- **Compensation in the z-domain**

Our goal is to find out the equivalent of phase lead or lag compensators in the z domain

Recall that in the s-domain we have

$$C(s) = A\left(\frac{1+\frac{s}{\omega_o}}{1+\frac{s}{\omega_p}}\right)$$ where A is the DC gain

If $\omega_o < \omega_p \Rightarrow$ phase lead
$\omega_o > \omega_p$ phase lag

To find the discrete time equivalent, we can start by using the bilinear transformation:

$$C(z) = C(s)\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{A\,\omega_p}{\omega_o}\left(\frac{\omega_o + \frac{2}{T}\frac{z-1}{z+1}}{\omega_p + \frac{2}{T}\frac{z-1}{z+1}}\right) = A_o\frac{\omega_p}{\omega_o}\frac{\left(\omega_o+\frac{2}{T}\right)}{\left(\omega_p+\frac{2}{T}\right)}\frac{\left(z-\frac{\frac{2}{T}-\omega_o}{\frac{2}{T}+\omega_o}\right)}{\left(z-\frac{2/T-\omega_p}{2/T+\omega_p}\right)}$$

$$= \boxed{K_d\left(\frac{z-z_o}{z-z_p}\right)}$$

where
$$\left.\begin{aligned} z_o &= \frac{\frac{2}{T}-\omega_o}{\frac{2}{T}+\omega_o} \\ z_p &= \frac{\frac{2}{T}-\omega_p}{\frac{2}{T}+\omega_p}\end{aligned}\right\} \Rightarrow \begin{aligned} &\text{if } \omega_o < \omega_p \Rightarrow z_o > z_p \text{ (phase lead)} \\ &\quad\;\; \omega_o > \omega_p \Rightarrow z_o < z_p \text{ (phase lag)}\end{aligned}$$

Note that for this to work we need
$$\left.\begin{aligned}\omega_o &< \frac{2}{T} \\ \omega_p &< \frac{2}{T}\end{aligned}\right\}$$ i.e both pole & zero must be below the Nyquist frequency

Recap:

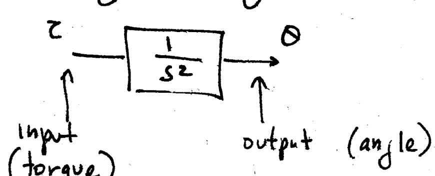
- phase lead $\quad C(z) = k\left(\frac{z-z_o}{z-z_p}\right) \quad 1 > z_o > z_p > 0$
- phase lag $\quad C(z) = k\left(\frac{z-z_o}{z-z_p}\right) \quad 1 > z_p > z_o > 0$

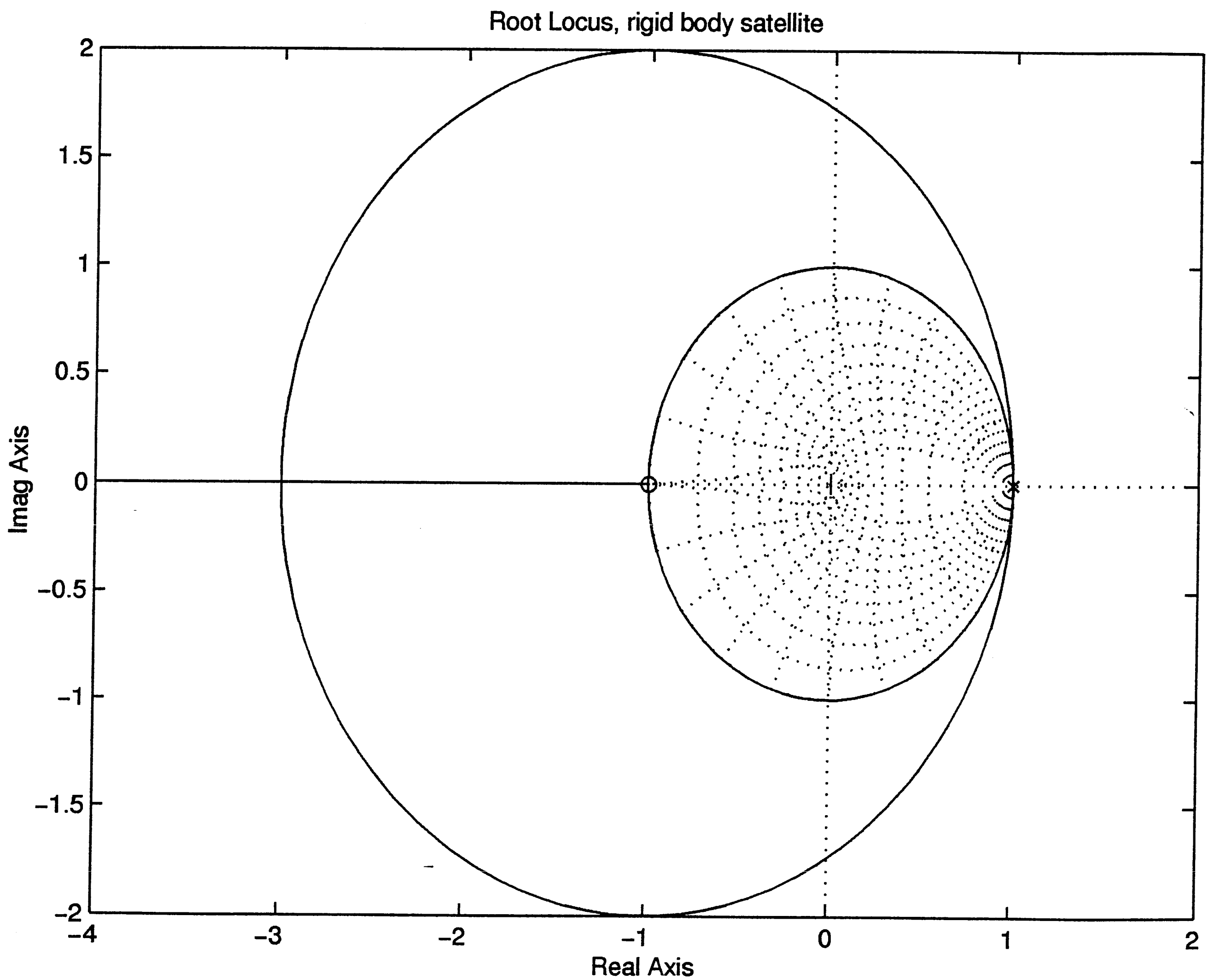
- **Example of phase lead compensation:**

