DEVOIR QFT: Théorème de Wick

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<u>Question</u>: Montrer que l'on a:

$$\begin{aligned}\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \mathcal{T}\{\phi(x_1)\phi(x_2)\}\mathcal{T}\{\phi(x_3)\phi(x_4)\} + \mathcal{T}\{\phi(x_1)\phi(x_2)\}\mathcal{T}\{\phi(x_3)\phi(x_4)\} \\ + \mathcal{T}\{\phi(x_1)\phi(x_2)\}\mathcal{T}\{\phi(x_3)\phi(x_4)\}\end{aligned}$$

<u>Réponse</u>: On connait le théorème de Wick qui dit que:

$$\begin{aligned}\mathcal{T}\{\prod_{k=1}^{m}\phi(x_k)\} &= N\{\prod_{k=1}^{m}\phi(x_k)\} + \sum_{\alpha,\beta}\overbracket{\phi(x_\alpha)\phi}(x_\beta)N\{\prod_{k\neq\alpha,\beta}^{m}\phi(x_k)\} \\ &+ \sum_{(\alpha,\beta),(\gamma,\delta)}\overbracket{\phi(x_\alpha)\phi}(x_\beta)\overbracket{\phi(x_\gamma)\phi}(x_\delta)N\{\prod_{k\neq\alpha,\beta,\gamma,\delta}^{m}\phi(x_k)\} \\ &+ ...\end{aligned}$$

Simplifions notre écriture en faisant

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\} = \mathcal{T}\{\phi_1\phi_2\}$$

Ainsi pour 4 opérateurs de champ, on a:

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}$$

On peut donc écrire de la manière suivante grace au théorème de Wick:

$$\begin{aligned}
\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} = {} & N\{\phi_1\phi_2\phi_3\phi_4\} + N\{\wick{\c1\phi_1\c1\phi_2\phi_3\phi_4}\} + N\{\wick{\c1\phi_1\phi_2\c1\phi_3\phi_4}\} + N\{\wick{\c1\phi_1\phi_2\phi_3\c1\phi_4}\} \\
& + N\{\wick{\phi_1\c1\phi_2\c1\phi_3\phi_4}\} + N\{\wick{\phi_1\c1\phi_2\phi_3\c1\phi_4}\} + N\{\wick{\phi_1\phi_2\c1\phi_3\c1\phi_4}\} + N\{\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}\} \\
& + N\{\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}\} + N\{\wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\}
\end{aligned} \tag{1}$$

On définit $N\{\wick{\c1\phi_1\c1\phi_2\phi_3\phi_4}\}$ de telle sorte que:

$$N\{\wick{\c1\phi_1\c1\phi_2\phi_3\phi_4}\} = \Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\}$$

Donc on réécrit l'équation (1) comme suit:

$$\begin{aligned}
\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} = {} & N\{\phi_1\phi_2\phi_3\phi_4\} + \Delta_F(x_1 - x_2)\mathbb{1}N\{\phi_3\phi_4\} + \Delta_F(x_1 - x_3)\mathbb{1}N\{\phi_2\phi_4\} \\
& + \Delta_F(x_1 - x_4)\mathbb{1}N\{\phi_2\phi_3\} + \Delta_F(x_2 - x_3)\mathbb{1}N\{\phi_1\phi_4\} + \Delta_F(x_2 - x_4)\mathbb{1}N\{\phi_1\phi_3\} \\
& + \Delta_F(x_3 - x_4)\mathbb{1}N\{\phi_1\phi_2\} + \Delta_F(x_1 - x_2)\mathbb{1}\Delta_F(x_3 - x_4)\mathbb{1} \\
& + \Delta_F(x_1 - x_3)\mathbb{1}\Delta_F(x_2 - x_4)\mathbb{1} + \Delta_F(x_1 - x_4)\mathbb{1}\Delta_F(x_2 - x_3)\mathbb{1}
\end{aligned} \tag{2}$$

Or on sait que on peut poser:

$$\begin{aligned}
\phi_x &= \phi_x^+ + \phi_x^- \\
\phi_y &= \phi_y^+ + \phi_y^-
\end{aligned}$$

Ce qui nous permet d'écrire d'après le cours:

$$\mathcal{T}\{\phi_x\phi_y\} = \Theta(x^0 - y^0)([\phi_x^+, \phi_y^-] + N'\{\phi_x\phi_y\}) + \Theta(y^0 - x^0)([\phi_y^+, \phi_x^-] + N'\{\phi_y\phi_x\}) \tag{3}$$

Aussi que:

$$\Delta_F(x - y) = \Theta(x^0 - y^0)\left\langle 0 \middle| [\phi_x^+\phi_y^-] \middle| 0 \right\rangle + \Theta(y^0 - x^0)\left\langle 0 \middle| [\phi_y^+\phi_x^-] \middle| 0 \right\rangle \tag{4}$$

On peut identifier les équations (3) et (4) pour obtenir:

$$\mathcal{T}\{\phi_x\phi_y\} = \Delta_F(x - y)\mathbb{1} + N'\{\phi_x\phi_y\} + N'\{\phi_y\phi_x\} \tag{5}$$

Où on définit $N'\{\phi_y\phi_x\}$ comme le normal ordering de $\phi_y\phi_x$ sans les termes $\phi_x^+\phi_y^-$ et avec un terme en plus $\phi_y^-\phi_x^+$.

