- **Disturbance Rejection:** (section 5.4)

In any control system, in addition to the input used to control the plant we may have other inputs that also influence the plant and that are beyond our control. These inputs are called "disturbances"

Examples: turbulence, wind gusts, sensor noise

Example of a control system: Keck astronomical telescope, (Mauna Kea, Hawaii)

Objective of a telescope: collect and focus starlight using a large concave mirror (the larger the mirror the better, since we can observe fainter stars)

The diameter of the mirror on the Keck telescope is 10 meter (~33 ft)

However: to make such a mirror from a single piece of glass is both extremely difficult (imperfections) and costly

Solution: Make the mirror using smaller mirrors and a control system to keep them focused and with the proper shape

(Keck's mirror has 36 hexagonal small mirrors)

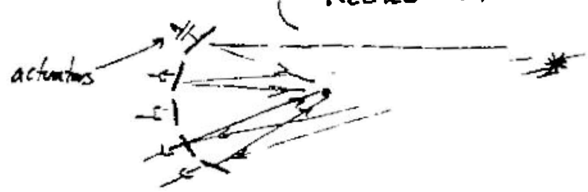
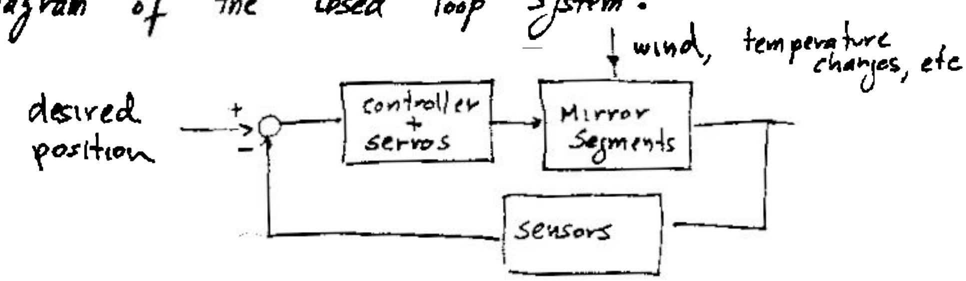
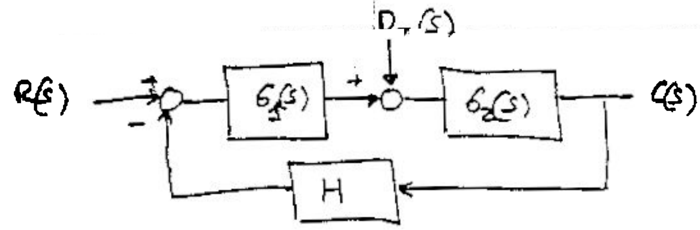


Diagram of the closed loop system:



The telescope is subject to disturbances due to wind gusts, temperature changes, etc

Suppose that we model the servo as a simple first order system and the telescope as a second order. Then we have something like:

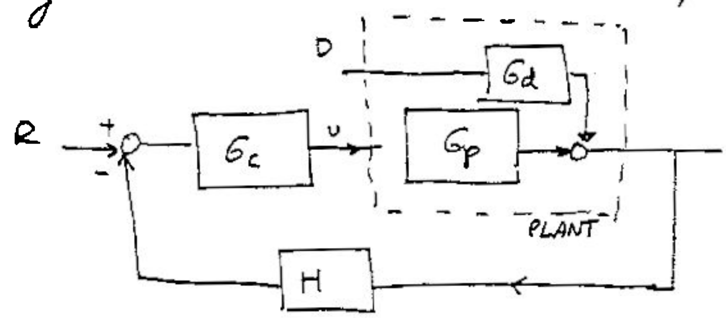


$G_1 = \frac{k}{(R+Ls)}$, $G_2 = \frac{1}{(Js^2+Bs)}$

Since the system is linear we can use superposition to find the output due to both the command $R(s)$ and the disturbance $D_T(s)$

$$C(s) = \frac{G_1(s)\,G_2(s)}{1+G_1(s)G_2(s)H}\,R(s) + \frac{G_2}{1+G_1G_2H}\,D_T(s) = \underbrace{T(s)}_{\text{command T.F.}}R(s) + \underbrace{T_D(s)}_{\text{disturbance transf. function}}\,D_T(s)$$

- In general we will have a situation of the form:



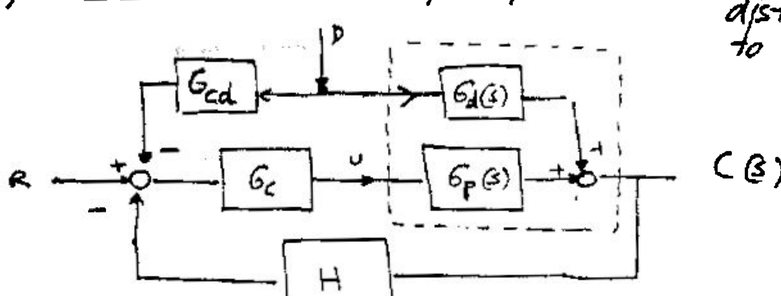
If we want to reject the disturbance $\Rightarrow$ we want T_D to be "small".

To achieve this (assuming that you can't change G_p, G_d)

1) You may try to increase G_c

For the example $T_d = \frac{G_2}{1+G_1G_2H}$ $\Rightarrow$ increase G_1 by increasing K

2) Use disturbance feedforward: If you can measure the disturbance you may try to cancel its effects:

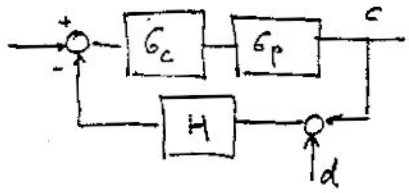


T_{CR} remains unchanged

$$T_D = \frac{G_d - G_{cd}\,G_c\,G_p}{1+G_cG_pH} \Rightarrow T_d \text{ small if } G_d - G_{cd}\,G_c\,G_p \cong 0$$

$\Rightarrow$ Try to make $G_d(j\omega) - G_{cd}(j\omega)\, G_c(j\omega)\, G_p(j\omega) \sim 0$ in the interval of interest

- Assume that we have the following system:



(sensor noise)

d = sensor noise

$$\frac{c}{R} = \frac{G_c G_p}{1 + G_c G_p H}$$

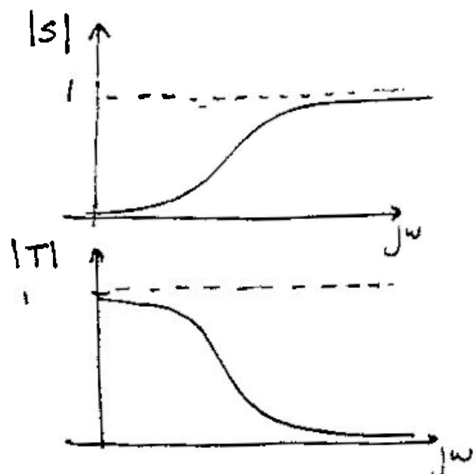
$$S = S^{T}_{G_p} = \frac{1}{1 + G_c G_p H} \quad \text{(sensitivity)}$$

$$\frac{c}{d} = \frac{G_c G_p H}{1 + G_c G_p H} = T \quad \text{(complementary sensitivity)}$$

Want: S small
T small

but $S + T = 1$!!

<u>Solution</u>: make S small, T large at low frequency (operating range)
S large, T small at high frequency



Consider the case where we want to minimize $M_1 = \max\limits_{\omega_1 \le \omega \le \omega_2} |S(j\omega)|$

(i.e.: want to minimize the worst case sensitivity)
$\Rightarrow$ good sensitivity if $M_1 \ll 1$

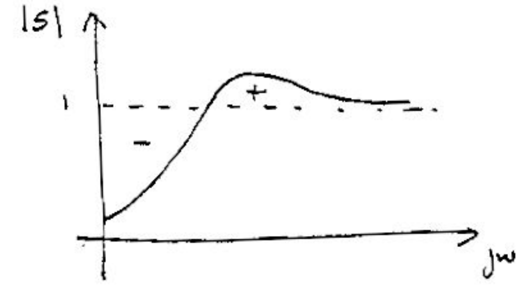
On the other hand, we want $M_2 = \max\limits_{\omega} |S(j\omega)|$ not too large

$\Rightarrow$ We'd like to solve the following problem:

minimize M_1 subject to M_2 not too large.

However, some plants exhibit the "water-bed" effect:

if $|S|$ is pushed-down in some frequency range, it pops-up somewhere else

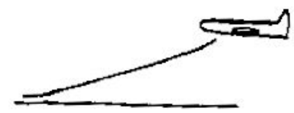


((the total area under $|S|$ has to remain constant)

- Steady state Accuracy (s.s)

It has to do with the steady-state error, i.e. the difference between the input & the output in steady-state

(The amount of steady-state error that we can tolerate depends on the application.

