Control Systems Specifications

The typical steps in designing a control system are:

iterate

(a) model the system (chapters 2-5)

(b) decide performance specifications ← You are here

(c) analyze the behavior of the system

(d) (if necessary) design a controller to modify the behavior so that these specs are met (or show that no such controller exists)

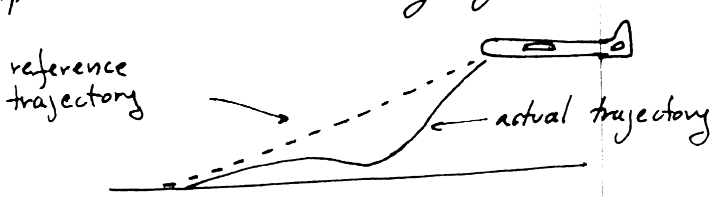
(e) check performance via simulations (using a detailed model) or in a prototype

(f) check performance in the actual plant

(g) done: go to the beach

- General features that we require from a control system:

Example: automatic landing system



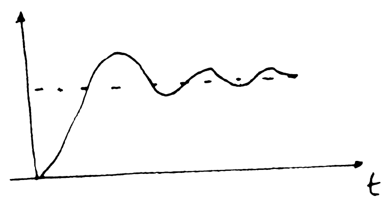
inputs: { reference trajectory, disturbances (wind) }

output: actual trajectory

(a) **Stability**: We want the system to remain "stable", in the sense that the output corresponding to any bounded input should also be bounded.

(b) Steady state accuracy: The final error (after all transients decay) must be stable.
In some cases we require zero steady state error ⇒ need to impose special structure on the controller to achieve this.

(c) <u>Transient response</u> (or dynamic response): Impose specifications upon overshoot, rise-time, settling-time.

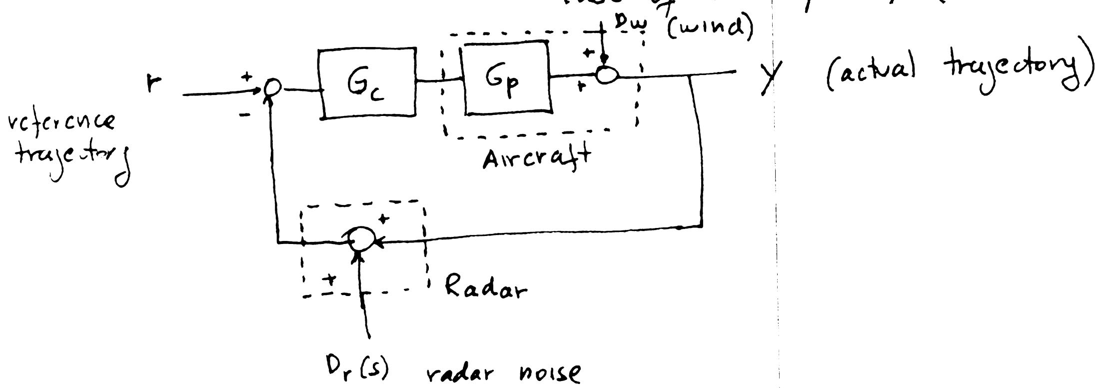


These specs are related to both how the steady state is achieved <u>and</u> the ability of the system to track a time varying input.

(d) <u>Sensitivity to parameter changes</u>: We want the overall behavior of the system (or at least some key properties) to remain invariant even if some of the parameters of the system change (related to "robust" control)

New aspects of the problem <u>specific</u> to digital control: effects of the sampling rate, round-off errors, etc.

(e) <u>Disturbance rejection</u>: Ability of the system to reject unwanted disturbances (e.g. wind gusts in the case of an airplane) (or instrument noise)



(f) Minimization of some "cost": peak control effort, energy, time to intercept, ...

