• Control Systems Characteristics : (Chapter 5)

Recall the steps in designing a control system :

1) Study the system, decide sensors & actuators

2) Model the system

3) Analyze the model

you are here → 4) Decide on performance specifications

⋮

Performance Specs: General features that we require from a control system.

a) <u>Stability</u> (bounded input - bounded output stability)

We want the output to remain bounded at all times for every bounded input

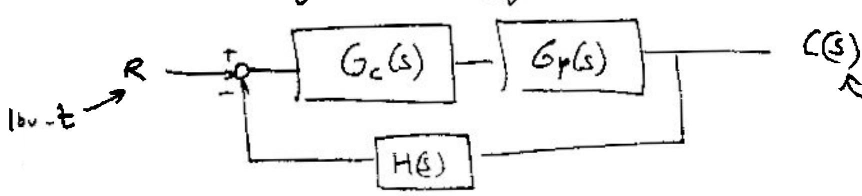
b) We want the closed loop system to be "insensitive" to changes in the plant (or to model uncertainty). This is related to the concepts of <u>robustness</u> and <u>sensitivity</u>

c) disturbance rejection

d) stedy - state accuracy

e) transient response

Recall the general configuration of a control system :



output variable (controlled variable)

Where : $G_c(s)$ is the controller
$G_p(s)$ is the plant
$H(s)$ are sensors

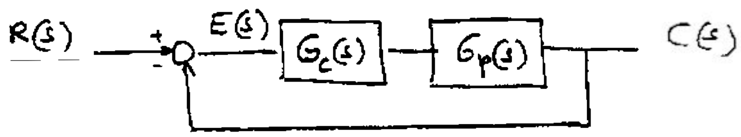
From Mason's formula we have: $T(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)\,G_p(s)}{1 + G_c(s)\,G_p(s)}$

Usually the sensors respond much faster than the plant $\Rightarrow$ $H(s)$ can be approximated by a pure gain (i.e. $H(s) = H$ constant)

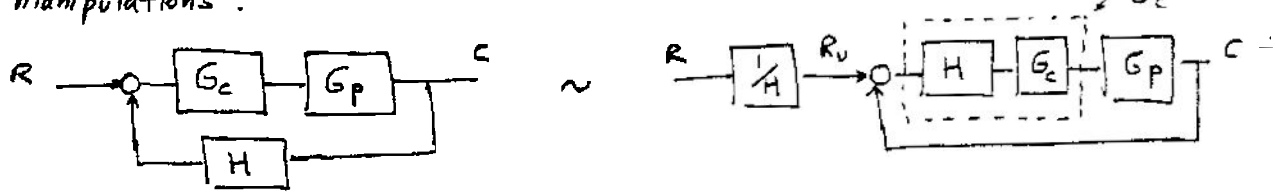
- Unity feedback system: The gain in the feedback path = 1

In this case we assume that the input $r(t)$ is also the desired value of the output (i.e. we want $c(t)$ to "track" $r(t)$)

In this case we define: $e(t) = r(t) - c(t)$ = system error



- What happens if our system does not have unity feedback? We can transform it to an "equivalent" system with unity feedback using block manipulations:



- Concept of Stability:

We want the output to remain bounded for any bounded input (BIBO stability)

There are many definitions of stability. Turns out that for LTI all of these are equivalent

Necessary and sufficient condition for stability:

$G(s) \overset{\mathcal{L}^{-1}}{\longleftrightarrow} g(t)$

$$Y(s) = G(s)\,U(s) \overset{\mathcal{L}^{-1}}{\longleftrightarrow} y(t) = \int_0^t g(\tau)\,u(t-\tau)\,d\tau$$

suppose that $u(t)$ is bounded $\Rightarrow \exists\, M \ni |u(t)| < M < \infty$ for all t

$$|y(t)| = \left|\int_0^t g(\tau)\,u(t-\tau)\,d\tau\right| \le \int_0^\infty |g(\tau)\cdot u(t-\tau)|\,d\tau = \int_0 |g(\tau)|\cdot\underbrace{|u(t-\tau)|}_{\le M}\,d\tau \le$$

$$M\int_0 |g(\tau)|\,d\tau \le M\int_0 |g(\tau)|\,d\tau \qquad (\text{since } |g(\tau)| \ge 0)$$

$\Rightarrow$ A <u>sufficient</u> condition for $|y(t)|$ to be bounded is that $\int_0^\infty |g(\tau)|d\tau$ be bounded.

