Performance Modeling of Computer Systems and Networks

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The multi-server queue

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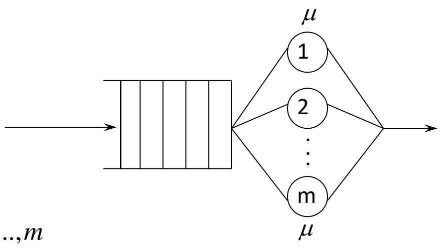
Erlang, 1917
M/M/m abstract scheduling

$$E(N_Q)_{Erlang}$$

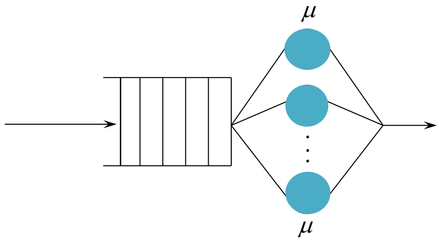


$$p(n) = \begin{cases} \frac{1}{n!}(m\rho)^n p(0) & for\ n = 1,\dots,m \\ \frac{m^m}{m!}\rho^n p(0) & for\ n > m \end{cases}$$

$$p(0) = \left[\sum_{i=0}^{m-1} \frac{(m\rho)^i}{i!} + \frac{(m\rho)^m}{m!(1-\rho)} \right]^{-1}$$

2

The *Erlang*-C formula



$$P_Q \cong \Pr\{n \geq m\} = \sum_{n=m}^{\infty} p(n)$$

$$= \sum_{n=m}^{\infty} \frac{m^m}{m!} \rho^n p(0) = \frac{m^m}{m!} p(0) \sum_{n=m}^{\infty} \rho^n$$

$$= \frac{m^m}{m!} p(0) \sum_{n=0}^{\infty} \rho^{n+m} = \frac{m^m}{m!} p(0) \rho^m \underbrace{\sum_{n=0}^{\infty} \rho^n}_{\frac{1}{1-\rho}}$$

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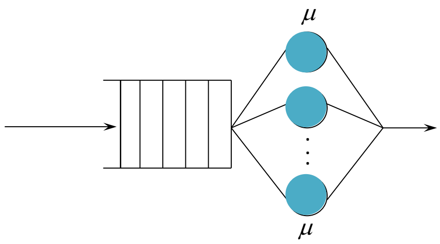
The *Erlang*-C formula



$$P_Q = \frac{(m\rho)^m}{m!(1-\rho)} p(0)$$

$$E(N_Q)_{Erlang} = P_Q \frac{\rho}{1-\rho} \qquad E(N_S) = P_Q \frac{\rho}{1-\rho} + m\rho$$

Little's law

$$E(T_Q) = \frac{E(N_Q)}{\lambda} \qquad E(T_Q) = P_Q \frac{\rho}{\lambda(1-\rho)} = \frac{P_Q E(S)}{1-\rho}$$

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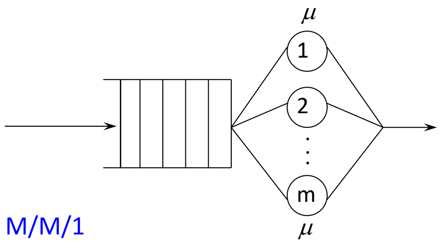
The *Erlang* formula



M/M/m

$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho}$$

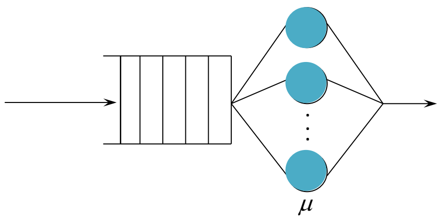
M/M/1

$$E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho} = \frac{E(S_{rem})}{1-\rho}$$