These specs lead to problems of the form:

$$\min \left\{ \max_t |u(t)| \right\} \qquad \text{(an "}L_\infty\text{" type problem) (solved in early 90's)}$$

$$\min \int_0^\infty |u|\, dt \qquad \text{("}L_1\text{" optimal control) (solved in mid 80's)}$$

$$\min \int_0^\infty [y^2(t) + u^2(t)]\, dt \qquad \text{(LQR control) (solved in mid 60's)}$$

Solving these problems requires tools beyond the scope of 5610. Some are covered in ECEG 7214 (Optimal control problems)

(Preliminary) Stability Analysis:

- **Def:** Bounded Input Bounded Output (BIBO) Stability

  A system is BIBO stable if and only if the output is bounded for every possible bounded input.

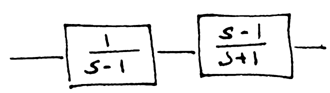
(There are many different definitions of stability: BIBO, asymptotic, Lyapunov, exponential, ... Turns out that for LTI systems all these definitions are equivalent)

Aside: "internal" versus input-output or "external" stability

"external" stability: purely input/output, does not care about internal signals that do not show up at the output

"internal" stability: all internal signals must remain bounded, even if they do not show up at the output

Important case: unstable pole-zero cancellation: we get a system that is input-output stable but not internally stable (in practice it will not work)



These issues are related to the concepts of controllability / observability and minimal realizations. More on this latter and in ECE 7200

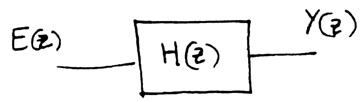
Back to stability analysis:

Our definition of BIBO stability provides a nice conceptual definition but it is useless as a practical tool.

According to that definition, in order to assess whether or not a system is BIBO stable we would need to try every possible bounded input (an infinite number!) and check if the corresponding output is bounded.

Obviously this is not feasible: we need to find an equivalent, practically implementable definition of stability

- Necessary and Sufficient Conditions for stability:



Suppose that we know the pulse transfer function $H(z)$. What conditions on $H(z)$ guarantee stability?

Let's look at the output $Y(z)$ corresponding to a generic input $E(z)$:

$$Y(z) = H(z)\, E(z) \Leftrightarrow y(nT) = \sum_{k=0}^{n} h(nT)\cdot e\left[(k-n)T\right]$$

where $h(nT) = \mathcal{Z}^{-1}\{H(z)\}$ (the impulse response, also known as the Markov parameters)

- Assume that the following condition holds:

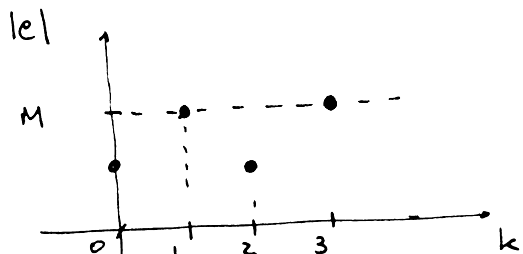
$$\sum_{n=0}^{\infty} |h(nT)| \leq K < \infty \quad \text{(for some } K \text{ large enough)}$$

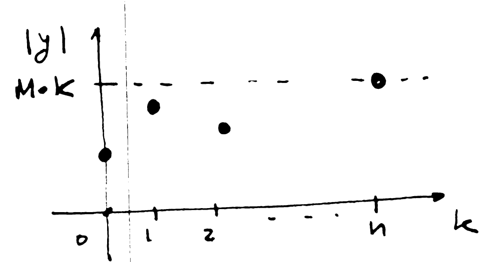
Then:

$$|y(nT)| = \left|\sum_{0}^{n} h(kT)\, e\left[(n-k)T\right]\right| \leq \sum_{0}^{n} |h(kT)|\left|e[(n-k)T]\right|$$
$$\leq \sum_{0}^{\infty} |h(kT)|\cdot\left|e[(n-k)T]\right| \leq \sup_k |e(kT)|\cdot \sum_{0}^{\infty} |h(kT)|$$

If the input $E(z)$ is bounded, i.e. $|e(kT)| < M$, all k

then: $|y(nT)| \leq \underbrace{\sup_k |e(kT)|}_{\leq M}\cdot \underbrace{\sum_{0}^{\infty} |h(kT)|}_{\leq K} \leq M\cdot K \Rightarrow$ also bounded





$\Rightarrow$ The system is BIBO stable!