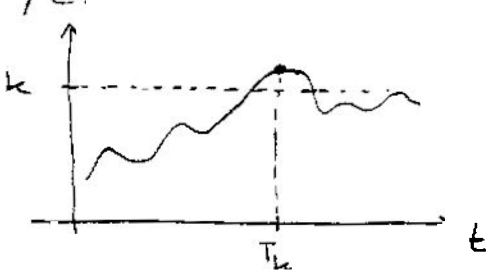
On the other hand, assume that $\int_0^\infty |g(\tau)|\,d\tau$ is <u>not</u> bounded. This means that for any number K, there exists T_k such that $\int_0^{T_k} |g(\tau)|\,d\tau > K$

Consider now the following input:

$$u(T_k-\tau) = \begin{cases} 1 & \text{if } g(\tau) > 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases} \quad \Rightarrow g(\tau)u(T_k-\tau) = |g(\tau)|$$

$$y(T_k) = \int_0^{T_k} g(\tau)u(T_k-\tau)\,d\tau = \int_0^{T_k} |g(\tau)|\,d\tau > K$$

This means that for <u>any</u> number k (no matter how large) we can find a <u>bounded</u> input ($|u| \le 1$) such that the output $y(t)$ exceeds k (at $t = T_k$) $\Rightarrow$ output not bounded $\Rightarrow$ system not BIBO stable



$\Rightarrow$ $\boxed{\text{The system is BIBO stable iff } \int_0^\infty |g(\tau)|\,d\tau \text{ is bounded}}$

- *Example*: $G(s) = \dfrac{1}{(s+1)(s-2)} = \dfrac{1}{3(s-2)} - \dfrac{1}{3(s+1)}$

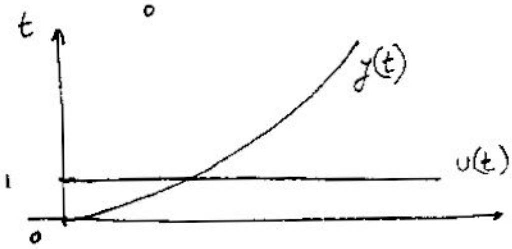
$$g(t) = \frac{1}{3}(e^{2t} - e^{-t}) \qquad |g(t)| = \frac{1}{3}(e^{2t} - e^{-t})$$

$$\int_0^\infty |g(t)|dt = \frac{1}{3}\left(\frac{1}{2}e^{2t} + e^{-t}\right)\Big|_0^\infty = \infty \Rightarrow \text{not BIBO stable}$$

What is the worst-case input here? Since $e^{2t} - e^{-t} > 0$,

if we take $u(t) = 1$, the output is given by

$$y(t) = \frac{1}{3}\int_0^t \left(e^{2t} - e^{-t}\right) dt \quad \Rightarrow y(t) \to \infty \quad \text{as } t \to \infty$$



- The problem with the condition we just found is that it may be hard to check. We'd like to have something simpler

- <u>Relationship between BIBO stability and the location of the poles</u>:

Suppose: $$G(s) = \frac{C(s)}{R(s)} = \frac{N(s)}{D(s)} = \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

assume for simplicity that all roots are simple. Then

$$G(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots \frac{k_n}{s-p_n} = \sum \frac{k_i}{s-p_i}$$

$$\overset{\mathcal{L}^{-1}}{\Longleftrightarrow} \quad g(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + = \sum k_i e^{p_i t}$$

$\Rightarrow \int_0^\infty |g(t)|\, dt$ bounded <u>iff</u> all real part $\{p_i\} < 0$ i.e. all poles <u>must</u> be in the left half of the s-plane

(if we have multiple roots then we have terms of the form $k_i \frac{t^\ell}{\ell!} e^{+p_i t}$

and the conclusion still stands)

$$\boxed{\text{System stable} \Longleftrightarrow \text{all poles to the left of } j\omega \text{ axis}}$$

Q: What happens if we have poles on the $j\omega$-axis?

