Supplementary Material to:

Incremental Process for Reliable SLAM under Constrained Optimization Formulation

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***Abstract*— This is the supplementary material to the paper entitled "Incremental Process for Reliable SLAM under Constrained Optimization Formulation". Some fundamentals about $SO(3)$ and the linearization of the constrained SLAM formulation are presented in this documentation.**

I. PRELIMINARIES ON SO(3) GROUP

A. A Brief Introduction to SO(3) Group

The definition of special orthogonal group ($SO(3)$ group) is the set of valid rotation matrices

$$SO(3) \overset{\text{def}}{=} \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}\mathbf{R}^{\mathsf{T}} = \mathbf{I}, \mathbf{det}(\mathbf{R}) = 1\}.$$

The Lie algebra of $SO(3)$ (denoted by $\mathfrak{so}(3)$), is usually referred as a set of skew-symmetric matrices which can be identified by an element in $\mathbb{R}^3$. Denote the skew-symmetric matrix associated with $\Theta \in \mathbb{R}^3$ as $[\Theta]_\times \in \mathfrak{so}(3)$, then

$$[\Theta]_\times = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

For each rotation $\mathbf{R}$, there is an axis-angle representation Θ. The rotation matrix $\mathbf{R}$ and its corresponding axis-angle Θ are connected by exponential mapping $\mathbf{Exp}(\cdot)$ and logarithm mapping $\mathbf{Log}(\cdot)$ as below.

$$SO(3) \ni \mathbf{R} = \mathbf{Exp}(\Theta) = \mathbf{I} + \frac{[\Theta]_\times}{\|\Theta\|}\sin\|\Theta\| + \frac{[\Theta]_\times^2}{\|\Theta\|^2}(1-\cos\|\Theta\|)$$

$$\mathbb{R}^3 \ni \Theta = \mathbf{Log}(\mathbf{R}) = \frac{\theta}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

where $\theta = \cos^{-1}(\frac{trace(\mathbf{R})-1}{2})$, $r_{ij} \in \mathbb{R}$ is the ij-th elements of the rotation matrix $\mathbf{R}$.

The BCH formula to incorporate the perturbation on $SO(3)$ into the axis-angle space is presented as,

$$\mathbf{Exp}(\Theta_1)\cdot\mathbf{Exp}(\Theta_2) \approx \begin{cases} \mathbf{Exp}[\mathbf{J}_r^{-1}(-\Theta_2)\cdot\Theta_1 + \Theta_2] & \Theta_1 \to \mathbf{0} \\ \mathbf{Exp}[\Theta_1 + \mathbf{J}_r^{-1}(\Theta_1)\cdot\Theta_2] & \Theta_2 \to \mathbf{0} \end{cases}$$

where $\mathbf{J}_r(\cdot)$ is the so-called right-hand Jacobian given by

$$\mathbf{J}_r(\Theta) = \mathbf{I} - \frac{1-\cos\|\Theta\|}{\|\Theta\|^2}[\Theta]_\times + \frac{\|\Theta\| - \sin\|\Theta\|}{\|\Theta\|^3}[\Theta]_\times^2.$$

If $\Theta = \mathbf{0}$, $\mathbf{J}_r(\Theta) = \mathbf{I}$.

An identity as below derived from the adjoint operation of $SO(3)$ is

$$\mathbf{R}\cdot\mathbf{Exp}(\Theta)\cdot\mathbf{R}^{\mathsf{T}} = \mathbf{Exp}(\mathbf{R}\Theta).$$

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B. Properties

Some properties for $SO(3)$ group are listed as below.

$$[\Theta_1]_\times \cdot \Theta_2 = -[\Theta_2]_\times \cdot \Theta_1, \quad \forall \Theta_1, \Theta_2 \in \mathbb{R}^3$$

$$\mathbf{R}\cdot[\Theta]_\times\cdot\mathbf{R}^T = [\mathbf{R}\Theta]_\times, \quad \forall\Theta\in\mathbb{R}^3, \mathbf{R}\in SO(3)$$

$$\mathbf{Exp}(\Theta) \approx \mathbf{I} + [\Theta]_\times + \frac{1}{2}[\Theta]_\times^2, \quad if \quad \mathbb{R}^3 \ni \Theta \to 0$$

$$\mathbf{R}\cdot\mathbf{Exp}(\Theta) = \mathbf{Exp}(\mathbf{R}\Theta)\cdot\mathbf{R}, \quad \forall\Theta\in\mathbb{R}^3, \mathbf{R}\in SO(3)$$

II. LINEARIZATION OF THE CONSTRAINED SLAM FORMULATION

A. Linearization of the Translational Equation

The linearization of the translational equation can be attained by firstly linearizing terms like

$$\mathbf{F}_c = [\prod_{i=1}^{m}\mathbf{R}_i]\mathbf{y} \quad (\mathbf{R}_i \in SO(3), \mathbf{y} \in \mathbb{R}^3)$$

then collecting the linearized terms together to obtain a full linearized version.