Recap: $\boxed{\text{If } \sum_{0}^{\infty} |h(kT)| < \infty \Rightarrow \text{BIBO stable}}$

So we have found a <u>sufficient</u> condition for stability
(i.e. if it holds then the system is stable)

Q: What about <u>necessity</u>? i.e: are there stable systems that do not satisfy this condition?

A: <u>No</u>: <u>All</u> LTI stable systems must satisfy this
(i.e if the condition fails the system cannot possibly be stable)

<u>Proof</u>: Assume that the condition fails, i.e for any $K>0$ we can find some n such that:

$$\sum_{k=0}^{n} |h(kT)| > K$$

Take now the following input: $e[(n-k)T] = \text{signum}\{h(kT)\}$

$$\left(\text{signum}\{x\} = \begin{matrix} 1 & x>0 \\ -1 & x<0 \\ 0 & x=0 \end{matrix}\right)$$

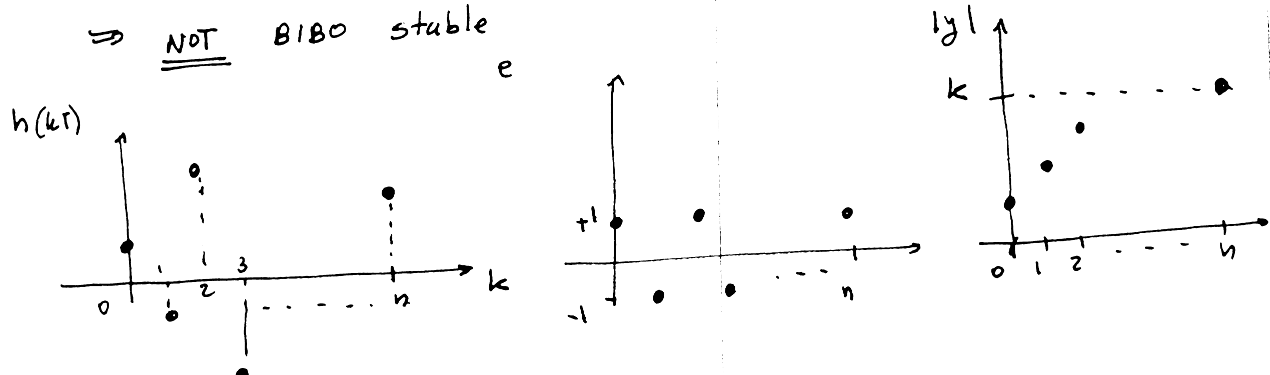
Obviously e is bounded by 1. Let's look now at the output

$$|y(nT)| = \left|\sum_{0}^{n} h(kT)\, e[(n-k)T]\right| = \left|\sum_{0}^{n} h(kT)\,\text{signum}\{h(kT)\}\right| = \left|\sum_{0}^{n} |h(kT)|\right|$$

$$= \sum_{0}^{n} |h(kT)| > K \quad \text{(by assumption)}$$

$\Rightarrow$ Given <u>any level K</u>, we can always find an input bounded by one and such that $|y(nT)| > K$
(i.e we can make the peak value of the output as large as desired)

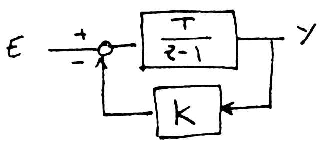
$\Rightarrow$ <u>NOT</u> BIBO stable



<u>Note</u>: as a byproduct we also found out the <u>worst-case</u> signal (the one that will do you in):
Is an input that it is "aligned" with the impulse response