$$E(S) = \frac{E(S_i)}{m}$$

The Multi Server Queue

c ≅ busy servers

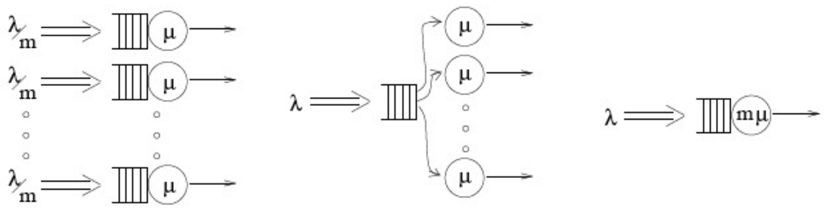


$$E(c) = \sum_{n=0}^{m-1} np(n) + \sum_{n=m}^{\infty} mp(n) = m\rho$$

$$\rho = \sum_{n=0}^{m-1} \frac{n}{m} p(n) + \sum_{n=m}^{\infty} p(n) = \sum_{n=0}^{m-1} \frac{n}{m} p(n) + P_Q \Rightarrow \rho \geq P_Q$$

$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho} \leq E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho}$$

Server Organizations



$$\rho = \frac{\lambda}{m\mu}$$

Communication systems

1
2
3
communication line
m

independent Poisson packet streams

each with an arrival rate of λ/m packets per second

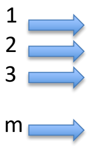
the transmission time for each packet Exponential($1/\mu$)

Frequency-division Multiplexing

Statistical Multiplexing

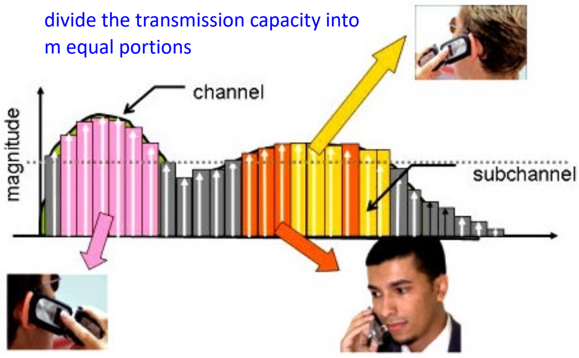
Communication systems

Frequency-division Multiplexing



separated streams

divide the transmission capacity into
m equal portions



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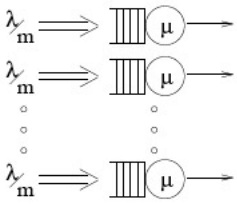
Communication systems

Frequency-division Multiplexing



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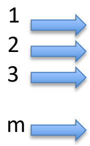
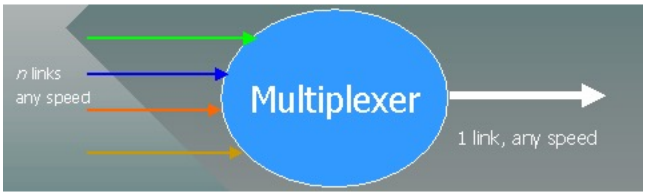
Communication systems

Statistical Multiplexing

keep the transmission capacity as a whole



merge into a single stream



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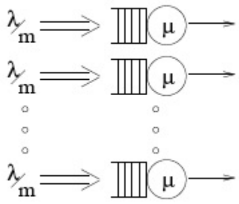
Communication systems

Frequency-division Multiplexing



Statistical multiplexing

How do the two approaches compare with respect to mean response time?

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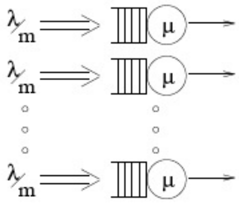
Communication systems

Frequency-division Multiplexing



Statistical multiplexing

$$E(T_S) = \frac{\rho E(S)}{1-\rho} + E(S) = \frac{E(S)}{1-\rho}$$

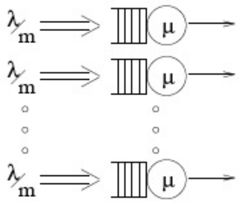
$$E(T_S) = \frac{1}{\mu\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu-\lambda}$$

