- ## Jury's stability test (section 7.5 book)

Jury's test is similar to Routh Hurwitz in the sense that it counts the number of unstable roots of the (discrete time) char. equation.

You form an array using the coefficients of the polynomial, starting with two rows of length n. From these you compute a succesor row of length $n-1$, then another one of length $n-2$ and so on, until we get a row of length 1. Stability is related to the entries of the first column, as follows:

Assume that the characteristic polynomial is given by:

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots \quad a_1 z + a_0, \qquad a_n > 0$$

- Step 1: Form Jury's Array:

$$\begin{array}{cccccc}
z^0 & z^1 & z^2 & \cdots & z^{n-1} & z^n \\
a_0 & a_1 & a_2 & \cdot & a_{n-1} & a_n \\
a_n & a_{n-1} & a_{n-2} & & a_1 & a_0 \\
b_0 & b_1 & b_2 & & b_{n-1} & \\
b_{n-1} & b_{n-1} & & & b_0 & \\
c_0 & c_1 & \cdots & & c_{n-2} & \\
c_{n-2} & c_{n-3} & & & c_0 & \\
\vdots & & & & & \\
\ell_0 & \ell_1 & \ell_2 & \ell_3 & & \\
\ell_3 & \ell_2 & \ell_1 & \ell_0 & & \\
m_0 & m_1 & m_2 & & &
\end{array}$$

Remark: the elements of the even numbered rows are the elements of the preceeding row in reverse order

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}; \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \ldots$$

Step 2 : Check the following conditions (necessary and sufficient for having all roots in $|z|<1$)

(a) $Q(1) > 0$

(b) $(-1)^n Q(-1) > 0$

(c) $|a_0| < a_n$,

$|b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

$|d_0| > |d_{n-3}|$

$\vdots$

$|m_0| > |m_2|$)

Remark : Check first $\varphi(1) > 0$, $(-1)^n \varphi(-1) > 0$, $a_n > |a_0|$

If any of these conditions fails the system is unstable and there is no need to proceed any further

Example 1 : (Laser example with $T = 0.1$)

$$\varphi(z) = 1 + K G(z) = z^2 + (0.0085K - 1.5752) z + (0.0072K + 0.6065)$$

Conditions: $\varphi(1) = 1 + (0.0085K - 1.5752) + (0.0072K + 0.6065) = 0.0314 + 0.0157\,K > 0$

$(-1)^2 \varphi(-1) = 3.1817 - 0.0013\,K > 0$

$a_2 > |a_0| \Rightarrow |0.0072K + 0.6065| < 1 \Leftrightarrow -1 < 0.0072K + 0.6065 < 1$

From these conditions we have:

$K > -2$

$K < 2.4475 \cdot 10^3$

$K < 54.713$

$K > -223.39$

$\Rightarrow \boxed{-2 < K < 54.713}$

same conditions as before

Example 2 (a robust stability example)

Consider the following second order system:

$$P(z) = z^2 + \alpha z + \beta \qquad \text{where } \alpha \,\&\, \beta \text{ are parameters}$$

The Jury array is:

z^0	z^1	z^2
β	α	1
1	α	β
β^2-1	$\alpha(\beta-1)$	

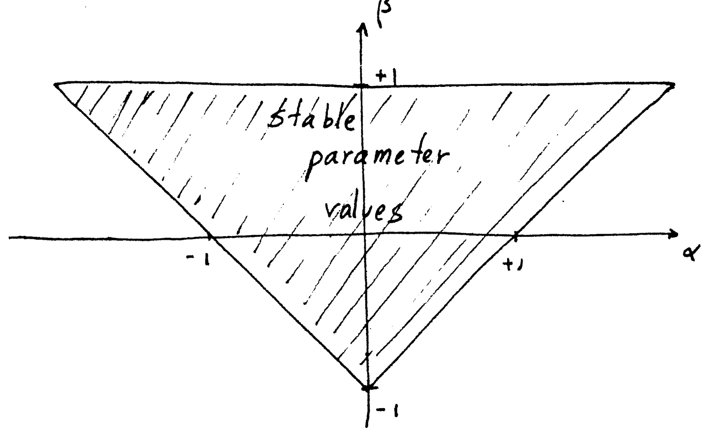
conditions:

$$P(1) > 0 \Rightarrow \alpha+\beta+1>0$$
$$(-1)^2 P(-1) > 0 \qquad \beta+1-\alpha>0$$

$$|a_0| < a_n \Rightarrow |\beta|<1 \Leftrightarrow -1<\beta<1$$
$$|b_0| > |b_{n-1}| \Rightarrow |\beta^2-1| > |\alpha(\beta-1)|$$
$$|(\cancel{\beta-1})(\beta+1)| > |\alpha|\cdot|\cancel{\beta-1}| \;\;//\;\; |(\beta+1)| > |\alpha|$$

or $1+\beta > \alpha$
$1+\beta > -\alpha$

Graphically we get the following region:



Example 3, a third order system: $\varphi(z) = z^3 - 1.8z^2 + 1.05z - 0.20$

$\varphi(1) = 1-1.8+1.05-0.20 = 0.05 > 0$ ✓, $(-1)^3\varphi(-1) = -1\cdot(-1-1.8-1.05-0.2) = 4.05 > 0$ ✓