A satellite modeled using only the rigid body mode:

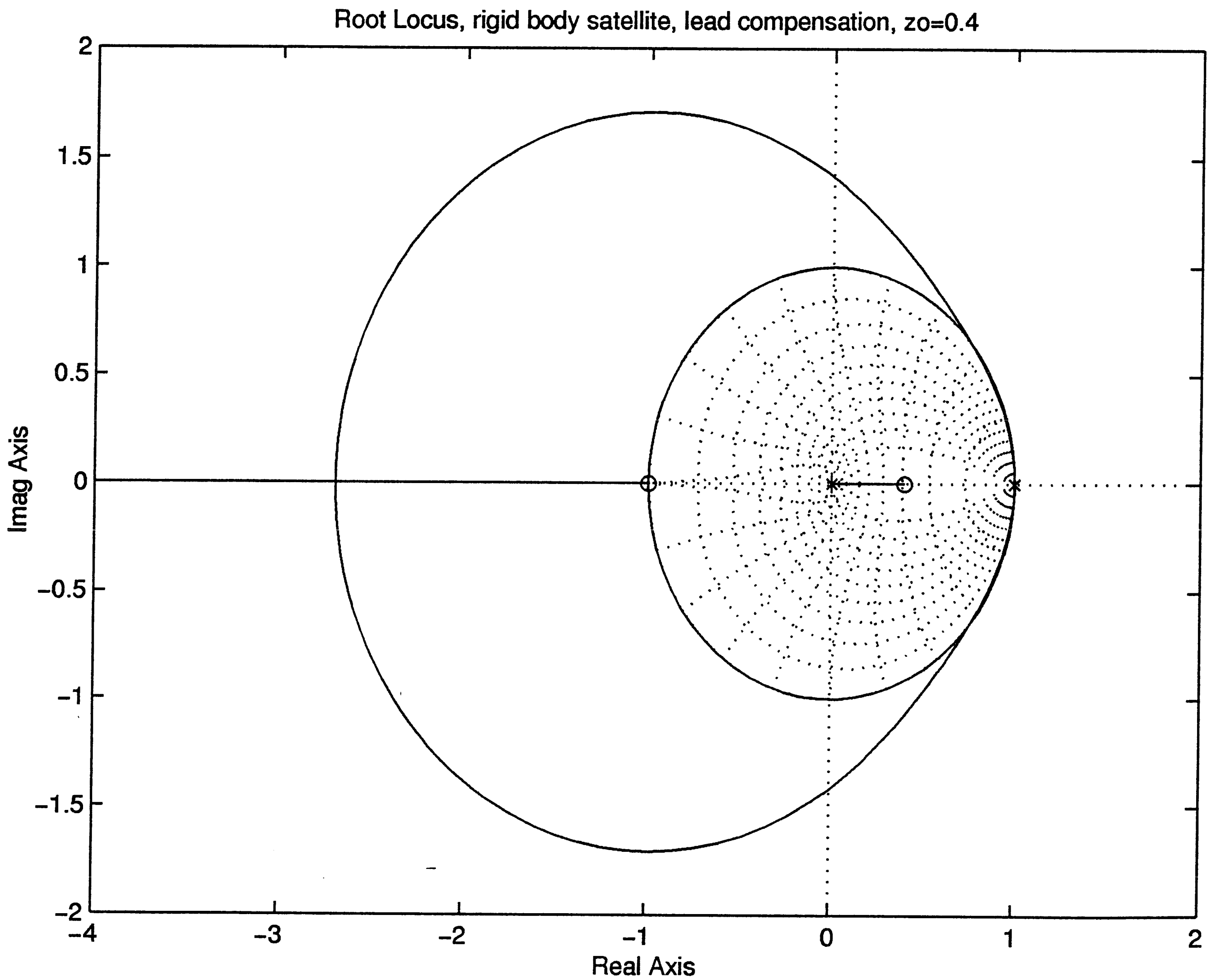
$J\ddot{\theta} = \tau \Rightarrow \theta s^2 = \frac{1}{J}\tau(s)$ i.e

