Performance Modeling of Computer Systems and Networks

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Generating Discrete Random Variates

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Prerequisite

We assume the knowledge of discrete random variables (sect.6.1).
In particular:

- *Equilakely*(a,b)
- *Geometric*(p)
- *Bernoulli*(p)
- *Binomial*(n,p)
- *Pascal*(n,p)
- *Poisson*(μ)

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sis2.c

```c
#include <stdio.h>
#include "rng.h"

#define MINIMUM   20
#define MAXIMUM   80
#define STOP     100      /* 100 weeks = about 2 years*/
#define sqr(x)    ((x) * (x))

long Equilikely(long a, long b)
{  return (a + (long) ((b - a + 1) * Random()));}

long GetDemand(void)
{
        return (Equilikely(10, 50)); }
```

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ssq2.c

distribution-driven simulation

```c
#include <stdio.h>
#include <math.h>
#include "rng.h"
#define LAST          10000L    /* number of jobs processed */
#define START         0.0

double Exponential(double m)                /* ------*
{return (-m * log(1.0 - Random())); }          m > 0.0
                                              ------ */
double Uniform(double a, double b)          /* ------*
{return (a + (b - a) * Random());     }          a < b
                                            * ------*/

                double GetArrival(void)
           {static double arrival = START;
             arrival += Exponential(2.0);
                  return (arrival);}

                double GetService(void)
             {return (Uniform(1.0, 2.0));}
```

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Preliminary Definitions

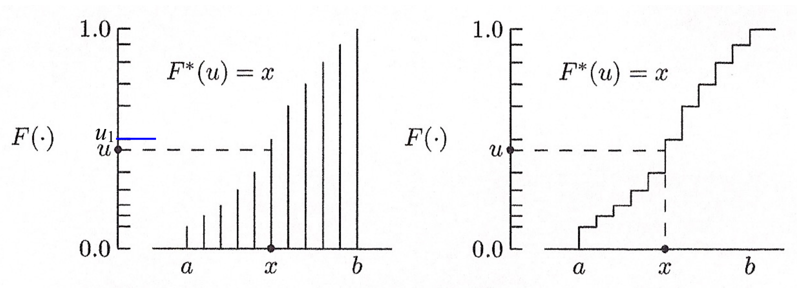
X random variable, $F(\cdot)$ is the cdf of X

The *inverse distribution function* (idf) of X is the function
$F^* : (0, 1) \rightarrow \chi, \forall\, u \in (0, 1)$

$$F^*(u) = \min_x \{x : u < F(x)\}$$

that is, if $F^*(u)=x$, x is the smallest possible value of X for which $F(x)$ is greater than u

- $\chi = \{a, a+1, \ldots, b\}$, where b may be ∞, $F(\cdot)$ is the cdf of X,
- $F(x)=\text{Prob}\{X \leq x\}=u_1 > u$ $\qquad F^*(u) = \min_x \{x : u < F(x)\}$



Theorem
- if $u < F(a)$, $F^*(u) = a$
- else $F^*(u) = x$ where $x \in \chi$ is the unique possible value of X for which $F(x-1) \leq u < F(x)$

Algorithm 1

```
x = a;
while (F(x) <= u)
        x++;
return x;              /*x is F*(u)*/
```

Average case analysis:

- let Y be the number of `while loop` passes
- $Y = X - a$
- $\mathrm{E}[Y] = \mathrm{E}[X-a] = \mathrm{E}[X] - a = \mu - a$

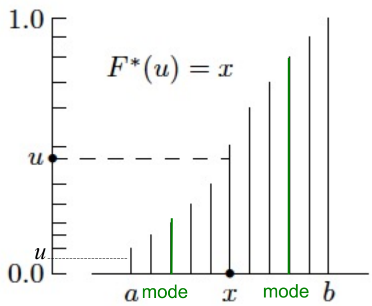
Linear search algorithm !

Idea: start at a more likely point

For $\chi=\{a, a+1, \ldots, b\}$, a more efficient linear search algorithm defines $F^*(u)$

Algorithm 2

```
x = mode;            /*initialize with the mode of X */
if (F(x) <= u)
    while (F(x) <= u)
        x++;
else if (F(a) <= u)
        while (F(x-1) > u)
          x--;
    else
        x = a;
return x;            /*x is F*(u)*/
```



For large χ, consider binary search

Idf Examples

- In some cases $F^*(u)$ can be determined explicitly
- If X is *Bernoulli*(p) and $F(x) = u$,
  then x=0 iff $0 < u < 1\text{-}p$

$$F^*(u) = \begin{cases} 0 & 0 < u < 1 - p \\ 1 & 1 - p \le u < 1 \end{cases}$$

Random Variate Generation By Inversion

- X is a discrete random variable with idf $F^*(\cdot)$
- continuous random variable U is *Uniform*(0,1)
- Z is the discrete random variable defined by $Z = F^*(U)$

Theorem
Z and X are identically distributed

this Theorem allows any discrete random variable (with known idf) to be generated with one call to `Random()`