M/M/1

$\lambda \quad \mu$

Communication systems

Frequency-division Multiplexing



Statistical multiplexing

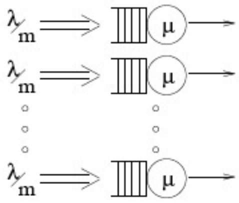
$$E(T_S)^{FDM} = \frac{1}{\mu-\frac{\lambda}{m}} = \frac{m}{m\mu-\lambda}$$

$$E(T_S)^{SM} = \frac{1}{m\mu-\lambda}$$

FDM shows a response time *m* times greater then for *SM* !

Communication systems

Frequency-division Multiplexing



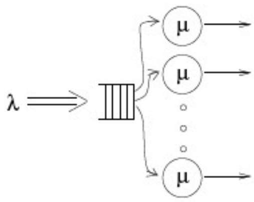
Statistical multiplexing

1 QoS guarantee for each stream: a specific service rate to each stream

No QoS guarantee

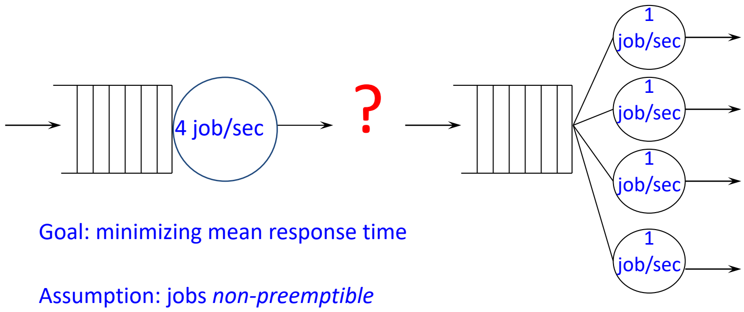
2 If the original m streams were very regular (not Poisson), i.e., they were much less variable than Poisson, by merging them, we introduce lots of variability into the arrival stream. This leads to problems if the application requires a low variability in delay, e.g., voice or video.

Server Organizations



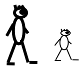
Modelling power: Design tool

Design Example 4: One machine or many?



Goal: minimizing mean response time

Assumption: jobs *non-preemptible*
each job must be run to completion

high variability 

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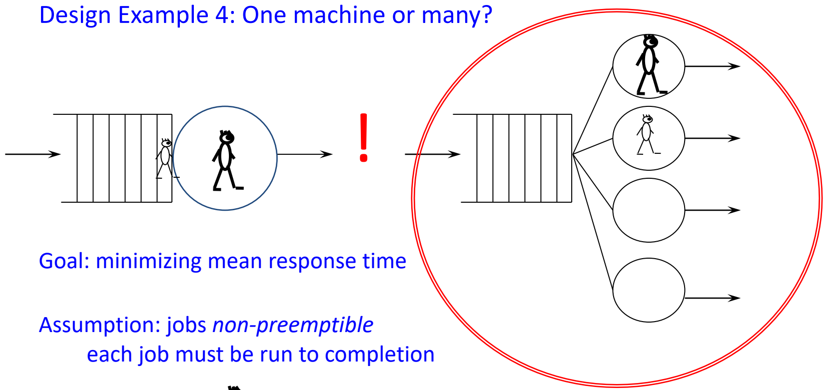
Modelling power: Design tool

Design Example 4: One machine or many?



Goal: minimizing mean response time

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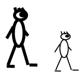
high variability 

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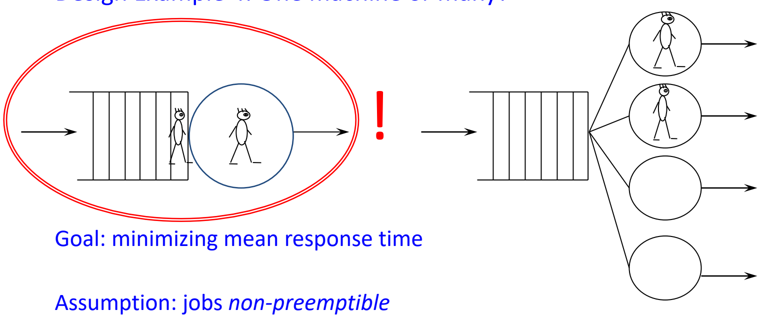
Modelling power: Design tool

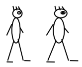
Design Example 4: One machine or many?