Ainsi chaque terme de l'équation (2) peut s'écrire comme suit:

$$\begin{aligned}
\Delta_F(x_1-x_2)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1-x_2)N'\{\phi_1\phi_2\} - \Theta(x_2-x_1)N'\{\phi_2\phi_1\}\\
\Delta_F(x_1-x_3)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_3\} - \Theta(x_1-x_3)N'\{\phi_1\phi_3\} - \Theta(x_3-x_1)N'\{\phi_3\phi_1\}\\
\Delta_F(x_1-x_4)\mathbb{1} &= \mathcal{T}\{\phi_1\phi_4\} - \Theta(x_1-x_4)N'\{\phi_1\phi_4\} - \Theta(x_4-x_1)N'\{\phi_4\phi_1\}\\
\Delta_F(x_2-x_3)\mathbb{1} &= \mathcal{T}\{\phi_2\phi_3\} - \Theta(x_2-x_3)N'\{\phi_2\phi_3\} - \Theta(x_3-x_2)N'\{\phi_3\phi_2\}\\
\Delta_F(x_2-x_4)\mathbb{1} &= \mathcal{T}\{\phi_2\phi_4\} - \Theta(x_2-x_4)N'\{\phi_2\phi_4\} - \Theta(x_4-x_2)N'\{\phi_4\phi_2\}\\
\Delta_F(x_3-x_4)\mathbb{1} &= \mathcal{T}\{\phi_3\phi_4\} - \Theta(x_3-x_4)N'\{\phi_3\phi_4\} - \Theta(x_4-x_3)N'\{\phi_4\phi_3\}
\end{aligned}$$

Prenons pour exemple le second terme du membre de droite de l'équation (2):

$$\Delta_F(x_1-x_2)\mathbb{1}N\{\phi_3\phi_4\} = (\mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1-x_2)N'\{\phi_1\phi_2\} - \Theta(x_2-x_1)N'\{\phi_2\phi_1\})(N\{\phi_3\phi_4\})$$

Or de par la définition du normal ordering, on sait que les termes contenant les normal ordering disparaissent sous $\langle 0|.|0\rangle$, donc on a:

$$\langle 0|\Delta_F(x_1-x_2)\mathbb{1}N\{\phi_3\phi_4\}|0\rangle = 0 \tag{6}$$

Donc si on effectue le calcul $\langle 0|.\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$ et en tenant compte de (6),on obtient:

$$\begin{aligned}
&\langle 0|\,\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle\\
&= \langle 0|\,(\mathcal{T}\{\phi_1\phi_2\} - \Theta(x_1-x_2)N'\{\phi_1\phi_2\} - \Theta(x_2-x_1)N'\{\phi_2\phi_1\})\times\\
&\quad (\mathcal{T}\{\phi_3\phi_4\} - \Theta(x_3-x_4)N'\{\phi_3\phi_4\} - \Theta(x_4-x_3)N'\{\phi_4\phi_3\})\,|0\rangle\\
&+ \langle 0|\,(\mathcal{T}\{\phi_1\phi_3\} - \Theta(x_1-x_3)N'\{\phi_1\phi_3\} - \Theta(x_3-x_1)N'\{\phi_3\phi_1\})\times\\
&\quad (\mathcal{T}\{\phi_2\phi_4\} - \Theta(x_2-x_4)N'\{\phi_2\phi_4\} - \Theta(x_4-x_2)N'\{\phi_4\phi_2\})\,|0\rangle\\
&+ \langle 0|\,(\mathcal{T}\{\phi_1\phi_4\} - \Theta(x_1-x_4)N'\{\phi_1\phi_4\} - \Theta(x_4-x_1)N'\{\phi_4\phi_1\})\times\\
&\quad (\mathcal{T}\{\phi_2\phi_3\} - \Theta(x_2-x_3)N'\{\phi_2\phi_3\} - \Theta(x_3-x_2)N'\{\phi_3\phi_2\})\,|0\rangle
\end{aligned} \tag{7}$$

En nous rappelant que les termes contenant les normal ordering disparaissent sous $\langle 0|.|0\rangle$, on obtient:

$$\begin{aligned}
&\langle 0|\,\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle\\
&= \langle 0|\,\mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\}\,|0\rangle\\
&+ \langle 0|\,\mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\}\,|0\rangle\\
&+ \langle 0|\,\mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\}\,|0\rangle
\end{aligned} \tag{8}$$

On peut regrouper chaque terme sous $\langle 0|.|0\rangle$ pour obtenir:

$$\begin{aligned}&\langle 0|\,\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle\\&=\langle 0|\,(\mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\}+\mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\}+\mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\})\,|0\rangle\end{aligned} \tag{9}$$

Finalement, on obtient le résultat attendu qui est :

$$\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}=\mathcal{T}\{\phi_1\phi_2\}\mathcal{T}\{\phi_3\phi_4\}+\mathcal{T}\{\phi_1\phi_3\}\mathcal{T}\{\phi_2\phi_4\}+\mathcal{T}\{\phi_1\phi_4\}\mathcal{T}\{\phi_2\phi_3\} \tag{10}$$