ASSIGNMENT - 1

① Let $(\mathbb{C}, +, \cdot)$ be the field of all complex numbers $\mathbb{C}$ closed over binary operations + and $\cdot$. Prove that every subfield of $(\mathbb{C}, +, \cdot)$ must contain every rational number.

Ans: ~~Let F be a subfield of $(\mathbb{C}, +, \cdot)$.~~

Let F be a subset of $\mathbb{C}$ and thus, $F \subset \mathbb{C}$.

Further, let $(F, +, \cdot)$ be a ~~subfield~~ ~~field~~ subfield of $(\mathbb{C}, +, \cdot)$ closed over binary operations + and $\cdot$.

Therefore, there exists a unique element (0) in F such that $x + 0 = x$, $x \in F$. Thus, 0 is in F.

Further, there exists a unique element(1) in F such that $x \cdot 1 = x$, $x \in F$. Thus, 1 is in F.

If 1 is in F and recalling that addition is a binary operation (thus implying it is closed by definition), $1 + 1 = 2 \in F$.

Via the same property, $1 + 2 = 3 \in F$

$1 + 3 = 4 \in F$.

$\vdots$

Therefore, ~~it can be proved that~~ every positive integer is in F.

Recalling that, in a field, $\forall x \in F$, there exists a unique element $-x$ in F such that $x + (-x) = 0$.

Hence, for every positive integer x, its additive inverse $-x$ is also in F.

Thus, every negative integer is also in F as every positive integer is in ~~in~~ F.

Recalling that for each non-zero element $x \in F$, $(x \neq 0)$, there corresponds a unique element $x^{-1} \in F$ such that $x \cdot x^{-1} = 1$

$$\therefore \text{for every } x \in F, x \neq 0,$$
$$\frac{1}{x} \in F.$$

As every non-zero integer is in F, every number x, $x \in [-1, 1]$ is in F.

Furthermore, as multiplication is a binary operation and is, by definition of a binary operator, closed,

$$m \times \frac{1}{n} = \frac{m}{n} \in F, \; m, n \in F, n \neq 0.$$

A rational number is defined as a number that can be expressed as a ratio of two integers with the denominator being a non-zero integer.

As $m, n \in \mathbb{Z}$ (the set of integers), $n \neq 0$, $\frac{m}{n}$

refers to ~~any~~ a rational number. As m can be any integer and n can be any non-zero integer, F contains every rational number.

Hence proved.

② Prove that the set of all complex numbers of the form $x + y\sqrt{2}$, where x and y are rational, is a subfield of $(\mathbb{C}, +, \cdot)$

Ans: Let F be the set of all complex numbers of the form $x + y\sqrt{2}$.

For ~~(F, +, ·)~~ F to be a subfield of $(\mathbb{C}, +, \cdot)$, F must be a subset of C and F must itself be a field under ~~add~~ + and ·.

As F is a set of a form of complex numbers, it is therefore a subset of C.

Moving onto proving F forms a field $(F, +, \cdot)$. First, we must prove F is closed under addition and multiplication.

Let $A = x_1 + y_1\sqrt{2}$ and
$B = x_2 + y_2\sqrt{2}$

$$A + B = x_1 + y_1\sqrt{2} + x_2 + y_2\sqrt{2}$$
$$= (x_1 + x_2) + (y_1 + y_2)\sqrt{2}$$

Let $x_1 + x_2 = a$ and $y_1 + y_2 = b$
$$\Rightarrow A + B = a + b\sqrt{2} \in F$$

Thus, F is closed under addition.

Next, we must prove F is closed under multiplication

$$A \cdot B = (x_1 + y_1 \sqrt{2}) \cdot (x_2 + y_2 \sqrt{2})$$
$$= x_1 x_2 + x_2 y_1 \sqrt{2} + x_1 y_2 \sqrt{2} + 2 y_1 y_2$$
$$= (x_1 x_2 + 2 y_1 y_2) + (x_2 y_1 + x_1 y_2) \sqrt{2}$$

Let $x_1 x_2 + 2 y_1 y_2 = c$ and $(x_2 y_1 + x_1 y_2) = d$

$$\Rightarrow \quad A \cdot B = c + d\sqrt{2} \in F$$

Having proved ~~a~~ F is closed under addition and multiplication and therefore proving + and . are indeed binary operators, we can move onto proving the axioms of field formation.

i) Let $A = x_1 + y_1 \sqrt{2}$
$\quad B = x_2 + y_2 \sqrt{2}$

$$A + B = x_1 + y_1 \sqrt{2} + x_2 + y_2 \sqrt{2}$$
$$B + A = x_2 + y_2 \sqrt{2} + x_1 + y_1 \sqrt{2}$$
$$\Rightarrow A + B = B + A$$
$\Rightarrow$ addition in F is commutative

ii) Let C be defined as: $C = x_3 + y_3 \sqrt{2}$

$$A + (B + C) = x_1 + y_1 \sqrt{2} + (x_2 + y_2 \sqrt{2} + x_3 + y_3 \sqrt{2})$$
$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)\sqrt{2}$$
$$(A + B) + C = (x_1 + y_1 \sqrt{2} + x_2 + y_2 \sqrt{2}) + x_3 + y_3 \sqrt{2}$$
$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)\sqrt{2}$$
$$\Rightarrow A + (B + C) = (A + B) + C$$
$\Rightarrow$ addition is associative

