Homework -1

2.2-2

$$\frac{Y(z)}{X(z)} = \frac{(T/2)(z+1)}{z-1}$$

Solution:

(a) $y(k+1) = y(k) + T\dfrac{x(k)+x(k+1)}{2}$

(b) $zY(z) = Y(z) + \dfrac{T}{2}\left[X(z) + zX(z)\right] \Rightarrow Y(z) = \dfrac{T}{2}\,\dfrac{z+1}{z-1}X(z)$

2.4-1

Solution:

(a) $\mathfrak{z}[e(t-2T)u(t-2T)] = \dfrac{(z^3-2z)z^{-2}}{z^4-0.9z^2+0.8}$

(b) $e(0)=0,\ e(1)=1$

$\therefore \mathfrak{z}[e(t+T)u(t)] = z[E(z)-e(0)-e(1)z^{-1}]$

$$= z\left[\frac{z^3-2z}{z^4-0.9z^2+0.8}-\frac{1}{z}\right] = \frac{-1.1z^2+0.8}{z^4-0.9z^2+0.8}$$

(c) $\mathfrak{z}[e(t-T)u(t-2T)] = e(T)z^{-2}+e(2T)z^{-3}+\cdots$

$= z^{-1}[E(z)-e(0)] = z^{-1}E(z)$, since $e(0)=0$

$$= \frac{z^2-z}{z^4-0.9z^2+0.8}$$

2.6-2. Given the difference equation

$$y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = e(k)$$

where $y(0) = y(1) = 0,\ e(0) = 0$, and $e(k) = 1,\ k = 1,\ 2, \ldots$.

(a) Solve for $y(k)$ as a function of k, and give the numerical values of $y(k),\ 0 \le k \le 4$.

(b) Solve the difference equation directly for $y(k),\ 0 \le k \le 4$, to verify the results of part (a).

(c) Repeat parts (a) and (b) for $e(k) = 0$ for all k, and $y(0) = 1,\ y(1) = -2$.

Solution:

(a) $E(z) = \mathfrak{z}[u(k-1)] = z^{-1}\left[\dfrac{z}{z-1}\right] = \dfrac{1}{z-1}$

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right]Y(z) = E(z)$$

$$\frac{Y(z)}{z} = \frac{1}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \cdot \frac{1}{z-1} = \frac{-8}{z} + \frac{8/3}{z-1} + \frac{-16}{z-\frac{1}{2}} + \frac{64/3}{z-1/4}$$

$$\therefore\ y(k) = -8\delta(0) + \frac{8}{3} - 16\left(\frac{1}{2}\right)^k + \frac{64}{3}\left(\frac{1}{4}\right)^k$$

$$\therefore\ y(0) = 0;\ \ y(1) = 0;\ y(2) = 0;\ y(3) = 1;\ y(4) = \frac{7}{4}$$

(b) $y(k+2) = e(k) + \dfrac{3}{4}y(k+1) - \dfrac{1}{8}y(k)$

$$y(2) = 0 + \frac{3}{4}(0) - \frac{1}{8}(0) = 0$$

$$y(3) = 1 + \frac{3}{4}(0) - \frac{1}{8}(0) = 1$$

$$y(4) = 1 + \frac{3}{4}(1) - \frac{1}{8}(0) = 7/4$$

(c) (a) $y(k+2)-\frac{3}{4}y(k+1)+\frac{1}{8}y(k)=0$

$$\therefore z^2[Y(z)-y(0)-y(1)z^{-1}]-\frac{3}{4}z[Y(z)-y(0)]+\frac{1}{8}Y(z)=0$$

$$\therefore \left[z^2-\frac{3}{4}z+\frac{1}{8}\right]Y(z)=z^2-2z-\frac{3}{4}z$$

$$\therefore Y(z)=z\left[\frac{z-1/4}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]=z\left[\frac{-9}{z-\frac{1}{2}}+\frac{10}{z-\frac{1}{4}}\right]\Rightarrow y(k)=-9\left(\frac{1}{2}\right)^k+10\left(\frac{1}{4}\right)^k$$

