- **Inverse z-transform**

Our method for solving difference equations works as follows

$$\text{difference equation(s)} \xrightarrow{\text{z transf}} \text{algebraic equation(s)} \xrightarrow{\text{solve}} X(z) \xrightarrow{z^{-1}} \{x_k\}$$

So for this to work we need to figure a way of carrying out the last step : inverse z transform

Three methods
(1) partial fraction expansion

(2) power series

(3) inversion formula : $e_k = \frac{1}{2\pi j} \oint E(z) z^{k-1} dz$

- **Partial fraction expansion :**

Idea similar to the continuous time case : break a complex transfer function into simpler blocks that we can either solve for or look-up in tables

However, note that __most__ z transforms are of the form

$$E(z) = z \frac{P(z)}{Q(z)} \qquad \left(\text{for instance} \quad Z(a^k) = \frac{z}{z-a}\right)$$

$\Rightarrow$ since we need that extra factor of z, is better to do a partial fraction expansion of $\frac{E(z)}{z}$ (instead of $E(z)$)

__Example :__
$$E_1(z) = \frac{z}{(z-1)(z-2)} \quad ; \quad \frac{E_1(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow E(z) = -\frac{z}{z-1} + \frac{z}{z-2} \xrightarrow{z^{-1}} e_k = -1 + 2^k$$

In general :
$$F(z) = \frac{E(z)}{z} = \frac{N(z)}{D(z)} = b_0 \frac{\Pi(z-z_i)}{\Pi(z-p_i)} = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_n}{z-p_n}$$

(assuming all p_i different) where $k_i = (z-p_i) F(z) \Big|_{z=p_i}$

If we have repeated poles the situation gets more complicated.
Suppose that the pole p_1 has multiplicity $r \Rightarrow$ we have r terms
associated with this pole:

$$\frac{E(z)}{z} = F(z) = \frac{N(z)}{(z-p_1)^r (z-p_2)\cdots(z-p_m)} = \frac{k_{11}}{(z-p_1)} + \frac{k_{12}}{(z-p_1)^2} + \cdots \frac{k_{1r}}{(z-p_1)^r} + \frac{k_2}{(z-p_2)} + \cdots \frac{k_m}{(z-p_m)}$$

where:
$$k_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{dz^{(r-j)}}\left[(z-p_1)^r F(z)\right]\Big|_{z=p_1}$$

Example:
$$E(z) = \frac{z}{(z-1)^2(z-2)} \Rightarrow \frac{E(z)}{z} = \frac{1}{(z-1)^2(z-2)} = \frac{k_{11}}{(z-1)} + \frac{k_{12}}{(z-1)^2} + \frac{k_2}{(z-2)}$$

$$k_2 = 1; \quad k_{11} = \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2 \frac{1}{(z-1)^2(z-2)}\right]\Big|_{z=1} = 1\cdot\frac{d}{dz}\left(\frac{1}{z-2}\right)\Big|_{z=1} = \frac{-1}{(z-2)^2}\Big|_{z=1} = -1$$

$$k_{12} = \frac{1}{0!}(z-1)^2\frac{1}{(z-1)^2(z-2)}\Big|_{z=1} = -1$$

$$E(z) = \frac{-z}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{(z-2)} \Rightarrow e_k = -k - 1 + 2^k \quad \#$$

- **Power Series method:** Divide the denominator into the numerator and obtain a power series of the form

$$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots +$$

The values of $\{e_k\}$ are the coefficients of this expansion.

Example:
$$E(z) = \frac{z}{(z-1)(z-2)} = \frac{z}{z^2 - 3z + 2}$$

$$
\begin{array}{r}
\frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \cdots \\
z^2 - 3z + 2 \overline{)z} \\
\underline{z - 3 + \frac{2}{z}} \\
3 - \frac{2}{z} \\
\underline{3 - \frac{9}{z} + \frac{6}{z^2}} \\
\frac{7}{z} - \frac{6}{z^2}
\end{array}
$$
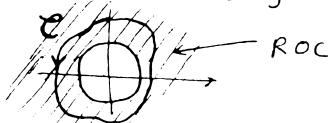
So the first terms of the sequence are: $\{0, 1, 3, 7 \cdots$
(consistent with the formula $-1 + 2^k$ that we found earlier)