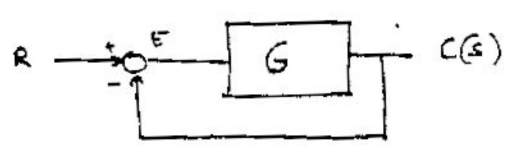


For an aircraft landing control, we want very small or zero steady-state error. For a home temperature control we can tolerate a few degrees)

- Implicit assumptions

1) System is closed-loop stable

2) Unity feedback (i.e. want output to track the input)



$G(s) = G_c(s) \cdot G_p(s)$

controller

plant

$$C(s) = \frac{G(s)}{1+G(s)} R(s) := \frac{P(s)}{Q(s)} = \frac{P_1(s)}{s^N Q_1(s)}$$

where neither P_1 nor Q_1 have zeros at $s=0$ (i.e. $P_1(0) \neq 0$, $Q_1(0) \neq 0$)

Example:

$G(s) = \dfrac{s-1}{s(s+1)} \Rightarrow N = 1$

$G(s) = \dfrac{s-1}{s^2+1} \Rightarrow N = 0$

Since $\frac{1}{s}$ is the T.F. of an integrator, N is the number of "free"

N is called the "system type"

Let's now look at the error: $E(s) = R(s) - C(s)$
$e(t) = r(t) - c(t)$

Steady-state value of the error:

$$e_{ss} = \lim_{t \to \infty} e(t) \underset{\substack{\uparrow \\ \text{Final value theorem}}}{=} \lim_{s \to 0} s E(s)$$

provided that $e(t)$ has a final value (i.e. closed-loop system must be stable)

$G_{ce} = \dfrac{G}{1+G}$ $\qquad C(s) = \dfrac{G}{1+G} R(s)$

$E(s) = \dfrac{1}{1+G} R(s)$

$$\boxed{e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} \frac{s R(s)}{1 + G(s)}}$$

- We are going to consider now the steady state error to 3 particular inputs: i) step; ii) ramp; iii) parabolic input

Recall that: (a) these are the most commonly used inputs
(b) any other input can be approximated by these three through the use of a Taylor expansion

$$r(t) = \underset{\text{step}}{r(0)} + \underset{\text{ramp}}{\left.\frac{dr}{dt}\right|_{t=0}} t + \frac{1}{2}\underset{\text{parabolic}}{\left.\left(\frac{d^2 r}{dt^2}\right)\right|_{t=0}} t^2 + \cdots$$

- Step input: $R(s) = \frac{1}{s} \Rightarrow e_{ss} = \lim_{s \to 0} \dfrac{s \cdot 1/s}{1 + G(s)} = \dfrac{1}{1 + \lim_{s \to 0} G(s)}$

Definition: $K_p = \lim_{s \to 0} G(s)$ is called the position error constant

(Reason for this name: when the system being controlled is mechanical, we can interpret the step as a change in a desired reference position)

$$e_{ss} = \frac{1}{1 + K_p}$$

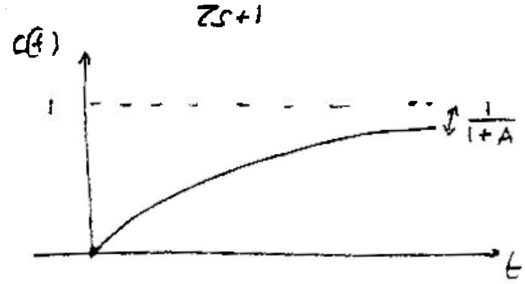
Now we can explore different situations based on the type of the system:

1) <u>Type zero systems</u>: $N=0 \rightarrow G(s) = \frac{P(s)}{Q(s)}$, $\quad P(0) \neq 0$, $G(0) \neq 0$

$\Rightarrow G(0) = k_p$ is a <u>finite</u> number and

$e_{ss} = \frac{1}{1+k_p}$ also a finite, non-zero, number

<u>Example</u>: $G(s) = \frac{A}{\tau s+1}$ $\qquad G(0) = A = k_p \Rightarrow e_{ss} = \frac{1}{1+A}$ #



