Appendix B
Design Equations

This appendix presents a general design method that applies to both root-locus design and frequency-response design. It is shown that the same design equations can be used with either design technique. First a general development is given; then the development is shown to apply to either root-locus design or to frequency-response design.

DEVELOPMENT

Consider the control system shown in Figure B.1, in which $G_c(s)$ is the compensator and $G_p(s)$ is the plant. We assume that the compensator has the first-order transfer function

$$G_c(s) = \frac{a_1 s + a_0}{b_1 s + 1} \tag{B-1}$$

Hence a_0 is the dc gain, $-a_0/a_1$ is the zero, and $-1/b_1$ is the pole for the compensator.

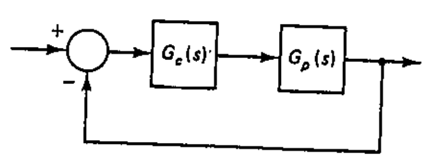

Figure B.1

In this development we find the values of a_1 and b_1 that, for a given value of s_1, satisfy the equation

$$G_c(s_1)G_p(s_1) = \alpha e^{j\gamma} \tag{B-2}$$

where α and γ may have any value but are given, and the compensator dc gain a_0 in (B-1) is also given. It is shown in this appendix that the solution of (B-2) is useful in designing control systems by both the root-locus procedure and the frequency-response procedure.

The basic design procedure is now developed. Equation (B-2) may be written as

$$a_0 + a_1 s_1 = \frac{\alpha e^{j\gamma}}{G_p(s_1)}(1 + b_1 s_1) \tag{B-3}$$

Since, in general, s_1 can be complex, we can express it in both rectangular form and polar form.

$$s_1 = \sigma_1 + j\omega_1 = |s_1|e^{j\beta} \tag{B-4}$$

where

$$|s_1| = [\sigma_1^2 + \omega_1^2]^{1/2}, \qquad \beta = \tan^{-1}\left(\frac{\omega_1}{\sigma_1}\right)$$

In a like manner, $G_p(s_1)$ can be written in polar form as

$$G_p(s_1) = |G_p(s_1)|e^{j\psi}, \qquad \psi = \angle G_p(s_1)$$

Thus the reciprocal of the ratio of functions in (B-3) can be expressed as

$$\frac{G_p(s_1)}{\alpha e^{j\gamma}} = \frac{|G_p(s_1)|e^{j\psi}}{\alpha e^{j\gamma}} = Me^{-j\theta} \tag{B-5}$$

where

$$M = \frac{|G_p(s_1)|}{\alpha}, \qquad \theta = \gamma - \psi$$

With the preceding definitions (B-3) can be expressed as

$$a_0 + a_1|s_1|(\cos\beta + j\sin\beta) = \frac{1}{M}(\cos\theta + j\sin\theta) + \frac{b_1|s_1|}{M}[\cos(\theta+\beta) + j\sin(\theta+\beta)] \tag{B-6}$$

Equating real to real and imaginary to imaginary in this equation results in the following equations:

$$\begin{bmatrix} |s_1|\cos\beta - \dfrac{|s_1|}{M}\cos(\theta+\beta) \\ |s_1|\sin\beta - \dfrac{|s_1|}{M}\sin(\theta+\beta) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\cos\theta}{M} - a_0 \\ \dfrac{\sin\theta}{M} \end{bmatrix} \tag{B-7}$$

We will now solve (B-7) for b_1. By Cramer's rule,

$$b_1 = \frac{\begin{vmatrix} |s_1| \cos \beta & \dfrac{\cos \theta}{M} - a_0 \\ |s_1| \sin \beta & \dfrac{\sin \theta}{M} \end{vmatrix}}{\begin{vmatrix} |s_1| \cos \beta & -\dfrac{|s_1|}{M} \cos (\theta + \beta) \\ |s_1| \sin \beta & -\dfrac{|s_1|}{M} \sin (\theta + \beta) \end{vmatrix}} = \frac{N}{D}$$

First the numerator, denoted as N, will be evaluated.

$$N = \frac{|s_1|}{M} [\sin \theta \cos \beta - \cos \theta \sin \beta + M a_0 \sin \beta]$$