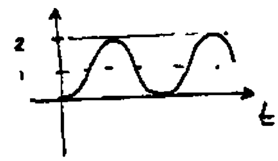
A: In this case the output is a <u>sustained oscillation</u> <u>unless</u> the input is a sinusoid with frequency equal to the $j\omega$ axis root, in which case the output becomes unbounded (resonance)

A system with poles on the $j\omega$-axis is called <u>marginally stable</u>

Example: $G(s) = \frac{1}{s^2+1}$ (marginally stable)

If the input is: $R(s) = \frac{1}{s}$ $\Rightarrow C(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$c(t) = u(t) - \cos(t)$



However, if the input is $R(t) = \cos t \Leftrightarrow R(s) = \frac{s}{s^2+1}$

$\Rightarrow C(s) = \frac{s}{(s^2+1)^2} \Rightarrow c(t) = \frac{1}{2} t \sin t$

- **Main point:** Stability determined by the roots of the characteristic equation (the poles of the system)

  $\Rightarrow$ To assess stability you don't need to find the complete T.F
  It is enough to find Δ using Mason's formula.

- **Sensitivity:** We want to find out how the change in some parameter (say mass, or gain) affects the T.F.

Suppose that we have a system (a T.F.) that depends on a parameter b. Then we can find how sensitive the system is to changes in b by considering the ratio:

$$S = \frac{\frac{\Delta T(s,b)}{T(s)}}{\frac{\Delta b}{b}} = \frac{\text{\% change in T.F}}{\text{\% change in } b} = \frac{\Delta T}{\Delta b} \cdot \frac{b}{T} \Rightarrow \Delta T = \frac{T}{b} \cdot$$

We define the sensitivity as the limit of this ratio when $\Delta b \to 0$, i.e.:

sensitivity of T

$$S_b^T = \lim_{\Delta b \to 0} \frac{\Delta T(s,b)}{\Delta b} \frac{b}{T(s,b)} = \boxed{\frac{\partial T}{\partial b} \cdot \frac{b}{T}}$$

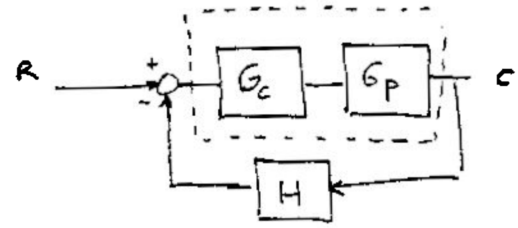
sensitivity with respect to b

In general S_b^T is a function of the complex frequency $s = \sigma + j\omega$

To plot S as a function of the frequency, we replace s by $j\omega$.
(This amounts to considering sinusoidal inputs, more on this latter)

The change $s \to j\omega$ allows us to talk about sensitivity at high or low frequencies

Example:



$$T_{CR} = \frac{G_c G_p}{1 + G_c G_p H}$$

Suppose that we want to find out the sensitivity w.r.t the plant G_p

$$S_{G_p}^{T} = \frac{\partial T}{\partial G_p} \frac{G_p}{T} = \frac{G_c(1 + G_c G_p H) - G_c G_p G_c H}{(1 + G_c G_p H)^2} \cdot \frac{G_p}{\frac{G_c G_p}{\cancel{1 + G_c G_p H}}} = \frac{1}{1 + G_c G_p H} \quad \#$$

$$S_{G_p}^{T}(j\omega) = \frac{1}{1 + G_c(j\omega) G_p(j\omega) H(j\omega)}$$

The product $G_c G_p H$ is called the loop gain (L) (recall graph)

The term $1 + G_c G_p H$ is called the return difference

(roughly speaking: difference between the signal we send in and the one that comes back)

Since $S_{G_p}^{T} = \frac{1}{1+L}$, to reduce the sensitivity we'd like the loop gain L to be high

Suppose now that we want to find the sensitivity w.r.t the sensors

$$S_H^T = \frac{\partial T_{CR}}{\partial H} \cdot \frac{H}{T_{CR}} = G_c G_p \frac{(-1) \cdot G_c G_p}{(1 + G_c G_p H)^{\cancel{2}}} \cdot \frac{H}{\frac{G_c G_p}{\cancel{(1 + G_c G_p H)}}} = -\frac{G_c G_p H}{1 + G_c G_p H}$$

$$\Rightarrow |S_H^T(j\omega)| = \left| \frac{G_c G_p H}{1 + G_c G_p H} \right| \Rightarrow$$ To get small sensitivity we need small L

(if we take $G_c G_p H = L$ large then $S_H^T \cong 1$)

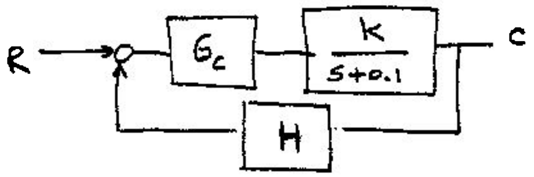
We have conflicting specifications.

