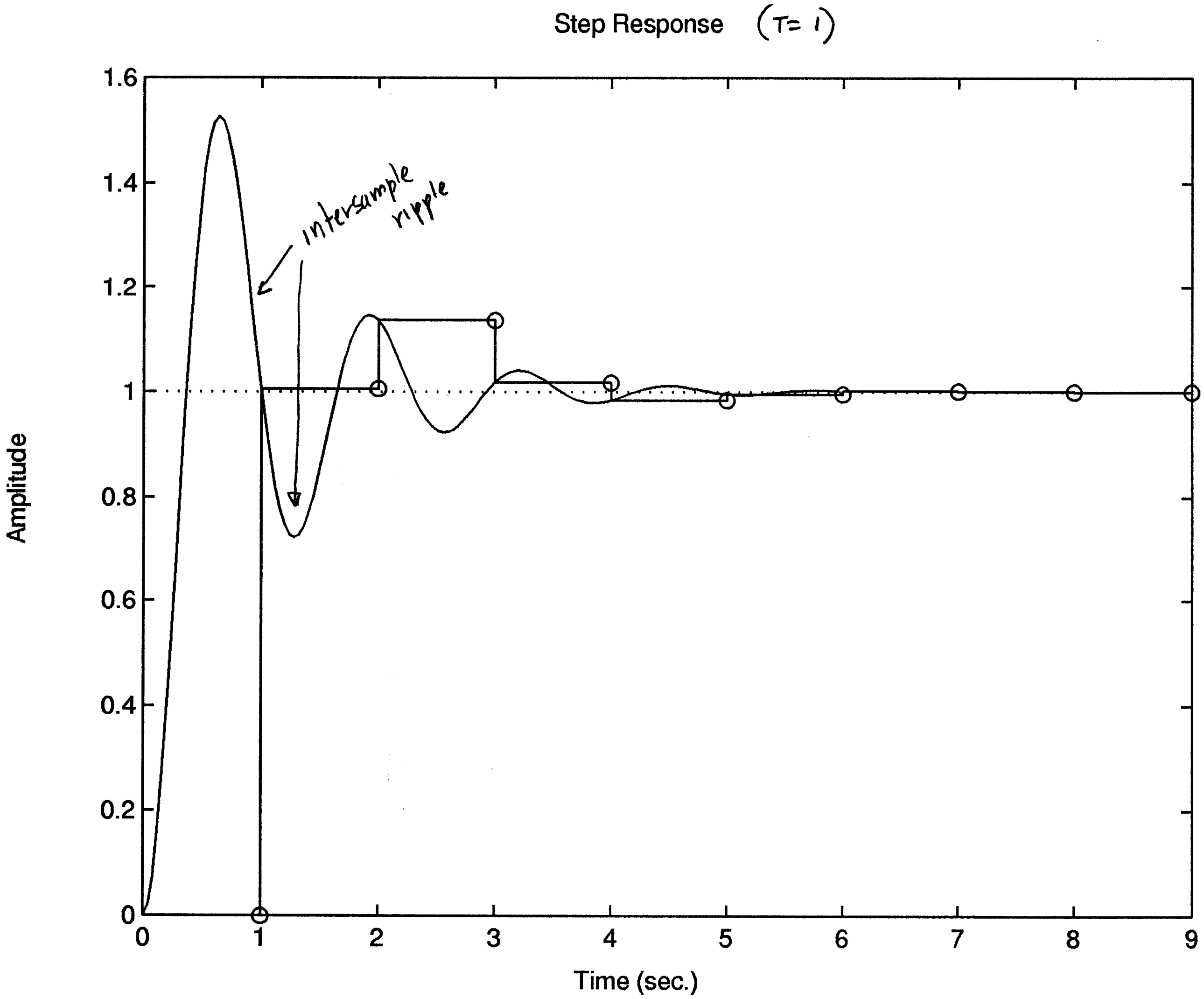
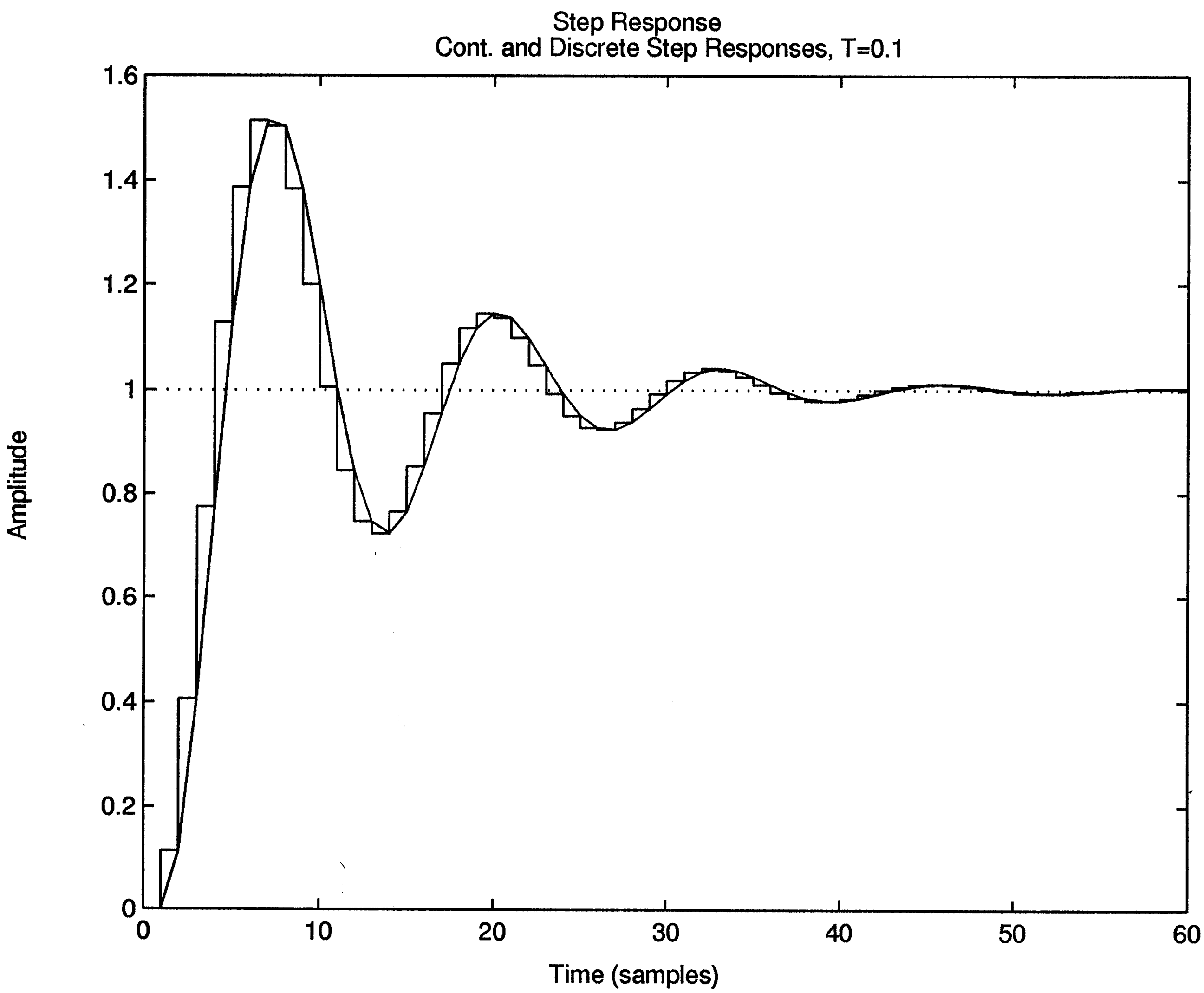


