Models: (Chapter 2, text)

Given a physical system we want to obtain its mathematical model, i.e. a mathematical relantionship between the input and the output.

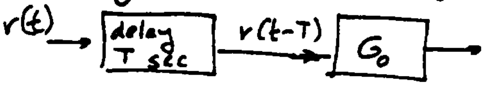
In this class we will deal only with FDLTI (finite dimensional linear, time-inv) systems. Hence the mathematical models will be either

1) an ordinary differential equation (O.D.E) with constant coeff

or

2) a rational Transfer function

Only exception: systems containing a time delay



$$G(s) = G_0(s)\, e^{-st}$$

- **Section 2.2:** Electrical Circuits read textbook

- **Mechanical Translational Systems:** (section 2.5)

Basic law Newton's second law $M\ddot{x} = \Sigma F$

Elements: a) Spring

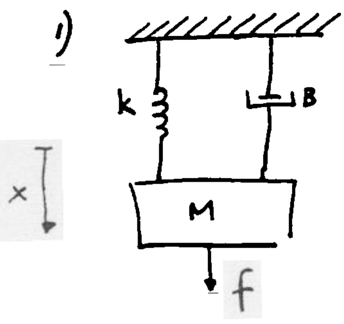
b) Viscous Damping & Friction

- Model of an ideal spring (Hooke's law):

$$F = -\underset{\substack{\uparrow\\ \text{spring}\\ \text{constant}}}{k}\,(\underset{\substack{\uparrow\\ \text{actual}\\ \text{length}}}{x} - \overset{\text{natural length}}{x_\circ})$$

- Model of viscous damping & friction: $F = -b\dot{x}$ always opposes motion

Examples: 1)



$$M\ddot{x} = -kx - b\dot{x} + f$$

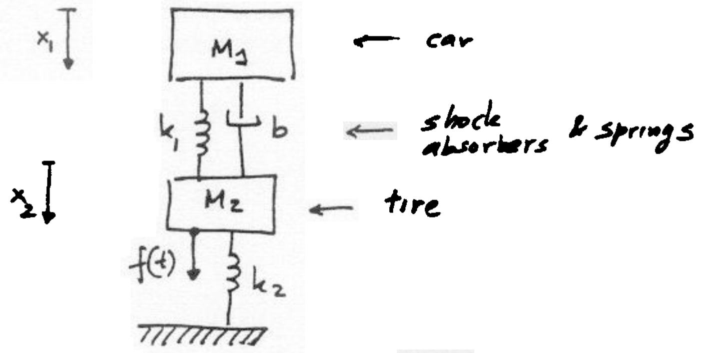
$$M\ddot{x} + b\dot{x} + kx = f$$

Taking Laplace Transforms on both sides yields:

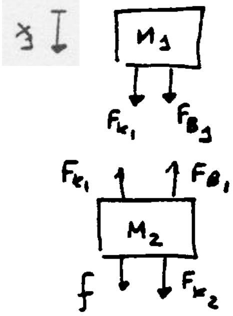
$$(Ms^2 + bs + k)\,X(s) + (I.C.) = F(s)$$

$\Rightarrow$ Transfer function $G(s) = \left.\dfrac{X(s)}{F(s)}\right|_{I.C.=0} = \dfrac{1}{Ms^2+bs+k}$

Example 2: Simplified model of an automobile suspension:



Free body diagrams:



$$M_1\ddot{x}_1 = -k_1(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2)$$

$$M_2\ddot{x}_2 = -k_1(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) - k_2x_2 + f$$

Taking Laplace Transforms (assume 0 Initial conditions) yields:

$$(M_1s^2 + b_1s + k_1)\,x_1 - (b_1s + k_1)\,x_2 = 0$$

$$-(b_2s + k_2)\,x_2 + (M_2s^2 + b_1s + k_1 + k_2)\,x_2 = F(s)$$

If we want to find the T.F from $F(s)$ to $X_1(s)$ we need to solve these simultaneous equations. For instance, we can use Cramer's formula. Later on we will see an alternative solution method using signal flow graphs

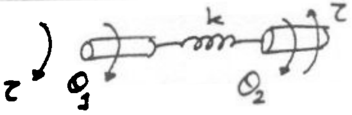
- **Mechanical Rotational systems** (section 2.6)

Elements: moment of Inertia (similar to mass)
friction
torsion

Basic Law: Newton's equation for rotational systems:

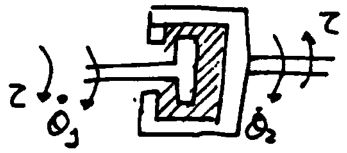
$$J\ddot{\theta} = \Sigma \text{ Torques}$$

Torsion Spring:



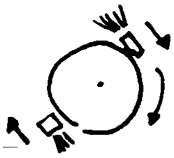
$\tau = k(\theta_2 - \theta_1)$

Viscous friction:



$\tau = B(\dot{\theta}_2 - \dot{\theta}_1)$

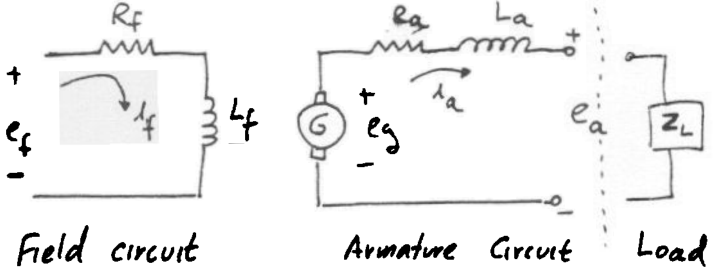
- **Exemple**: A rotating rigid satellite; with torque applied by 2 thrusters



$$J\frac{d^2\theta}{dt^2} = \tau(t)$$

- **Electromechanical Systems**: (2.7)

  - **Generator** (rotating at constant speed)



input: e_f

output E_a

1) Field circuit: $e_f = R_f i_f + L_f \frac{di_f}{dt}$ // $E_f(s) = (R_f + L_f s)\, I_f(s)$

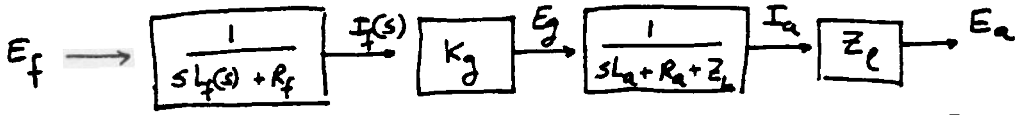
2) Generator circuit: $e_g = R_a i_a + L_a \frac{di_a}{dt} + E_a$ // $E_g(s) = E_a(s) + (R_a + sL_a) I_a(s)$

3) Electromechanical: equation $e_g = k\Phi\dot{\theta}$

linear assumption: $\Phi = k_\Phi i_f$

$e_g = k_g i_f$ (constant velocity)

Block Diagram of the generator:



Transfer function: $G(s) = \dfrac{Z_\ell K_g}{(R_f + sL_f)(sL_a + R_a + Z_L)}$

Note that G(s) depends on Z_ℓ!!

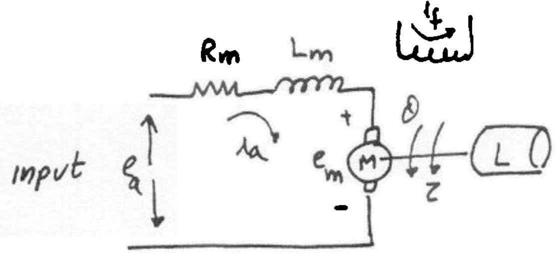
$$\left.\begin{aligned} E_g(s) &= E_a(s) + (sL_a + R_a)\, I_a(s) \\ E_a(s) &= Z_L\, I_a \end{aligned}\right\}$$

$$E_g = (Z_L + sL_a + R_a)\, I_a(s)$$

$$I_a(s) = \left(\frac{1}{Z_L + sL_a + R_a}\right) E_g$$

$$E_a = Z_L\, I_a$$

- <u>DC Motor</u> (with independent excitation)



back emf

1) Electrical equation: $e_a = R_m i_a + L_m \frac{d i_a}{dt} + e_m$ // $E_a(s) = (s L_m + R_m) I_a(s) + E_m(s)$

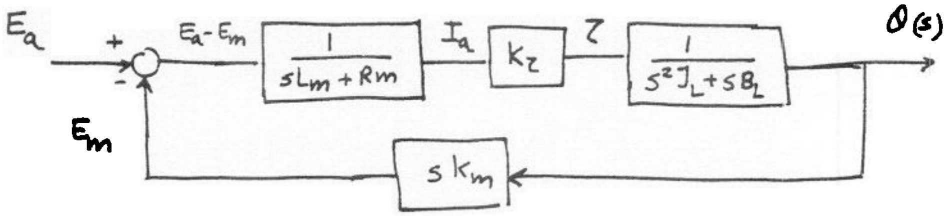
2) Back emf: $e_m = k_m \dot{\theta}$ // $E_m(s) = k_m s\, \theta(s)$