$y(0)=1,\ y(1)=-2,\ y(2)=-13/8,\ y(3)=-31/32,\ y(4)=-67/128$

(b) $y(k+2)=\frac{3}{4}\,y(k+1)-\frac{1}{8}y(k)$

$$y(2)=\frac{3}{4}(-2)-\frac{1}{8}(1)=-13/8$$

$$y(3)=\frac{3}{4}\left(-\frac{13}{8}\right)-\frac{1}{8}(-2)=-31/32$$

$$y(4)=\frac{3}{4}\left(-\frac{31}{32}\right)-\frac{1}{8}\left(-\frac{13}{8}\right)=-\frac{67}{128}$$

2.6-3. Given the difference equation

$$x(k)-x(k-1)+x(k-2)=e(k)$$

where $e(k)=1$ for $k\geq 0$.

(a) Solve for $x(k)$ as a function of k, using the z-transform. Give the values of $x(0)$, $x(1)$, and $x(2)$.

(b) Verify the values $x(0)$, $x(1)$, and $x(2)$, using the power-series method.

(c) Verify the values $x(0)$, $x(1)$, and $x(2)$ by solving the difference equation directly.

(d) Will the final-value property give the correct value for $x(\infty)$?

Solution:

(a) $[1 - z^{-1} + z^{-2}]X(z) = E(z) = \dfrac{z}{z-1}$

$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 1)}, \qquad \text{poles: } z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1\angle \pm 60°$$

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{k_1}{z - p_1} + \frac{k_1^*}{z - p_1^*} \quad \text{with } p = 1\angle 60°$$

$$k_1 = \left.\frac{z^2}{(z-1)(z - 1\angle -60°)}\right|_{z=1\angle 60°} = \frac{1\angle 120°}{(.5 + j.866 - 1)(.5 + j.866 - .5 + j.866)}$$

$$= \frac{1\angle 120°}{1\angle 120° \, [j2(0.866)]} = 0.5774\angle -90°$$

$$\therefore aT = \ln(|p_1|) = 0; \; bT = \arg p_1 = \frac{\pi}{3}$$

$$A = 2|k_1| = 1.155; \; \theta = \arg \; k_1 = -90°$$

$$\therefore x(k) = 1 + 1.155 \cos\left(\frac{\pi}{3}k - 90°\right) = 1 + 1.155 \sin\left(\frac{\pi}{3}k\right)$$

$$x(0) = 1, \; x(1) = 2, \; x(2) = 2$$

(b)

$$\begin{array}{r|l} & 1 + 2z^{-1} + 2z^{-2} + \cdots \\ \hline z^3 - 2z^2 + 2z - 1 & z^3 \\ & \underline{z^3 - 2z^2 + 2z - 1} \\ & \quad 2z^2 - 2z + 1 \\ & \quad \underline{2z^2 - 4z + 4 - 2z^{-1}} \\ & \qquad 2z + \cdots \end{array}$$

$\therefore x(0) = 1$
$x(1) = 2$
$x(2) = 2$

(c) $x(k) = 1 + x(k-1) - x(k-2)$

$x(0) = 1 + 0 - 0 = 1$

$x(1) = 1 + 1 - 0 = 2$

$x(2) = 1 + 2 - 1 = 2$

(d) *No*, 3 poles for $X(z)$ on the unit circle.

2.9-1. Find two different state-variable formulations that model the system whose difference equation is given by:

(a) $y(k+2)+6y(k+1)+5y(k)=2e(k)$

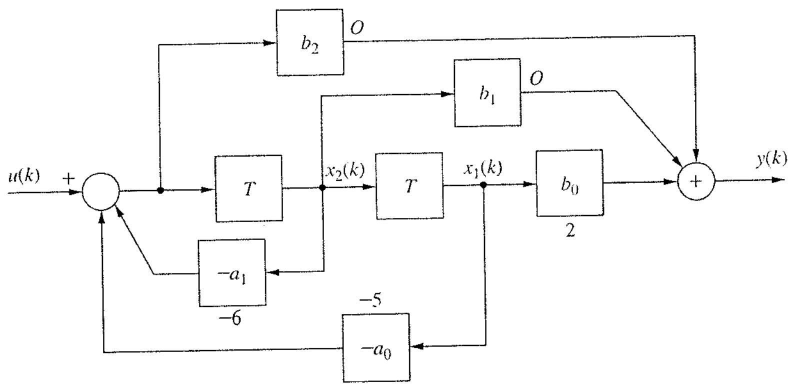
(b) $y(k+2)+6y(k+1)+5y(k)=e(k+1)+2e(k)$

(c) $y(k+2)+6y(k+1)+5y(k)=3e(k+2)+e(k+1)+2e(k)$

Solution:

(a) $\dfrac{Y(z)}{U(z)}=\dfrac{2}{z^2+6z+5}$

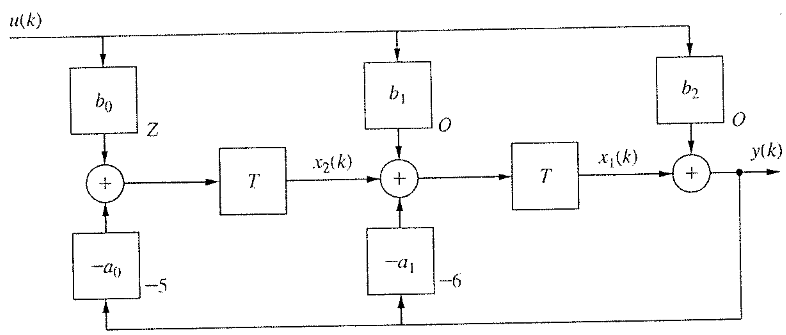
(1) control canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 2 & 0 \end{bmatrix}\mathbf{x}(k)$$

(2) observer canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)$$

(b) $\dfrac{Y(z)}{U(z)} = \dfrac{z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) =$ same as (a)
$y(k) = [2 \quad 1]x(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)$$

(c) $\dfrac{Y(z)}{U(z)} = \dfrac{3z^2+z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) =$ same as (a)
$y(k) = [-13 \quad -17]\mathbf{x}(k) + 3u(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) + 3u(k)$$

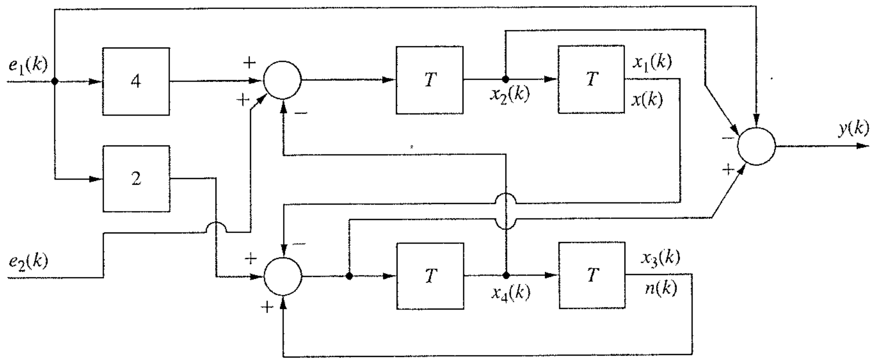
2.10-1. Find a state-variable formulation for the system described by the coupled second-order difference equations given. The system output is $y(k)$, and $e_1(k)$ and $e_2(k)$ are the system inputs. *Hint:* Draw a simulation diagram first.

$$x(k+2) + v(k+1) = 4e_1(k) + e_2(k)$$

$$v(k+2) - v(k) + x(k) = 2e_1(k)$$

$$y(k) = v(k+2) - x(k+1) + e_1(k)$$

Solution:



$$\mathbf{x}(k+1)=\begin{bmatrix}0&1&0&0\\0&0&0&-1\\0&0&0&1\\-1&0&1&0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0&0\\4&1\\0&0\\2&0\end{bmatrix}e(k)$$

$$y(k)=x_4(k+1)-x_2(k)+e_1(k)=-x_1(k)+x_3(k)-x_2(k)+e_1(k)$$

$$\therefore y(k)=\begin{bmatrix}-1&-1&1&0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1&0\end{bmatrix}e(k)$$