Drawback of the method : We don't get a closed-form expression for $\{e_k\}$

- Inversion Formula method :

We will see that a closed form expression for $\{e_k\}$ is given by:

$$e_k = \frac{1}{2\pi j} \oint_{\mathcal{C}} E(z)\, z^{k-1}\, dz$$

ROC
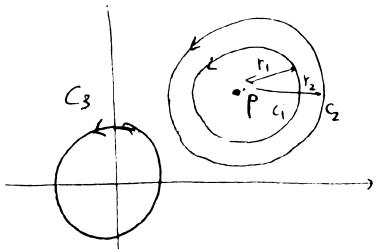
where $\mathcal{C}$ denotes a closed path <u>encircling the origin</u> and <u>inside</u> the ROC. Obviously, this method works only if there is a convenient way of computing this integral. We will see that this is the case, but first we need to introduce some concepts from complex analysis and analytic function theory

Motivation :   Consider the function $\frac{1}{z-p}$ (single pole at $z=p$)

Lets compute $\frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{1}{(z-p)}\, dz$ where $\mathcal{C}$ is a curve that may o may not enclose p

(for simplicity will take circles)

$(z-p)$ is a vector from a generic point on $\mathcal{C}$ to the point z

$\Rightarrow$ we can write it as: $(z-p) = r\,e^{j\theta}$

$\Rightarrow \frac{1}{(z-p)} = \frac{1}{r}\, e^{-j\theta}$

$$\frac{1}{2\pi j} \oint_{C_1} \frac{1}{(z-p)}\, dz = \frac{1}{2\pi j} \int_0^{2\pi} \frac{1}{r} e^{-j\theta}\, d\left[r e^{j\theta}\right] = \frac{1}{2\pi j} \int_0^{2\pi} \frac{1}{r} \cdot e^{-j\theta}\, r\, j\, e^{j\theta}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta = \frac{2\pi}{2\pi} = \boxed{1}$$

Note that the answer is 1, regardless of the <u>radius</u>. (In fact it can be shown that we get this answer for <u>any</u> curve encircling p)

On the other hand, it can be shown that $\oint_{C_3} = 0$

$\Rightarrow$ If C $\begin{cases} \text{encircles } p \Rightarrow \frac{1}{2\pi} \oint = 1 \\ \text{does not encircle } p \Rightarrow \frac{1}{2\pi} \oint = 0 \end{cases}$

This is a special case of Cauchy's Theorem:

Facts:
1) A function $F(z)$ is analytic at a point z_0 if it is continuously differentiable at z_0 (i.e. $F'(z)$ cont. at z_0)

2) $\oint_C F(z)\, dz = 0$ if $F(z)$ is analytic in the region enclosed by C

3) If $F(z)$ is analytic in the region enclosed by C _except_ at a finite number of isolated singularities z_i (and has no singularities on C) then

$$\frac{1}{2\pi j} \oint_C F(z) = \sum_i \text{Res}(z_i)$$

where $\text{Res}(z_i)$ (the "residues") are given by:

$$\text{Res}(z_i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_i)^n F(z)\right]\Big|_{z=z_i} \qquad \text{if } f(z) \text{ has a singularity of order } n \text{ at } z_i$$
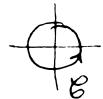
$$\text{Res}(z_i) = (z-z_i) F(z)\Big|_{z=z_i} \qquad \text{for singularities of order 1}$$

Examples (a) $F(z) = z^k$ $\Rightarrow$ analytic everywhere $\Rightarrow$ $\oint z^k dz = 0$

(b) $F(z) = \frac{1}{z}$ $\Rightarrow$ isolated singularity at $z=0$

$\Rightarrow \frac{1}{2\pi j} \oint_C \frac{1}{z} dz = \text{Res}(z=0) = z\cdot\frac{1}{z}\Big|_{z=0} = \boxed{1}$ as before

(c) $F(z) = \frac{1}{z^k}$ $\Rightarrow$ singularity of order k at $z=0$

$$\Rightarrow \frac{1}{2\pi j} \oint_C \frac{1}{z^k} dz = \text{Res}(z=0) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{(k-1)}}\left(z^k \frac{1}{z^k}\right)$$

$$= \frac{1}{(k-1)!}\frac{d^{k-1}(1)}{dz^{(k-1)}} = \boxed{0}$$

Question: Is this relevant to us at all?
Answer: Yes! we can use this both to _prove_ the inversion formula _and_ to compute the $\oint$ in an efficient way.