Step Response (T= 1)
intersample ripple
Amplitude
Time (sec.)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
1
2
3
4
5
6
7
8
9


Step Response
Cont. and Discrete Step Responses, T=0.1
Amplitude
Time (samples)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
10
20
30
40
50
60

Key assumptions: If we have more than one sampler then

(a) They work at the same rate ("single rate sampling")
(b) They are synchronized

In real applications these assumptions do not necessarily hold: we can have multirate / nonsynchronous sampling.

Q: How do we deal with these cases

A: We need a new tool: the "modified" z-transform.

- **Modified z transform**

So far we have been considering systems with integer number of sampling periods delays and synchronous sampling.
In order to analyze systems with non-integer delays and/or nonsynchronous sampling we need a new tool: the "modified" z-transform

The modified z-transform method is a modification of the regular z transform using fictitious time delays.

Consider a function $f(t)$ and delay it by an amount ΔT, $0 < \Delta \leq 1$

$$f(t) \xrightarrow{\text{delay}} f(t-\Delta T)\, u(t-\Delta T) \xrightarrow{\mathcal{L}} F(s)\, e^{-s\Delta T}$$

$$f(kT) \longrightarrow f(kT-\Delta T)$$

$$\Rightarrow \mathcal{Z}\{f(t-\Delta T)u(t-\Delta T)\} = \sum_{n=1}^{\infty} f(nT-\Delta T)\, z^{-n}$$ delayed z transform

$$F(z,\Delta) \triangleq \mathcal{Z}\left[f(t-\Delta T)\, u(t-\Delta T)\right] = \mathcal{Z}\left[F(s)\, e^{-s\Delta T}\right]$$