Goal: minimizing mean response time

Assumption: jobs *non-preemptible*
each job must be run to completion

low variability



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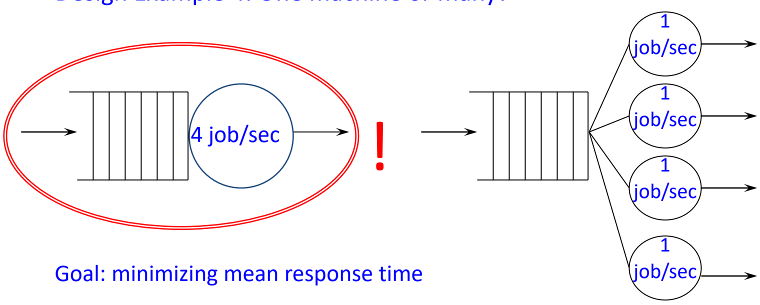
Modelling power: Design tool

Design Example 4: One machine or many?



Goal: minimizing mean response time

Assumption: jobs *preemptible*
each job can be stopped and restarted where they left off

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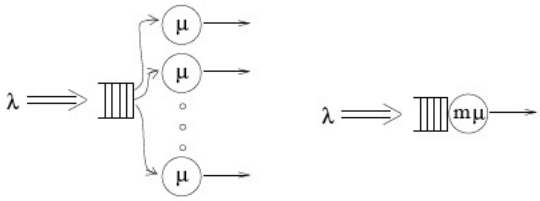
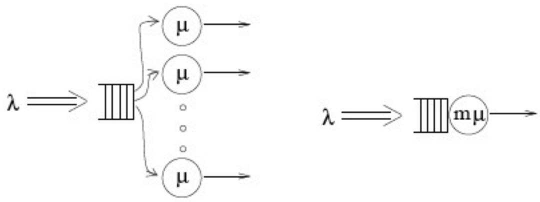
Server Organizations



$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho} \qquad E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho}$$

$\rho \geq P_Q$ → from the waiting time perspective the distributed capacity solution produces an improvement in the user perceived QoS

Server Organizations

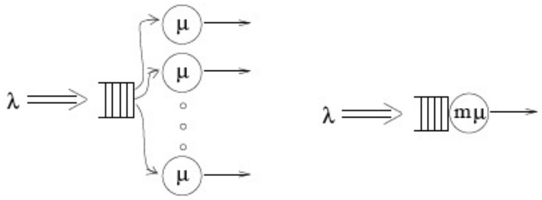


What about the response time perspective??

$$E(T_S)_{Erlang} = \frac{P_Q E(S)}{1-\rho} + E(S_i) \qquad E(T_S)_{KP} = \frac{\rho E(S)}{1-\rho} + E(S)$$

$$E(S_i) = \frac{1}{\mu} = m\frac{1}{m\mu} = mE(S)$$

Server Organizations



What about the response time perspective??

$$E(T_S)_{Erlang} = \frac{P_Q E(S)}{1-\rho} + mE(S)$$

Decreasing less than linear $\wedge$ $\vee$ linear growth

$$E(T_S)_{KP} = \frac{\rho E(S)}{1-\rho} + E(S)$$

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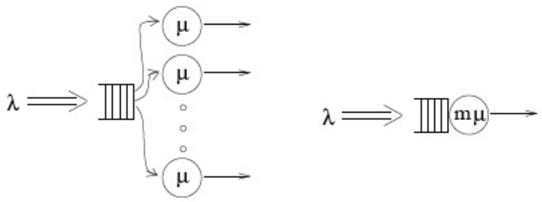
Server Organizations