Note that if we want to find e_{ss} the "old" way, then we have to proceed as follows:

$$E(s) = \frac{1}{1+G(s)} R(s) = \frac{1}{1+\frac{A}{1+\tau s}} R(s) = \frac{\tau s+1}{\tau s+A+1} \frac{1}{s}$$

$$= \left(\frac{1}{1+A}\right) \cdot \frac{1}{s} + \frac{A}{(1+A)} \cdot \frac{1}{s+\left(\frac{1+A}{\tau}\right)}$$

$$e(t) = \frac{1}{1+A} u(t) + \frac{A}{(1+A)} e^{-\left(\frac{1+A}{\tau}\right)}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \frac{1}{1+A} \qquad \text{(provided that } A > -1\text{)}$$

$\Rightarrow$ lots of work, even for a simple first-order system !!

2) Type 1 or higher systems:

if $N \geq 1$, $\quad G(s) = \frac{P_1(s)}{s^N Q_1(s)} \Rightarrow k_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{1}{s^N} \frac{\overbrace{P_1(0)}^{\neq 0}}{Q_1(0)} = \infty$

$$\Rightarrow \boxed{e_{ss} = \frac{1}{1+\infty} = 0}$$

- If a system is type 1 or higher it has <u>zero</u> steady state error to a step input.

- <u>Ramp input</u>: $r(t) = t\,u(t) \Rightarrow R(s) = \frac{1}{s^2}$

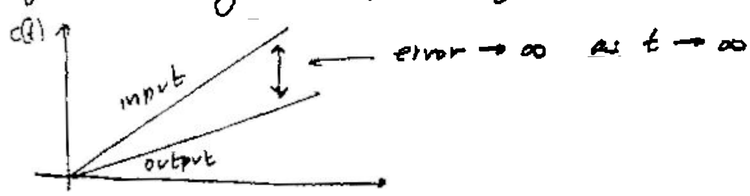
$$e_{ss} = \lim_{s\to 0} \cancel{s}\cdot\frac{1}{1+G(s)}\cdot\frac{1}{s^{\cancel{2}}} = \lim_{s\to 0}\frac{1}{\cancel{s}+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$$

Definition: $K_v = \lim_{s\to 0} sG(s) =$ <u>velocity error constant</u>

(Again the name harks back to mechanical systems where a ramp can be interpreted as a change in the desired velocity of the output)

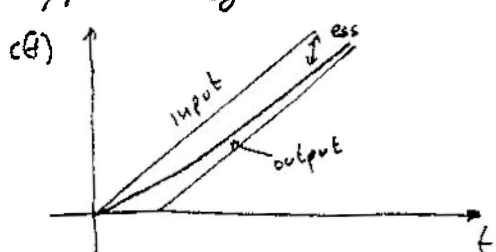
<u>Type 0 system:</u> $G(0)$ finite $\Rightarrow K_v = \lim_{s\to 0} sG(s) = 0 \Rightarrow e_{ss} = \frac{1}{K_v} = \infty$!

Physical meaning: a type 0 system <u>can't</u> track a ramp input



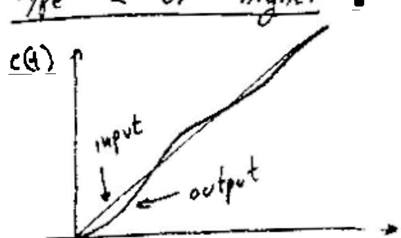
Type 0

<u>Type 1 system:</u> $K_v = \lim_{s\to 0}\frac{\cancel{s}P_1(s)}{\cancel{s}\varphi_1(s)} = \frac{P_1(0)}{\varphi_1(0)} =$ finite $\Rightarrow e_{ss} = \frac{1}{K_v}$ <u>finite</u>



Type 1

<u>Type 2 or higher</u>: $K_v = \lim_{s\to 0}\frac{sP_1(s)}{s^2\varphi_1(s)} = \infty \Rightarrow e_{ss} = 0$



Type 2

- <u>Parabolic input</u>: $r(t) = \frac{1}{2}t^2 u(t) \Rightarrow R(s) = \frac{1}{s^3}$

$$e_{ss} = \lim_{s\to 0} s\,\frac{1}{[1+G(s)]}\cdot\frac{1}{s^3} = \lim_{s\to 0}\frac{1}{s^2G(s)} = \frac{1}{K_a}$$

Where: $K_a = \lim_{s \to 0} s^2 G(s)$ is called the <u>acceleration error constant</u>