<u>Recap:</u> $\boxed{\text{BIBO stable} \iff \sum_{0}^{\infty} |h(kT)| < \infty}$

So now we have a testable condition:

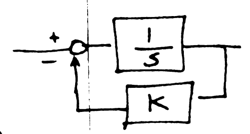
Example: 

$$H(z) = \frac{\frac{T}{z-1}}{1 + \frac{kT}{z-1}} = \frac{T}{z-1+kT}$$

$$H(z) = \frac{T}{z-(1-kT)} = \frac{T}{z}\,\frac{1}{\left(1-\left(\frac{1-kT}{z}\right)\right)} \iff h[nT] = \begin{cases} 0 & n = 0 \\ T\cdot(1-kT)^{n-1} & n \geq 1 \end{cases}$$

$$\Rightarrow \sum_{n=0}^{\infty} |h(nT)| = T\sum_{0}^{\infty} |(1-kT)|^{n} = \begin{cases} T\cdot\frac{1}{1-|1-kT|} < \infty & \text{if } |1-kT| < 1 \\ \infty & \text{if } |1-kT| \geq 1 \end{cases}$$

$\Rightarrow$ System is BIBO stable <u>iff</u>: $|1-kT| < 1$ or $kT < 2$

(Compare to the continuous time case: 

: $H(s) = \frac{1}{s+k}$

$$h(t) = e^{-kt}, \qquad \int_{0}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} e^{-kt}\,dt = \frac{1}{k} < \infty \quad \text{all } k > 0$$

$\Rightarrow$ cont. time system is stable for all k)

Now we have a testable condition for stability. However it is hard to use: You need to (1) find $\mathcal{Z}^{-1}[H]$

(2) compute $\sum_{0}^{\infty} |h_k|$

We'd like to have something simpler. Turns out that if your system is finite dimensional linear time invariant (FDLTI) (as always the case in 429) we can assess stability by looking at the <u>location of the poles</u>

- <u>Relationship between BIBO stability and the location of the poles:</u>

Suppose $G(z) = \frac{C(z)}{E(z)} = \frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}$

Assume for simplicity that all roots are simple. Then:

$$G(z) = \sum_i k_i \frac{z}{z-p_i} = \sum_i k_i \frac{1}{1-\frac{p_i}{z}} \overset{\mathcal{Z}^{-1}}{\Longleftrightarrow} g_k = \sum k_i (p_i)^k$$

Note that $|p_i|^k \to \infty$ if $|p_i| > 1$

In fact, it can be shown that $\sum |p_i|^k < \infty \Leftrightarrow |p_i| < 1$

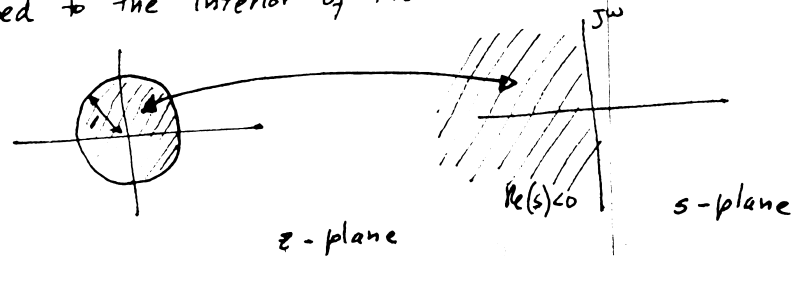
$\Rightarrow \sum_{0}^{\infty} |g(k)|$ bounded $\Leftrightarrow |p_i| < 1$, i.e. all poles <u>must</u> be inside the unit disk.