- We will define the modified z transform from the delayed z transform

Definition: Modified z-transform: $\boxed{F(z,m) = F(z,\Delta)\Big|_{\Delta=1-m}}$

$$\Rightarrow F(z,m) = \mathcal{Z}\left[F(s)\, e^{-s\Delta T}\Big|_{\Delta=1-m}\right] = f(mT)\cdot\frac{1}{z} + f[(1+m)T]\frac{1}{z^2} + \cdots +$$

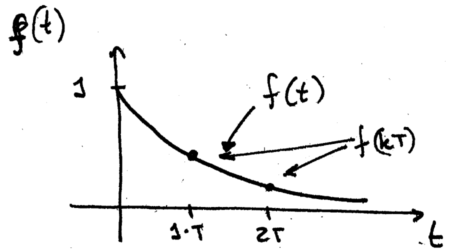
$$= \boxed{\sum_{i=0}^{\infty} f[(m+i)T]\, z^{-(i+1)}}$$

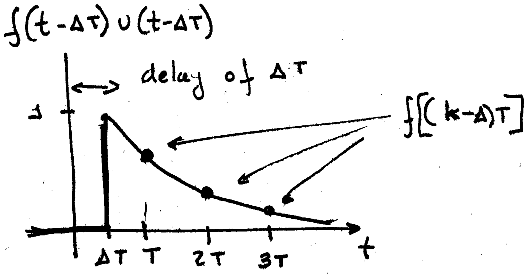
Example 1: Suppose that $f(t) = e^{-t} \Rightarrow f(kT) = e^{-kT}$

and that we want to find out the z transform of the signal delayed by ΔT

$$F(z,m) = \sum_{i=0}^{\infty} f\left[(m+i)T\right] z^{-(i+1)} = \sum_{0}^{\infty} e^{-(k+m)T} z^{-(k+1)} = \frac{e^{-mT}}{z} \cdot \frac{1}{1 - \frac{1}{z \cdot e^{T}}}$$

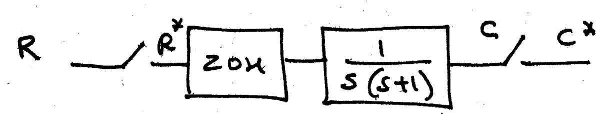
$$\boxed{F(z,m) = \frac{e^{-mT}}{z - e^{-T}}}$$



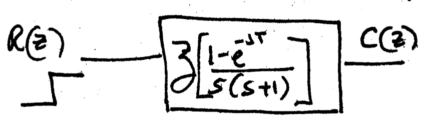


So we can use this technique to "peek" at the signal in between sampling instants (provided that we can handle that extra $\Delta \cdot T$ delay at the beginning)

Example 2:



We may want to look at $c(kT - \Delta T)$ to make sure that there is no intersample ripple. In this case we have



From previous lectures we know that $\mathcal{Z}\left[\frac{1-e^{-sT}}{s(s+1)}\right] = \frac{1-e^{-T}}{z-e^{-T}}$

and that $C(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right) R(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right)\frac{z}{z-1} = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$

$$c(kT) = 1 - e^{-kT}$$

What about the modified z transform?

$$G(s) = \frac{1-e^{-sT}}{s(s+1)} = \underbrace{\frac{1}{s(s+1)}}_{C_1(s)} - \underbrace{\frac{e^{-sT}}{s(s+1)}}_{C_2(s)} =$$

$\Rightarrow$ We can find out $C_1(z,m)$ proceeding as before.

Q: What about $C_2(z,m)$?

A: <u>Trouble</u>: We need to deal with a time delay e^{-sT}

- <u>Facts</u>:

(1) Tables of ordinary z transforms <u>do not</u> work for the modified z-transform (bummer!)

(2) <u>However</u> some of the properties are still valid. In particular, the <u>time-shift</u> theorem still valid, i.e.

$$\boxed{\mathcal{Z}_m\left[e^{-kTs}E(s)\right] = z^{-k}\mathcal{Z}_m\left[E(s)\right]}$$

Proof: $\mathcal{Z}_m\left[e^{-kTs}E(s)\right] = \mathcal{Z}\left[e^{-kTs}\cdot e^{-\Delta Ts}E(s)\right] = z^{-k}\mathcal{Z}\left[e^{-\Delta T\cdot s}E(s)\right] = z^{-k}\mathcal{Z}_m\left[E(s)\right]$

(3) <u>Additional properties</u>:

$$E(z,1) = E(z,m)\big|_{m=1} = e(1)z^{-1} + \dots \qquad = E(z) - e(0)$$
$$E(z,0) = E(z,m)\big|_{m=0} = e(0)z^{-1} + \dots \qquad = \frac{1}{z}E(z)$$

