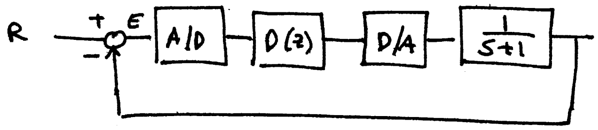
# Root Locus (section 7.6)

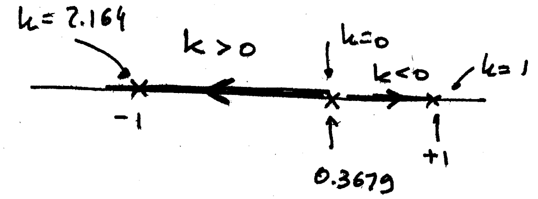
Consider the system:




The characteristic equation is: $1 + D(z)\,\frac{1-e^{-T}}{z-e^{-T}} = 0$

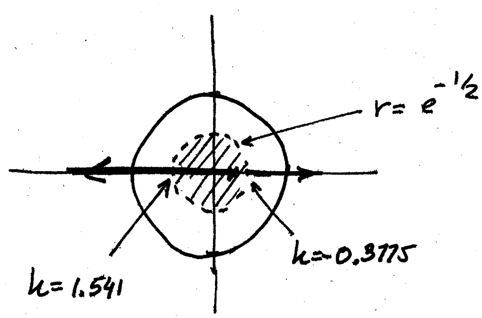
If we take $k = D(z)$, $T=1 \Rightarrow$ single pole at $z_0 = e^{-1} - k(1-e^{-1})$

$\cong 0.3679 - 0.6321\,k$




By changing the value of k we change the location of the pole $\Rightarrow$ change $T_r$, $T_s$, overshoot, etc.

Suppose that we want $T_s \le 8\,sec \Rightarrow \tau \le 2 \Rightarrow |r| \le e^{-\frac{1}{2}}$




By looking at the intersection of the region where $|r| \le 1/2$ with the region of achievable closed-loop poles we get the region of admissible values of k

(in this case $-0.38 < k < 1.541$

Now we can select a value of k to optimize some other performance measure such as $e_{ss}^{step}$: in this case pick $k = 1.541 \Rightarrow e_{ss} \cong \frac{1}{1+k} \cong 0.40$

Suppose we want to achieve $e_{ss} = 0.1 \Rightarrow \frac{1}{1+k} = 0.1$ // $k \ge 9$

but this value of k renders the system unstable. $\Rightarrow$ we can try a controller of the form:

$$D(z) = k\frac{(z+a)}{(z-1)}$$

to render the system type I (i.e $e_{ss}^{step} = 0$).

<u>Problem</u>: how do we pick the value of k? First we need to guarantee stability $\Rightarrow$ find char. eq:

$$(z-1)(z-e^{-T}) + k(z+a)(1-e^{-T}) = 0$$

$$z^2 + (0.6321\,k - 1.3679)z + 0.3679 + 0.6321\,ka = 0$$

If we are interested <u>only</u> in stability we can use Jury's test:

$$\varphi(1) > 0 \quad \Rightarrow \quad 0.6321\,k + 0.3679 + 0.6321\,ka > 0$$

$$\varphi(-1) > 0 \qquad 2.7358 + 0.6321\cdot(1-a)\cdot k > 0$$

$$|a_0| < a_n \quad \Rightarrow \quad 0.6321\,ka < 1$$

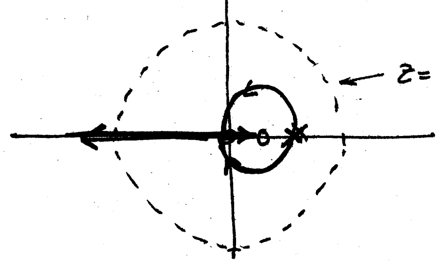
However, if we are also interested in performance, we need to look at the location of the <u>closed-loop</u> poles. In this case,

since it is a second order system we can, conceivably, solve explicitly:

$$z_{1,2} = \frac{(1.3679 - 0.6321\,k) \pm \sqrt{(0.6321\,k - 1.3679)^2 - 4(0.3679 + 0.6321\,ka)}}{2}$$

<u>Quite messy</u>. For larger order polynomials this approch is unfeasible

<u>Alternative</u>: Draw a "reasonably" accurate sketch without having to solve the char. equation for each value of K. Once you have this plot (the root locus), you can pick the value of K to meet the design specs.




<u>Definition</u>: The Root Locus of a system is the plot of the roots of the systems characteristic equation as some parameter of the system changes.