3) Mechanical equation: $\tau = K_\tau i_a$

4) Finally: Newton's second equation: $J_L \frac{d\ddot{\theta}}{dt^2} + B_L \dot{\theta} = \tau$ // $(s^2 J_L + s B_L)\theta = \tau(s)$

Putting everything together yields the following block diagram

From (1): $I_a = \left(\frac{E_a(s) - E_m(s)}{sL_m + R_m}\right)$; $E_m(s) = s\, K_m\, \theta(s)$



New feature: the system has <u>built-in</u> feedback (through the back emf)

If we want to find $\theta(s)$ as a function of $E_a(s)$ we need to solve:

$$\begin{aligned} E_a &= (sL_m + R_m)\, I_a + s K_m\, \theta(s) \\ 0 &= -K_\tau I_a + (s^2 J_L + s B_L)\,\theta \end{aligned}$$

or, in matrix form:
$$\begin{bmatrix} sK_m & sL_m + R_m \\ s^2J_L + sB_L & -K_2 \end{bmatrix} \begin{pmatrix} \Theta \\ I_a \end{pmatrix} = \begin{pmatrix} E_q \\ 0 \end{pmatrix}$$

Using Cramer's rule we have:

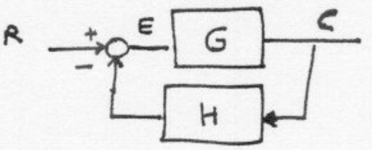
$$\Theta = \frac{\Delta_\Theta}{\Delta}$$

$$\Delta = -k_2 k_m s - (R_m + sL_m)(sJ_L + B_L)s$$

$$\Delta_\Theta = -K_2 E_a$$

$$\Rightarrow \Theta = \frac{\Delta_\Theta}{\Delta} = \frac{k_2}{k_m k_2 s + (R+sL)(sJ_L+B_L)s} E_a \quad \#$$

Note that the system has the general form:



$$\left.\begin{aligned} E &= R - HC \\ C &= GE \end{aligned}\right\} \quad C = GR - GHC$$

or $\boxed{\frac{C}{R} = \frac{1}{1+GH}}$ #

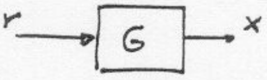
This is a special case of <u>Mason's Formula</u>

- Signal Flow Diagrams (sections 2.3, 2.4)

They provide an alternative representation of Transfer Function relationships. They also provide an alternative to Cramer's rule for computing T.F.

Rules:

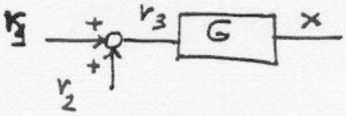
- Each signal is represented by a node
- Each T.F. is represented by a branch (arrow)

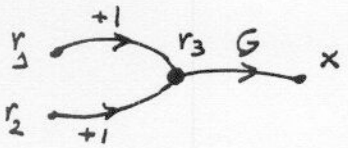
Block Diagram:


$x(s) = G(s)\, r(s)$

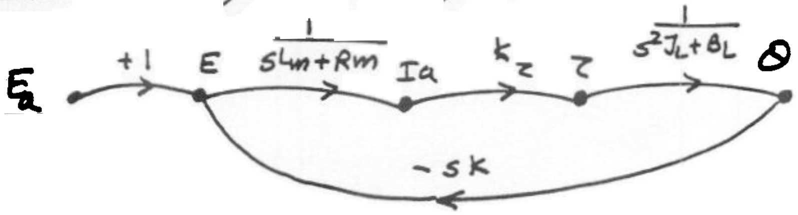
Signal flow:


- Summing junctions are represented implicitly: all the inputs converging to a node are added together



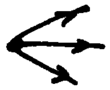


Example: signal flow graph representation of the DC motor



Some Terminology:

source node: A node that has all signals flowing away from it.



sink node: A node with incoming signals only

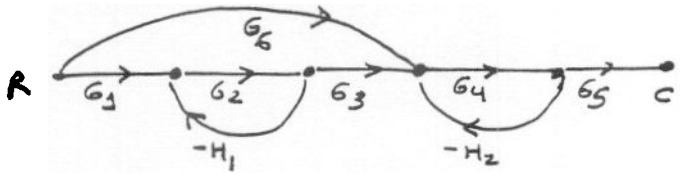
Path: Continuous connection of branches between 2 nodes (directed)

Loop: Closed path (i.e starting node = finishing node)

Path (loop) Gain: Product of all T.F. of all the branches in the path (loop)

Non Touching loops (paths): Loops that do not have any nodes in common.