- <u>How to compute the modified z transform</u>

(Recall that usual tables do not apply)

$$E(z,m) = \mathcal{Z}\left[E(s)\,e^{-(1-m)Ts}\right] = \mathcal{Z}\left[E(s)\,e^{mTs}\cdot e^{-Ts}\right]$$

Let $E_1(s) = E(s)e^{mTs} \Rightarrow E(z,m) = \mathcal{Z}\left[e^{-Ts}E_1(s)\right] = z^{-1}\mathcal{Z}\left[E_1(s)\right]$

Now we can use the residue's formula for $\mathcal{Z}[E_1(s)] \Longrightarrow$

$$E(z,m) = z^{-1}\left[\sum_{\substack{\text{poles}\\ E_1(\lambda)}} \text{residues } E_1(\lambda)\frac{1}{1-\frac{e^{\lambda T}}{z}}\right] = z^{-1}\left[\sum_{\substack{\text{poles}\\ E(\lambda)}} \text{residues } E(\lambda)\,e^{mT\lambda}\frac{1}{1-\frac{e^{\lambda T}}{z}}\right]$$

(similarly: $E^*(s,m) = \frac{1}{T}\sum_{n=-\infty}^{n=+\infty} E(s+jn\omega_s)\,e^{-(1-m)(s+jn\omega_s)T}$

Example: modified z transform of $e(t) = e^{-t} \Leftrightarrow E(s) = \frac{1}{s+1}$

$$E(z,m) = \frac{1}{z} \sum_{\substack{\text{poles}\\ E(\lambda)}} \left\{ \text{res } E(\lambda) e^{mT\lambda} \frac{1}{1 - \frac{e^{\lambda T}}{z}} \right\} = \frac{1}{z} \frac{\cancel{(1+\lambda)}}{\cancel{(1+\lambda)}} e^{mT\lambda} \frac{1}{\left(1 - \frac{e^{\lambda T}}{z}\right)} \Bigg|_{\lambda = -1}$$

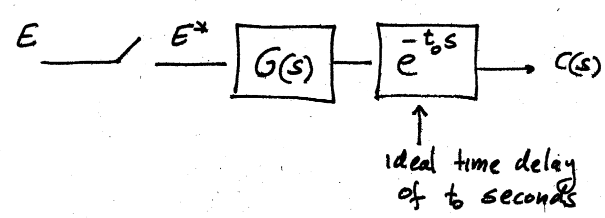
$$= \frac{1}{z} e^{-mT} \frac{1}{1 - \frac{e^{-T}}{z}} = \boxed{\frac{e^{-mT}}{z - e^{-T}}} \quad \# \quad \text{(same as before)}$$

Now we will see how to use the modified z transform to deal with
(a) systems with time delays
(b) non-synchronous sampling
(c) multirate sampling

- **Systems with Time-Delays**

Q: why do we care?

A: A common situation: computer controlled systems where the computation time is not negligible (w.r.t. the time constants of the plant)



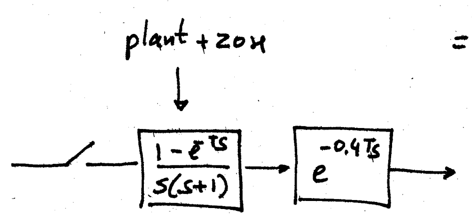
Let $t_o = kT + \Delta T \quad 0 < \Delta < 1 \Rightarrow$

$$C(s) = G(s)\, e^{-t_o s} E^*(s)$$

$$C(z) = \mathcal{Z}\left[G(s) e^{-t_o s}\right] E(z) = \mathcal{Z}\left[G(s) e^{-\Delta T s} e^{-kTs}\right] E(z)$$

$$= \boxed{z^{-k} G(z,m) E(z)} \quad \text{where } m = 1 - \Delta$$

Example:



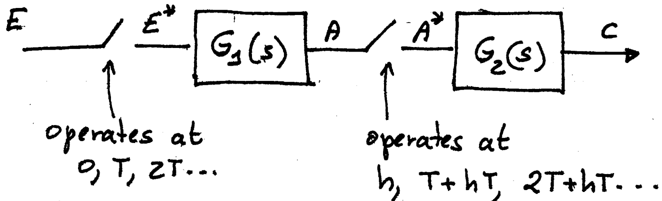
$$G(z,m) = \mathcal{Z}_m\left[\frac{1 - e^{-Ts}}{s(s+1)}\right] \underset{\uparrow \text{ shifting theorem}}{=} \left(1 - \frac{1}{z}\right) \mathcal{Z}_m\left[\frac{1}{s(s+1)}\right] \overset{\text{from tables}}{=} \left(\frac{z-1}{z}\right) \left[\frac{z(1 - e^{-mT}) + e^{-mT} - e^{-T}}{(z-1)(z - e^{-T})}\right]$$