Performance goal:

Waiting time perspective

Distributed capacity

Response time perspective

$\rho \rightarrow 0$ distrib. capac. gives an *m* times slower organization

$\rho \rightarrow 1$ approximately the same response time

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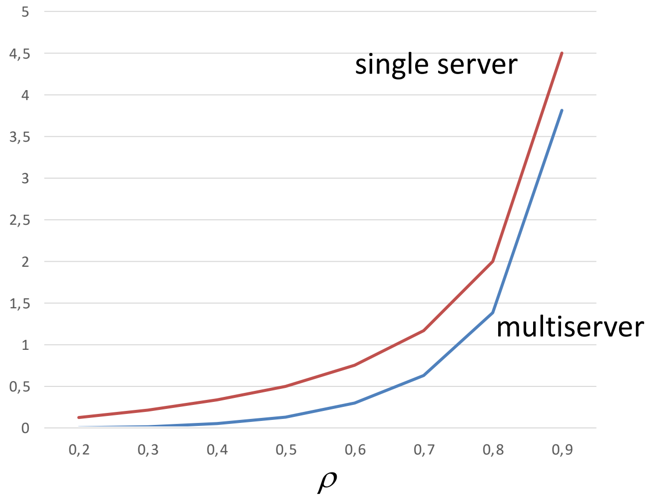
Waiting time perspective

E(S)=0,5 s



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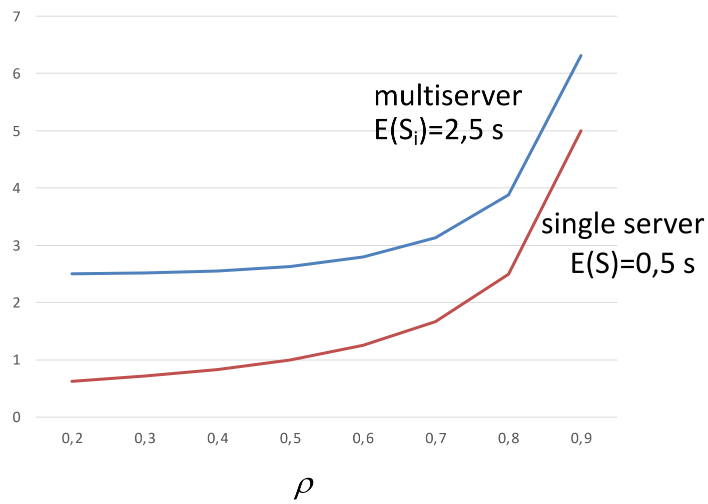
Response time perspective



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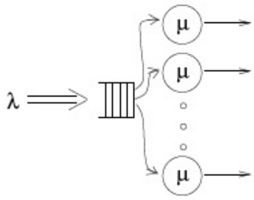
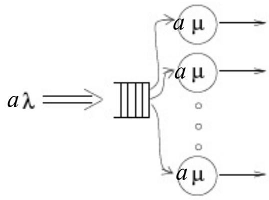
Scaling factor





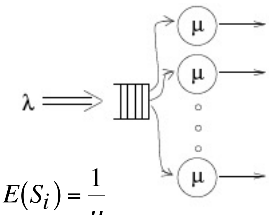
What about waiting and response time?

$$\rho = \frac{\lambda}{m\mu} \qquad \rho = \frac{a\lambda}{ma\mu} = \frac{\lambda}{m\mu}$$

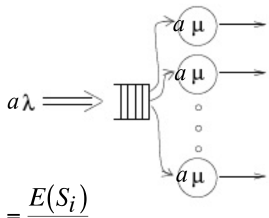
$$E(S_i) = \frac{1}{\mu} \qquad E(S_i) = \frac{1}{a\mu} \qquad E(S) = \frac{E(S_i)}{m}$$

