

DEVOIR QFT: Théorème de Wick

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Question: Montrer que l'on a:

$$\begin{aligned}\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = & \mathcal{T}\{\phi(x_1)\phi(x_2)\}\mathcal{T}\{\phi(x_3)\phi(x_4)\} + \mathcal{T}\{\phi(x_1)\phi(x_3)\}\mathcal{T}\{\phi(x_2)\phi(x_4)\} \\ & + \mathcal{T}\{\phi(x_1)\phi(x_4)\}\mathcal{T}\{\phi(x_2)\phi(x_3)\}\end{aligned}$$

Réponse: On connaît le théorème de Wick qui dit que:

$$\begin{aligned}\mathcal{T}\left\{\prod_{k=1}^m \phi(x_k)\right\} = & N\left\{\prod_{k=1}^m \phi(x_k)\right\} + \sum_{\alpha, \beta} \overline{\phi(x_\alpha)}\phi(x_\beta)N\left\{\prod_{k \neq \alpha, \beta}^m \phi(x_k)\right\} \\ & + \sum_{(\alpha, \beta), (\gamma, \delta)} \overline{\phi(x_\alpha)}\overline{\phi(x_\beta)}\overline{\phi(x_\gamma)}\phi(x_\delta)N\left\{\prod_{k \neq \alpha, \beta, \gamma, \delta}^m \phi(x_k)\right\} \\ & + \dots\end{aligned}$$

Simplifions notre écriture en faisant

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\} = \mathcal{T}\{\phi_1\phi_2\}$$

Ainsi pour 4 opérateurs de champ, on a:

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}$$

On peut donc écrire de la manière suivante grâce au théorème de Wick:

$$\begin{aligned}
\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} &= N\{\phi_1\phi_2\phi_3\phi_4\} + N\{\overline{\phi_1}\phi_2\phi_3\phi_4\} + N\{\phi_1\overline{\phi_2}\phi_3\phi_4\} + N\{\phi_1\phi_2\overline{\phi_3}\phi_4\} \\
&\quad + N\{\phi_1\overline{\phi_2}\overline{\phi_3}\phi_4\} + N\{\phi_1\phi_2\overline{\phi_3}\overline{\phi_4}\} + N\{\phi_1\overline{\phi_2}\phi_3\overline{\phi_4}\} \\
&\quad + N\{\overline{\phi_1}\overline{\phi_2}\phi_3\phi_4\} + N\{\overline{\phi_1}\phi_2\overline{\phi_3}\phi_4\}
\end{aligned} \tag{1}$$

On définit $N\{\phi_1\phi_2\phi_3\phi_4\}$ de telle sorte que:

$$N\{\phi_1\phi_2\phi_3\phi_4\} = \Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\}$$

Donc on réécrit l'équation (1) comme suit:

$$\begin{aligned}
\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} &= N\{\phi_1\phi_2\phi_3\phi_4\} + \Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\} + \Delta_F(x_1 - x_3)\mathbb{1}N\{\phi_2\phi_4\} \\
&\quad + \Delta_F(x_1 - x_4)\mathbb{1}N\{\phi_2\phi_3\} + \Delta_F(x_2 - x_3)\mathbb{1}N\{\phi_1\phi_4\} + \Delta_F(x_2 - x_4)\mathbb{1}N\{\phi_1\phi_3\} \\
&\quad + \Delta_F(x_3 - x_4)\mathbb{1}N\{\phi_1\phi_2\} + \Delta_F(x_1 - x_2)\mathbb{1}\Delta_F(x_3 - x_4)\mathbb{1} \\
&\quad + \Delta_F(x_1 - x_3)\mathbb{1}\Delta_F(x_2 - x_4)\mathbb{1} + \Delta_F(x_1 - x_4)\mathbb{1}\Delta_F(x_2 - x_3)\mathbb{1}
\end{aligned} \tag{2}$$

Or on sait que on peut poser:

$$\begin{aligned}
\phi_x &= \phi_x^+ + \phi_x^- \\
\phi_y &= \phi_y^+ + \phi_y^-
\end{aligned}$$

Ce qui nous permet d'écrire d'après le cours:

$$\mathcal{T}\{\phi_x\phi_y\} = \Theta(x^0 - y^0)([\phi_x^+, \phi_y^-] + N\{\phi_x\phi_y\}) + \Theta(y^0 - x^0)([\phi_y^+, \phi_x^-] + N\{\phi_y\phi_x\}) \tag{3}$$

Aussi que:

$$\Delta_F(x - y) = \Theta(x^0 - y^0) \langle 0 | [\phi_x^+ \phi_y^-] | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | [\phi_y^+ \phi_x^-] | 0 \rangle \tag{4}$$

