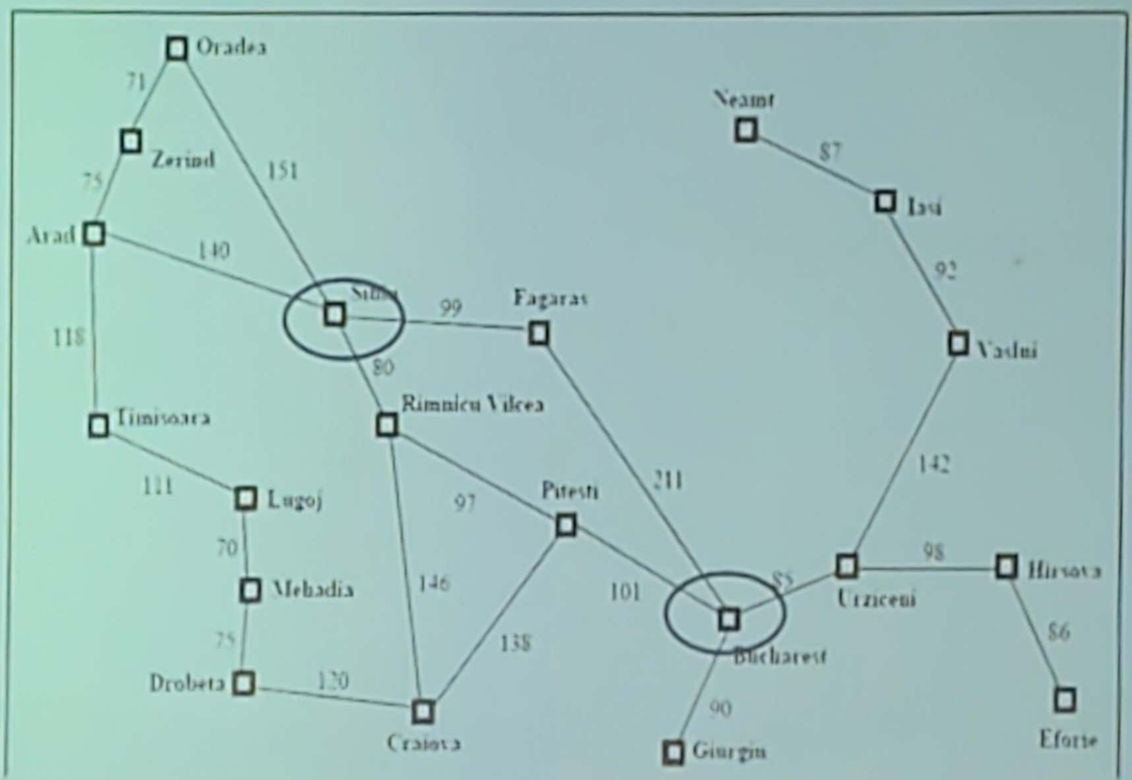
A brief intro

A pen is being sold in a shop for 100 rupees

- What makes you decide to buy this pen ?
- Would it be the same value for every one of you ?
- On what factors does it depends on ?
- How should you capture it for an automated agent?

Techniques to take decisions in the face of uncertainty:

A Map of Romania



Which route to take ?

- Route 1: Sibiu → Fagaras → Bucharest : 310
- Route 2: Sibiu → Rimnicu Vilcea → Pitesi → Bucharest : 278
- Ideally would take the shortest route
- What if traffic is a factor now ?

- Suppose you have accurate information that
  - Route 1 has fast moving traffic while route 2 has slow moving traffic

- A logical agent may reason
  - *Slow(Route 2), fast(Route 1)*
  - *Slow(x) => Avoid (x)*
  - *Avoid(x) ^ fast(y) => select(y)*
  - *Agent selects Route 1*

- **Do you think this is a good way to reason?**
- Why or why not?