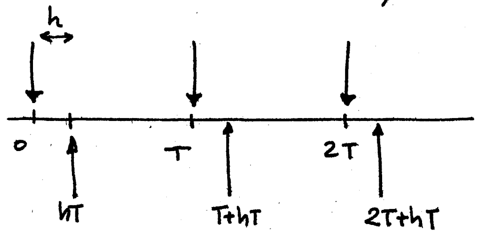
with $m = 1 - \Delta = 0.6$

$$G(z,m) = \frac{z-1}{z}\left[\frac{z(1 - e^{-0.6T}) + e^{-0.6T} - e^{-T}}{(z-1)(z - e^{-T})}\right]; \qquad C(z) = G(z,m) E(z) = \frac{z(1 - e^{-0.6T}) + e^{-0.6T} - e^{-T}}{(z-1)(z - e^{-T})}$$

- **Non synchronous sampling**

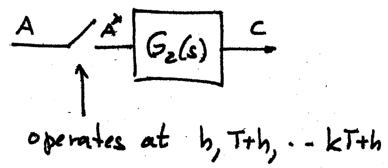
We consider now the case where we have several samplers, operating at the same rate, but they are not synchronized.



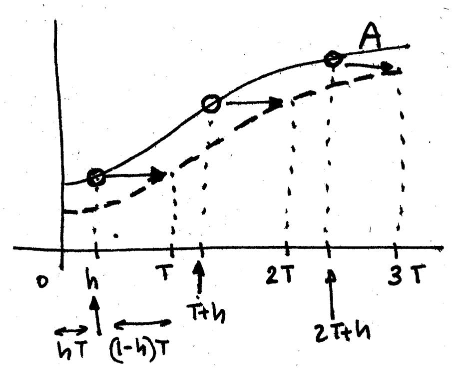


The idea is to try to convert the second sampler to something that operates at: 0, T, ...

Suppose we have:

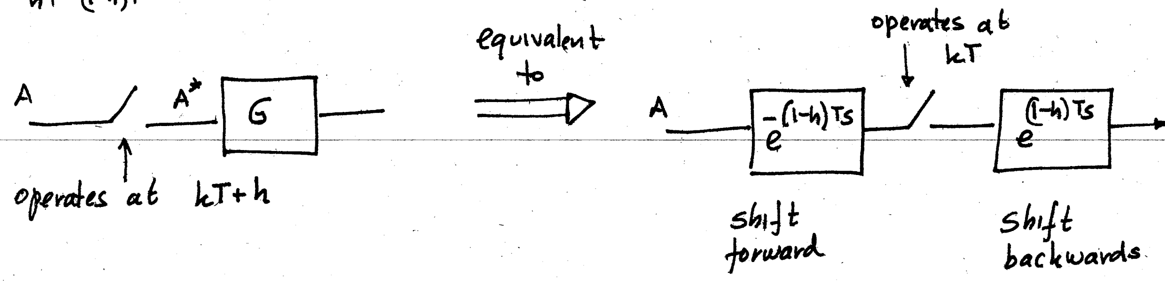


We want to convert it to something that operates at 0, T, 2T ...

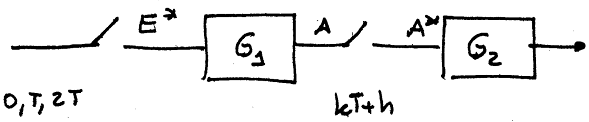


To get the same information, we could delay A by $(1-h)T$ and sample at kT

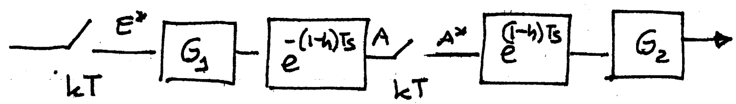
This way we get the information we want, but at the "wrong" time. We still need to "undo" the time shift to get the information at the right instants.



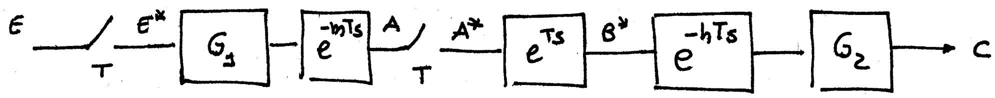
Now, if we have something like this:



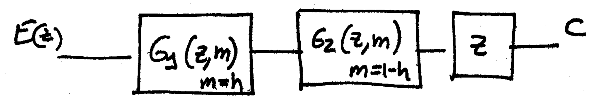
we can rewrite it as:



or:



or in the z-domain:



Proof:

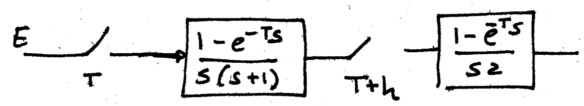
$$A = e^{-mTs} G_1 E^* \Rightarrow A^* = \left(e^{-mTs} G_1\right)^* E^* \Rightarrow A(z) = G_1(z,m)\big|_{m=h} E(z)$$

$$B = e^{sT} A^* \Rightarrow C = \left(G_2 e^{-hTs} B\right) = G_2 e^{-hTs} \cdot e^{sT} \cdot A^*$$

$$C^* = \left[G_2 e^{-hTs} e^{sT}\right]^* A^* = e^{sT} \left[G_2 e^{-hTs}\right]^* A^*$$

$$C(z) = z \cdot G_2(z,m)\big|_{m=1-h} \cdot A(z) = \boxed{z \cdot G_2(z,m)\big|_{m=1-h} \cdot G_1(z,m)\big|_{m=h} E(z)}$$

Example:



$T = 0.05$

$hT = 0.01$ (or $h = 0.2\,T$)

$$G_1(z,m) = \mathcal{Z}_m\left[\frac{1-e^{-Ts}}{s(s+1)}\right] = \left(\frac{z-1}{z}\right) \mathcal{Z}_m\left[\frac{1}{s(s+1)}\right] = \left(\frac{z-1}{z}\right)\left[\frac{z\left(1-e^{-mT}\right) + e^{-mT} - e^{-T}}{(z-1)\left(z-e^{-T}\right)}\right]$$

$mT = 0.01$

$$G_2(z,m) = \left(\frac{z-1}{z}\right) \mathcal{Z}_m\left(\frac{1}{s^2}\right)$$

Need $\mathcal{Z}_m\left(\frac{1}{s^2}\right)$:

$$\mathcal{Z}_m\left(\frac{1}{s^2}\right) = \mathcal{Z}\left[\frac{e^{-(1-m)Ts}}{s^2}\right] = \mathcal{Z}\left[\frac{e^{mTs}}{s^2} e^{-Ts}\right] = \frac{1}{z} \mathcal{Z}\left[\frac{e^{mTs}}{s^2}\right]$$

$$\mathcal{Z}\left[\frac{e^{mTs}}{s^2}\right] = \sum_{\text{poles of } \left(\frac{1}{s^2}\right)} \mathrm{Res}\left\{\frac{e^{mT\lambda}}{\lambda^2} \frac{1}{\left(1-\frac{e^{\lambda T}}{z}\right)}\right\} = \frac{d}{d\lambda}\left\{\frac{e^{mT\lambda}}{1-\frac{e^{\lambda T}}{z}}\right\}\Bigg|_{\lambda=0} =$$

$$= \frac{mT\left(1-\frac{1}{z}\right) - 1\cdot\left(-\frac{T}{z}\right)}{\left(1-\frac{1}{z}\right)^2} = z \cdot \frac{mTz - mT + T}{(z-1)^2} \Rightarrow$$

$$\mathcal{Z}_m\left(\frac{1}{s^2}\right) = \frac{mTz - mT + T}{(z-1)^2}$$

$$\left(\frac{z-1}{z}\right)\mathcal{Z}_m\left(\frac{1}{s^2}\right)\Big|_{m=1-h} = \frac{0.04z+0.01}{z(z-1)} \quad \Rightarrow$$

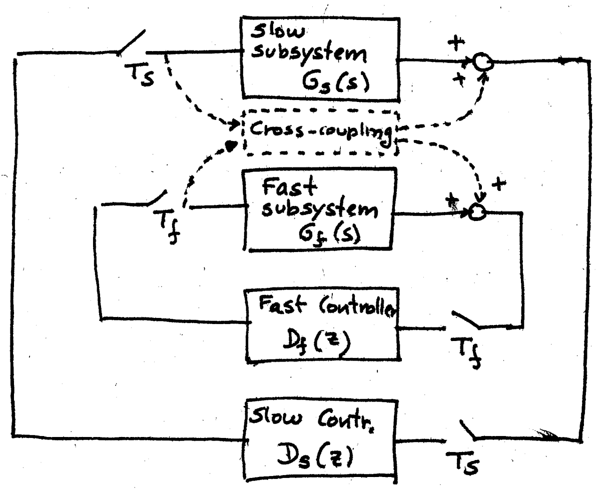
$$C(z) = \cancel{z}\cdot\left(\frac{\cancel{z-1}}{\cancel{z}}\right)\left[\frac{z\left(1-e^{-0.01}\right)+e^{-0.01}-e^{-0.05}}{(\cancel{z-1})\left(z-e^{-0.05}\right)}\right]\left(\frac{0.04z+0.01}{z(z-1)}\right)\frac{z}{(z-1)}$$

Approximation: $e^{-0.01} \sim 1-0.01$, $e^{-0.05} \sim 1-0.05$ $\Rightarrow e^{-0.01} - e^{-0.05} = 0.04$

$$\Rightarrow C(z) \sim \frac{\cancel{z}}{\cancel{z}(z-1)^2}\,\frac{(0.04z+0.01)(0.01z+0.04)}{\left(z-e^{-0.05}\right)}$$

- **Multirate Sampling**

When there are significant time constant differences in some natural modes or control loops, improvement in performance can be obtained by sampling at different rates.