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iii) 0 can be expressed as $0 + 0\sqrt{2}$, where $a=0$ and $b=0$ in the expression $a+b\sqrt{2}$

Further, $A+0 = x + y\sqrt{2} + 0$

$= x + y\sqrt{2}$

$= A$

$\Rightarrow A + 0 = A, \ \forall A \in F$

Thus, ~~an identi~~ there exists a unique element 0 in F such that $x + 0 = x, \forall x \in F$

iv) For $A = x_1 + y_1\sqrt{2}$,

$-A = -x_1 + (-y_1)\sqrt{2}$

$\Rightarrow -A \in F$

Thus, there exists a unique element $-x$ in F such that $x + (-x) = 0, \forall x \in F$.

v) Assumming definitions of A ~~B~~ and B from (i),

$AB = (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})$

$= x_1x_2 + x_1y_2\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2$

$BA = (x_2 + y_2\sqrt{2})(x_1 + y_1\sqrt{2})$

$= x_1x_2 + x_2y_1\sqrt{2} + x_1y_2\sqrt{2} + 2y_2y_1$

$\Rightarrow AB = BA$

$\Rightarrow$ multiflication is commutative

vi) Taking $A = x_1 + y_1\sqrt{2}$,

$B = x_2 + y_2\sqrt{2}$,

$C = x_3 + y_3\sqrt{2}$

$$A(BC) = (x_1 + y_1\sqrt{2})[(x_2 + y_2\sqrt{2})(x_3 + y_3\sqrt{2})]$$
$$= (x_1 + y_1\sqrt{2})[x_2x_3 + x_3y_2\sqrt{2} + x_2y_3\sqrt{2} + 2y_2y_3]$$
$$= x_1x_2x_3 + x_1x_3y_2\sqrt{2} + x_1x_2y_3\sqrt{2} + 2x_1y_2y_3$$
$$+ x_2x_3y_1\sqrt{2} + 2x_3y_2y_1 + 2x_2y_1y_3 + 2\sqrt{2}y_1y_2y_3$$

$$(AB)C = [(x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})](x_3 + y_3\sqrt{2})$$
$$= [x_1x_2 + x_2y_1\sqrt{2} + x_1y_2\sqrt{2} + 2y_1y_2](x_3 + y_3\sqrt{2})$$
$$= x_1x_2x_3 + x_1x_3y_2\sqrt{2} + x_1x_2y_3\sqrt{2} + 2x_1y_2y_3$$
$$+ x_2x_3y_1\sqrt{2} + 2x_3y_2y_1 + 2x_2y_1y_3 + 2\sqrt{2}y_1y_2y_3$$

$\Rightarrow A(BC) = (AB)C$

$\Rightarrow$ Multiplication F is associative

vii) 1 can be represented as $1 + 0\sqrt{2}$ and

$$\cancel{A \times 1}\ A.1 = (x_1 + y_1\sqrt{2}) \cdot 1$$
$$= x_1 + y_1\sqrt{2}$$
$$= A$$
$$\Rightarrow A.1 = A$$

$\Rightarrow$ There exists a unique element 1 in F such that $x \cdot 1 = x, \forall x \in F$

viii) $$\frac{1}{x_1 + y_1\sqrt{2}} = \frac{1}{x_1 + y_1\sqrt{2}} \cdot \frac{x_1 - y_1\sqrt{2}}{x_1 - y_1\sqrt{2}}$$
$$= \frac{x_1 - y_1\sqrt{2}}{x_1^2 - 2y_1^2}$$
$$= \frac{x_1}{x_1^2 - 2y_1^2} + \frac{(-y_1\sqrt{2})}{x_1^2 - 2y_1^2} \in F$$
$$= x_2 + y_2\sqrt{2} \in F, \quad x_2 = \frac{x_1}{x^2 - 2y_1^2},\ y_2 = \frac{-y_1}{x_1^2 - 2y_1^2}$$

$\Rightarrow$ For every non-zero $x \in F$ $(x \neq 0)$, there corresponds a unique element such that $x . x^{-1} = 1$

$$\left[\because\ x_1 + y_1\sqrt{2} \cdot \frac{1}{x_1 + y_1\sqrt{2}} = 1 \right]$$

ix) Assume A, B from (i) and C from (ii)

$$A(B+C) = (x_1 + y_1\sqrt{2})[x_2 + y_2\sqrt{2} + x_3 + y_3\sqrt{2}]$$
$$= x_1x_2 + x_1y_2\sqrt{2} + x_1x_3 + x_1y_3\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2 + x_3y_1\sqrt{2} + 2y_1y_3$$

$$AB + BC = x_1x_2 + x_1y_2\sqrt{2} + x_1x_2 + x_1y_2\sqrt{2} + \cancel{x_2}\ x_1x_3 + x_1y_3\sqrt{2} + x_3y_1\sqrt{2} + 2y_1y_3$$

$\Rightarrow A(B+C) = AB+AC$

$\Rightarrow$ Multiplication is distributive over addition.

Thus, $(F, +, \cdot)$ is a field and $F \subset \mathbb{C}$ and thus, F is a subfield of $(\mathbb{C}, +, \cdot)$.

Hence proved