(if we have repeated poles we get terms of the form $n p_i^n$ and the conclusion still stands)

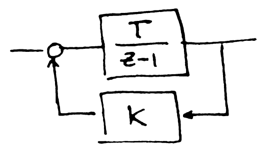
$$\boxed{\text{System BIBO stable} \Leftrightarrow \text{all poles inside the unit disk}}$$

This is not surprising. Recall from the HW that $G(z)$ has a pole at $z = z_o$ $\Leftrightarrow$ $G(s)$ has a pole at s_o where $z_o = e^{s_o T}$

Thus the stable region in the s plane $(Re(s) < 0)$ gets mapped to the interior of the unit disk



Example revisted:

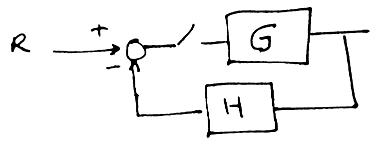


$\Rightarrow H(z) = \dfrac{T}{z-1+kT}$

has a single pole at $z = 1-kT$

$\Rightarrow$ stable $\Leftrightarrow |1-kT|<1 \Leftrightarrow kT<2$

- So now we have an easy way of checking stability

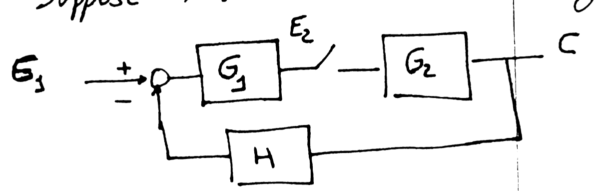


$\Leftrightarrow$ Char equation: $1+G(z)H(z)=0$
poles: roots of this equation

(a) find all the poles of the transfer function (i.e roots of the characteristic equation: Mason's $\Delta = 0$)

(b) stable $\Leftrightarrow$ all poles inside unit disk

But trouble: Not all sampled data systems have a Transfer Function
Suppose that we have something like



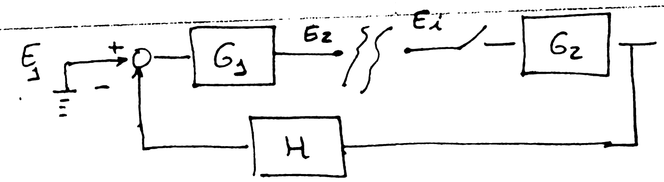
We know that in this case the pulse transfer function from $E(z)$ to $C(z)$ does not exist. So: how do we handle these cases?

A: It turns out that we can still define a characteristic equation

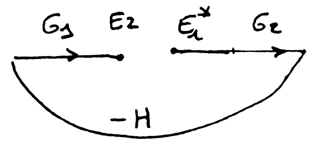
Recall that stability is an intrinsic property of the system, i.e. it does not depend on which signals we choose as inputs and outputs (provided that there are no pole/zero cancellations)

$\Rightarrow$ Instead of considering the signal E_1 in the diagram above, we could take one that is more convenient (i.e one such that the TF exists)

Specifically, we can open the system before the sampler (at E_2) (and set $E_1 = 0$)



Now we can find the TF from E_i to E_2. (we will call this TF the <u>"open loop" function</u>)



$$\Rightarrow \quad G_{op} = \frac{E_2^*}{E_i^*} = -(G_1 H G_2)^*$$

When you close the loop, you get:

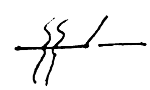
$$E_i(z) = E_2(z)$$

and since $E_2(z) = G_{op}(z) E_i(z) \Rightarrow \left[1 - G_{op}(z)\right] E_i = 0 \Rightarrow \left[1 - G_{op}(z)\right] E_o = 0$

Solutions: $\begin{cases} E_o = 0 & \text{or} \\ 1 - G_{op} = 0 & \text{(in which case } E_o \text{ can be arbitrarily large)} \end{cases}$

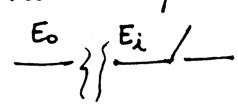
This is a general result: $\boxed{1 - G_{op}(z) = 0 \text{ is the characteristic equation}}$

<u>Recap</u>: If a TF does not exist, we can still find the char. equation as follows:

(1) Open the system in front of a sampler 

(2) Set all inputs to zero

(3) find the TF between the points where the loop was opened

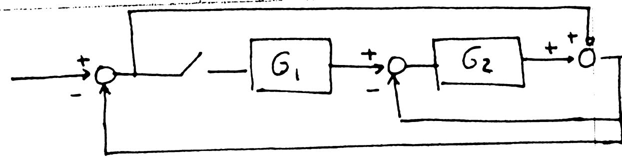


$$G_{op} = \frac{E_o(z)}{E_i(z)}$$

(4) Char equation: $1 - G_{op} = 0$

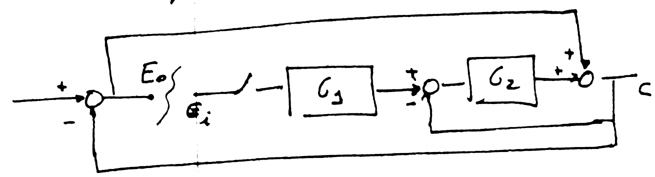
(5) "poles": roots of Char equation

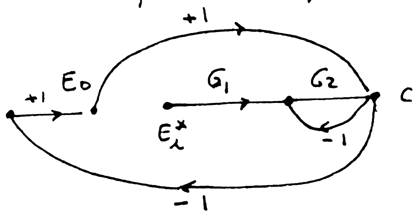
Example 2:



We saw that this system does not have a TF $\Rightarrow$ can't find the characteristic equation by finding the poles of $G_{ce}(z)$

$\Rightarrow$ (1) set $R = 0$
(2) open the loop in front of the sampler



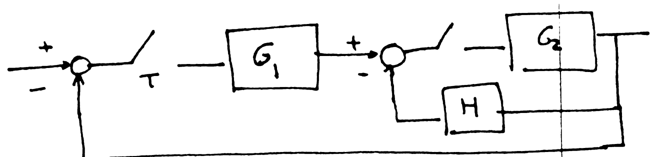


$$\Delta = 1 + G_2 + 1 = 2 + G_2$$

$$\frac{E_o}{E_i^*} = -\frac{G_1 G_2}{2 + G_2} \Rightarrow \frac{E_o^*}{E_i^*} = -\left[\frac{G_1 G_2}{2 + G_2}\right]^*$$

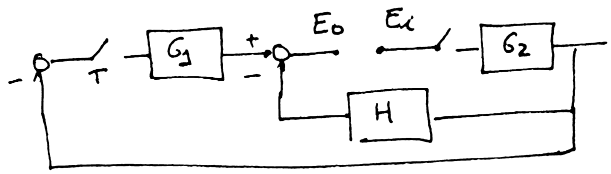
$$\Rightarrow G_{op} = -\mathcal{Z}\left[\frac{G_1 G_2}{2+G_2}\right] \Rightarrow \text{Char equation: } 1 + \mathcal{Z}\left[\frac{G_1 G_2}{2+G_2}\right] = 0$$

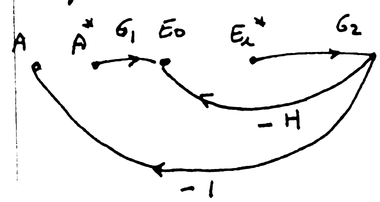
Example 3:



Here we have <u>two</u> samplers: Q: which one do we open
A: it doesn't matter (you get the same result)

The textbook opens the first one, so let's open the second:





$$E_o = G_1 A^* - H G_2 E_i^*$$
$$A = -G_2 E_i^* \Rightarrow A^* = -G_2^* E_i^*$$

$$E_o = -G_1 G_2^* E_i^* - (H G_2) E_i^*$$
$$E_o^* = -\left(G_1^* G_2^* + (H G_2)^*\right) E_i^*$$

$$\Rightarrow G_{op} = -\left[G_1(z) G_2(z) + \mathcal{Z}[H G_2]\right]$$

$\Rightarrow$ Char equation: $1 + G_1(z) G_2(z) + \mathcal{Z}[H G_2] = 0$

(same as in the book)