Solution: Choose high quality stable components for your sensors (at least in the frequency range of interest), so ΔH remains

Note that in general you don't have control over the variations in the plant

- <u>Examples of sensitivity to a specific parameter</u>:

Suppose that we have a gain K that might change:



Let's find out the sensitivity with respect to k:

a) <u>open loop</u>: $T = \frac{K}{s+0.1}$ $\qquad S_k^T = \frac{\partial T}{\partial k}\cdot\frac{k}{T} = \frac{1}{(s+0.1)}\frac{k}{\frac{k}{s+0.1}} = 1$ #

(ie a variation ΔK translates into the same variation of the output)

b) <u>closed loop</u>: $T = \frac{G_c k}{s+0.1+G_c kH}$

$$S_k^T = \frac{\partial T}{\partial k}\cdot\frac{k}{T} = \frac{\partial T}{\partial G_p}\cdot\frac{\partial G_p}{\partial K}\cdot\frac{K}{T} \text{ (chain rule)} = \underbrace{\frac{\partial T}{\partial G_p}\cdot\frac{G_p}{T}}_{S_{G_p}^T}\ \underbrace{\frac{\partial G_p}{\partial k}\ \frac{k}{G_p}}_{S_k^{G_p}}$$

$$\Rightarrow S_k^T = \frac{1}{1+G_cG_pH} \qquad \underset{\substack{\| \\ 1}}{S_k^{G_p}} = \frac{1}{1+G_cG_pH} \ \#$$

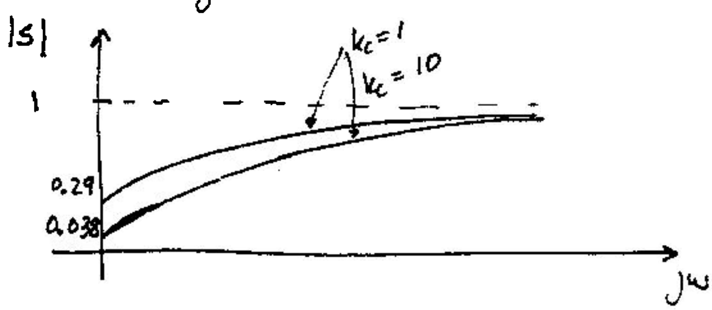
(Note that the <u>open loop</u> sensitivity is multiplied by $\frac{1}{1+L}$

$\Rightarrow$ large loop gain L reduces the sensitivity)