Analytical models
server organizations

Scaling factor



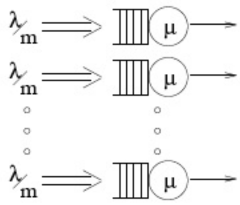
$$E(S_i) = \frac{1}{\mu}$$

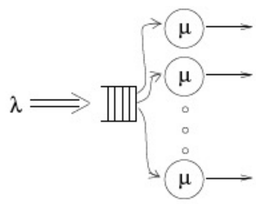


$$E(S) = \frac{E(S_i)}{m} \qquad E(S_i) = \frac{1}{a\mu}$$

Mean waiting time

$$E(T_Q)_{m,a} = \frac{P_Q E(S)_{m,a}}{1-\rho} = \frac{P_Q}{ma\mu(1-\rho)} = \frac{1}{a}\frac{P_Q E(S)_{m,1}}{(1-\rho)} = \frac{1}{a}E(T_Q)_{m,1}$$





$\lambda = 4$ j/s,   m=4, μ=1.5 j/s E(S)=0.666667 s

$$E(S) = \frac{E(S_i)}{4} = 0.1667$$

$$\rho = 0.666667$$

$$E(T_S) = \frac{1}{\mu - \lambda} = 2\ s$$

$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336$$

$$\rho = 0.666667$$

$$p(0) = \left[\sum_{i=0}^{3} \frac{(4\rho)^i}{i!} + \frac{(4\rho)^4}{4!(1-\rho)} \right]^{-1}$$

$$= \left[1 + 4\rho + \frac{(4\rho)^2}{2} + \frac{(4\rho)^3}{6} + \frac{(4\rho)^4}{24(1-\rho)} \right]^{-1} = 0.059857$$

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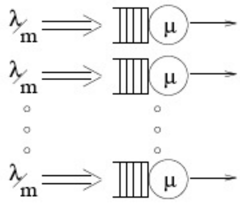

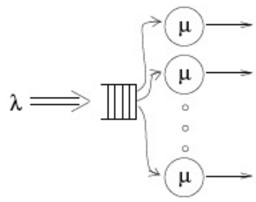

$\lambda = 4$ j/s,   m=4, μ=1.5 j/s E(S)=0.666667 s

$$E(S) = \frac{E(S_i)}{4} = 0.1667$$

$E(S_i)$=0.666667 s

$$\rho = 0.666667$$

$$E(T_S) = \frac{1}{\mu - \lambda} = 2\ s$$

$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336$$

$$\rho = 0.666667$$

$$P_Q = \frac{(4\rho)^4}{4!(1-\rho)} p(0) = 0.37847$$

$$E(T_S) = \frac{P_Q E(S)}{1-\rho} + E(S_i) = 0.855992$$

$$E(T_Q) = 0.189292$$

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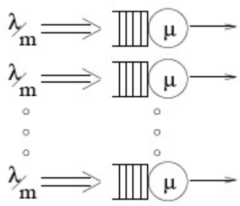
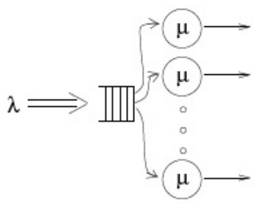
$\lambda = 4$ j/s, $m\mu = 4\text{x}1.5 = 6$ j/s E(S)=0.166667 s

$\rho = 0.666667$

$$E(T_S) = \frac{1}{m\mu - \lambda} = 0.5$$

$$E(T_Q) = 0.3334$$





$\rho = 0.666667$

$$E(T_S) = \frac{1}{\mu - \lambda} = 2 \qquad E(T_S) = 0.855992 \qquad \boxed{E(T_S) = \frac{1}{\mu - \lambda} = 0.5\ s}$$

$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336 \qquad \boxed{E(T_Q) = 0.189292} \qquad E(T_Q) = 0.3334$$

Esercizio proposto: $\rho = 0.533334 \quad \rho = 0.8$