MIDSEM ANSWER KEY

Q1-a)

let,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 1 & 4 & \lambda \end{bmatrix} \xrightarrow{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 0 & 0 & \lambda - 6 \end{bmatrix}$$

$$\Big\downarrow R_1' = R_2 - R_1$$

Now

upto this reduction Marks = 1.

$$\begin{bmatrix} 0 & 3 & 5 \\ 1 & 4 & 6 \\ 0 & 0 & \lambda - 6 \end{bmatrix} = B \text{ (say)}.$$

Now $\det B = 0$

$5 \cdot 4 \cdot (\lambda - 6) = 0$

if $\lambda \neq 6$ then the solution is unique.

upto this Marks = 2

$\lambda = 6.$

no solution

infinite many solutions.

$$\bar{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \cancel{\lambda}\,6 & \phi \end{bmatrix} \xrightarrow{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 6 & 20 \\ 0 & 0 & 0 & \phi - 20 \end{bmatrix}$$

if $\phi - 20 \neq 0$

then $\text{rank}(A) \neq \text{rank}(\bar{A})$

Hence $\phi \neq 20$ and $\lambda = 6$ implies that the system of linear equation have no solutions.

upto this marks = 4.

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Q1-b)

Q. If $AX = b$ always has at least one soln. show that the only soln to $A^T Y = 0$ is $Y = 0$.

→ ∵ $AX = b$ always has at least one soln

$\Rightarrow$ A is non-zero.

$\Rightarrow$ A^T is also non-zero.

To satisfy the eqn $A^T Y = 0$, the only possibility is for y to be zero.

If y is non-zero then $A^T Y$ would also be non-zero contradicting the eqn.

∴ The only soln that satisfy the eqn is $Y = 0$.

MIDSEM ANSWER KEY

Q2)

Q.2

Let $v \in \ker(T) \cap \mathrm{Im}(T)$,
ker $\Rightarrow$ kernel or nullspace, Im $\Rightarrow$ image or range

$\Rightarrow T(v) = 0$

Now, as $v \in \mathrm{Im}(T)$, there exists $u \in V$, such that $T(u) = v$.

(1M) Now, $T(T(u)) = T(v)$.
$\Rightarrow T(T(u)) = 0$
$\Rightarrow u \in \ker(T^2)$.

We know, $\ker(T) \subseteq \ker(T^2)$.
(1M) As $rk(T) = rk(T^2)$, by rank-nullity theorem,
$\mathrm{Null}(T^2) = \mathrm{Null}(T)$
$\Rightarrow \ker(T) = \ker(T^2)$

(1M) $\Rightarrow u \in \ker(T) \Rightarrow T(u) = 0$
$\Rightarrow v = 0$
Thus, only the 0 vector is common for the range & nullspace of T.

MIDSEM ANSWER KEY

Q3)

Let U and V be vector spaces. Then they are isomorphic iff there is a bijection from a basis of U to a basis of V. The isomorphism is the basis changer function.

This means that if U and V are finite-dimensional vector spaces, they are isomorphic iff $\dim(U) = \dim(V)$.

This also means that a finite-dimensional vector space cannot be isomorphic to an infinite-dimensional vector space, since there cannot be a bijection from a finite basis to an infinite basis.

Proof of 'only-if' part

Let P be a basis of U and Q be a basis of V. Let ϕ be a bijection from P to Q. This means that a basis-changer $T : U \mapsto V$ exists from P to Q. The basis changer is an isomorphic linear transformation. Therefore, U is isomorphic to V.

Proof of 'if' part

Let B be a basis of U. Let $T : U \mapsto V$ be an isomorphism. This means T is one-to-one, therefore, T is a bijection from B to $T(B)$.

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Part 1: $\mathrm{span}(T(B)) = T(U)$

Let $T(u) \in T(U)$, where $u \in U$. Since every element in U is representable as a linear combination of B, $u = \sum_{b \in B} a_b b$.

$$
\begin{aligned}
T(u) &= T\left(\sum_{b \in B} a_b b\right) \\
&= \sum_{b \in B} a_b T(b) && (\because T \text{ is a linear transformation}) \\
&\in \mathrm{span}(T(B))
\end{aligned}
$$

Therefore, $T(U) \subseteq \mathrm{span}(T(B))$.

Let $\sum_{b \in B} a_b T(b)$ be an element of $\mathrm{span}(T(B))$. Then

$$
\sum_{b \in B} a_b T(b) = T\left(\sum_{b \in B} a_b b\right) \in T(U)
$$

Therefore, $\mathrm{span}(T(B)) \subseteq T(U) \Rightarrow \mathrm{span}(T(B)) = T(U)$.

Part 2: $T(B)$ is linearly independent

$$
\sum_{b \in B} a_b T(b) = 0 \Rightarrow T\left(\sum_{b \in B} a_b b\right) = 0
$$

$$
T(0) = T(0+0) = T(0) + T(0) \implies T(0) = 0
$$

Since T is a bijection, $\sum_{b \in B} a_b b = 0$. Since B is linearly independent, $\forall b \in B, a_b = 0$. Therefore, $T(B)$ is linearly independent.