<u>Important point</u>: Feedback <u>reduces</u> sensitivity

Take for instance the following values: $K=5$, $H=0.05$, $G_c(s)=K_c$

$$\Rightarrow |S_k^T| = \left|\frac{0.1+j\omega}{j\omega+0.1+0.25K_c}\right|$$

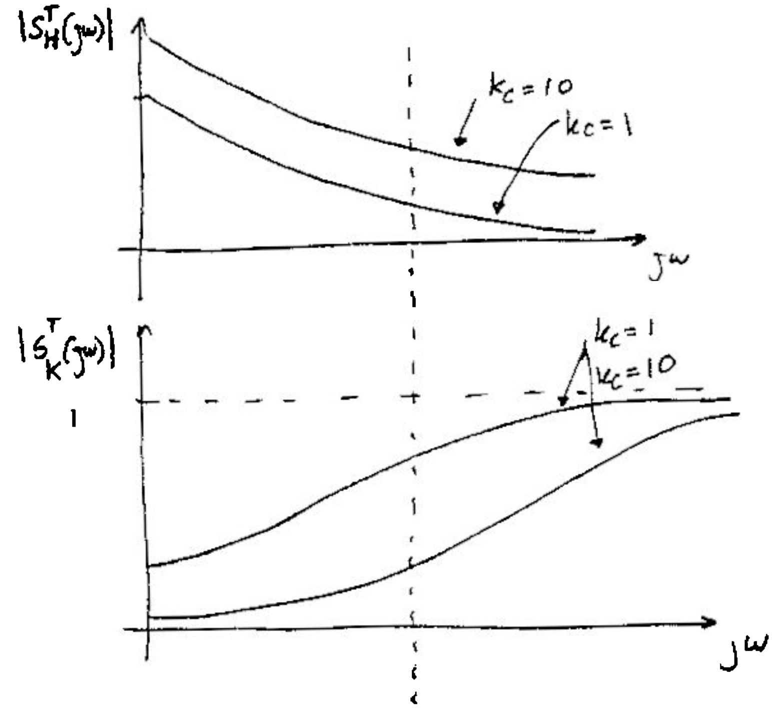


ZNAMER

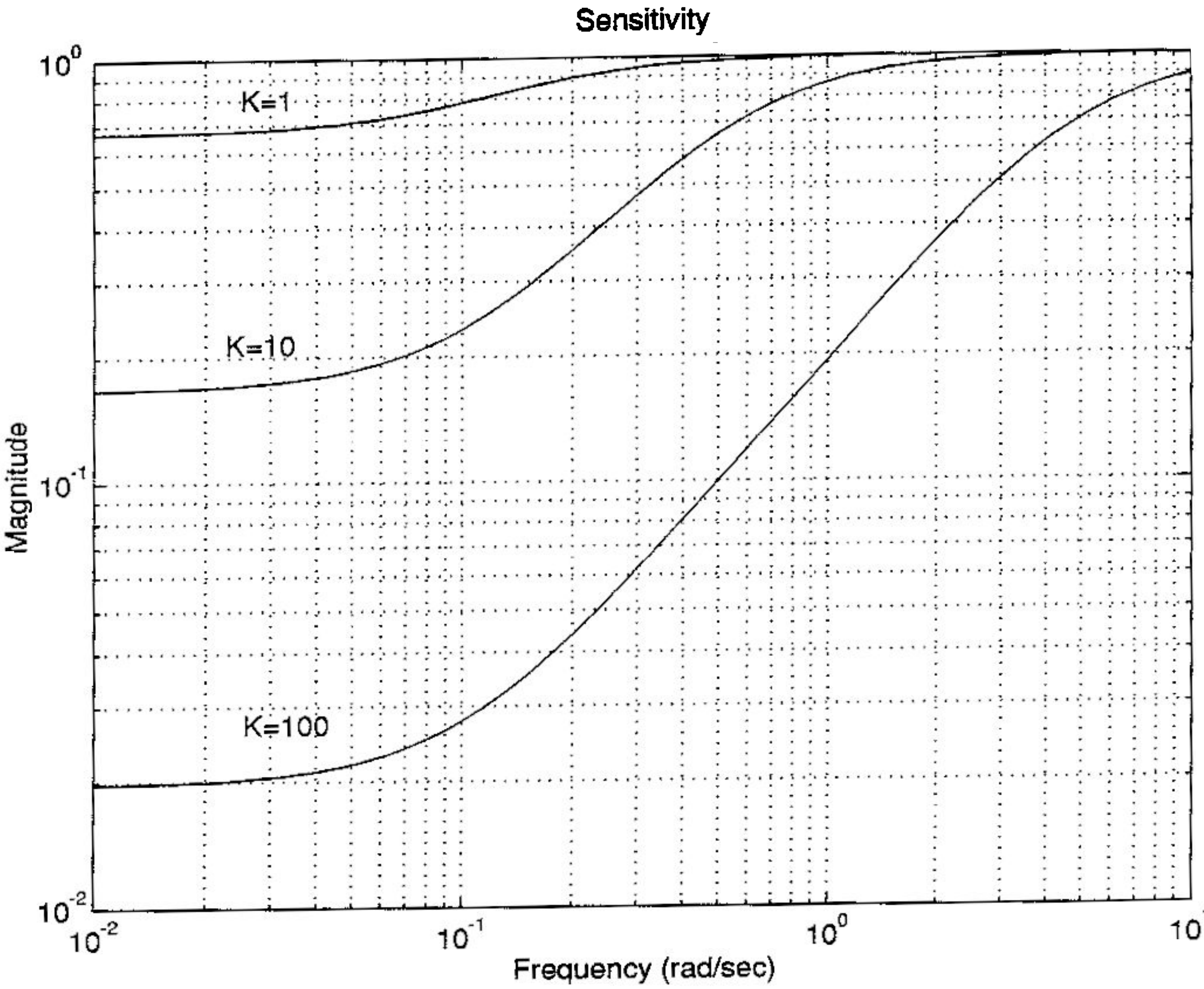
Similarly: $\quad S_H^T = \frac{\partial T}{\partial H}\frac{H}{T} = \dfrac{\dfrac{-G_c k}{(s+0.1)} \quad H}{1 + \dfrac{G_c k H}{s+0.1}} = \dfrac{-G_c k H}{s+0.1+G_c k H}$