Example:



2 loops: $-G_2H_1$ (L_1)
$-G_4H_2$ (L_2)

Path $G_6G_4G_5$ does not touch L_1
Path $G_1G_2G_3G_4G_5$ touches both L_1 and L_2

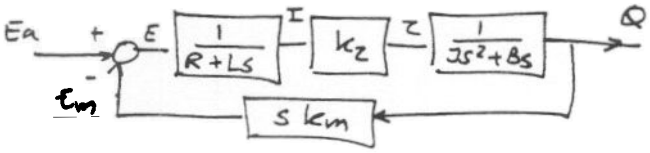
- Mason's Formula (section 2.4) Provides an alternative to Cramer's rule or elimination for finding Transfer Functions

$$T_{CR} = \frac{1}{\Delta}\sum_{k=1}^{P} M_k\Delta_k = \frac{1}{\Delta}\left(M_1\Delta_1 + \qquad\qquad + M_p\Delta_p\right)$$

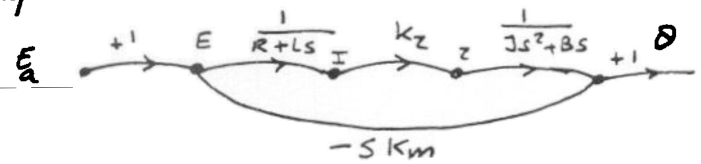
Where:

- $\Delta = 1 - \left(\Sigma \text{ gains individual loops}\right) + \Sigma\left(\text{products of non-touching pairs of loops}\right)$
  $- \Sigma\left(\text{products of triplets of non-touching loops}\right)$
  $+ \cdots$
- $M_k =$ Gain of the k^{th} path between R and C
- $\Delta_k =$ Value of Δ when the nodes in the path M_k are removed from the graph

Example 1: DC motor:



Signal flow graph:



1 loop: $L_1 = \frac{-k_z k_m s}{(R+Ls)(Js+B)s} \Rightarrow \Delta = 1 + \frac{k_z k_m s}{(R+Ls)(Js+B)s}$

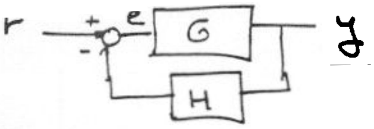
Only 1 path from E_a to Θ: $M_1 = \frac{k_z}{(R+Ls)(Js+B)s}$

$\Delta_1 = 1$

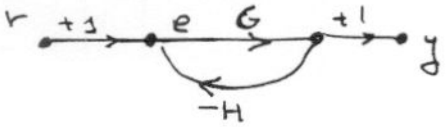
$$T_{\Theta E_a} = \frac{1}{\Delta}\cdot M_1\Delta_1 =$$

$$= \frac{\frac{k_z}{(R+Ls)(Js+B)s}}{1 + \frac{k_z k_m s}{(R+Ls)(Js+B)s}} = \frac{k_z}{(R+Ls)(Js+B)s + k_m k_z} \quad \#$$

Note: This is a special case of:



In signal flow form:

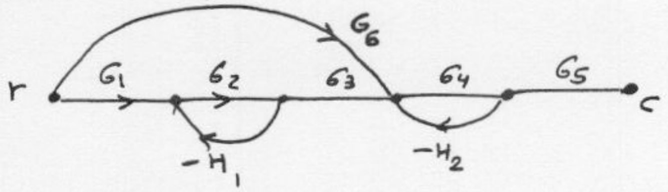


$L_1 = -GH$

$\Delta = 1 - L_1 = 1 + GH$

$T_{yr} = \frac{M}{\Delta} = \frac{G}{1+GH} \quad \#$

Example 2: (Example 2.14 text)



Number of paths between r & c: 2 — $M_1 = G_1 G_2 G_3 G_4 G_5$

$M_2 = G_6 G_4 G_5$

Number of loops: 2 $L_1 = -G_2 H_1$ (non touching)