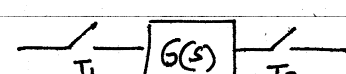


Typical case

One approach to dealing with multirate systems is to reduce it to an equivalent single-rate system operating at the largest sampling period (slowest)

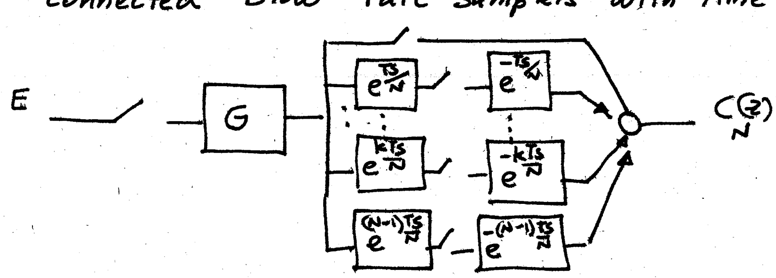
This is called the "switch decomposition" method

Consider for example the case:

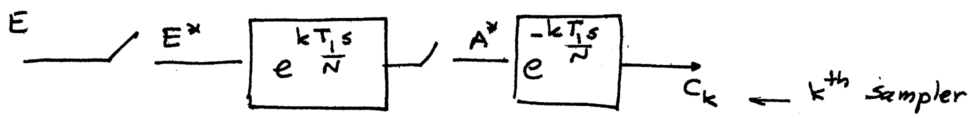


$T_1 > T_2$

this is called "slow-fast" multirate. Assume for convenience that $T_1 = NT_2$
$\Rightarrow$ The idea is to write the fast rate sampler as N parallel connected slow rate samplers with time delay and advance units



For each one of the fictitious samplers we have: $\left(T_2 = \frac{T_1}{N}\right)$



$$A = \left[e^{kT_2 s} G(s)\right] E^*(s) \qquad A^* = \left[e^{kT_2 s} G(s)\right]^* E^*(s)$$

since $e^{-\frac{kT_1}{N}s} = e^{-kT_2 s}$ is a pure delay, we have:

$$C_k^* = e^{-kT_2 s}\left[e^{kT_2 s} G(s)\right]^* E^*(s) = e^{-\frac{kT_1}{N}s}\left[e^{\frac{kT_1}{N}s} G(s)\right]^* E^*(s)$$

In the $\mathcal{Z}$ domain:

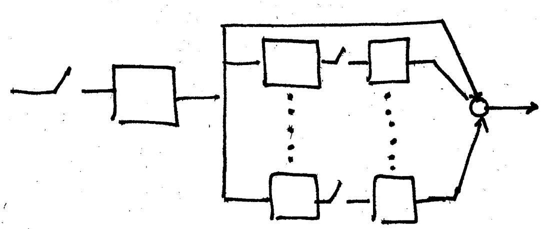
$$C_k(z) = C_k^*(s)\Big|_{z=e^{sT_1}} = z^{-\frac{k}{N}}\,\mathcal{Z}\left[e^{\frac{k}{N}T_1 s} G(s)\right] E(z) \Rightarrow$$

$$C_k(z) = z^{-\frac{k}{N}}\,\mathcal{Z}\left[e^{-(1-\frac{k}{N})T_1 s} \underset{\substack{\uparrow \\ \text{Time} \\ \text{advance}}}{e^{T_1 s}} G(s)\right] E(z) = z^{-\frac{k}{N}} \cdot z\,\mathcal{Z}\left[e^{-(1-\frac{k}{N})T_1 s} G(s)\right] E(z)$$

But $\mathcal{Z}\left[e^{-(1-\frac{k}{N})T_1 s} G(s)\right] = G(z,m)\Big|_{m=\frac{k}{N}}$ ("modified" z transf.)

$$\Rightarrow \boxed{C_k(z) = z^{1-\frac{k}{N}}\, G(z,m)\Big|_{m=\frac{k}{N}} E(z)}$$