$|a_0| = 0.2 < a_3 = 1$ ✓

Jury array:

z^0	z^1	z^2	z^3
-0.2	1.05	-1.8	1
1	-1.8	1.05	-0.2
-0.96	1.59	-0.69	

$\Rightarrow |b_0| = 0.96 > |b_2| = 0.69$ ✓

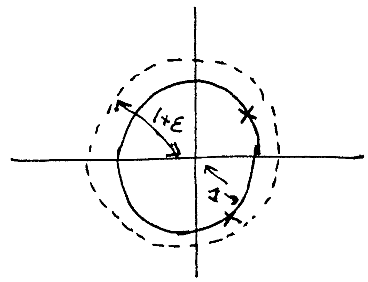
All conditions hold: <u>System is stable</u>

$$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = -0.96; \quad b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = 1.59, \quad b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix} = -0.69$$

• **Singular cases:** What do we do if we get an equality (rather than strict inequality) in any of the conditions?

A: Use the transformation $z = (1+\epsilon)\hat{z}$ $\quad (\epsilon > 0)$

This transformation amounts to expanding the unit circle (i.e if we have a root at $z = z_0$, $P(\hat{z})$ will have a root at $\hat{z}_0 = \frac{z_0}{1+\epsilon}$)



Let $P(z) = \sum a_i z^i \Rightarrow P(\hat{z}) = \sum_i a_i (1+\epsilon)^i \hat{z}^i$

Since for ϵ small $(1+\epsilon)^n \sim (1+n\epsilon)$ then the transformation is equivalent to multiplying the coefficients of the original polynomial times $(1+k\epsilon)$ i.e:

$$a_k \longrightarrow (1+k\epsilon)\, a_k$$

• **Raible's Tabular Form of Jury's Criterion**

(Ref: "A simplification of Jury's Tabular Form," IEEE Trans. Autom Control, AC-19, June 1974, pp 248-250)

Write Jury's array as:

$$\begin{array}{lllllll|l}
 & a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & a_0 \quad & k_a = \frac{a_0}{a_n} \\
 & a_0 k_a & a_1 k_a & a_2 k_a & & a_{n-2} k_a & a_{n-1} k_a & & \\
\hline
\rightarrow & b_0 & b_1 & b_2 & & b_{n-2} & b_{n-1} & & k_b = \frac{b_{n-1}}{b_0} \\
 & b_{n-1} k_b & b_{n-2} k_b & & & b_1 \cdot k_b & & & \\
\hline
\rightarrow & c_0 & c_1 & & & c_{n-2} & & & k_c = c_{n-2}/c_0 \\
 & \vdots & \vdots & & & \vdots & & & \vdots \\
\rightarrow & q_0 & q_1 & & & & & & k_q = q_1/q_0 \\
 & q_1 k_q & & & & & & & \\
\hline
\rightarrow & r_0 & & & & & & &
\end{array}$$

Where $b_i = a_{n-i} - k_a a_i$, i.e you obtain the rows marked with an arrow by subtracting the previous row from the one immediately above it

Then: # of positive elements in the first column of the calculated rows (the ones marked with an arrow) =

roots inside the unit disk, i.e.

$$\boxed{\text{system stable iff all the elements in the first column of the marked rows are positive}}$$

Example 3 revisited: $\varphi(z) = z^3 - 1.8z^2 + 1.05z - 0.20$

	1	-1.8	1.05	-0.2	$k_a = -0.2$
	0.04	-0.21	0.36		
→	0.96	-1.59	0.69		$k_b = 0.7188$
	0.4959	-1.1428			
→	0.4641	-0.4472			$k_c = -0.9636$
	0.4309				
→	0.0152				

$b_0, c_0, d_0 > 0 \Rightarrow$ System stable

Example 2 revisited: $\varphi(z) = z^2 + \alpha z + \beta$

	1	α	β	$k_a = \beta$
	β^2	$\alpha\beta$		
→	$1-\beta^2$	$\alpha(1-\beta)$		$k_\beta = \frac{\alpha(1-\beta)}{(1-\beta^2)} = \frac{\alpha}{1+\beta}$
	$\alpha^2\frac{(1-\beta)}{(1+\beta)}$			
→	$(1-\beta^2) - \alpha^2\frac{(1-\beta)}{(1+\beta)}$			

Stable if $1-\beta^2 > 0 \Rightarrow \beta^2 < 1 \Leftrightarrow \boxed{|\beta| < 1}$

$$(1-\beta^2) - \alpha^2\frac{(1-\beta)}{1+\beta} > 0 \;//\; (1+\beta) - \frac{\alpha^2}{1+\beta} > 0 \;//\; (1+\beta)^2 > \alpha^2 \Leftrightarrow \boxed{\begin{matrix} 1+\beta > \alpha \\ 1+\beta > -\alpha \end{matrix}}$$

Same conditions as before. Note that these conditions are redundant and can be obtained from

$\varphi(1) > 0 \Rightarrow \beta + 1 > -\alpha$

$\varphi(-1) > 0 \quad \beta + 1 > \alpha$

$|a_0| < a_n \Rightarrow |\beta| < 1$