Algorithm 3

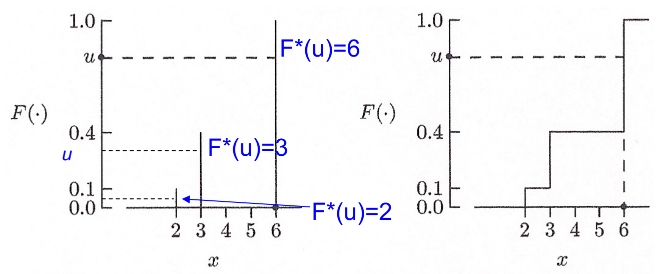
```
u = Random();
return F*(u);
```

Inversion Examples

- Consider X with pdf

$$f(x) = \begin{cases} 0.1 & x = 2 \\ 0.3 & x = 3 \\ 0.6 & x = 6 \end{cases}$$

- The cdf for X is plotted using two formats



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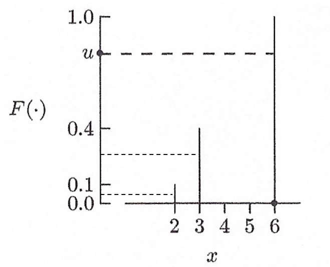
```
if (u < 0.1)
    return 2;
else if (u < 0.4)
        return 3;
    else
        return 6;
```



This algorithm returns
2 with probability 0.1,
3 with probability 0.3 and
6 with probability 0.6.
This corresponds to the pdf of X.

more efficiency: check the ranges for *u* associated with $x = 6$ first (the mode), then $x = 3$, then $x = 2$

- problems may arise when $|\mathcal{X}|$ is large or infinite

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More inversion examples

Generating a *Bernoulli*(*p*) random variate

```
u = Random();
if (u < 1-p)
        return 0;
else
        return 1;
```

Generating an *Equilikely*(*a*,*b*) random variate

```
u = Random();
return a + (long) (u * (b - a + 1));
```


Library `rvgs`

- Includes 6 discrete random variate generators (as below) and 7 continuous random variate generators
  - `long Bernoulli(double p)`
  - `long Binomial(long n, double p)`
  - `long Equilikely(long a, long b)`
  - `long Geometric(double p)`
  - `long Pascal(long n, double p)`
  - `long Poisson(double μ)`
- Functions Bernoulli, Equilikely, Geometric use inversion; essentially ideal
- Functions Binomial, Pascal, Poisson do not use inversion

Library `rvms`

- Provides accurate pdf, cdf, idf functions for many random variates
- Idfs can be used to generate random variates by inversion
- Functions `idfBinomial`, `idfPascal`, `idfPoisson` may have high marginal execution times
- Not recommended when many observations are needed due to time inefficiency
- Array of cdf values with inversion may be preferred

Truncation

Sometimes, the realistic values of a variable are restricted to a subset

X random variable with possible values $\chi=\{0, 1, 2, \dots\}$ and cdf $F(x)=\Pr(X \le x)$

- want to restrict X to the finite range $0 \le a \le x \le b < \infty$

- if $a > 0$, $\alpha = \Pr(X < a), \quad \beta = \Pr(X > b)$

$$\alpha = \Pr(X < a) = \Pr(X \le a\text{-}1) = F(a\text{-}1)$$

$$\beta = \Pr(X > b) = 1\text{-}\Pr(X \le b) = 1\text{-}F(b)$$

$$\Pr(a \le X \le b) = \Pr(X \le b) - \Pr(X < a) = F(b) - F(a\text{-}1)$$

essentially, always true iff $F(b) \cong 1.0$ and $F(a\text{-}1) \cong 0.0$

Specifying truncation points

- if a and b are specified

Left-tail, right-tail probabilities α and β obtained using cdf

$$\alpha = \Pr(X < a) = F(a\text{-}1) \quad \text{and} \quad \beta = \Pr(X > b) = 1\text{-}F(b)$$

transformation is exact

- if α and β are specified

idf can be used to obtain a and b

$$a = F^*(\alpha) \quad \text{and} \quad b = F^*(1\text{-}\beta)$$

transformation is not exact because X is discrete

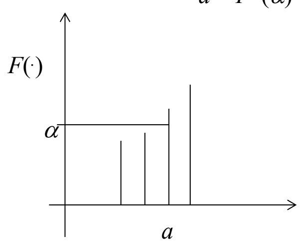
$$\Pr(X < a) \leq \alpha \quad \text{and} \quad \Pr(X > b) < \beta$$

$F(x-1) \leq u < F(x)$

Specifying truncation points

- if α and β are specified

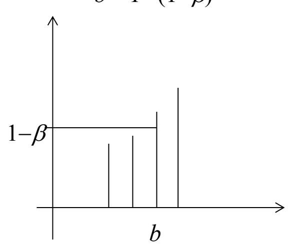
$a = F^*(\alpha)$



$F(a) > \alpha$

$\Pr(X < a) \leq \alpha$

$b = F^*(1\text{-}\beta)$



$F(b) > 1\text{-}\beta$

$\Pr(X \leq b) > 1\text{-}\beta$

$\text{-}\Pr(X \leq b) < \beta\text{-}1$

$1\text{-}\Pr(X \leq b) < \beta$

$\Pr(X > b) < \beta$

Effects of truncation

sometimes truncation is <u>insignificant</u>:
truncated and un-truncated random variables have (essentially) the same distribution

Truncation is useful for efficiency:

- When idf is complex, inversion requires cdf search
- cdf values are typically stored in an array
- Small range gives improved space/time efficiency

Truncation is useful for realism:

- Prevents arbitrarily large values possible from some variates

In some applications, truncation is <u>significant</u>

- Produces a new random variable
- Must be done correctly !