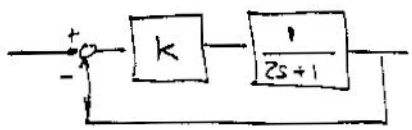
and the same type of reasoning applies, i.e:

if the system type is: 0 then $K_a = 0$ and $e_{ss} = \infty$
1 $K_a = 0$ $e_{ss} = \infty$
2 K_a finite $e_{ss} = 1/K_a$
≥ 3 $K_a = \infty$ $e_{ss} = 0$

Remember: $r(t) = r(0) + \left.\frac{dr}{dt}\right|_0 t + \frac{1}{2}\left(\frac{d^2 r}{dt^2}\right) t^2 +$

Trade-off: the higher the type the more accurate is the steady-state
<u>however</u>, it is more difficult to control

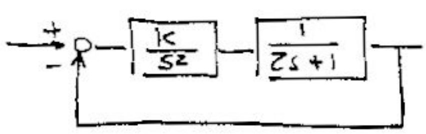
Example: A position control system:



Type 0: $\Rightarrow K_p = K$, $e_{ss}(\text{step}) = \frac{1}{1+K}$

$K_v = K_a = 0$ $e_{ss}(\text{ramp}) = e_{ss}(\text{parabolic}) = \infty$

So we decide to make it type 2 in order to get 0 steady state to a <u>step</u> and a ramp:



$K_p = \infty \Rightarrow e_{step} = 0$
$K_v = \infty$ $e_{ramp} = 0$
$K_a = K$ $e_{parabola} = 1/K$

WRONG! $T_{ER} = \frac{1}{1+G} = \frac{1}{1 + \frac{K}{s^2(\tau s+1)}} = \frac{s^2(\tau s+1)}{\tau s^3 + s^2 + K}$

and it can be shown (more on this in about 1 week) that

$\tau s^3 + s^2 + K = 0$ has always one root in RHP regardless of K
$\Rightarrow$ <u>system unstable</u> (no steady state values)

Remember: Always need to check stability before checking steady state values.

- <u>General property</u>: A system of type N can follow, without steady-state error an input of the form $\frac{A}{s^k}$, with $k \le N$
and with a <u>finite</u> (non-zero) error an input $\frac{A}{s^{N+1}}$

(For $k > N+1$ we get ∞ steady state error)

- <u>Summary</u>:

N \ $R(s)$	$1/s$	$\frac{1}{s^2}$	$\frac{1}{s^3}$
0	$\frac{1}{1+k_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

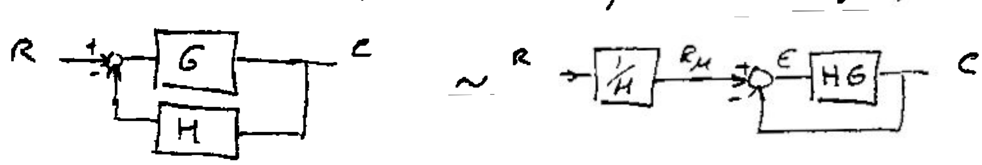
where
$$k_p = \lim_{s \to 0} G(s)$$
$$K_v = \lim_{s \to 0} s\,G(s)$$
$$K_a = \lim_{s \to 0} s^2 G(s)$$

(Note: in general, increasing the gain <u>reduces</u> steady-state errors)

- <u>Non-unity</u> feedback systems:

In this case the output and the input do <u>not</u> necessarily have the same units. (Recall the HW problem with input in Volts & output in RPMs)

$\Rightarrow$ need to transform to an <u>equivalent</u> unity feedback system



Hence, for non-unity feedback systems the type is determined by <u>GH</u>

We'd like to have: $e_{d_{ss}} = \lim_{t\to\infty} e_d(t) = 0 \Leftarrow \lim_{s\to 0} \dfrac{s\, G_d(s)}{1+G_c(s)G_p(s)} D(s) = 0$

Assume that the disturbance can be modelled as a step (examples are: constant wind pushing against an antenna or constant offset in a sensor)

Then $D(s) = \dfrac{D}{s}$

$$e_{d\,ss} = \lim_{s\to 0} \frac{\cancel{s}\, G_d(s)}{1+G_c(s)G_p(s)} \frac{D}{\cancel{s}}$$

We need to consider now the system type for G_c, G_d, G_p

If G_c, G_d, G_p Type 0 $\Rightarrow e_{d_{ss}} = \dfrac{G_d(0)}{1+G_c(0)G_p(0)} D$

