Performance Modeling of Computer Systems and Networks

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Memoryless property
and
probability distributions

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Memoryless property

Informally:
the RV does not "remember" the past,
it behaves as a new variable

the future depends only on relevant information about the current time, not on information from further in the past

Example:
X is the time elapsed in a shop from 9 am on a certain day until the arrival of the first customer

X is the time a server waits for the first customer

The "memoryless" property makes a comparison between the probability distributions of the time a shop has to wait from 9 am onwards for his first customer, and the time that the shop still has to wait for the first customer on those occasions when no customer has arrived by any given later time:

the property of memorylessness is that these distributions of "time from now to the next customer" are exactly the same.

exponential
geometric

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The remaining service time

A post office has 2 clerks.
Customer B is being served by one clerk, and customer C is being served by the other clerk, when A walks in.
All service times are exponentially distributed.

What is *Pr {*A *is the last to leave}*?

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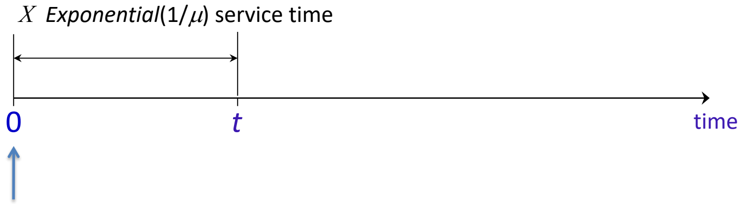
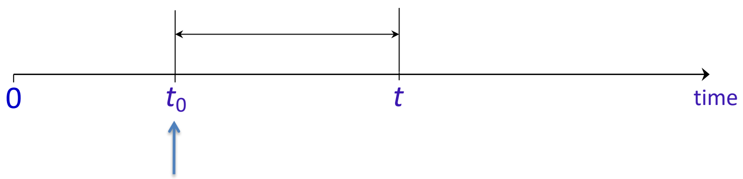
Note that one of B and C will leave first.
Let us say B leaves first.
Then C and A will have the same distribution on their remaining service time.
It doesn't matter that C has been serving for a while.

The remaining service time

X *Exponential*($1/\mu$) service time



service start $Prob\{X \le t\} = 1 - e^{-\mu t}$



$Prob\{X \le t_0 + t \mid X > t_0\}$? $X - t_0$ is the *remaining service time*

The remaining service time

$$Prob\left\{X \le t_0 + t \middle| X > t_0\right\} = \frac{Prob\left\{t_0 < X \le t_0 + t\right\}}{Prob\left\{X > t_0\right\}} = \frac{Prob\left\{X \le t_0 + t\right\} - Prob\left\{X \le t_0\right\}}{Prob\left\{X > t_0\right\}}$$

$$= \frac{\cancel{1}-e^{-\mu(t_0+t)} - \left(\cancel{1}-e^{-\mu t_0}\right)}{\cancel{1}-\left(\cancel{1}-e^{-\mu t_0}\right)} = \frac{e^{-\mu t_0} - e^{-\mu(t_0+t)}}{e^{-\mu t_0}}$$

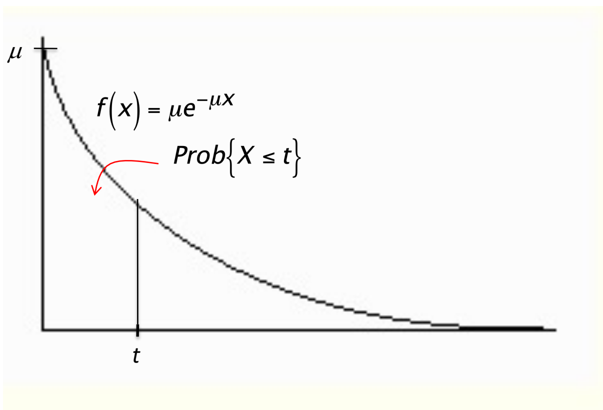
$$= 1 - \frac{\cancel{e^{-\mu t_0}}e^{-\mu t}}{\cancel{e^{-\mu t_0}}} = 1 - e^{-\mu t}$$

$$Prob\left\{X \le t_0 + t \middle| X > t_0\right\} = Prob\left\{X \le t\right\}$$

$$Prob\left\{\underbrace{X - t_0}_{\text{remaining service time}} \le t \middle| X > t_0\right\} = Prob\left\{\underbrace{X \le t}_{\text{service time}}\right\}$$

The two distributions are exactly the same.