Decision Theory
(How to make decisions)

Decision Theory

= Probability theory + Utility Theory

(deals with chance) *(deals with outcomes)*

- *Fundamental idea:*
  - *The* ***MEU (Maximum expected utility)*** *principle*
  - *Agent is rational if and only if it chooses the action that yields the highest expected utility, averaged over all possible outcomes of the action*
  - *Weigh the utility of each outcome by the probability that it occurs*

Uncertainty

- May not have categorical answers
  - *May not know exactly which route is slow*
  - *50% chance that one route is slow*
  - *Even if we know it is slow, how slow ?*

- **Uncertainty** changes the way an agent makes a decision

- Probability theory used in helping agents reason with uncertainty
  - *Summarize uncertainty due to our ignorance or laziness*
  - *May be very very costly to remove such uncertainty*

- **Rational decision** depends on
  - Relative importance of various goals
  - Likelihood or degree to which they will be achieved

Utility Theory – von Neumann (Game Theory)

Let X be the set of outcomes. $\succ$ be the preference of a player over the set of outcomes

Axioms

- **Completeness**: every pair of outcomes is ranked
- **Transitivity**: If $x_1 \succ x_2$ and $x_2 \succ x_3$ then $x_1 \succ x_3$
- **Substitutability**: If $x_1 \sim x_2$ then any lottery in which x_1 is substituted by x_2 is equally preferred
- **Decomposability**: two different lotteries assign same probability to each outcome, then player is indifferent between these two lotteries
- **Monotonicity**: If $x_1 \succ x_2$ and $p > q$ then $[x_1 : p, x_2 : 1-p] \succ [x_1 : q, x_2 : 1-q]$
- **Continuity**: If $x_1 \succ x_2 \succ x_3$, $\exists p \ni x_2 \sim [x_1 : p, x_3 : 1-p]$

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Von Neumann and Morgenstern

Theorem

Given a set of outcomes X and a preference relation on X that satisfies above six axioms, there exists a utility function $u : X \to [0, 1]$ with the following properties:

1. $u(x_1) \geq u(x_2)$ *iff* $x_1 \succ x_2$
2. $U([x_1 : p_1; x_2 : p_2; \ldots; x_m : p_m]) = \sum_{j=1}^{m} p_j u(x_j)$

Conditional Probability

- What if agent has some evidence?
- What if agent knows that if the last 2 days are sunny there is higher probability that it will be sunny today ?

- Posterior or conditional probability
- P(A|B) probability of A given all we know is B
  - *P(X=sunny|Last 2 days were sunny)*

- If we know B and also know C, then $P(A|\ B \wedge C)$

Diagnosis domain: Introducing Cavity Variable

- Doctor diagnosing a dental patient's toothache
- Propositional logic rules
- Toothache → Cavity [Toothache implies Cavity] (Incomplete rule)
- Toothache →Cavity ∨ Gum Problem ∨ Abcess ... (Lots of such problems)
- Turning the rule around, Cavity → Toothache

Joint Probability Distribution

- Notation for distributions on multiple variables
  - P*(Weather, Cavity) i.e. P(Weather ∧ Cavity)*
- Joint probability distribution is a table
  - *Assigns probabilities to all possible assignment of values for combinations of variables*
  - *Weather has values {sunny, rain, cloudy, snow}*
  - *Cavity has values {true, false}*
  - *4 * 2 table of probabilities called joint probability distribution*

- P($X_1, X_2, \ldots X_n$) assigns probabilities to all possible assignment of values to variables $X_1, X_2, \ldots X_n$

Inference using full joint distribution

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- 3 variables : {Cavity, Toothache, Catch}
- Dentist probe catches or not
- 2 * 2 * 2 table

Marginalization

- Denominator can be viewed as a normalization constant 1/P(toothache)
- **P(Cavity|Toothache)** = α P(Cavity, toothache) = α[P(Cavity, toothache, catch) + P(Cavity, toothache, ¬catch)] = .108 + .012 = α 0.12
- α P(¬Cavity, toothache) = α[.016 + .064>] = α 0.08
- Normalizing α <.12, .08> gives <.6, .4>
- **Issue:** For a domain with n Boolean variables, input table size of O(2^n)
- Full joint distribution in tabular form not a practical tool for building reasoning systems

Independence

- Helps in reducing size of domain representation and complexity of inference problem
- Let's add weather into the example : sunny, rain, cloudy, snow → 2 * 2 * 2 * 4 entries in table
- P(toothache, catch, cavity, cloudy) = P(cloudy | toothache, catch, cavity) P(toothache, catch, cavity)
- P(cloudy | toothache, catch, cavity) = P(cloudy)
- **32 element table becomes 8 element + 4 element table**

Bayes' Rule

- *Given that*
  - $P(A \wedge B) = P(A|B)*P(B)$
  - $P(A \wedge B) = P(B|A)*P(A)$
  - $P(B|A) = (P(A|B)*P(B)) / P(A)$