$L_2 = -G_4 H_2$

$$\Delta = 1 - \Sigma \text{ loops} + \underset{(N.T.)}{\Sigma \text{ pairs}} = 1 + G_2 H_1 + G_4 H_2 + G_2 G_4 H_1 H_2$$

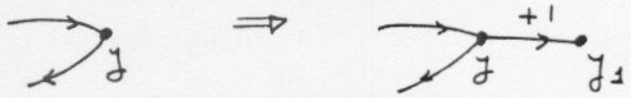
Path M_1 touches both loops $\Rightarrow \Delta_1 = 1$

Path M_2 touches only L_2 $\Rightarrow \Delta_2 = 1 + G_2 H_1$

$$T_{cr} = \frac{1}{\Delta}(M_1 \Delta_1 + M_2 \Delta_2) = \frac{G_1 G_2 G_3 G_4 G_5 + G_6 G_4 G_5 (1 + G_2 H_1)}{1 + G_2 H_1 + G_4 H_2 + G_2 G_4 H_1 H_2}$$

Important: Technically, Mason's formula is valid to compute the TF ONLY from a source node to a sink node

The restriction of the output being a sink node is easy to remove: add an extra branch with gain 1



(This essentially says $y = y_1$)

However, the restriction of the input being a source node can't be dealt with in this form.

- Basic operations on systems (Block Diagrams)

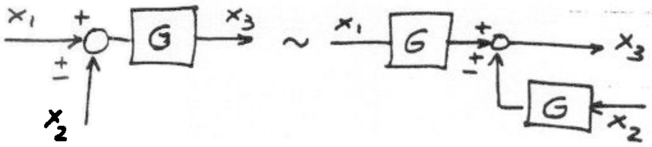
Here we are going to learn how to operate on block diagrams. This will allow us to obtain simpler (hopefully) diagrams, easier to solve

The basic operations are:

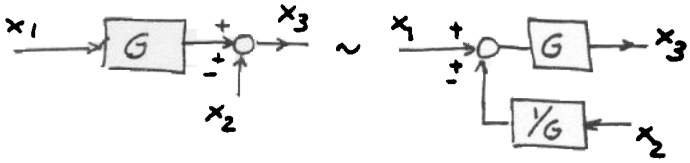
1) 

(cascade)

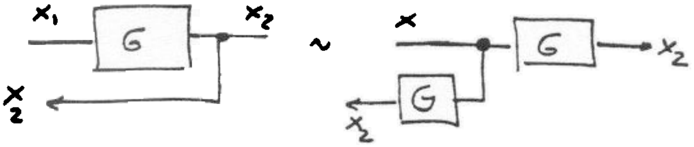
2) Moving a summing junction behind a block



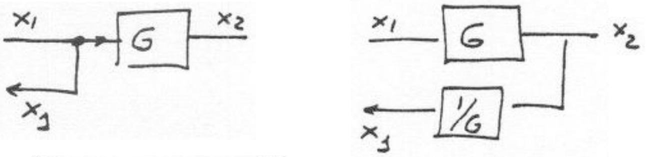
3) Moving a summing junction ahead of a block



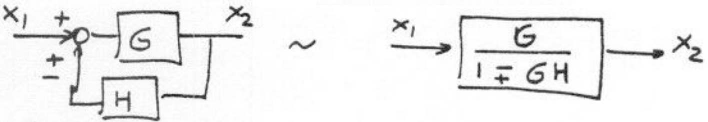
4) Moving a "tap" ahead of a block



5) Moving a "tap" behind a block

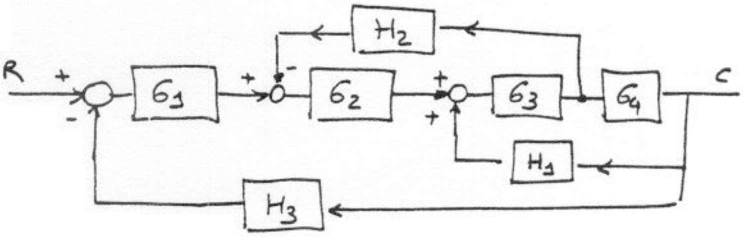


6) Eliminating a feedback loop:



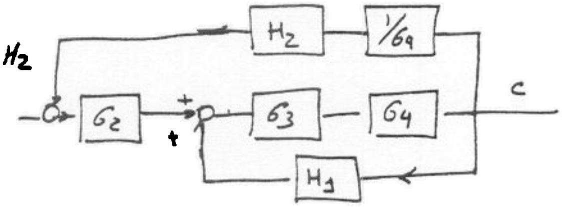
(This follows from Mason's formula)