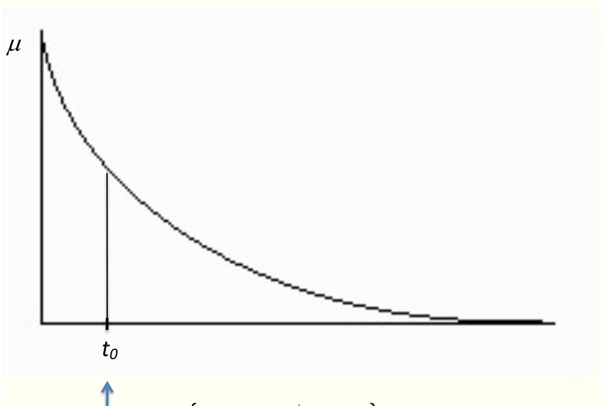
Exponential$(1/\mu)$



Analytical models
Memoryless property

The remaining service time

Exponential$(1/\mu)$



$Prob\{X - t_0 \le t | X > t_0\}$

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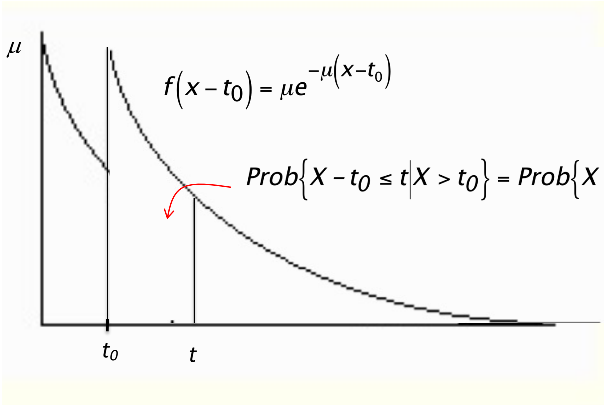
Analytical models
Memoryless property

The remaining service time

Exponential$(1/\mu)$



$f(x - t_0) = \mu e^{-\mu(x - t_0)}$

$Prob\{X - t_0 \le t | X > t_0\} = Prob\{X \le t\}$

The exponential is the only continuous distr memoryless

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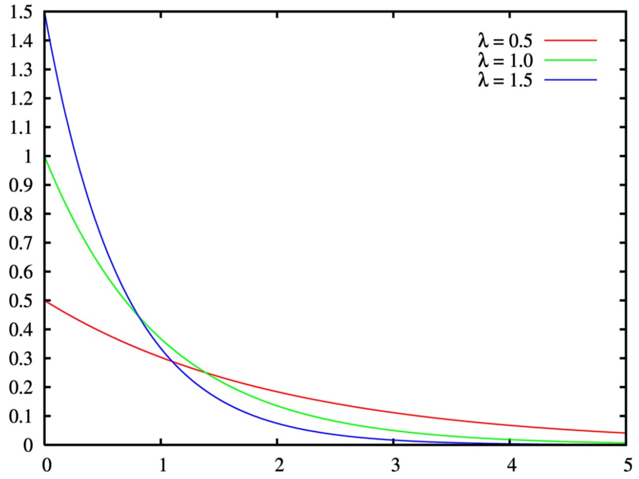
exponential distribution



$$f(x) = \lambda e^{-\lambda x}$$

1. For each unit step, the "quantity" is reduced of $e^{-\lambda}$

λ is the *decay constant*

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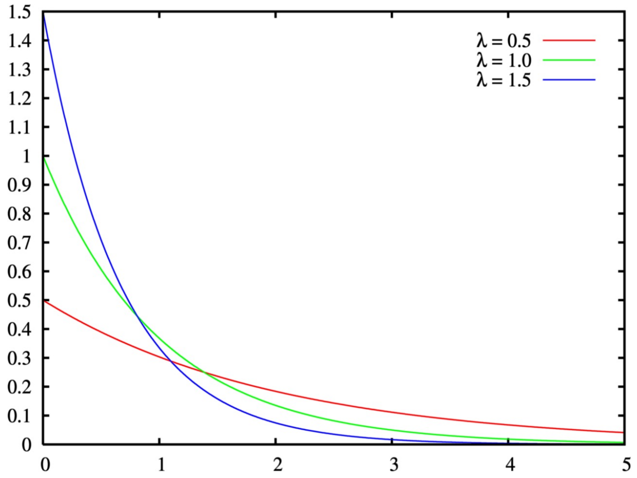
exponential distribution



$$f(x) = \lambda e^{-\lambda x}$$

2. Independently of the λ value, $1/\lambda$ is called the exponential *time constant* : is the time to reduce the initial value of about 63.21%