Using the trigonometric identity,

$$\sin a \cos b - \cos a \sin b = \sin (a - b)$$

the numerator becomes

$$N = \frac{|s_1|}{M} [M a_0 \sin \beta + \sin (\theta - \beta)]$$

The denominator, D, can be expressed as

$$D = \frac{|s_1|^2}{M} [-\sin (\theta + \beta) \cos \beta + \sin \beta \cos (\theta + \beta)]$$

Using the trigonometric identity,

$$\sin a \cos b = \frac{1}{2} [\sin (a + b) + \sin (a - b)]$$

the denominator term becomes

$$\begin{aligned} \text{denominator} &= \frac{|s_1|^2}{2M} [-\sin (\theta + 2\beta) - \sin \theta + \sin (\theta + 2\beta) + \sin (-\theta)] \\ &= \frac{|s_1|^2}{2M} (-2 \sin \theta) = -\frac{|s_1|^2}{M} \sin \theta \end{aligned}$$

Thus,

$$b_1 = \frac{N}{D} = \frac{M a_0 \sin \beta + \sin (\theta - \beta)}{-|s_1| \sin \theta}$$

The coefficient a_1 can be found from (B-7) in the same manner, resulting in the two equations

$$\begin{aligned} a_1 &= \frac{\sin \beta - a_0 M \sin (\theta + \beta)}{M |s_1| \sin \theta} \\ b_1 &= \frac{M a_0 \sin \beta + \sin (\theta - \beta)}{-|s_1| \sin \theta} \end{aligned} \qquad \text{(B-8)}$$

We refer to these as the design equations, and we now show that they may be applied in the design of control systems.

ROOT-LOCUS DESIGN

For the system of Figure B.1, the characteristic equation is given by

$$1 + G_c(s)G_p(s) = 0 \tag{B-9}$$

Hence the root locus for this system is obtained by allowing the gain of $G_p(s)$ to vary and plotting the roots of (B-9).

Suppose that it is desired that the root locus pass through the point $s = s_1$ for a given gain. Then, for that gain, from (B-9),

$$G_c(s_1)G_p(s_1) = -1 \tag{B-10}$$

Comparing (B-10) with (B-2) and (B-5), we see that for

$$\begin{aligned} \alpha &= 1 \\ \gamma &= 180° \\ \theta &= \gamma - \psi = 180° - \underline{/G_p(s_1)} \end{aligned} \tag{B-11}$$

the design equations (B-8) become

$$\begin{aligned} a_1 &= \frac{\sin\beta + a_0|G_p(s_1)|\sin(\beta - \psi)}{|s_1||G_p(s_1)|\sin\psi} \\ b_1 &= \frac{\sin(\beta + \psi) + a_0|G_p(s_1)|\sin\beta}{-|s_1|\sin\psi} \end{aligned} \tag{B-12}$$

Given the compensator dc gain a_0 and the plant transfer function $G_p(s)$, the coefficients a_1 and b_1 in (B-12) result in the closed-loop transfer function of the system having a pole at $s = s_1$. The angle of $G_p(s_1)$, ψ, cannot be 0° or 180°, or a_1 and b_1 become unbounded. For the case that ψ is 180°, the equations of (B-7) are dependent and may be written as

$$a_1|s_1|\cos\beta - \frac{b_1|s_1|}{M}\cos\beta = \frac{1}{M} - a_0 \tag{B-13}$$

For this case, b_1 may be assigned arbitrarily and (B-13) solved for a_1, or else some other design criteria may be used to form a second equation in a_1 and b_1.