Let $E(z) = \sum_0^\infty e_k z^{-k} = e_0 + \dfrac{e_1}{z} + \cdots + \dfrac{e_k}{z^k} + \cdots$

Multiply both sides by z^{k-1} and integrate along a closed curve $\mathcal{C}$ enclosing the origin

$$\frac{1}{2\pi j} \oint_\mathcal{C} E(z) z^{k-1} dz = \frac{1}{2\pi j} \oint \left(\sum_0^\infty e_n z^{-n} \right) z^{k-1} dz = \frac{1}{2\pi j} \oint \left(e_0 z^{k-1} + e_1 z^{k-2} + \cdots \frac{e_k}{z} + \frac{e_{k+1}}{z^2} + \cdots \right) dz$$

$$\overset{\underset{\uparrow}{=}}{} \quad \frac{1}{2\pi j} \left[e_0 \oint z^{k-1} dz + e_1 \oint z^{k-2} dz + \cdots e_{k-1} \oint dz + \right\} = 0$$

$$+ e_k \oint \frac{1}{z} dz \qquad\qquad \left. \right\} = e_k$$

$$+ e_{k+1} \int \frac{1}{z^2} dz + e_{k+2} \int \frac{1}{z^3} dz + \cdots \left. \right\} = 0$$

Assuming that we can interchange $\sum$ and $\oint$

$$\Rightarrow \boxed{\frac{1}{2\pi j} \oint_\mathcal{C} E(z) z^{k-1} dz = e_k}$$

provided that $\mathcal{C}$ is inside the region of convergence so that indeed we can interchange $\sum$ and $\oint$
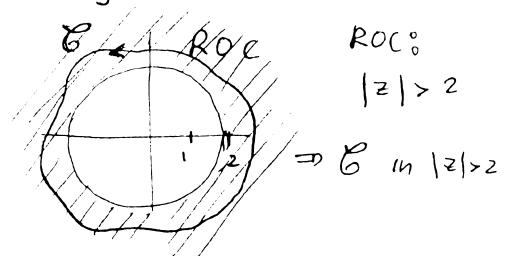
We have proved the inversion formula!

Q: what about the second issue (how to compute the $\oint$ on the left)?

A: Let's use the residue formula:

$$e_k = \frac{1}{2\pi j} \oint_\mathcal{C} E(z) z^{k-1} dz = \sum \operatorname{Res} \left\{ E(z) z^{k-1} \right\}$$

Example: $E(z) = \dfrac{z}{(z-1)(z-2)}$ poles at $z = 1, 2$

ROC: $|z| > 2$

$\Rightarrow \mathcal{C}$ in $|z| > 2$

$$E(z) z^{k-1} = \frac{z^k}{(z-1)(z-2)}$$

$\operatorname{Res}\left[E(z) z^{k-1} \right]$ at $z=1$: $\left. \dfrac{(z-1) z^k}{(z-1)(z-2)} \right|_{z=1} = \boxed{-1}$

$\operatorname{Res}\left[\right]$ at $z=2$: $\left. \dfrac{(z-2) z^k}{(z-1)(z-2)} \right|_{z=2} = \boxed{2^k}$

$$\Rightarrow \boxed{e_k = -1 + 2^k} \quad \underline{\text{as before}}!$$

Example 2:   Assume $f_3(k) = f_1(k) f_2(k)$

Want $F_3(z)$

$$F_3(z) = \sum_0^\infty f_3(k) z^{-k} , \qquad f_3(k) z^{-k} = f_1(k) \cdot f_2(k) z^{-k}$$

$$= f_1(k) \cdot \frac{1}{2\pi j}\left(\oint_C F_2(\lambda) \lambda^{k-1} d\lambda\right) z^{-k}$$

$$= f_1(k) \frac{1}{2\pi j} \oint_C F_2(\lambda) \left(\frac{\lambda}{z}\right)^k \frac{d\lambda}{\lambda}$$

$$F_3(z) = \sum_0^\infty f_3(k) \frac{1}{2\pi j} \oint_C F_2(\lambda)\left(\frac{\lambda}{z}\right)^k \frac{d\lambda}{\lambda} = \frac{1}{2\pi j} \oint_C \left[\underbrace{\sum_0^\infty f_1(k)\left(\frac{z}{\lambda}\right)^{-k}}_{F_3\left(\frac{z}{\lambda}\right)} F_2(\lambda) \frac{d\lambda}{\lambda}\right]$$

$$= \frac{1}{2\pi j} \oint_C F_3\left(\frac{z}{\lambda}\right) F_2(\lambda) \frac{d\lambda}{\lambda} \quad \#$$