$\theta = \frac{1}{Js^2}\tau(s)$

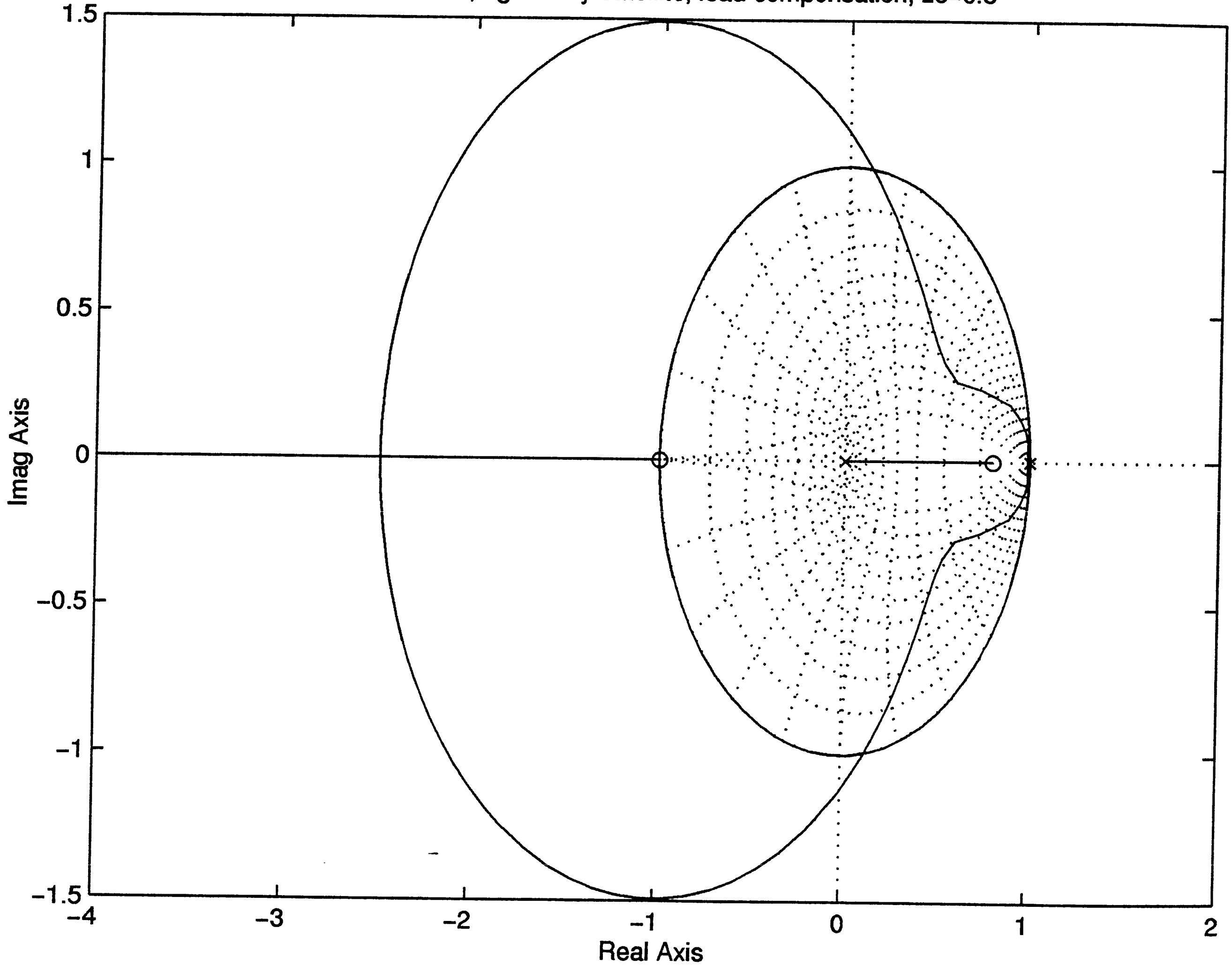


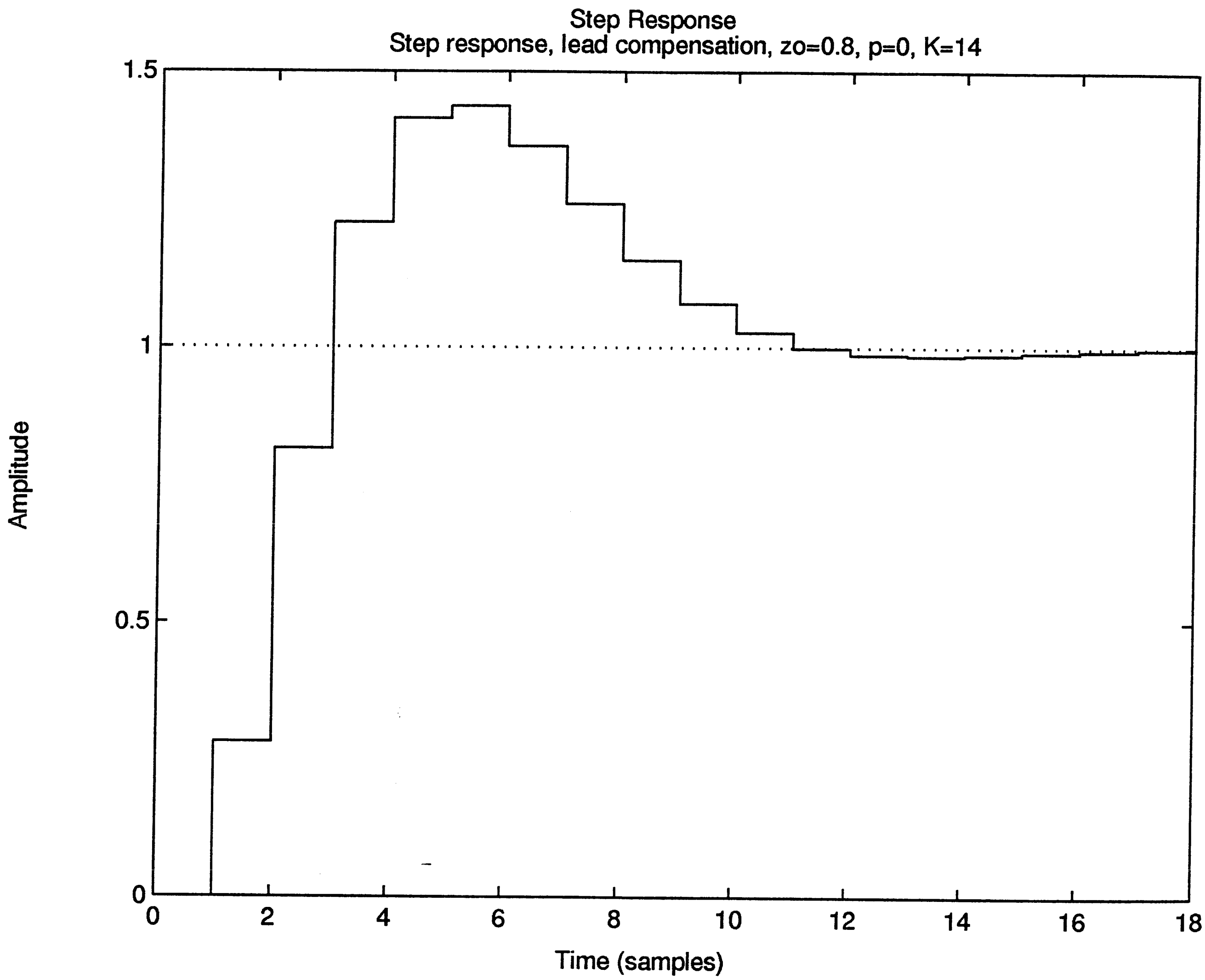


Root Locus, rigid body satellite
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−4
−3
−2
−1
0
1
2
Real Axis
Imag Axis


Root Locus, rigid body satellite, lead compensation, zo=0.4
Imag Axis
Real Axis
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−4
−3
−2
−1
0
1
2