Let $\Omega_m^n = \prod_{i=m}^n \mathbf{R}_i$, then $\mathbf{F}_c$ can be written as

$$\mathbf{F}_c = [\prod_{i=1}^{m}\mathbf{R}_i]\mathbf{y} = \Omega_1^{j-1}\cdot\mathbf{R}_j\cdot\Omega_{j+1}^m\mathbf{y}.$$

For $\mathbf{R}_j$ and $\mathbf{y}$, add a perturbation on their current estimate $\breve{\mathbf{R}}_j$ and $\breve{\mathbf{y}}$ respectively, then approximate the rotational perturbation with its second-order approximation, which results

$$\mathbf{R}_j = \breve{\mathbf{R}}_j\cdot\mathbf{Exp}(\Theta_j) \approx \breve{\mathbf{R}}_j\cdot(\mathbf{I} + [\Theta_j]_\times + \frac{1}{2}[\Theta_j]_\times^2)$$

$$\mathbf{y} = \breve{\mathbf{y}} + \mathbf{y}_m$$

where $\Theta_j \to \mathbf{0}$, $\mathbf{y}_m \to \mathbf{0}$.

By Taylor expansion, at point $\mathbf{y}_m = \mathbf{0}$, $\Theta_j = \mathbf{0}$ $(j = 1,\ldots,m)$, we have

$$\mathbf{F}_c = \mathbf{F}_c\Big|_{\mathbf{0}} + \frac{\partial\mathbf{F}_c}{\partial\mathbf{y}_m}\Big|_{\mathbf{0}}\mathbf{y}_m + \sum_{j=1}^{m}\frac{\partial\mathbf{F}_c}{\partial\Theta_j}\Big|_{\mathbf{0}}\Theta_j + \mathbf{o}$$

$$\begin{aligned}
\frac{\partial\mathbf{F}_c}{\partial\Theta_j}\Big|_{\mathbf{0}} &= \frac{\partial}{\partial\Theta_j}\{\Omega_1^{j-1}\cdot\breve{\mathbf{R}}_j\cdot(\mathbf{I}+[\Theta_j]_\times + \frac{1}{2}[\Theta_j]_\times^2)\cdot\Omega_{j+1}^m\mathbf{y}\}\Big|_{\mathbf{0}} \\
&= \frac{\partial}{\partial\Theta_j}\{\Omega_1^{j-1}\cdot\breve{\mathbf{R}}_j\cdot[\Theta_j]_\times\cdot\Omega_{j+1}^m\mathbf{y}\}\Big|_{\mathbf{0}} \\
&= \frac{\partial}{\partial\Theta_j}\{-\Omega_1^{j-1}\cdot\breve{\mathbf{R}}_j\cdot[\Omega_{j+1}^m\mathbf{y}]_\times\cdot\Theta_j\}\Big|_{\mathbf{0}} \\
&= -\Omega_1^{j-1}\cdot\breve{\mathbf{R}}_j\cdot[\Omega_{j+1}^m\mathbf{y}]_\times\Big|_{\mathbf{0}} \\
&= -\breve{\Omega}_1^{j-1}\cdot\breve{\mathbf{R}}_j\cdot[\breve{\Omega}_{j+1}^m\breve{\mathbf{y}}]_\times \\
&= -\breve{\Omega}_1^{j}[\breve{\Omega}_{j+1}^m\breve{\mathbf{y}}]_\times
\end{aligned}$$

$$\frac{\partial \mathbf{F}_c}{\partial \mathbf{y}_m}\Big|_{\mathbf{0}} = \frac{\partial}{\partial \mathbf{y}_m}\{[\prod_{i=1}^{m}\mathbf{R}_i](\check{\mathbf{y}}+\mathbf{y}_m)\}\Big|_{\mathbf{0}} = [\prod_{i=1}^{m}\mathbf{R}_i]\Big|_{\mathbf{0}} = [\prod_{i=1}^{m}\check{\mathbf{R}}_i] = \check{\Omega}_1^m$$

$$\mathbf{F}_c\Big|_{\mathbf{0}} = [\prod_{i=1}^{m}\mathbf{R}_i]\mathbf{y}\Big|_{\mathbf{0}} = [\prod_{i=1}^{m}\check{\mathbf{R}}_i]\check{\mathbf{y}} = \check{\Omega}_1^m\check{\mathbf{y}}$$

Note here we use notation $\check{\Omega}_m^n = \prod_{i=m}^{n}\check{\mathbf{R}}_i$.

Thus, after linearization

$$\mathbf{F}_c = \check{\Omega}_1^m\check{\mathbf{y}} + \check{\Omega}_1^m\mathbf{y}_m - \sum_{j=1}^{m}\check{\Omega}_1^j[\check{\Omega}_{j+1}^m\check{\mathbf{y}}]_\times\Theta_j + \mathbf{o}.$$

B. Linearization of the Rotational Equation

For the rotational equation, a term in the form like

$$\prod_{i=1}^{m}\mathbf{R}_i = \mathbf{R}_1\mathbf{R}_2\cdots\mathbf{R}_m = \mathbf{R}_0$$

needs to be linearized.