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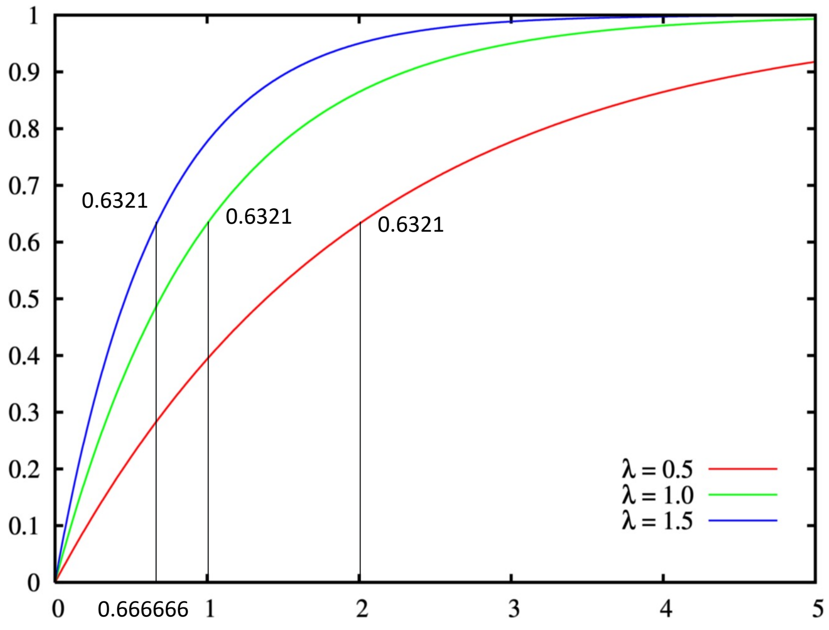

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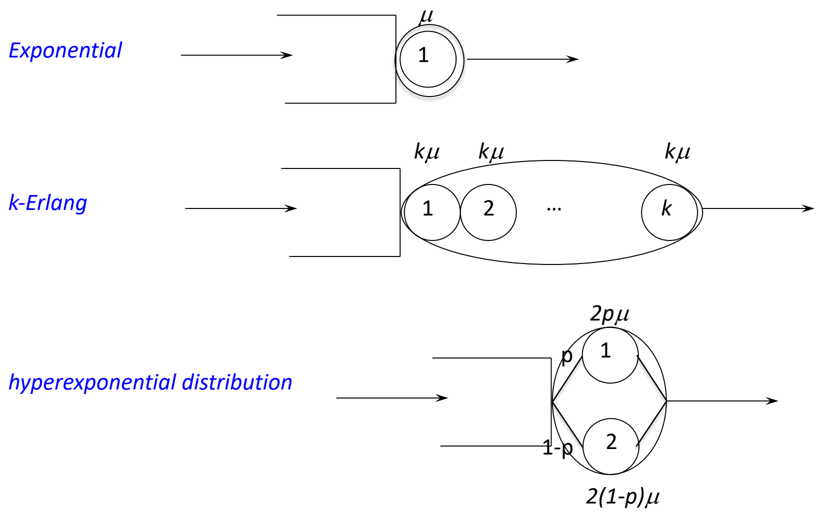
Phase-type distributions



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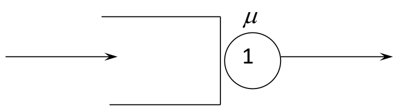
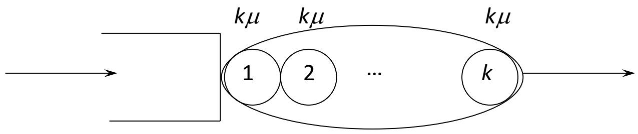
Phase-type distributions

Exponential



k-Erlang



- a job that requires the execution of k programs in series
- I/O unit that needs of a series of operations to serve a requirement (search for a cylinder, sector, read/write etc.)

Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

k-Erlang $f_i(x) = k\mu e^{-k\mu x}$ $E[X_i] = \frac{1}{k\mu}$ $\sigma^2(X_i) = \left(\frac{1}{k\mu}\right)^2$

$$f(x) = (k\mu)^k \frac{e^{-k\mu x}}{(k-1)!} x^{k-1} \qquad k \geq 1$$

$$E[X] = \sum_{i=1}^{k} E[X_i] = k\frac{1}{k\mu} = \frac{1}{\mu}$$

as the exponential!