If we let $z = 1$, $f_1 = f_2 = f$ we get

$$\sum_0^\infty f_3(k) = \sum_0^\infty f^2(k) = \frac{1}{2\pi j} \oint_C F\left(\frac{1}{\lambda}\right) F(\lambda) \frac{d\lambda}{\lambda}$$

Take now the circle $|\lambda| = 1$ as the contour C:

$$\sum_0^\infty f^2(k) = \frac{1}{2\pi j} \oint_{|\lambda|=1} F\left(\frac{1}{\lambda}\right) F(\lambda) \frac{d\lambda}{\lambda} = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\Theta}) F(e^{-j\Theta}) d\Theta$$

$$\uparrow$$
$$\lambda = e^{j\Theta}$$

$$\boxed{\sum_0^\infty f^2(k) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\Theta}) F(e^{-j\Theta}) d\Theta}$$

This is known as Parseval's theorem: energy in the time domain
= energy in the freq domain

Summary of the z-transform
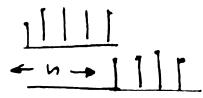
- **Definition:** $\quad E(z) = \sum\limits_{k=0}^{\infty} e_k z^{-k} \qquad$ for some region $r_o < |z| < R_o$

- **Properties:**
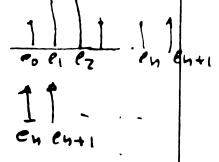
1) Linearity $\qquad Z\left\{\alpha e_1(k) + \beta e_2(k)\right\} = \alpha Z(e_1) + \beta Z(e_2)$

2) Time shift:

   a) $\qquad Z\left\{e(k-n)\right\} = z^{-n} E(z)$

   b) $\qquad Z\left\{e(k+n)\right\} = z^n\left[E(z) - \sum\limits_{0}^{n-1} e(k) z^{-k}\right]$

3) scaling in z-plane:
$$Z\left\{\left(r^{-k} e_k\right)\right\} = E(rz)$$

4) Initial Value Theorem:
$$e(o) = \lim\limits_{z \to \infty} E(z)$$

5) Final Value Theorem:
$$\lim\limits_{k \to \infty} e(k) = \lim\limits_{z \to 1} (z-1) E(z)$$

   Provided that the left hand side exist $\Leftrightarrow$ $(z-1) F(z)$ has all poles inside the unit circle

6) Convolution of Time Sequences:
$$f_k = e_1(k) * e_2(k) = \sum\limits_{\ell=0}^{k} e_1(\ell) \cdot e_2(k-\ell)$$
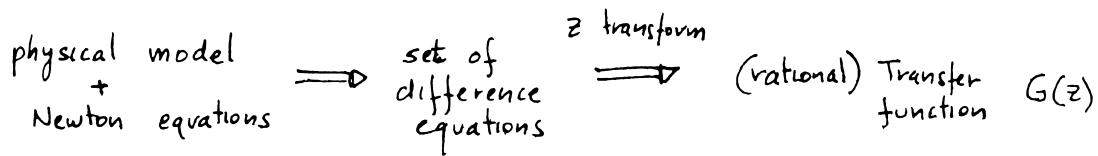
$$Z\left\{e_1 * e_2\right\} = E_1(z) E_2(z)$$

7) Inversion Formula: $\qquad$ Let $\quad E(z) = Z(e_k)$, then:
$$e_k = \frac{1}{2\pi j} \oint_C E(z) z^{k-1} dz$$

- ## Representation of Linear Time Invariant Discrete Time Systems

So far we have seen that a LTI system described by difference equations can also be described by a transfer function

$$\text{physical model} + \text{Newton equations} \implies \text{set of difference equations} \xrightarrow{\text{z transform}} \text{(rational) Transfer function } G(z)$$
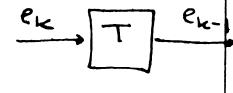
Question: Suppose that we are given a transfer function $C(z)$, can we find a system with that t.f? how?