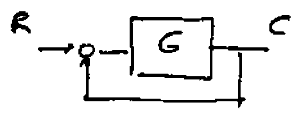
If $G_c(s)$ Type 1 $\Rightarrow \boxed{e_{d_{ss}} = 0}$

- <u>Transient Response</u> (5.6)

So far we have considered stability and steady-state characteristics (i.e. what happens as $t\to\infty$). However, in many cases we need to look also at the way that we approach steady-state (to make sure that we do not have unacceptable characteristics: overshoot, settling time, etc)

Recall the airplane:

It is not enough to guarantee zero steady state error. Here we need to make sure that we do not have any overshoot.



Let $T_{ce} = \dfrac{G}{1+GH} = \dfrac{P(s)}{\varphi(s)} = \dfrac{P(s)}{(s-p_0)(s-p_1)\cdots(s-p_n)}$

Then, for an input $r(t)$ we have:

$$C(s) = T_{ce}\, R(s) = \frac{P(s)}{\prod_{i=0}^{m-1}(s-p_i)} R(s) \overset{\text{partial fraction expansion}}{=} \underbrace{\frac{k_0}{(s-p_0)} + \frac{k_1}{(s-p_1)} + \cdots \frac{k_{n-1}}{(s-p_n)}}_{C_n(s)} + C_r(s)$$

$C_n(s)$: natural or transient response

$C_r(s)$: terms that originate in $R(s)$

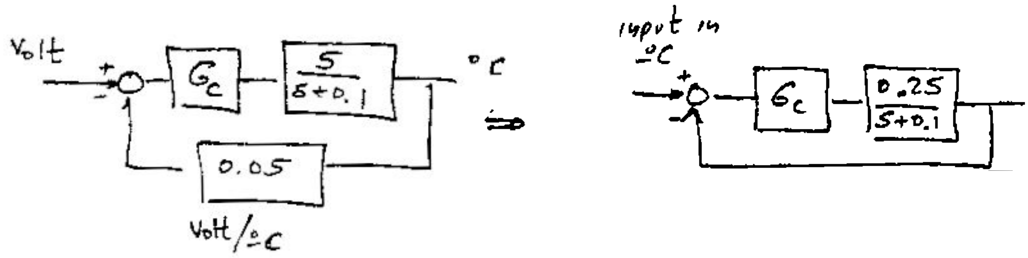
and the error coefficients are given by:

$$K_p = \lim_{s\to 0} GH$$

$$K_v = \lim_{s\to 0} s\,GH$$

$$K_a = \lim_{s\to 0} s^2 GH$$

Example:



if $G_c(s) = k \Rightarrow kGH = \dfrac{0.25k}{s+0.1} \Rightarrow$ Type 0 system

$K_p = \lim_{s\to 0} kGH = 2.5k \Rightarrow e_{ss}^{step} = \dfrac{1}{1+K_p} = \dfrac{1}{1+2.5k}$ (for a unit step)

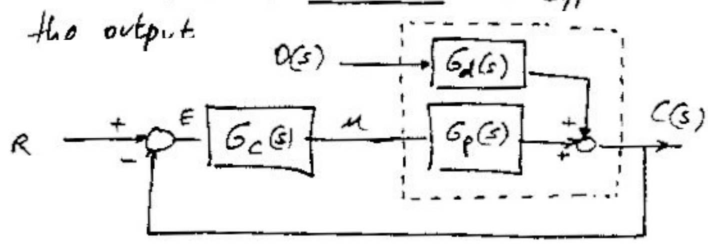
If we use a PI (proportional + integral) controller we get

$G_c = k_p + \dfrac{k_I}{s} \Rightarrow GH = \left[\dfrac{sk_p + k_I}{s}\right]\dfrac{0.25}{(s+0.1)} \Rightarrow$ system is now type 1

$\Rightarrow e_{ss} = 0$

- **Disturbance Input Errors:**

The idea is to minimize the effect that the disturbance has upon the output.



$C(s) = T(s)R(s) + T_D(s)D(s) = C_r(s) + C_d(s)$

output due to disturbance

(superposition)

output due to command

$E(s) = R(s) - C(s) = \underbrace{R(s) - C_R(s)}_{E_R(s)} - \underbrace{C_d(s)}_{E_D(s)}$

$e(t) = e_r(t) + e_d(t)$

error in the command

error due to disturbance