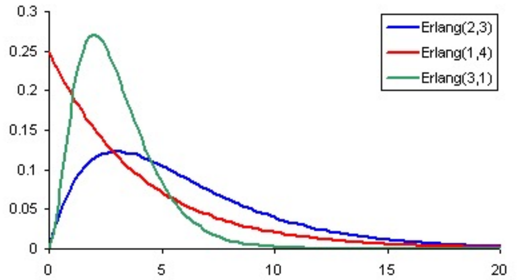
$$\sigma^2(X) = \sum_{i=1}^{k} \sigma^2(X_i) = k\left(\frac{1}{k\mu}\right)^2 = \frac{1}{k}\left(\frac{1}{\mu}\right)^2$$

k times less than the exponential!

Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

k-Erlang $f_i(x) = k\mu e^{-k\mu x}$ $E[X_i] = \frac{1}{k\mu}$ $\sigma^2(X_i) = \left(\frac{1}{k\mu}\right)^2$



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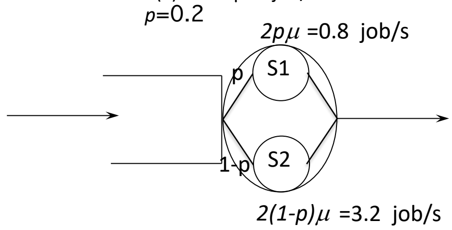
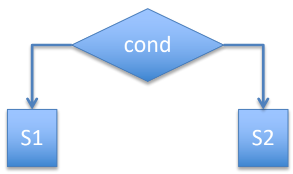
Phase-type distributions

E(S)=0.5 s μ=2 job/s
p=0.2

hyperexponential distribution





- a job that requires the execution of 2 programs as an alternative

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Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

hyperexponential distribution

$$f_1(x) = 2p\mu e^{-2p\mu x} \qquad f_2(x) = 2(1-p)\mu e^{-2(1-p)\mu x}$$

$$f(x) = pf_1(x) + (1-p)f_2(x)$$

$$E[X] = pE[X_1] + (1-p)E[X_2] = p\frac{1}{2p\mu} + (1-p)\frac{1}{2(1-p)\mu} = \frac{1}{\mu}$$ as the exponential!

$$\sigma^2(X) = g(p)\left(\frac{1}{\mu}\right)^2$$ where $$g(p) = \frac{1}{2p(1-p)} - 1$$ $g(p)$ times more than the exponential!

p=0.5 $\longrightarrow$ $g(p)$=1, hyperexpo=expo

p=0.2 $\longrightarrow$ $g(p)$=2.125

$p \to 0.5 \longrightarrow g(p) \to 1$ variance decreases

$p \to 0$ or $1 \longrightarrow g(p) \to \infty$ variance grows

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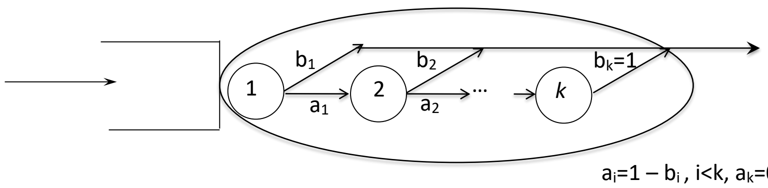
Cox distribution

How can we model a service demand with a different law, that is an arbitrary distribution?



$a_i = 1 - b_i$, $i<k$, $a_k=0$, $b_k=1$

- each stage is expo with mean $1/\mu_i$
- if $t_1, t_2, \ldots, t_k$, are the time spent in each stage the total time spent t is:
  - $t = t_1$ with probability b_1
  - $t = t_1+t_2$ with probability $a_1 b_2$
  - $t = t_1+t_2+t_3$ with probability $a_1 a_2 b_3$

  …
  - $t = t_1+\ldots+t_k$ with probability $a_1 a_2 \ldots a_{k-1} b_k$

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Arbitrary distribution

Case a) Arbitrary $f(t)$ with rational Laplace transform

$\longrightarrow$ $C_k(t) = f(t)$ for a given k,

exact, or with known precision

Case b) Arbitrary $g(t)$ without rational Laplace transform

$\longrightarrow$ $f(t) \approx g(t)$ approximate, with known precision

$C_k(t) \approx g(t)$