A point in the z-plane belongs to the RL iff

$$\boxed{1 + K_{sp}\,G_{oz}(z) = 0} \quad \Leftrightarrow \quad |K\,G(z)| = 1 \quad \text{(Magnitude Criterion)}$$

two equations

$$\angle K\,G(z) = 180 \quad \text{(angle criterion)}$$

<u>Remarks</u>:
1) Usually (but not always), K is taken to be positive
2) If we have an "n th" order system the RL has n - branches

3) note that the angle criterion does not depend on k:

If $k>0$, a point $z_1 \in RL \iff \angle G(z_1) = 180 + 2\pi \cdot \ell \quad \ell = 0, \pm 1, \ldots$

$\Rightarrow$ can use the angle criterion to determine whether or not a point $z_1$ is in the RL. If $z_1 \in RL$ then the actual value of $k$ can be found from the magnitude criterion.

- Rules for sketching the RL

These rules are the same as for continuous-time systems (EE 428) with minor modifications:

1) Mark the open loop poles (origin) & zeros (endind)

2) Draw the R.L on the real axis: points to the left of an odd number of real poles & zeros

3) Draw $n-m$ asymptotes leaving at angles

$$\theta = (2\ell+1)\frac{180}{n-m} \qquad \ell = 0, 1 \cdots n-m-1$$

where $n = \#$ open loop poles
$m = \#$ open loop zeros

If $n-m>1$ the asymptotes intersect the real axis at the point

$$\sigma = \frac{\Sigma p_i - \Sigma z_i}{n-m} \qquad \text{(centroid)}$$

4) Break away (break-in) points:

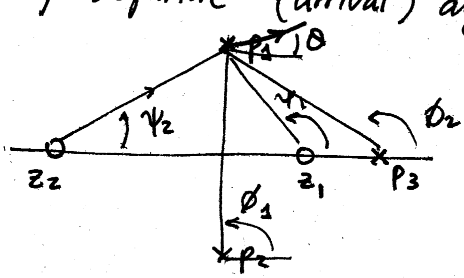
Solve $\frac{d}{dz}\left\{\frac{1}{G(z)}\right\} = 0 \quad \left(\text{or } \frac{d}{dz} G(z) = 0\right)$

Note: this is only a necessary condition. Need to go back and check that the solutions indeed are in the RL. Solutions not in the RL are discarded

Different from EECE 5580

5) Unit circle intersect: Use Jury's method to find K. Plug these values of k in the Char. eq. and solve for the roots

6) Departure (arrival) angle:




$$\theta_d = \Sigma \psi_i - \Sigma \phi_i - (2\ell+1)180$$

$$\theta_a = \Sigma \phi_i - \Sigma \psi_i + (2\ell+1)180$$

# Frequency Response Analysis (section 7.8)

- Most "clasical" control design tools are based upon the use of frequency domain information (ie the way the system responds to signals of various frequencies)

These methods clearly display the trade-offs that go into control systems design.

From 1960-1980 most of the theoretical work was based on time domain ("state space") methods (however practicing control engineers kept using frequency domain)

Now we use a mixture: some freq. domain motivated methods but implemented using state-space techniques

- Frequency response of discrete-time systems: start by considering the output of a system excited by a sampled sine wave

$$E(z) \longrightarrow \boxed{G(z)} \longrightarrow C(z)$$

(where for the time being $G(z)$ is assumed stable)

$E(z)$: sampled sine wave $\Rightarrow$ $E(z) = \mathcal{Z}[\sin\omega t] = \mathcal{Z}\left[\dfrac{e^{j\omega T} - e^{-j\omega T}}{2j}\right]$

$$= \frac{1}{2j}\mathcal{Z}\left[e^{j\omega T} - e^{-j\omega T}\right] = \frac{1}{2j}\left[\frac{1}{1-\frac{e^{j\omega T}}{z}} - \frac{1}{1-\frac{e^{-j\omega T}}{z}}\right] = \frac{z}{2j}\,\frac{e^{j\omega T} - e^{-j\omega T}}{(z-e^{j\omega T})(z-e^{-j\omega T})} =$$

$$= \boxed{\frac{z\,\sin\omega T}{(z-e^{j\omega T})(z-e^{-j\omega T})}}$$

The corresponding output is given by: $C(z) = G(z)\,E(z) = \dfrac{G(z)}{2j}\left(\dfrac{z}{z-e^{j\omega T}} - \dfrac{z}{z-e^{-j\omega T}}\right)$

$$= k_1\frac{z}{z-e^{j\omega T}} + k_1^*\frac{z}{z-e^{-j\omega T}} + C_g(z)$$

partial fraction expansion

terms due to the stable poles of $G(z)$

where $k_1 = G\left[e^{j\omega T}\right]\left(\dfrac{\sin\omega T}{e^{j\omega T} - e^{-j\omega T}}\right) = \dfrac{G\left[e^{j\omega T}\right]}{2j}$; $\quad k_1^* = -\dfrac{G\left[e^{-j\omega T}\right]}{2j}$