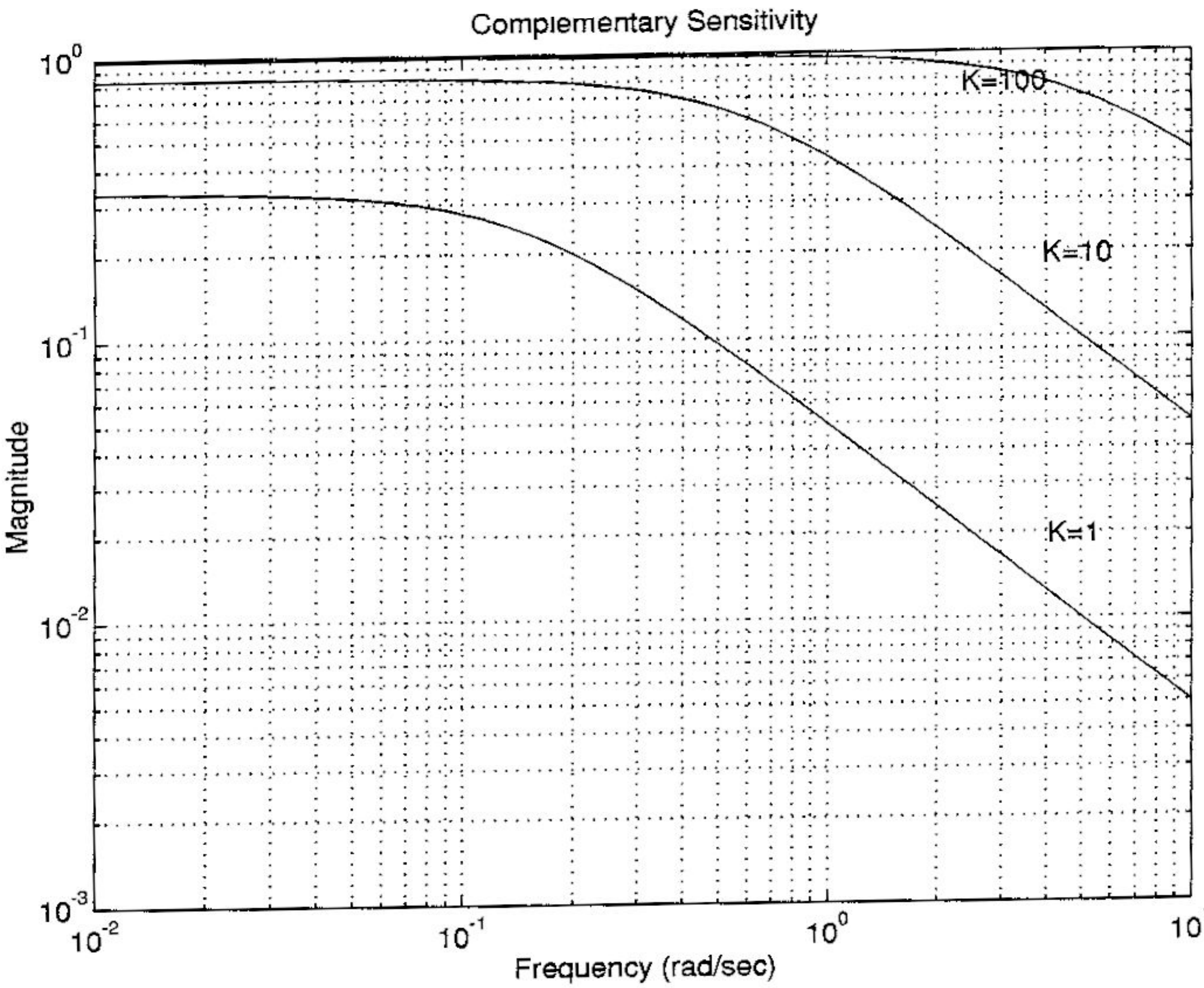
If we consider the same case as before we get:

$$\left| S_H^T(j\omega) \right| = \frac{0.25\ k_c}{|j\omega + 0.1 + 0.25 k_c|}$$

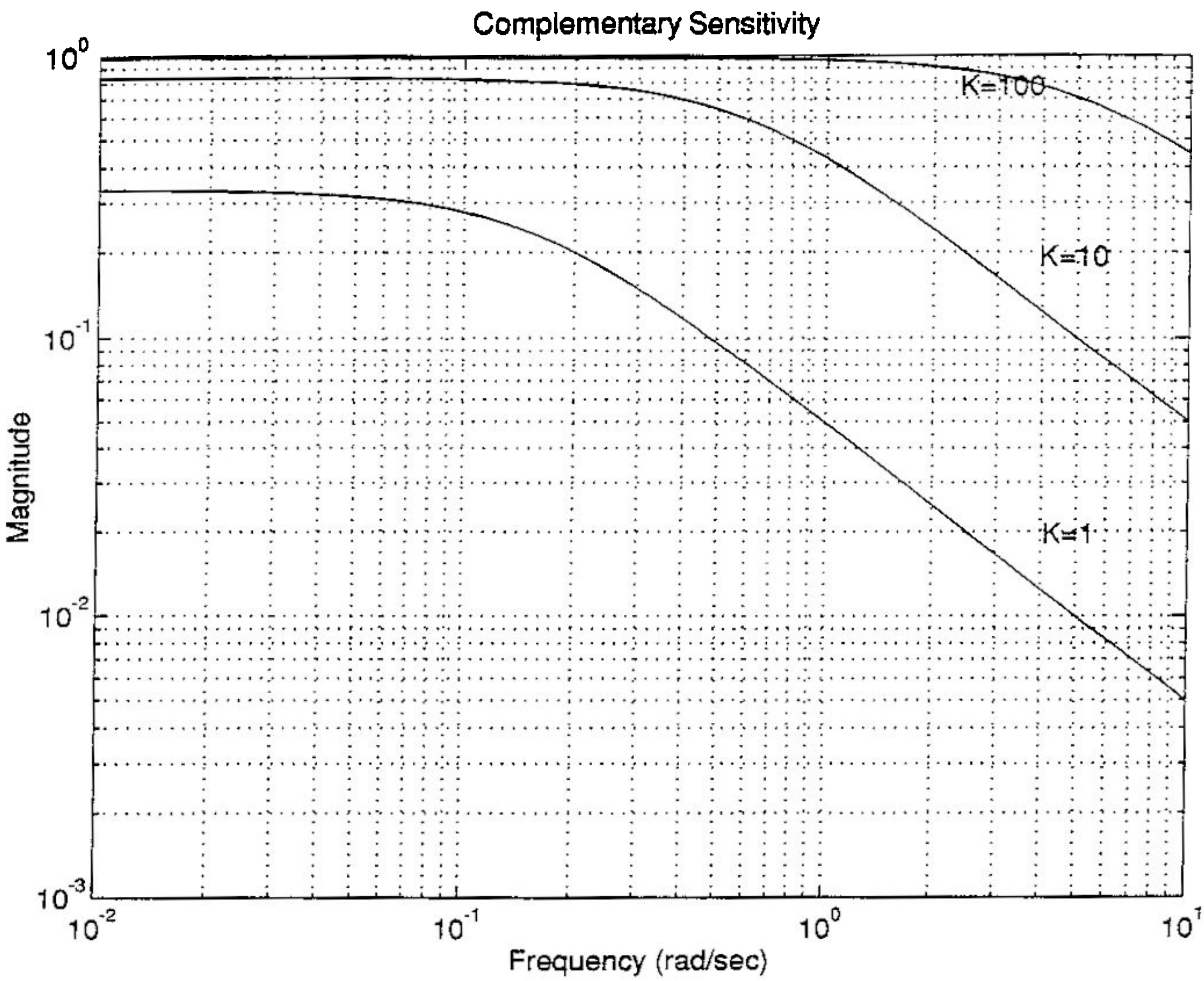


- To get good sensitivity characteristics, we need to design the controller such that S^o is low within the freq. interval of interest. (and make sure that our sensors are decent in that range)