On peut identifier les équations (3) et (4) pour obtenir:

$$\mathcal{T}\{\phi_x\phi_y\} = \Delta_F(x - y)\mathbb{1} + N\{\phi_x\phi_y\} + N\{\phi_y\phi_x\} \tag{5}$$

Où on définit $N\{\phi_y\phi_x\}$ comme le normal ordering de $\phi_y\phi_x$

Ainsi chaque terme de l'équation (2) peut s'écrire comme suit:

$$\begin{aligned}
\Delta_F(x_1 - x_2)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1 - x_2)N\{\phi_1\phi_2\} - \Theta(x_2 - x_1)N\{\phi_2\phi_1\} \\
\Delta_F(x_1 - x_3)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_3\} - \Theta(x_1 - x_3)N\{\phi_1\phi_3\} - \Theta(x_3 - x_1)N\{\phi_3\phi_1\} \\
\Delta_F(x_1 - x_4)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_4\} - \Theta(x_1 - x_4)N\{\phi_1\phi_4\} - \Theta(x_4 - x_1)N\{\phi_4\phi_1\} \\
\Delta_F(x_2 - x_3)\mathbb{1} &= \mathcal{T}\{\phi_2\phi_3\} - \Theta(x_2 - x_3)N\{\phi_2\phi_3\} - \Theta(x_3 - x_2)N\{\phi_3\phi_2\} \\
\Delta_F(x_2 - x_4)\mathbb{1} &= \mathcal{T}\{\phi_2\phi_4\} - \Theta(x_2 - x_4)N\{\phi_2\phi_4\} - \Theta(x_4 - x_2)N\{\phi_4\phi_2\} \\
\Delta_F(x_3 - x_4)\mathbb{1} &= \mathcal{T}\{\phi_3\phi_4\} - \Theta(x_3 - x_4)N\{\phi_3\phi_4\} - \Theta(x_4 - x_3)N\{\phi_4\phi_3\}
\end{aligned}$$

Prenons pour exemple le second terme du membre de droite de l'équation (2):

$$\Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\} = (\mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1 - x_2)N\{\phi_1\phi_2\} - \Theta(x_2 - x_1)N\{\phi_2\phi_1\})(N\{\phi_3\phi_4\})$$

Or de par la définition du normal ordering, on sait que les termes contenant les normal ordering disparaissent sous $\langle 0| . |0\rangle$, donc on a:

$$\langle 0| \Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\} |0\rangle = 0 \quad (6)$$

Donc si on effectue le calcul $\langle 0| . \mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} |0\rangle$ et en tenant compte de (6), on obtient:

$$\begin{aligned}
&\langle 0| \mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle \\
&= \langle 0| (\mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1 - x_2)N\{\phi_1\phi_2\} - \Theta(x_2 - x_1)N\{\phi_2\phi_1\}) \times \\
&\quad (\mathcal{T}\{\phi_3\phi_4\} - \Theta(x_3 - x_4)N\{\phi_3\phi_4\} - \Theta(x_4 - x_3)N\{\phi_4\phi_3\}) |0\rangle \\
&+ \langle 0| (\mathcal{T}\{\phi_1\phi_3\} - \Theta(x_1 - x_3)N\{\phi_1\phi_3\} - \Theta(x_3 - x_1)N\{\phi_3\phi_1\}) \times \quad (7) \\
&\quad (\mathcal{T}\{\phi_2\phi_4\} - \Theta(x_2 - x_4)N\{\phi_2\phi_4\} - \Theta(x_4 - x_2)N\{\phi_4\phi_2\}) |0\rangle \\
&+ \langle 0| (\mathcal{T}\{\phi_1\phi_4\} - \Theta(x_1 - x_4)N\{\phi_1\phi_4\} - \Theta(x_4 - x_1)N\{\phi_4\phi_1\}) \times \\
&\quad (\mathcal{T}\{\phi_2\phi_3\} - \Theta(x_2 - x_3)N\{\phi_2\phi_3\} - \Theta(x_3 - x_2)N\{\phi_3\phi_2\}) |0\rangle
\end{aligned}$$

En nous rappelant que les termes contenant les normal ordering disparaissent sous $\langle 0| . |0\rangle$, on obtient:

$$\begin{aligned}
&\langle 0| \mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle \\
&= \langle 0| \mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\} |0\rangle \\
&+ \langle 0| \mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\} |0\rangle \quad (8) \\
&+ \langle 0| \mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\} |0\rangle
\end{aligned}$$

On peut regrouper chaque terme sous $\langle 0|.|0\rangle$ pour obtenir:

$$\begin{aligned} & \langle 0| \mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle \\ &= \langle 0| (\mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\} + \mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\} + \mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\}) |0\rangle \end{aligned} \quad (9)$$

Finalement, on obtient le résultat attendu qui est :

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\} + \mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\} + \mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\} \quad (10)$$