Part 3: Conclusion

Since $\mathrm{span}(T(B)) = T(U)$ and $T(B)$ is linearly independent, $T(B)$ is a basis of $T(U)$. Since T is onto, $T(U) = V$.

Therefore, T is a bijection from a basis of U (B) to a basis of V ($T(B)$).

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Q4-a)

1:27 Sat, 2 Mar 77%

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Midsem Q4 a

02 March 2024 13:02

Q4.

(a) • To prove range(T) is a subspace of W.

We need to show

① $0 \in \text{range}T$).

② if $w_1, w_2 \in \text{range}\,T$ then $w_1 + w_2 \in \text{range}\,T$

③ if $w \in \text{range}$ then $\lambda w \in \text{range}\,T \ \forall \lambda \in \mathbb{F}$

① $\text{range}\,T = \{T(v) : v \in V\}$ 1

$T : V \to W$.

V is VS $\Rightarrow 0 \in V$. Then,

$$T(0) = T(0+0) = T(0) + T(0)$$

$$\Rightarrow T(0) = 0$$

$$\Rightarrow 0 \in \text{range}$$

② 1

Let $w_1, w_2 \in \text{range}\,T$

$\Rightarrow \exists\ v_1 \in V$ such that $T(v_1) = w_1$

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(2) Let $w_1, w_2 \in \text{range } T$ 1

$\Rightarrow \exists\ v_1 \in V$ such that $T(v_1) = w_1$

and $\exists\ v_2 \in V$ such that $T(v_2) = w_2$.

$v_1 + v_2 \in V$ as V is a VS.

Then, $T(v_1 + v_2) = T(v_1) + T(v_2)$

$= w_1 + w_2$

$\Rightarrow w_1 + w_2 \in \text{range } T$.

$\Rightarrow$ range T is closed under addition.

(3) If $w \in \text{range } T$ then 0.5

$\exists\ v \in V$ such that $T(v) = w$.

$v \in V \Rightarrow \lambda v \in V$.

$T(\lambda v) = \lambda T(v) = \lambda w \in \text{range } T$.

$\Rightarrow T$ is closed under scalar multiplication.

From (i), (ii) and (iii) $\Rightarrow$ range T is a subspace of

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- $T: V \to W$ is one-one iff $\text{null } T = \{0\}$

T is one-one then $\text{null } T = \{0\}$. 1-25

We know from above $T(0) = 0$

$\Rightarrow 0 \in \text{null } T.$

$\Rightarrow \{0\} \subset \text{null } T.$

Suppose $v \in \text{null } T.$

$\Rightarrow T(v) = 0 = T(0)$

$\because$ T is injective (one-one).

$T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

$\Rightarrow v = 0.$

$\Rightarrow \text{null } T = \{0\}.$

$\text{null } T = \{0\}$ then T is injective: 1-25

Let $v_1, v_2 \in \text{null } \{0\}.$

Then, $T(v_1) = 0$

$T(v_2) = 0.$

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1:28 Sat, 2 Mar 77%

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null $T = \{0\}$ then T is injective: 1.25

Let $v_1, v_2 \in$ null $\{0\}$.

Then, $T(v_1) = 0$

$T(v_2) = 0$.

Consider $T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 \in \text{null } T$

But null of T contains only 0.

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$

We show $T(v_1) = T(v_2)$

$\Rightarrow v_1 = v_2$

Thus, T is injective.

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Q4-b)

7:06 Tue, 5 Mar 99%

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$$A = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix} \right\} \qquad B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

$$\quad |A_1\rangle \quad |A_2\rangle \qquad\qquad |B_1\rangle \quad |B_2\rangle$$

* using the notation $|a\rangle$ to denote that a is a vector instead of $\vec{a}$.

let $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Rightarrow 1 = 2\alpha_1 + \beta_1$ (1) $\qquad \Rightarrow 3 = 6\alpha_1 + 3\beta_1$ (3)

$2 = \alpha_1 + 3\beta_1$ (2) $\qquad \Rightarrow 4 = 2\alpha_1 + 6\beta_1$ (4)

(3) − (2) $\qquad$ (4) − (1)

$1 = 5\alpha_1 \Rightarrow \alpha_1 = \frac{1}{5} \qquad 3 = 5\beta_1 \Rightarrow \beta_1 = \frac{3}{5}$

$\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$-2 = 2\alpha_2 + \beta_2$ (1) $\qquad -6 = 6\alpha_2 + 3\beta_2$ (3)

$-3 = \alpha_2 + 3\beta_2$ (2) $\qquad -6 = 2\alpha_2 + 6\beta_2$ (4)

(3) − (2) $\qquad$ (4) − (1)

$-3 = 5\alpha_2 \Rightarrow \alpha_2 = \frac{-3}{5} \qquad -4 = 5\beta_2 \Rightarrow \beta_2 = \frac{-4}{5}$

Thus, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = |A_1\rangle$

& $\begin{bmatrix} -2 \\ -3 \end{bmatrix} = -\frac{3}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = |A_2\rangle$

1 mark for finding how to represent A's basis as LC of B's.