FREQUENCY-RESPONSE DESIGN

For the system of Figure B.1, the Nyquist diagram is obtained by plotting $G_c(j\omega)G_p(j\omega)$ for $0 \leq \omega \leq \infty$. The system phase margin, ϕ_m, occurs at the frequency ω_1 for which $|G_c(j\omega_1)G_p(j\omega_1)| = 1$ and is defined by the equation

$$G_c(j\omega_1)G_p(j\omega_1) = e^{j(180° + \phi_m)} \tag{B-14}$$

Comparing (B-14) with (B-2), (B-4), and (B-5), we see that for

$$\alpha = 1$$
$$\gamma = 180° + \phi_m$$
$$\theta = \gamma - \psi = 180° + \phi_m - \underline{/G_p(j\omega_1)} \quad \text{(B-15)}$$
$$s_1 = |s_1|e^{j\beta} = \omega_1 e^{j90°}, \qquad |s_1| = \omega_1, \qquad \beta = 90°$$

the design equations (B-8) become

$$a_1 = \frac{1 - a_0|G_p(j\omega_1)|\cos\theta}{\omega_1|G_p(j\omega_1)|\sin\theta}$$
$$b_1 = \frac{\cos\theta - a_0|G_p(j\omega_1)|}{\omega_1 \sin\theta} \quad \text{(B-16)}$$

Given the compensator dc gain a_0 and the plant transfer function $G_p(s)$, the coefficients a_1 and b_1 in (B-16) result in the closed-loop system having a phase margin ϕ_m at the frequency ω_1, provided the resulting system is stable.

PID COMPENSATORS

The procedure just given can also be used to develop general design equations for the PID compensator with the transfer function

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \quad \text{(B-17)}$$

From (B-2) and (B-5) we let

$$G_c(s_1) = \left(\frac{1}{M}\right)e^{j\theta} \quad \text{(B-18)}$$

Or, for (B-17) and (B-18),

$$K_D s_1^2 + K_P s_1 + K_I = \frac{s_1 e^{j\theta}}{M}, \qquad s_1 = |s_1|e^{j\beta} \quad \text{(B-19)}$$

Then

$$K_D|s_1|^2(\cos 2\beta + j\sin 2\beta) + K_P|s_1|(\cos\beta + j\sin\beta) + K_I$$
$$= \frac{|s_1|}{M}[\cos(\beta + \theta) + j\sin(\beta + \theta)] \quad \text{(B-20)}$$

Equating real to real and imaginary to imaginary in this equation yields

$$\begin{bmatrix} |s_1|^2\cos 2\beta & |s_1|\cos\beta \\ |s_1|^2\sin 2\beta & |s_1|\sin\beta \end{bmatrix} \begin{bmatrix} K_D \\ K_P \end{bmatrix} = \begin{bmatrix} \dfrac{|s_1|}{M}\cos(\beta + \theta) - K_I \\ \dfrac{|s_1|}{M}\sin(\beta + \theta) \end{bmatrix} \quad \text{(B-21)}$$

Solving these equations for K_P and K_D as a function of K_I yields

$$
\begin{aligned}
K_P &= \frac{\sin(\beta-\theta)}{M\sin\beta} - \frac{2K_I\cos\beta}{|s_1|} \\
K_D &= \frac{\sin\theta}{M|s_1|\sin\beta} + \frac{K_I}{|s_1|^2}
\end{aligned}
\tag{B-22}
$$

For root-locus design, we use the values of (B-11) substituted into (B-22) to obtain

$$
\begin{aligned}
K_P &= \frac{-\sin(\beta+\psi)}{|G_p(s_1)|\sin\beta} - \frac{2K_I\cos\beta}{|s_1|} \\
K_D &= \frac{\sin\psi}{|s_1||G_p(s_1)|\sin\beta} + \frac{K_I}{|s_1|^2}
\end{aligned}
\tag{B-23}
$$

For frequency-response design, we use the values of (B-15) substituted into (B-22) to obtain

$$
\begin{aligned}
K_P &= \frac{\cos\theta}{|G_p(j\omega_1)|} \\
K_D &= \frac{\sin\theta}{\omega_1|G_p(j\omega_1)|} + \frac{K_I}{\omega_1^2}
\end{aligned}
\tag{B-24}
$$

For PID compensator design, one of the three gains K_P, K_I, or K_D, must be computed from other design specifications, and (B-23) or (B-24) yields two equations in the two remaining gains. For PI or PD compensator design, the appropriate gain in (B-23) or (B-24) is set to zero, which then gives two equations in the two remaining gains.

For the case that s_1 is real, (B-23) cannot be employed, since $\sin\beta$ is zero. For this case, (B-21) yields only one equation in K_P, K_I, and K_D, and two of these gains must be determined from other design specifications.