Taking inverse z transform yields:

$$c(kT) = \mathcal{Z}^{-1}\left[k_1\frac{z}{z-e^{j\omega T}} + k_1^*\frac{z}{z-e^{-j\omega T}} + C_g(z)\right] = k_1 e^{jk\omega T} + k_1^* e^{-jk\omega T} + C_g(kT)$$

Since $c_g(kT)$ is the free response of a stable system, then $c_g(kT) \to 0$ as $k \to \infty$. Hence, in steady state we have

$$c(kT) = k_1 e^{jk\omega T} + k_1^* e^{-jk\omega T} = \frac{G(e^{j\omega T})}{2j} e^{jk\omega T} - \frac{G^*(e^{j\omega T})}{2j} e^{-jk\omega T}$$

write $G(e^{j\omega T}) = |G(e^{j\omega T})| \angle G \Rightarrow c(kT) = |G(e^{j\omega T})| \dfrac{e^{j(k\omega T+\theta)} - e^{-j(\omega kT+\theta)}}{2j}$

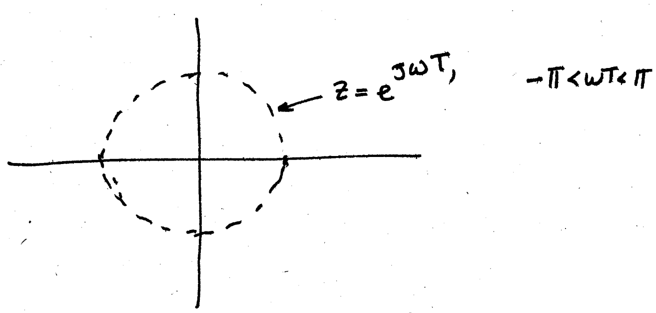
$$= \boxed{|G(e^{j\omega T})| \sin(k\omega T+\theta)}$$

Recap: input: sampled sin, freq $\omega$

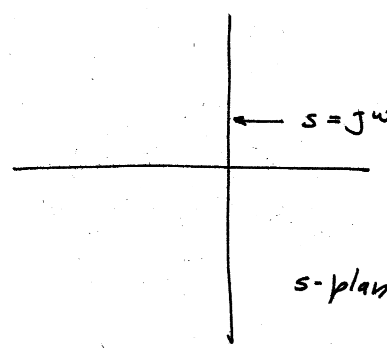
output: $c(kT) = |G(e^{j\omega T})| \cdot \sin(k\omega T + \theta)$ where $\theta = \angle G(e^{j\omega T})$

i.e: another sinusoidal of the same frequency, with a phase shift of $\theta$ and gain $|G(e^{j\omega T})|$

$G(e^{j\omega T})$ is the frequency response of the discrete time system (compare to $G(j\omega)$ for continuous time)




z - plane




s-plane

As in the continuous time case, we can represent the frequency response using either Bode plots or polar plots

- ## Nyquist Criterion:

We will use polar plots (freq domain data) to assess the stability properties of the closed-loop system

Advantages:

- We don't need to know the Transfer Function explicitly. We can get the required information from experimental data
- We get information about the type of compensation required to stabilize the system
- We get "robustness" information, i.e how much uncertainty you can tolerate before going unstable

Basic principle: Cauchy's argument principle, relating the number of poles and zeros of a function $F(z)$ in a certain region to its polar plot:

Consider a closed contour $C$ in the $z$-plane. Let $f(z)$ be an analytic function inside $C$ except at a finite number of poles, and such that it has neither poles nor zeros on $C$. Then the image $\Gamma$ of $C$ under the mapping $w = F(z)$ encircles the origin of the $w$ plane a number $\underline{\underline{N}}$ of times given by