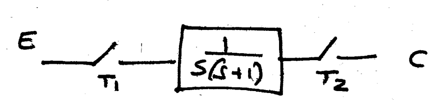
Collecting the outputs of the N samplers we get:



$$C_N = \sum_0^{N-1} C_k(z)$$

$$\boxed{C_N(z) = \left\{\sum_{k=0}^{N-1} z^{(1-\frac{k}{N})}\, G(z,m)\Big|_{m=\frac{k}{N}}\right\} E(z)}$$

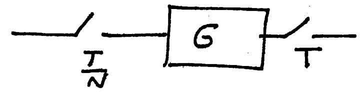
Example: $G(s) = \frac{1}{s(s+1)}$, $E(s) = \frac{1}{s}$, $T_1 = 3T_2$

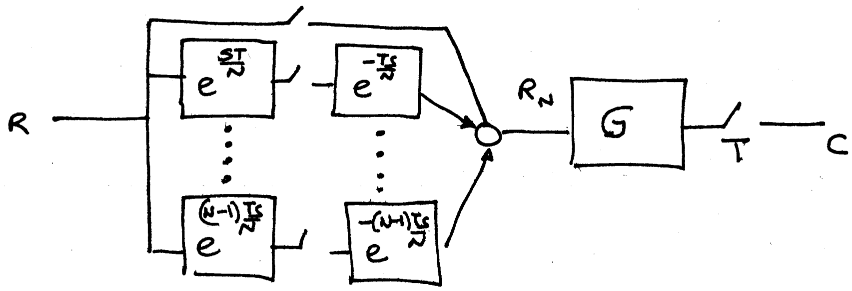


$$G(s) = \frac{1}{s(s+1)} \Rightarrow G(z,m) = \frac{1}{z-1} - \frac{e^{-m}}{z-e^{-1}}$$

$$C_3(z) = \left\{\sum_{k=0}^{2} z^{(1-\frac{k}{3})}\left[\frac{1}{z-1} - \frac{e^{-k/3}}{z-0.386}\right]\right\} \frac{z}{z-1}$$

- Dual case: Fast – slow system:

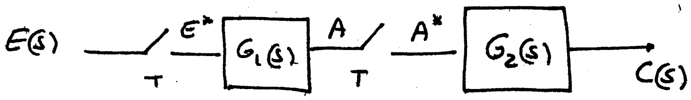




$$\boxed{C(z) = R(z)\,G(z) + \sum_{k=1}^{N-1} z\, R\left(z, \tfrac{k}{N}\right) G\left(z, 1-\tfrac{k}{N}\right)}$$

Summary of Chapter 4:

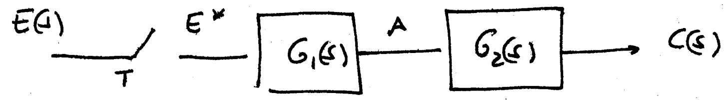
(a)



$$C(s) = G_2(s)A^* = G_2(s)G_1^*(s)E^*$$
$$C^*(s) = G_2^* G_1^* E^*$$

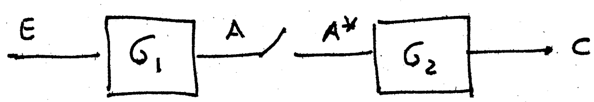
$$C(z) = G_2(z)\,G_1(z)\,E(z)$$

(b)



$$C^*(s) = \left[G_2(s)\,G_1(s)\right]^* E^*(s)$$
$$C(z) = \mathcal{Z}\left[G_1 \cdot G_2\right] E(z)$$

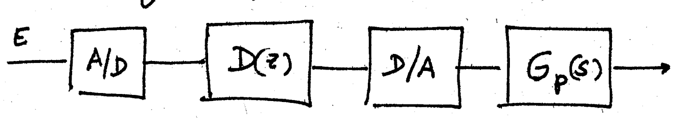
(c)



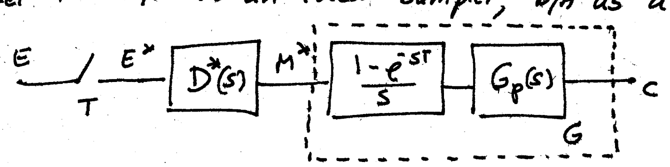
$$C(s) = G_2 A^* = G_2\left[G_1 E\right]^*$$
$$C^*(s) = G_2^*\left[G_1 E\right]^* \Rightarrow C(z) = G_2(z)\,\mathcal{Z}\left[G_1 \cdot E\right]$$

In this case a transfer function <u>does not</u> exist: $E(z)$ can't be factored out

(d) Systems with digital filters:



model the A/D as an ideal sampler, D/A as a data hold:



$$G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s} \cdot G_p(s)\right] = \frac{z-1}{z}\,\mathcal{Z}\left[\frac{G_p(s)}{s}\right]$$

$$C(z) = G(z) \cdot D(z) \cdot E(z)$$