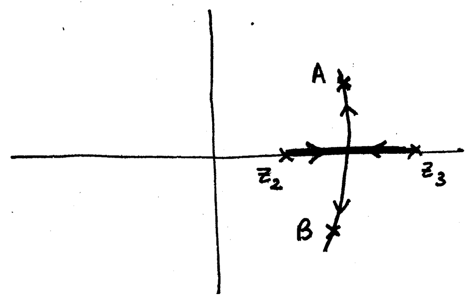
Root Locus, rigid body satellite, lead compensation, zo=0.8




Step Response
Step response, lead compensation, zo=0.8, p=0, K=14
1.5
1
0.5
0
Amplitude
0
2
4
6
8
10
12
14
16
18
Time (samples)

- Example of phase-lag compensation:

Suppose that we have the following uncompensated root locus:



and we select k so that the closed-loop poles end up at A and B, so that we meet some given transient performance specifications.

Suppose that the steady state error is not acceptable $\Rightarrow$ to improve it we can try increasing the gain k, but this will move the poles further away from the real axis $\Rightarrow$ less damping $\Rightarrow$ more overshoot, less stability

Ideally we would like to be able to increase k, while at the same time keeping the closed loop poles at z_a, z_b. We can try to add a pole and a zero:

Uncompensated:

$$G = K_v \frac{\Pi(z - z_i)}{\Pi(z - p_i)} \Rightarrow A \in RL \Leftrightarrow K_v \Pi \frac{|A - z_i|}{|A - p_i|} = 1 \Rightarrow$$

$$k_v = \frac{\Pi |A - p_i|}{\Pi |A - z_i|}$$

Compensated:

$$G_c = \underbrace{k' \frac{z - z_0}{z - z_p}}_{\text{compensator}} \cdot \frac{\Pi(z - z_i)}{(z - p_i)}$$

we still want A to be in the root locus $\Rightarrow$ $k' \left|\frac{z - z_0}{z - z_p}\right| \cdot \overbrace{\left|\Pi \frac{z - z_i}{z - p_i}\right|}^{\frac{1}{K_v}}\Bigg|_{z=A} = 1$

Assume that $\frac{A - z_0}{A - z_p} \sim 1$ (i.e z_0 close to z_p) $\Rightarrow \frac{k'}{k_v} = 1$

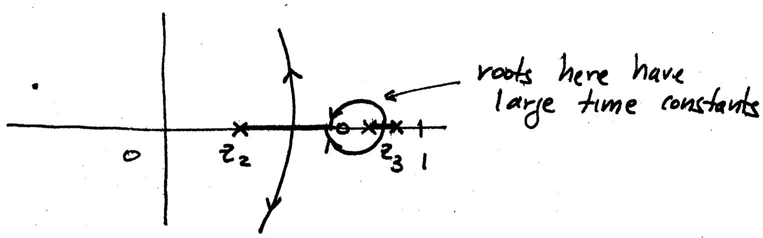
$$\Rightarrow k' \sim k_v \quad \text{(so no effect)}$$

On the other hand, look at the effect on the DC gain:

$$\text{DC gain} = \lim_{z \to 1} k\, D(z) \prod \frac{(z - z_i)}{(z - p_i)} = \text{DC}_{\text{gain-old}} \cdot \left(\frac{1 - z_o}{1 - z_p}\right)$$

So by taking z_o, z_p close to 1 and such that $1 - z_o \gg 1 - z_p$ we can increase the DC gain (ie improving steady state), while still maintaining the same transient.

- <u>Warning</u>: by adding the pole & zero close to 1 we could have a very slowly decaying transient:



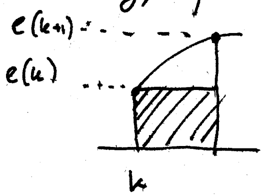
- <u>PID control design</u>:

As in the continuous time case, discrete time PID controllers are a combination of 3 types of action:

$$C(z) = \underbrace{k_p}_{\text{proportional}} + \underbrace{K_D\left(1 - \frac{1}{z}\right)}_{\text{derivative}} + \underbrace{K_I \frac{z}{z-1}}_{\text{integral}}$$

If we approximate $\dfrac{de}{dt} \simeq \dfrac{e(k) - e(k-1)}{T}$ $\left(\begin{array}{c}\text{backward}\\ \text{differences}\end{array}\right) \Rightarrow D(z) = \left(1 - \frac{1}{z}\right) \cdot \frac{1}{T}$

similarly, if we approximate the $\int$ by the area under the rectangle:

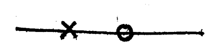


$$u(k) - u(k-1) = e(k)\, T \Rightarrow \left(1 - \frac{1}{z}\right) U(z) = T\, E(z)$$

$$\frac{U}{E} = \frac{Tz}{z-1}$$

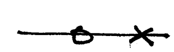
- <u>PD control</u>: $D(z) = k_p + k_D \dfrac{z-1}{z} = \dfrac{(k_p + k_D) z - k_D}{z} = k' \left(\dfrac{z - \alpha}{z}\right)$

similar to lead compensation:

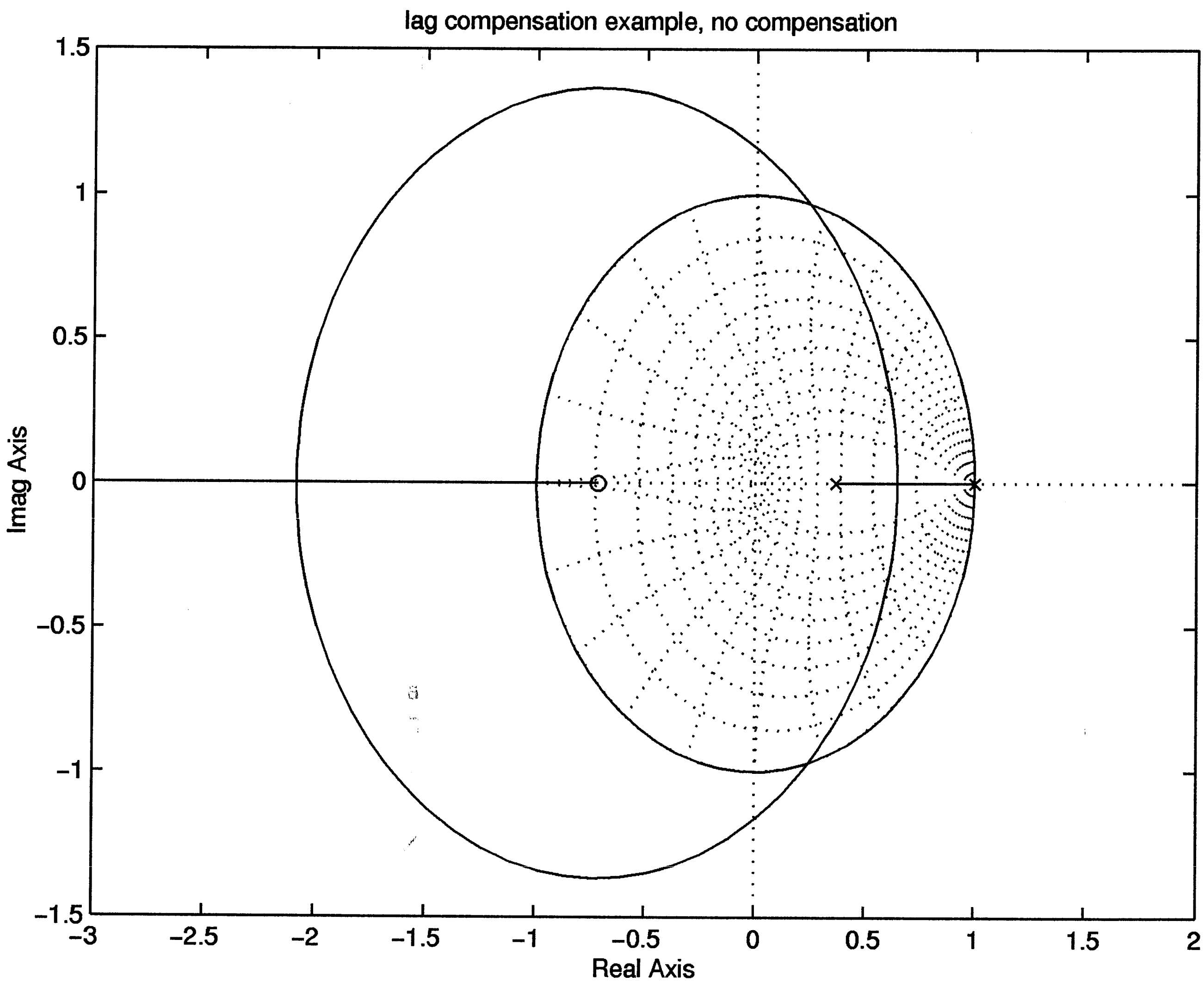


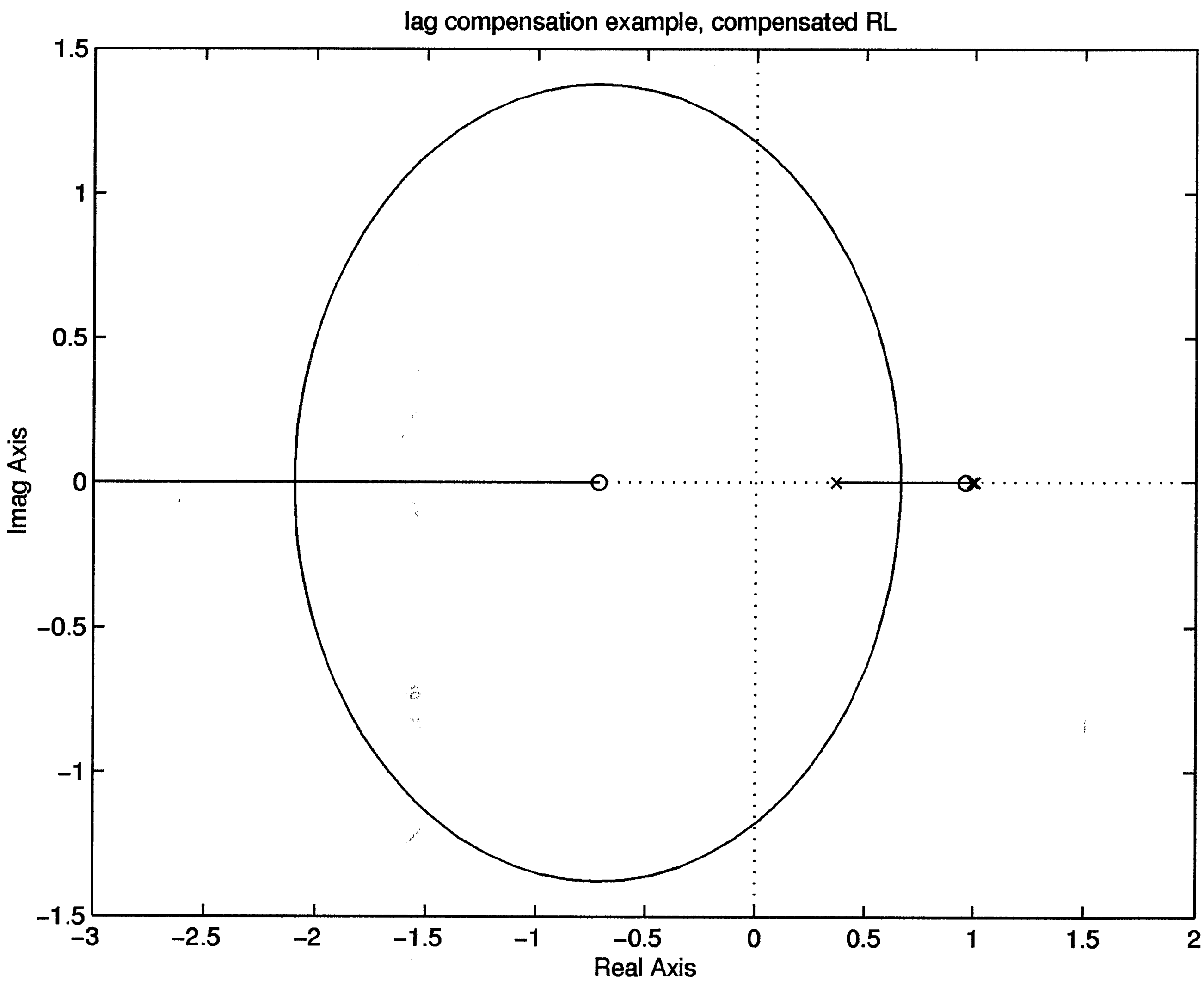
<u>PI control</u>: $D(z) = k_p + k_I \dfrac{z}{z-1} = \dfrac{(k_p + k_I) z - k_p}{(z-1)} = k' \dfrac{(z - \alpha)}{(z-1)}$

similar to phase lead (pole at 1, zero @ $\alpha < 1$)



PID control: both lead & lag action


lag compensation example, no compensation
Imag Axis
Real Axis
1.5
1
0.5
0
−0.5
−1
−1.5
−3
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


lag compensation example, compensated RL
Imag Axis
Real Axis
1.5
1
0.5
0
−0.5
−1
−1.5
−3
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Analytical Phase Lead (or Lag) Design

So far, we have selected the location of the poles & zeros of the controller by trial and error

Q: Is there a more systematic way to proceed?