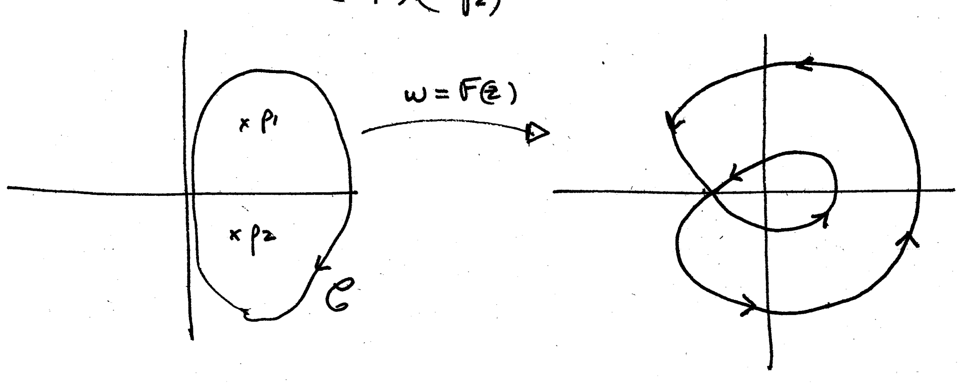
$$N = Z - P$$

where
- $N$: number of encirclements of the origin in the $w$-plane
- $Z$: number of zeros of $F(z)$ enclosed by $C$
- $P$: number of poles of $F(z)$ enclosed by $C$

Example $\quad F(z) = \dfrac{k}{(z+p_1)(z+p_2)}$

$Z = 0$
$P = 2$

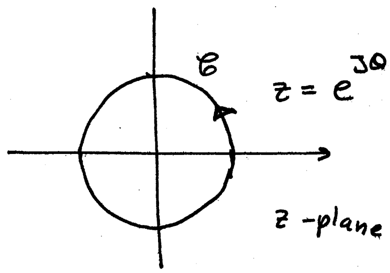
$N = -2$



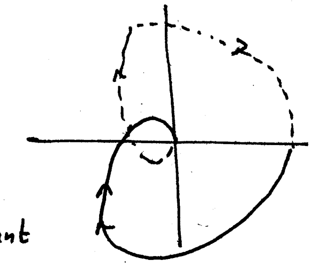
## Nyquist criterion for discrete time system:

In this case we are interested in finding out the number of zeros of the characteristic equation <u>outside</u> the unit disk

$\Rightarrow$ Take as $\mathcal{C}$ the unit circle (counter-clockwise)




$0 \le \theta < 2\pi$




open loop plant

**Steps:** (1) Plot the image of $G_{ol}(z)\Big|_{z=e^{j\theta}}$ (i.e the polar plot of $G_{ol}(z)$)

(2) Let $Z_i$ = # roots of char. eq inside unit circle
$N$ = # of encirclements of $-1$ (clockwise)
$P_i$ = # of open loop poles inside the unit disk

$\Rightarrow \quad N = -(Z_i - P_i)$

Now assume that $G(z) = \dfrac{num(z)}{den(z)}$, degree (num) = $m$, deg (den) = $n$, $m \le n$

$\Rightarrow$ $P_o$ = # poles <u>outside</u> the unit disk = $n - P_i$
$Z_o$ = # zeros of the char. eq. outside = $n - Z_i$

$\Rightarrow \quad Z_i - P_i = P_o - Z_o$ // $\boxed{N = Z_o - P_o}$

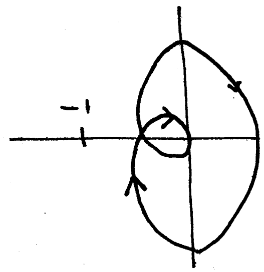
**Recap:** if $N$ = # encirclements of $(-1)$ (clockwise)
$P_o$ = # open loop poles outside the unit disk

$\Rightarrow$ $\boxed{Z_o = N + P_o}$

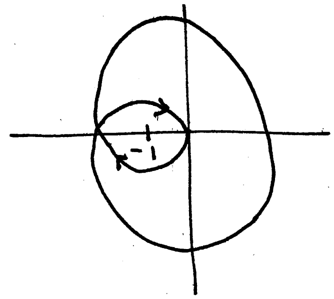
$Z_o$: number of <u>unstable</u> closed loop poles

$\Leftarrow$ stable iff $Z_o = 0$, ie
$N = -P_o$

- In the special case where the plant is <u>open loop stable</u> (ie $P_o=0$) then the closed loop system is stable <u>iff</u> the Nyquist plot does not enclose the $-1$ point