- <u>Example of application:</u>

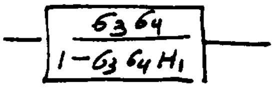


Want to find T_{CR}

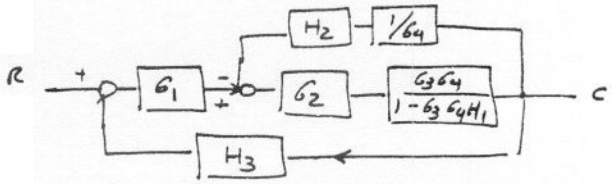
- Step 1: Move the tap for H_2 behind G_4



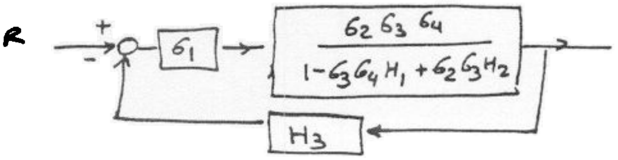
- Step 2: Eliminate the feedback loop $G_3 G_4 H_1$



Now we have:



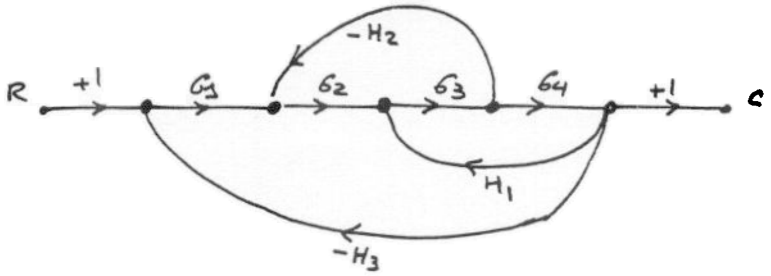
- Step 3: Eliminate the inner loop:



- Step 4: Collapse the final loop (i.e use Mason's again)

$$R \longrightarrow \boxed{\frac{G_1 G_2 G_3 G_4}{1 - G_3 G_4 H_1 + G_2 G_3 H_2 + G_2 G_3 G_4 G_1 H_3}} \longrightarrow C$$

Alternative solution: Transform to a signal flow graph and use Mason's



Loops: $-G_2 G_3 H_2$

$G_3 G_4 H_1$ (all touching)

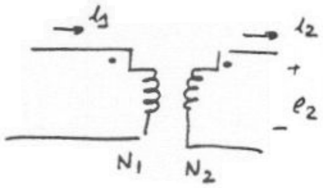
$-G_1 G_2 G_3 G_4 H_3$

$$\Rightarrow \frac{M}{\Delta} = \frac{1\ 2\ 3\ 4}{\cdot G_2 G_3 H_2 \cdot G_3\ {}_4 H_1 + G_1 G_2 G_3 G_4 H_3}$$

Forward path (only one) $M = G_1 G_2 G_3 G_4$

• 2 additional elements: Transformers & Gears (Section 2.11)

1) Electrical transformer



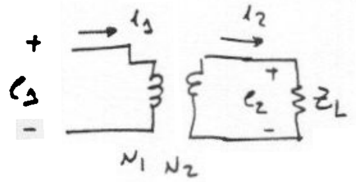
$\frac{N_1}{N_2}$ = turns ratio

Assuming ideal transformer (no magnetic flux lost) we get

$$N_1 i_1 = N_2 i_2$$

Since no power is lost: $e_1 i_1 = e_2 i_2 \Rightarrow \frac{e_1}{N_1} = \frac{e_2}{N_2}$ // $\boxed{\frac{e_2}{e_1} = \frac{N_2}{N_1} = \frac{i_1}{i_2}}$

Suppose now that we load the secondary with a load Z_e. What happens to the primary circuit?