Then, $|v\rangle = \underbrace{v^1|A_1\rangle + v^2|A_2\rangle}_{|v\rangle \text{ in A basis}} = \underbrace{v^{\bar{1}}|B_1\rangle + v^{\bar{2}}|B_2\rangle}_{|v\rangle \text{ in B basis}}$

$$\Rightarrow v^1\left[\frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] + v^2\left[-\frac{3}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right] = v^{\bar{1}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + v^{\bar{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \left(\frac{1}{5}v^1 - \frac{3}{5}v^2\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left(\frac{3}{5}v^1 - \frac{4}{5}v^2\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = v^{\bar{1}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + v^{\bar{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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$$\Rightarrow \left(\frac{1}{5}v^1 - \frac{3}{5}v^2\right)\begin{bmatrix}2\\1\end{bmatrix} + \left(\frac{3}{5}v^1 - \frac{4}{5}v^2\right)\begin{bmatrix}1\\3\end{bmatrix} = v^{\bar{1}}\begin{bmatrix}2\\1\end{bmatrix} + v^{\bar{2}}\begin{bmatrix}1\\3\end{bmatrix}$$

$$\Rightarrow \quad v^{\bar{1}} = \frac{1}{5}v^1 - \frac{3}{5}v^2, \qquad v^{\bar{2}} = \frac{3}{5}v^1 - \frac{4}{5}v^2 \quad \Rightarrow \begin{bmatrix}v^{\bar{1}}\\ v^{\bar{2}}\end{bmatrix} = \underbrace{\begin{bmatrix}1/5 & -3/5\\ 3/5 & -4/5\end{bmatrix}}_{P}\begin{bmatrix}v^1\\ v^2\end{bmatrix}$$

1 marks for final form

$$\begin{pmatrix}2\\1\end{pmatrix} = \alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \beta_1\begin{pmatrix}-2\\-3\end{pmatrix}$$

$$\Rightarrow \quad 2 = \alpha_1 - 2\beta_1 \quad (1) \qquad 4 = 2\alpha_1 - 4\beta_1 \quad (3)$$
$$1 = 2\alpha_1 - 3\beta_1 \quad (2)$$

$(2) - (3)$

$-3 = \beta_1$ Then $\alpha_1 = -4$

$$\begin{pmatrix}1\\3\end{pmatrix} = \alpha_2\begin{pmatrix}1\\2\end{pmatrix} + \beta_2\begin{pmatrix}-2\\-3\end{pmatrix}$$

$$\Rightarrow \quad 1 = \alpha_2 - 2\beta_2 \quad (1) \quad \Rightarrow 2 = 2\alpha_2 - 4\beta_2 \quad (3)$$
$$3 = 2\alpha_2 - 3\beta_2 \quad (2)$$

$(2) - (3) \Rightarrow 1 = \beta_2$ Then $\alpha_2 = 3$

Thus,
$$|B_1\rangle = \begin{pmatrix}2\\1\end{pmatrix} = -4\begin{pmatrix}1\\2\end{pmatrix} - 3\begin{pmatrix}-2\\-3\end{pmatrix}$$
$$|B_2\rangle = \begin{pmatrix}1\\3\end{pmatrix} = 3\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix}$$

same way 1 mark for this

$$|v\rangle = v^{\bar{1}}|B_1\rangle + v^{\bar{2}}|B_2\rangle = v^1|A_1\rangle + v^2|A_2\rangle$$

$$\Rightarrow v^{\bar{1}}\left(-4\begin{pmatrix}1\\2\end{pmatrix} - 3\begin{pmatrix}-2\\-3\end{pmatrix}\right) + v^{\bar{2}}\left(3\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix}\right) = v^1\begin{pmatrix}1\\2\end{pmatrix} + v^2\begin{pmatrix}-2\\-3\end{pmatrix}$$

$$\Rightarrow \left(-4v^{\bar{1}} + 3v^{\bar{2}}\right)\begin{pmatrix}1\\2\end{pmatrix} + \left(-3v^{\bar{1}} + v^{\bar{2}}\right)\begin{pmatrix}-2\\-3\end{pmatrix} = v^1\begin{pmatrix}1\\2\end{pmatrix} + v^2\begin{pmatrix}-2\\-3\end{pmatrix}$$

$$\Rightarrow \quad v^1 = -4v^{\bar{1}} + 3v^{\bar{2}}, \qquad v^2 = -3v^{\bar{1}} + 1v^{\bar{2}} \quad \Rightarrow \begin{bmatrix}v^1\\ v^2\end{bmatrix} = \underbrace{\begin{bmatrix}-4 & 3\\ -3 & 1\end{bmatrix}}_{Q}\begin{bmatrix}v^{\bar{1}}\\ v^{\bar{2}}\end{bmatrix}$$

1 mark