For each rotation $\mathbf{R}_i$ $(i=0,1,\ldots,m)$, adding a perturbation Θ_i in the axis-angle space at its current estimate $\check{\mathbf{R}}_i$, the constraint can be written as

$$\prod_{i=1}^{m}[\check{\mathbf{R}}_i\mathbf{Exp}(\Theta_i)] = \check{\mathbf{R}}_0\mathbf{Exp}(\Theta_0). \tag{1}$$

Using the identity

$$\mathbf{R}\cdot\mathbf{Exp}(\Theta) = \mathbf{Exp}(\mathbf{R}\Theta)\cdot\mathbf{R}$$

iteratively, the left-hand side of (1) becomes

$$\begin{aligned}\prod_{i=1}^{m}[\check{\mathbf{R}}_i\mathbf{Exp}(\Theta_i)] &= \{\prod_{i=1}^{m}\mathbf{Exp}([\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i)\}\cdot\prod_{i=1}^{m}\check{\mathbf{R}}_i\\ &\approx \mathbf{Exp}(\sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i)\cdot\prod_{i=1}^{m}\check{\mathbf{R}}_i.\end{aligned}$$

Hence (1) takes the form

$$\mathbf{Exp}(\sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i)\cdot\prod_{i=1}^{m}\check{\mathbf{R}}_i \approx \mathbf{Exp}(\check{\mathbf{R}}_0\Theta_0)\cdot\check{\mathbf{R}}_0.$$

Let $\eta = \mathbf{Log}(\check{\mathbf{R}}_0\cdot[\prod_{i=1}^{m}\check{\mathbf{R}}_i]^\top)$, then

$$\mathbf{Exp}(-\check{\mathbf{R}}_0\Theta_0 + \sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i)\cdot\mathbf{Exp}(-\eta) \approx \mathbf{I}.$$

Using BCH formula, it can be written as

$$\mathbf{Exp}(\mathbf{J}_r^{-1}(\eta)\{-\check{\mathbf{R}}_0\Theta_0 + \sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i\} - \eta) \approx \mathbf{I}.$$

The axis-angle space takes the form

$$\mathbf{J}_r^{-1}(\eta)\{-\check{\mathbf{R}}_0\Theta_0 + \sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i\} - \eta \approx \mathbf{0} \tag{2}$$

if Θ_i and η are small.

Equivalently, (2) can be written as

$$-\check{\mathbf{R}}_0\Theta_0 + \sum_{i=1}^{m}[\prod_{j=1}^{i}\check{\mathbf{R}}_i]\cdot\Theta_i \approx \mathbf{J}_r(\eta)\eta.$$

C. Linearization of the Objective Function

The only non-linear part of the objective lies in the logarithm mapping which can be linearized as follow.

$$\begin{aligned}\mathbf{Log}(\mathbf{Z}_{R_j^i}{}^\top\mathbf{R}_j^i) &= \mathbf{Log}\{\mathbf{Z}_{R_j^i}{}^\top\check{\mathbf{R}}_j^i\cdot\mathbf{Exp}(\Theta_j^i)\}\\ &= \mathbf{Log}\{\mathbf{Exp}(\eta_{R_j^i})\cdot\mathbf{Exp}(\Theta_j^i)\}\\ &\approx \mathbf{Log}\{\mathbf{Exp}[\eta_{R_j^i} + \mathbf{J}_r^{-1}(\eta_{R_j^i})\Theta_j^i]\}\\ &= \eta_{R_j^i} + \mathbf{J}_r^{-1}(\eta_{R_j^i})\Theta_j^i\end{aligned}$$

where $\eta_{R_j^i} = \mathbf{Log}(\mathbf{Z}_{R_j^i}^\top\check{\mathbf{R}}_j^i)$.

III. DISCUSSION

As mentioned in the paper, it is also possible to choose $SE(3)$ optimization techniques by considering the translational and rotational part together as $SE(3)$ transformations. The linearization on $SE(3)$ is very similar to the linearization of the rotational equation described in Section II-B. However, to generalize the implementation of pose-graph SLAM and feature-based SLAM, we stick to $SO(3)$ linearization techniques here.

It is worth mentioning that the covariance for $SE(3)$ and $SO(3)$ are slightly different. For the rotational part, the covariance are exactly the same. However for the translational part, the covariance for $SO(3)$ and $SE(3)$ are related by

$$R_z\cdot\Sigma_T\cdot R_z{}^\top = \Omega_T$$

Here R_z is the rotational part of the relative pose measurement data, and Σ_T is the covariance for the translational part in $SE(3)$ while Ω_T is that in $SO(3)$. This can be examined easily by considering the translational and rotational part in $SE(3)$ separately.