this logic recursively, $B_1 \cup \{b_1, b_2, \dots, b_{n-1}\}$ is also a linearly independent set

$$\Rightarrow B_1 \cup B_2 \text{ is linearly independent.}$$

Knowing that the standard basis of R^n contains n elements and $B_1 \cup B_2$ also contains n elements, it can be claimed that $B_1 \cup B_2$ is also basis of R^n (given that $B_1 \cup B_2$ is an independent set).

c) As any vector from R^n can be expressed as a linear combination of $B_1 \cup B_2$,

$$x = c_1 a_1 + c_2 a_2 + \dots + c_r a_r + c_{r+1} b_1 + \dots + c_n b_n \quad — (1)$$

As B_1 and B_2 are disjoint sets and $B_1 \cup B_2$ is the union of disjoint sets, examining (1) and splitting it,

$$c_1 a_1 + c_2 a_2 + \dots + c_r a_r = v \in V \quad — (2)$$

and,

$$c_{r+1} B_1 + c_{r+2} B_2 + \dots + c_n B_{n-r} = w \in W \quad — (3)$$

Substituting (2) and (3) in (1),

$$v + w = x, \; v \in V, \; w \in W.$$

d) This statement can be proved via contradiction.

Assume that the ~~x~~ representation is not unique.

i.e. $x = v_1 + w_1 = v_2 + w_2$

Add $(-v_1 - w_1)$ to both sides,

$$v_1 - v_2 = w_2 - w_1$$

As $v_1 - v_2 \in V$ and $w_1 - v_2 \in W$, the two ~~can be~~ sides can only be equal if

$$v_1 - v_2 \text{ and } w_1 - w_2 \in V \cap W = 0$$

i.e. $v_1 - v_2 = 0$ and $w_1 - w_2 = 0$

$\Rightarrow v_1 = v_2$ and $w_1 = w_2$

Hence, by contradiction, $x = v_1 + w_1$ is a unique representation.

Hence proved.