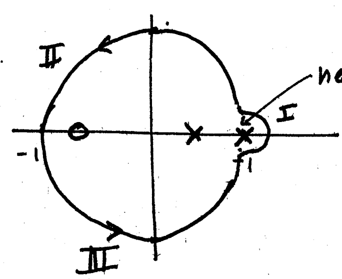

stable



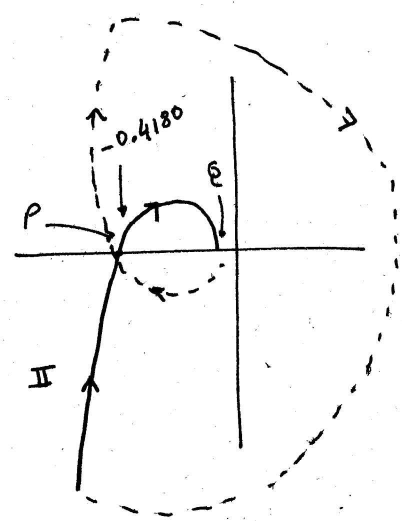

unstable (2 unstable roots)

- What happens if our open loop plant has a pole on the unit circle? Techically we cannot apply the criterion, since $F$ is not allowed to have poles on the contour $\mathcal{C}$ $\Rightarrow$ perturb the contour by taking a small detour:

Example: $G(z) = 0.3679 \dfrac{(z+0.7183)}{(z-1)(z-0.3679)}$




need to take a small detour here




In region II we have $G(z) = G(e^{j\omega T})$, $0 < \omega \le \frac{\pi}{T} = \frac{\omega_s}{2}$

Around I: $z = 1 + re^{j\theta} \Leftrightarrow \frac{1}{z-1} = \frac{1}{r} e^{-j\theta} \xRightarrow{r \ll 1}$ half circle with radius $\to \infty$, clockwise

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Region III: mirror image of II

- Finally to assess stability we need to find the points $P$ & $\rho$ $\Rightarrow$ 2 alternatives

  (a) Impose $G[e^{j\theta}]$ real $\Leftrightarrow \mathrm{Im}\left[G(e^{j\theta})\right] = 0$

  (b) use Jury's test

- Relative Stability (Concepts of Phase & Gain margin)

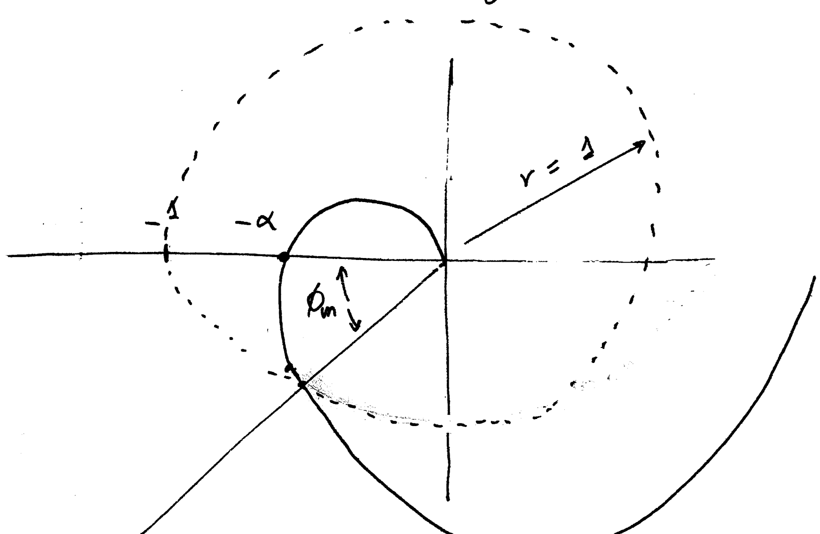
- Concept of "relative stability": How close is a given system to being unstable?

(ie: how much can we change GH and still have a stable system?) This gives a measure of the "robustness" of the system against model uncertainty

Classically 2 measures:

a) Gain margin: How much gain can we add to our system and still keep it stable? (equivalently: how much gain do I have to add to the system to make it unstable)

b) Phase margin: How much phase do I have to add to the system to make it marginally stable?




- Gain margin: Let $\alpha$ be the magnitude of $G(j\omega)$ at the $180°$ cross-over (ie $G(j\omega) = -\alpha$) $\Rightarrow$ Then, the gain margin is $1/\alpha$ (usually expressed in Db)

- Phase margin: magnitude of the minimum angle by which the Nyquist diagram must be rotated in order to go through the -1 point $\Rightarrow$ in the diagram: $\phi_m$

Gain & phase margins can be read (in most cases) from the Bode diagrams