(This is relevant because we will carry out the design of controllers in the z-domain, but then we will need to implement them)

Answer: Yes, using as an intermediate step yet another representation: Simulation diagrams.

- ## Simulation diagrams:

elements: (a) Time Delay (shift register) $e_k \to \boxed{T} \to e_{k-1}$

(b) Product by a constant $e_k \to \boxed{\alpha} \to \alpha e_k$

(c) Summing junction

Turns out that with these three elements we can built a simulation diagram that realizes any T.F.

Example: Suppose that we want to realize the T.F

$$G(z) = \frac{M(z)}{E(z)} = \frac{\beta_0 z + \beta_1}{z^2 + \alpha_1 z + \alpha_2}$$

Dividing numerator & denominator by $1/z^2$ yields:
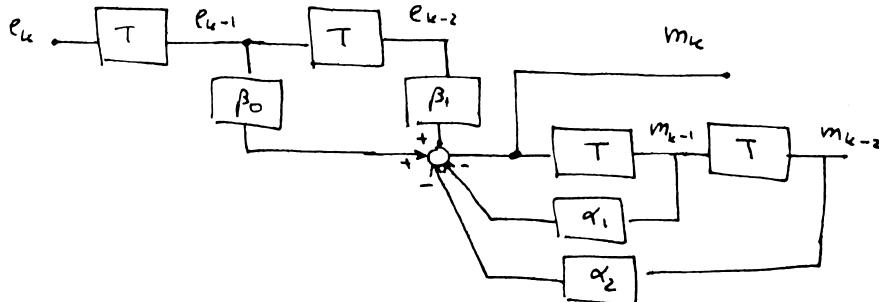
$$G(z) = \frac{M(z)}{E(z)} = \frac{\frac{\beta_0}{z} + \beta_1 \cdot \frac{1}{z^2}}{1 + \alpha_1 \cdot \frac{1}{z} + \frac{\alpha_2}{z^2}} \implies \left(1 + \alpha_1 \frac{1}{z} + \frac{1}{z^2}\alpha_2\right)M(z) = \left(\frac{\beta_0}{z} + \frac{\beta_1}{z^2}\right)E(z)$$

Now recall that $\frac{1}{z} \xleftrightarrow{z^{-1}}$ unit time delay
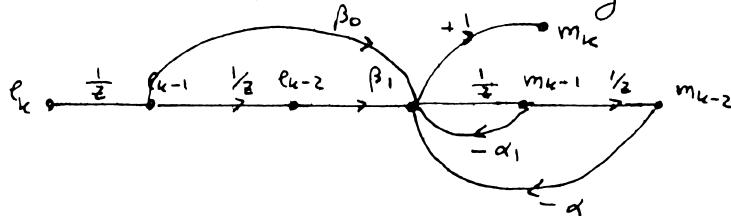
Taking inverse z transforms on both sides yields:

$$m_k + \alpha_1 m_{k-1} + \alpha_2 m_{k-2} = \beta_0 e_{k-1} + \beta_1 e_{k-2}$$

Now we can "build" a system that is described precisely by this equation:

Sanity check: We can transform this diagram back to the z-domain $\left(\boxed{T} \xrightarrow{} \frac{1}{z} \right)$ and check the T.F. using Mason's formula:

Exercise: check that indeed you get the right T.F.
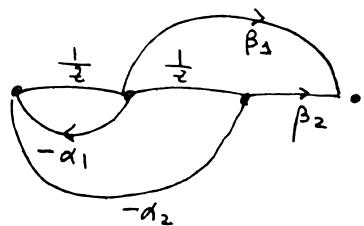
Q? Is this the "best" way to proceed? Is this the only way to proceed

A: Not necessarily. Note that we started out with a second order T.F. Hence we would expect to be able to realize it with just two delays. However our realization uses 4!