A: Yes: Recall that the controller has the general form: $G_c(z) = A\left(\frac{z-z_0}{z-z_p}\right)$

We want to select A, z_0, z_p so that the R.L. passes through a specific point, say $z = z_1$ $\Rightarrow$

$K\,G_c(z_1)\,G_p(z_1) = -1 \Rightarrow$ 4 unknowns (A, K, z_0, z_p)
2 equations (mag & angle criteria)

Assume that $G_p(z) = \frac{N(z)}{D(z)} \Rightarrow K G_c G_p = K A\left(\frac{z-z_0}{z-z_p}\right)\frac{N(z)}{D(z)} \Rightarrow$ char. eq:

$P(z) = (z-z_p)D(z) + kA\,(z-z_0)\,N(z)$

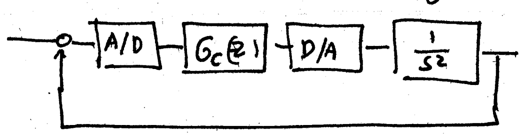
same

Since we want a root at $z = z_1$, then P should look like:

$P_1(z) = (z-z_1)(z-\bar{z}_1)\,Q_{n-2}(z)$ (assuming that z_1 is complex)

To get the equations, expand $P(z)$ & $P_1(z)$ and compare terms:

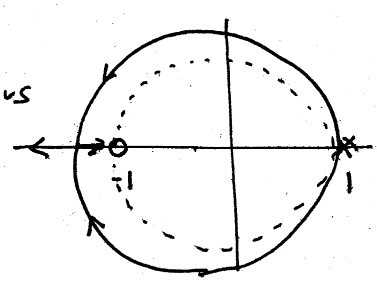
- **Example:** Recall the rigid satellite example: $G(s) = \frac{1}{s^2}$



Assume $T = 1 \Rightarrow$

$G_p(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s^3}\right] = 0.5\,\frac{z+1}{(z-1)^2}$

Uncompensated root locus



(unstable for all k)

Suppose that the performance specs call for: $M_p \le 15\% \Rightarrow \zeta \ge 0.5$
$T_r \le 1.8$ sec $\quad \omega_n \ge 1$

A suitable closed-loop pole location is $z_1 = 0.4 \pm j\,0.5$

$\Rightarrow$ We want the characteristic equation to look like:

$$P_1(z) = (z - 0.4 - j0.5)(z - 0.4 + j0.5)(z + p)$$

third pole since we have a third order system

$$P_1(z) = (z^2 - 0.8z + 0.3625)(z + p) = z^3 - (p + 0.8)z^2 + (0.3625 + 0.8p)z - 0.3625p$$

on the other hand:

$$P(z) = (z - z_p)(z-1)^2 + \overbrace{KA}^{A'} \cdot 0.5(z+1)(z - z_0) = (z - z_p)(z^2 - 2z + 1) + 0.5A'(z^2 + (1 - z_0)z - z_0)$$

$$= z^3 + [0.5A' - 2 - z_p]z^2 + [0.5A'(1 - z_0) + 2z_p + 1]z - 0.5A'(z_0) - z_p$$

Comparing $P_1(z)$ & $P(z)$ yields:

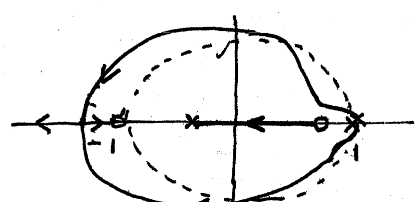
$$\left.\begin{aligned} -(p + 0.8) &= 0.5A' - 2 - z_p \\ 0.3625 + 0.8p &= 0.5A' - 0.5A'z_0 + 2z_p + 1 \\ -0.3625p &= -0.5A'z_0 - z_p \end{aligned}\right\} \quad 3 \text{ eq}, \quad 4 \text{ unknowns: } A', z_0, z_p, p$$

Since we have an additional degree of freedom, we can use it to fix the DC gain of the controller, by imposing $A\dfrac{(1 - z_0)}{(1 - z_p)} = 1$, or the location of the third pole p.

Assume that we want $p = 0$ (to get fast response) $\Rightarrow$

$$\left.\begin{aligned} (1) \quad -0.8 &= 0.5A' - 2 - z_p \\ (2) \quad 0.3625 &= 0.5A' - 0.5A'z_0 + 2z_p + 1 \\ (3) \quad 0 &= -0.5A'z_0 - z_p \end{aligned}\right\} \text{ solving (replace (3) in (2), solve for } z_p \text{ and } A) \text{ yields:}$$

$$z_p = -0.4594$$
$$z_0 = 0.6203$$
$$A' = 1.4812$$



compensated RL

Sanity check, the resulting closed-loop system has poles at $0.4 \pm j0.499$, 0

- Note that for higher order system this approach may be infeasible $\Rightarrow$ use state space based method that allow for placing the poles at desired location (procedure developed in mid 60's taught in EECE 7200)