- We can determine P(B|A) given P(A|B), P(B) and P(A)
- P(effect|cause) may be **causal knowledge**, P(cause|effect) **diagnostic knowledge**
- Often have conditional probabilities on causal relationships → P(symptoms|disease)
  - Need to infer diagnostic knowledge many times

More General Forms of Bayes' Rule

- P (Y | X) = P(X | Y) * P(Y) / P(X)

- Bayes' rule for multi-valued variables
- Generalize to some background evidence e

- P (Y | X, e) = P(X | Y, e) * P(Y | e) / P(X | e)

Conditional Independence

- Toothache and Catch are independent given the presence of Cavity
  - Each is directly caused by cavity, toothache is caused by cavity and a dentist catches cavity using probe
  - A toothache cannot be caught by a probe nor a probe results in a toothache
- Conditional independence captured as : **P(toothache ^ catch | cavity) = P(toothache | catch, cavity) P(catch | cavity)**
- P(toothache | catch, cavity) = P(A|B, C)
  - *If A and C are conditionally independent given B*
  - *Then, probability of A is not dependent on C*
  - *i.e.* $P(A | B \wedge C) = P(A | B)$
- Therefore P(toothache, catch | cavity) = P(toothache | cavity) P(catch | cavity)

Combining Evidence

- Revisiting the Meningitis example --

- S: Proposition that patient has stiff neck
- H: Proposition that patient has severe headache
- M: Proposition that patient has meningitis
- Meningitis causes stiff-neck, 50% of the time
- Meningitis causes head-ache, 70% of the time
- Probability of Meningitis should go up, if both symptoms reported -- how to combine such symptoms?

Application of MEU Principle

- EU(Plan1) = P(home-early | plan1) *U(home-early)

  + P(stuck1| plan1) * U(stuck1)

  = *0.8 * 100 + 0.2 * -1000 = -120*

- EU(Plan2) = P(home-somewhat-early | plan2) *U(home-somewhat-early)

  + P(stuck2 | plan2) * U(stuck2)

  = *0.7 * 50 + 0.3 * -10 = 32*

Revisiting Romania example

- If plan1 and plan2 are the two plans:
  - Plan 1 uses route 1
    - P(home-early|plan1) = .8, while P(stuck1|plan1) = .2
    - Route 1 will be quick if flowing, but stuck for 1 hour if slow
    - ➢ U(home-early) = 100, U(stuck1) = -1000
    - ➢ Assigned numerical values to outcomes!

$EU = -120$

100×0.8
$- 1000 \times 0.2$

  - Plan 2 uses route 2
    - P(home-somewhat-early|plan2) = .7, P(stuck2|plan2) = .3
    - Route 2 will be somewhat quick if flowing, but not bad even if slow
    - ➢ U(home-somewhat-early) = 50, U(stuck2) = -10

$EU = 50 \times 0.7$
$- 10 \times 0.3$
$= 32$

Application of MEU Principle

- EU(Plan1) = P(home-early | plan1) *U(home-early)

  + P(stuck1| plan1) * U(stuck1)

  = *0.8 * 100 + 0.2 * -1000 = -120*

- EU(Plan2) = P(home-somewhat-early | plan2) *U(home-somewhat-early)

  + P(stuck2 | plan2) * U(stuck2)

  = *0.7 * 50 + 0.3 * -10 = 32*

Risk Aversion

- We are **risk averse**
- Our utility functions for money are as follows (!!):
  - Our first million means a lot U($1M) = 10
  - Second million not so much U($2M) = 15 (NOT 20)
  - Third million even less so U($3M) = 18 (NOT 30)
  - ….

- Additional money is not buying us as much utility
- If we plot amount of money on the x-axis and utility on the y-axis, we get a concave curve

More Risk Aversion

- Key: Slope of utility function is **continuously decreasing**
  - We will refuse to play a monetarily fair bet
- Suppose we start with x dollars
  - We are offered a game:
    - 0.5 chance to win 1000 dollars ($c = 1000$)
    - 0.5 chance to lose 1000 dollars ($c = 1000$)
    - Expected monetary gain or loss is zero (hence monetarily fair)

  - Should be neutral to it, but seems we are not! Why?
    - $U(x + c) - U(x) < U(x) - U(x - c)$
    - $U(x + c) + U(x - c) < 2\ U(x)$
    - $[U(x + c) + U(x - c) / 2] < U(x)$
    - EU (playing the game) < EU (not playing the game)

Example: What SEQUENCE of actions should our agent take?

- Agent can take action N, E, S, W
- Each action costs –1/25