Let's look at the problem again and try reverse engineering: first find a signal flow graph and then the simulation diagram
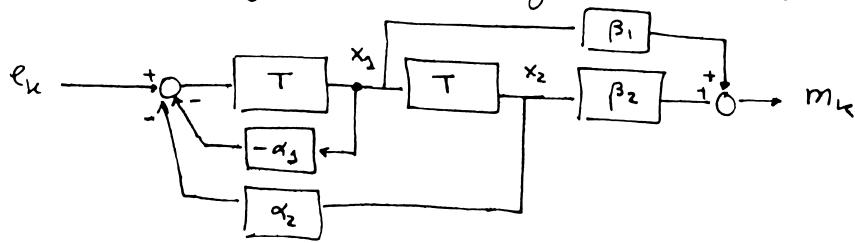
$$G(z) = \frac{\frac{\beta_0}{z} + \frac{\beta_1}{z^2}}{1 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2}}$$

$\implies$ Let's build something that has

$$\Delta = 1 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2}$$

and $\sum M_i \Delta_i = \frac{\beta_0}{z} + \frac{\beta_1}{z^2}$

(Note: we needed only two "$\frac{1}{z}$" blocks)

From here we get the following simulation diagram:

Q: Is this the "minimal" realization? Is it unique?
Do the intermediate variables "x_i" have any significance?
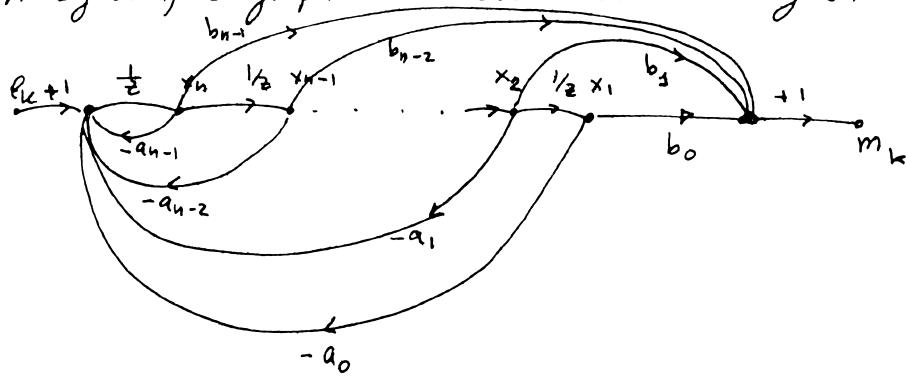
- Turns out that to answer these questions we need to introduce the concept of <u>state variables</u>

- <u>State Space Models</u>:

Consider a generic transfer function of the form

$$G(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z + \qquad + b_{n-1} z^{n-1}}{a_0 + a_1 z + \quad \cdot \quad a_{n-1} z^{n-1} + z^n}$$

(note that it is strictly proper)

Dividing by z^n yields:

$$G(z) = \frac{b_{n-1} z^{-1} + \cdot \quad \cdot \quad + b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdot \quad \cdot \quad a_0 z^{-n}}$$

A signal flow graph that realizes this T.F is given by:

Note that we have n loops (all touching) with gains $L_i = -a_{n-i} \left(\frac{1}{z}\right)^i$
and n forward paths, each with $\Delta_i = \Delta$ and $M_i = \frac{b_{n-i}}{z^i}$

According to Mason's formula: $G(z) = \frac{\sum M_i \Delta_i}{\Delta} = \frac{b_{n-1} z^{-1} + \cdot \cdot \quad b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdot \quad a_0 z^{-n}}$

(precisely what we wanted)

Q: Here e_k is the input and m_k is the output, but what are the (internal) variables x_k?

A: x_k are called the <u>states of the system</u>. These state variables represent the minimum amount of information that (together with the input) is necessary to determine the future evolution of the system. In other words, they encapsulate <u>all</u> the past history.

From the signal flow graph we get the following equations:

$$x_n(k+1) = e(k) - a_{n-1}x_n(k) - a_{n-2}x_{n-1}(k) \cdots - a_0 x_1(k)$$

$$x_{n-1}(k+1) = x_n(k)$$

$$\vdots$$

$$x_1(k+1) = x_2(k)$$

$$m(k) = b_0 x_1(k) + b_1 x_2(k) + \cdots + b_{n-1} x_n(k)$$

or, in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ -a_0 & & & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} e(k)$$

$$m(k) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

In compact notation:

$$x(k+1) = A x(k) + B e_k$$

$$m(k) = C x(k)$$

where $\begin{cases} A \text{ is a } n \times n \\ B \text{ is a } n \times n_u \\ C \text{ is a } n_z \cdot n \end{cases}$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ is an } n \times 1 \text{ vector}$$