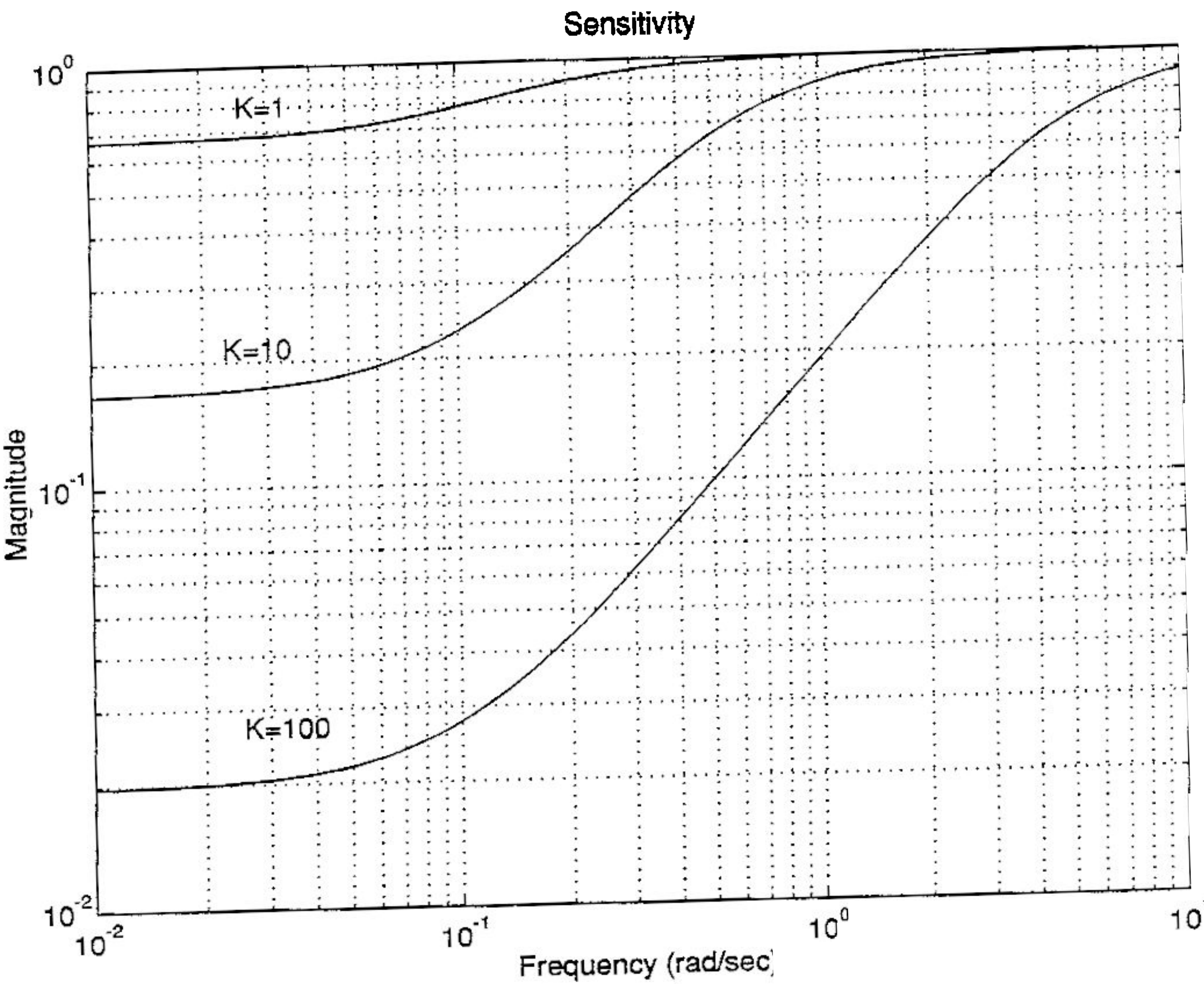

Sensitivity
K=1
K=10
K=100
Magnitude
10^0
10^{-1}
10^{-2}
10^{-2}
10^{-1}
10^0
10^1
Frequency (rad/sec)



Complementary Sensitivity
K=100
K=10
K=1
Magnitude
Frequency (rad/sec)



Complementary Sensitivity
Magnitude
Frequency (rad/sec)
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
K=100
K=10
K=1


Sensitivity
K=1
K=10
K=100
Magnitude
Frequency (rad/sec)
10^0
10^-1
10^-2
10^-2
10^-1
10^0
10