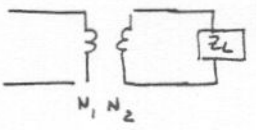
$$e_2 = Z_L i_2$$

$$\frac{N_2}{N_1} e_1 = Z_L \left(\frac{N_1}{N_2}\right) i_1 \quad // \quad e_1 = \left(\frac{N_1}{N_2}\right)^2 Z_L \, i_1$$

$$e_1 = Z_{eq} \, i_1$$

$$Z_{eq} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

The load is "reflected" in the primary circuit as an equivalent load equal to $\left(\frac{N_1}{N_2}\right)^2 Z_L$



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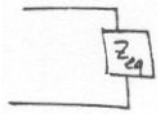
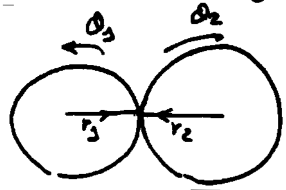


$Z_{eq} = \left(\frac{N_1}{N_2}\right)^2 Z_L$

2) Gear train: 2 gears with teeth meshing perfectly (i.e. no slipping)



if there is no slipping then

$r_1 \dot{\theta}_1 = r_2 \dot{\theta}_2$ (same linear velocity at the contact point)

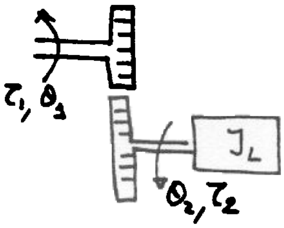
From power considerations:

$$\tau_1 \dot{\theta}_1 = \tau_2 \dot{\theta}_2 \Rightarrow \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$$

or $\boxed{\frac{\tau_2}{\tau_1} = \frac{r_2}{r_1} = \frac{\dot{\theta}_1}{\dot{\theta}_2}}$

Note τ transforms as e
$\dot{\theta}$ transforms as i

Example:



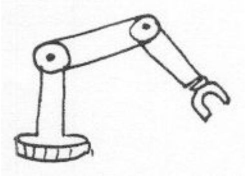
$$\frac{\tau_1}{\tau_2} = \frac{r_1}{r_2} \qquad \frac{\theta_1}{\theta_2} = \frac{r_2}{r_1}$$

$$\tau_2 = J_L \frac{d^2\theta_2}{dt^2} \Rightarrow \tau_1\left(\frac{r_2}{r_1}\right) = J_L \frac{d^2}{dt^2}\left(\frac{r_1}{r_2}\theta_1\right) = J_L\left(\frac{r_1}{r_2}\right)\frac{d^2\theta_1}{dt^2}$$

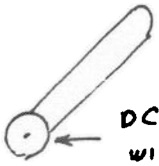
$$\Rightarrow \tau_1 = J_L\left(\frac{r_1}{r_2}\right)^2\frac{d^2\theta_1}{dt^2} \Rightarrow \boxed{J_{eq} = J_L\left(\frac{r_1}{r_2}\right)^2}$$

⇒ Inertia transforms following the same rules of impedances

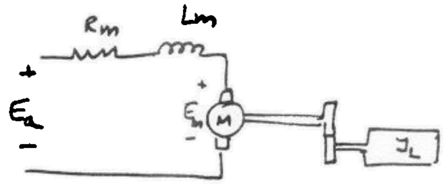
- **Example:** (A Robotic Control System) (Section 2.12)

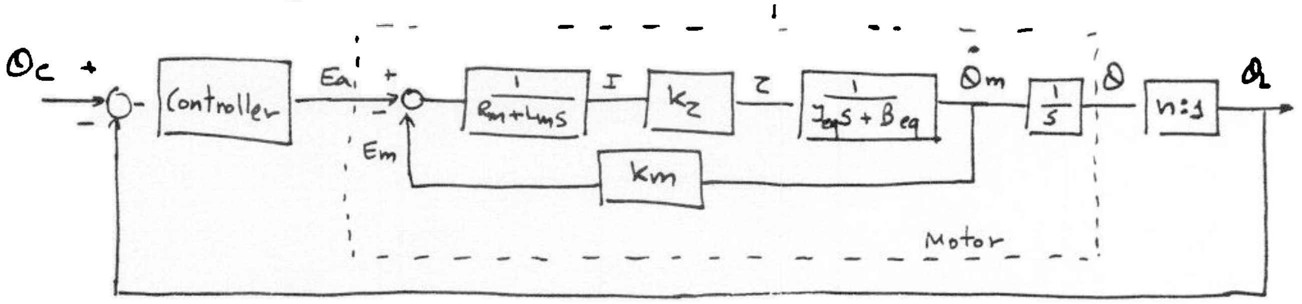


Single joint robotic arm:



DC motor and gear train with ratio $n = \frac{r_1}{r_2}$



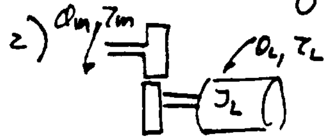


Here J_{eq} = DC motor inertia + (arm inertia)$\cdot n^2$ = $J_m + J_{arm}\cdot n^2$

$B_{eq} = B_m + B_{arm}\cdot n^2$

Notes 1) You get a third order system unless you neglect L_m

2)



$$J_L\ddot{\theta}_L + B_L\dot{\theta}_L = \tau_L \qquad \tau_L = \tau_m/n$$

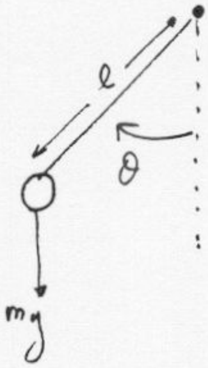
$$\theta_L = n\,\theta_m$$

$$J_L\cdot n\,\ddot{\theta}_m + B_L\,n\,\dot{\theta}_m = \tau_m/n$$

$$n^2 J_L\ddot{\theta}_m + n^2 B_L\dot{\theta}_m = \tau_m \Rightarrow \begin{aligned} J_{eq} &= J_L n^2 \\ B_{eq} &= B_L n^2 \end{aligned}$$

Linearization (section 2.14 book)

So far we have considered only linear models. What happens if we want to analyze (or control) a nonlinear system, for instance a pendulum:



Newton's equation for rotational motion yields:

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

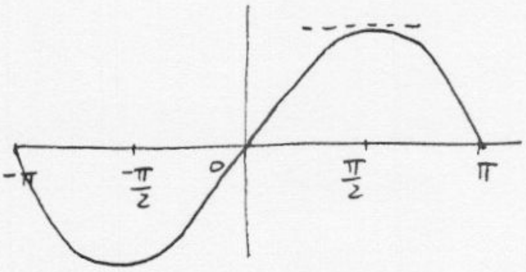
Assume we have an initial condition $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$ and we want to find $\theta(t)$

Q: can we use Laplace transform?

A: No! the right hand side of the equation is non-linear in the variable

Q: Can we get some sort of "linear approximation" so that we can use our linear toolbox?

A: Yes, depending on the value of θ.



If θ is small, then $\sin\theta \sim \theta$

On the other hand, if $\theta \sim \frac{\pi}{2}$ the natural approximation would be $\sin\theta \sim 0$

For small θ we get $\ddot{\theta} \approx -\frac{g}{\ell}\sin\theta \Rightarrow \theta(t) = \theta_0\cos\sqrt{\frac{g}{\ell}}\,t$

which matches quite closely the motion (as long as θ_0 is small)

On the other hand, for $\theta \sim \frac{\pi}{2}$ we get $\ddot{\theta} = 0$

$$\Rightarrow \theta(t) = \theta_0$$

which is clearly wrong!

In general, given an equation of the form:

$$\dot{x} = f(x)$$

suppose that we want to linearize it around an equilibrium point x_0 (i.e. $f(x_0) = 0$)

Taking a Taylor series expansion of $f(x)$ around x_0 yields:

$$\dot{x} \simeq \underset{0}{\cancel{f(x_0)}} + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \text{higher order terms}$$

$\Rightarrow$ if all derivatives (except the first) are small and if $(x - x_0)$ is small then

$$\dot{x} \cong f'(x_0)(x - x_0)$$

(or, if we define $\delta x = x - x_0$) $\qquad \dot{\delta x} = f'(x_0)\,\delta x$

(a linear approximation) valid for <u>small</u> δx

In the case of the pendulum:

- At $\theta = 0$ we have

$f(\theta) = \sin\theta$ $\qquad$ $f(0) = 0$

$f'(0) = \cos\theta$ $\qquad$ $f'(0) = 1$

$f''(0) = -\sin\theta$ $\qquad$ $f''(0) = 0$

$f'''(0) = -\cos\theta$ $\qquad$ $f'''(0) = 1$

$$\Rightarrow \quad \sin\theta \sim \delta\theta + \frac{1}{6}\delta\theta^3 \sim \delta\theta$$

- At $\theta = \pi/2$,

$$\sin\theta = \sin\left(\frac{\pi}{2}\right) + \underset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}}\,\delta\theta - \sin\left(\frac{\pi}{2}\right)(\delta\theta)^2 + \cdot$$

$$= 1 - \frac{\pi}{2}(\delta\theta)^2 \Rightarrow \text{linearization fails}$$