- State space formulations were introduced in Control theory in the early 60's

Advantages of state space methods:

1) can handle multiple-input/multiple-output plants

2) Can answer our earlier question on what is the minimum number of delays required to realize a given transfer function

   (minimal realizations) and analyze why some realizations are not minimal (unobservable and/or uncontrollable states)

   For instance, it can be shown that the minimum number of delays required in our earlier example is indeed two, except for some special values of $\beta_0, \beta_1, \alpha_1$ and α_2 where it may reduce to 1

3) Give a systematic way of designing controllers that place all closed-loop poles at desired locations

However: Dealing with state-space representations requires the use of tools from linear algebra and linear vector spaces that are beyond the scope of this course (at NU these tools are covered in ECE 7200)

Additional drawback of state space methods: "traditional" state space methods are less robust than frequency domain based methods (such as the ones learned in ECE 5580/5610)

Current state of the art: mix of both techniques. We use state space tools to design controllers based on a generalization of the methods covered in 5580/5610

TABLE 2-2 z-TRANSFORM TABLE

Number sequence, $\{e(k)\}$	z-Transform $E(z)$
$\{1\}$	$\dfrac{z}{z-1}$
$\{k\}$	$\dfrac{z}{(z-1)^2}$
$\{k^2\}$	$\dfrac{z(z+1)}{(z-1)^3}$
$\{a^k\}$	$\dfrac{z}{z-a}$
$\{ka^k\}$	$\dfrac{az}{(z-a)^2}$
$\{\sin ak\}$	$\dfrac{z \sin a}{z^2 - 2z \cos a + 1}$
$\{\cos ak\}$	$\dfrac{z\,(z-\cos a)}{z^2 - 2z \cos a + 1}$
$\{a^k \sin bk\}$	$\dfrac{az \sin b}{z^2 - 2az \cos b + a^2}$
$\{a^k \cos bk\}$	$\dfrac{z^2 - az \cos b}{z^2 - 2az \cos b + a^2}$

TABLE 2-3   TABLE OF COMMONLY USED z-TRANSFORMS

Laplace transform, $E(s)$	Time function, $e(t)$	z-transform, $E(z)$
$\dfrac{1}{s}$	$u(t)$	$\dfrac{z}{z-1}$
$\dfrac{1}{s^2}$	t	$\dfrac{Tz}{(z-1)^2}$
$\dfrac{1}{s+a}$	ϵ^{-at}	$\dfrac{z}{z - \epsilon^{-aT}}$
$\dfrac{a}{s(s+a)}$	$1 - \epsilon^{-at}$	$\dfrac{z(1 - \epsilon^{-aT})}{(z-1)(z - \epsilon^{-aT})}$
$\dfrac{1}{(s+a)^2}$	$t\epsilon^{-at}$	$\dfrac{Tz\epsilon^{-aT}}{(z - \epsilon^{-aT})^2}$
$\dfrac{a}{s^2(s+a)}$	$t - \dfrac{1 - \epsilon^{-at}}{a}$	$\dfrac{Tz}{(z-1)^2} - \dfrac{(1 - \epsilon^{-aT})z}{a(z-1)(z - \epsilon^{-aT})}$
$\dfrac{a}{s^2 + a^2}$	$\sin(at)$	$\dfrac{z \sin(aT)}{z^2 - 2z \cos(aT) + 1}$
$\dfrac{s}{s^2 + a^2}$	$\cos(at)$	$\dfrac{z(z - \cos(aT))}{z^2 - 2z \cos aT + 1}$
$\dfrac{1}{(s+a)^2 + b^2}$	$\dfrac{1}{b}\epsilon^{-at} \sin bt$	$\dfrac{1}{b}\left[\dfrac{z\epsilon^{-aT} \sin bT}{z^2 - 2z\epsilon^{-aT} \cos(bT) + \epsilon^{-2aT}}\right]$
$\dfrac{s+a}{(s+a)^2 + b^2}$	$\epsilon^{-at} \cos bt$	$\dfrac{z^2 - z\epsilon^{-aT} \cos bT}{z^2 - 2z\epsilon^{-aT} \cos bT + \epsilon^{-2aT}}$

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