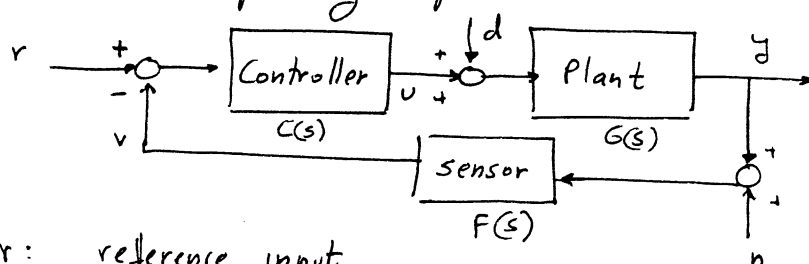


## SISO systems: Stability & Performance

Now we are going to analyze the stability and performance characteristics of SISO systems

Consider the following loop:



$r$ : reference input  
 $d$ : external disturbance  
 $u$ : control signal  
 $y$ : plant output  
 $n$ : sensor noise  
 $v$ : sensor output

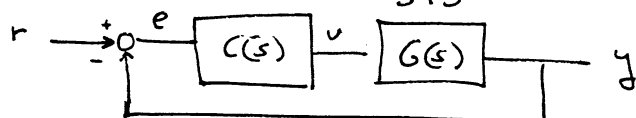
Control problem: keep  $y$  "close" to  $r$  in the face of the external disturbance  $d$  and sensor noise  $r$

However, before we can tackle this problem we need to analyze the interconnection of the different elements

Q: Assume that the plant, controller and sensors all have proper rational transfer functions. Can we interconnect them and obtain a well defined loop?

A: not necessarily

Example:  $C(s) = \frac{s+7}{s+5}$ ,  $G(s) = \frac{1-s}{2+s}$ ,  $F(s) = 1$



$$T_{er} = \frac{1}{1 + \left(\frac{s+7}{s+5}\right)\left(\frac{1-s}{2+s}\right)} = \frac{(s+2)(s+5)}{(s+17)}$$

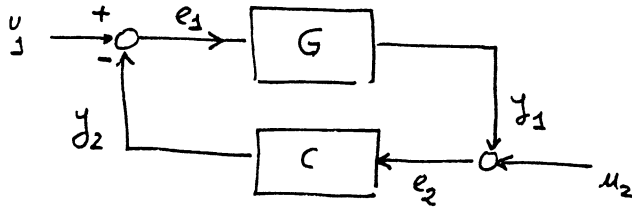
Trouble!!

$T_{er}$  not proper  $\Rightarrow$

$e$  is not well defined

To prevent this type of situation, we need to introduce the concept of well posedness

Assume for simplicity that  $F=1 \Rightarrow$



Definition: The feedback loop is well posed if and only if the 4 transfer functions between the inputs  $u_1$  and  $u_2$  and the outputs  $e_1$  and  $e_2$  exist and are proper

$$\begin{aligned} e_1 &= u_1 - y_2 = u_1 - C e_2 \\ e_2 &= u_2 + G e_1 \end{aligned} \Rightarrow \begin{pmatrix} 1 & C(s) \\ -G(s) & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$\Rightarrow$  problem is well posed iff  $\begin{pmatrix} 1 & C(s) \\ -G(s) & 1 \end{pmatrix}$  is non singular and its inverse is proper.

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & C \\ -G & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{1 + G(s)C(s)} \begin{bmatrix} 1 & -C \\ G & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Claim: The following are equivalent:

- i) loop is well posed
- ii)  $P$  has a proper inverse
- iii)  $1 + G(s)C(s)$  has a proper inverse
- iv)  $1 + G(\infty)C(\infty) \neq 0$  (i.e.  $1 + G(s)C(s)$  not strictly proper)

Proof: (i)  $\Leftrightarrow$  (ii) Since  $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 & C \\ -G & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  then

all the T.F. exist and are well defined if  $P^{-1}$  exists for almost all  $s$  and it is proper

(ii)  $\Leftrightarrow$  (iii) if  $G, C$  are proper then  $P^{-1}$  is proper  
iff  $(1 + GC)^{-1}$  is proper

(iii)  $\Leftrightarrow$  (iv)  $(1 + GC)^{-1}$  is proper and exists for almost all  $s$   
 $\Leftrightarrow$  is proper and exists for  $s \rightarrow \infty$   
 $\Leftrightarrow 1 + G(\infty)C(\infty) \neq 0$

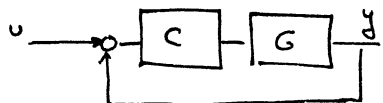
Finally, note that if  $1 + G(\infty)C(\infty) \neq 0$  then  $\frac{1}{1+GC}$  is proper.

Let  $G = \frac{P}{\Phi}$   $C = \frac{N}{M}$  with  $\deg(P) \leq \deg(\Phi)$   
 $\deg(N) \leq \deg(M)$

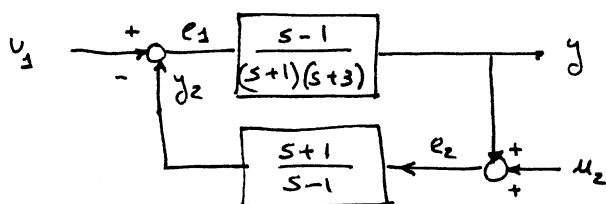
then  $\frac{1}{1+GC} = \frac{1}{1 + \frac{PN}{\Phi M}} = \frac{\Phi M}{\Phi M + PN}$

but  $\deg(\Phi M) \leq \deg(\Phi M + PN)$  unless the high order terms  
 cancel out and this will imply  $\frac{P(\infty)N(\infty)}{\Phi(\infty)M(\infty)} = -1 \Leftrightarrow 1 + G(\infty)C(\infty) = 0$

- Stability: Usual definition of BIBO stability:  $y$  bounded for all bounded inputs  $u$  (input/output stability)



However, we can have a situation like this:



$$y_1 = \frac{s-1}{(s+1)(s+3)} u_1 = \frac{s-1}{(s+1)(s+4)} u_1 \quad \text{stable}$$

but:

$$\frac{y_2}{u_2} = \frac{\frac{s+1}{s-1}}{1 + \frac{(s-1)(s+1)}{(s+1)(s+3)(s-1)}} = \frac{(s+1)(s+3)}{(s-1)(s+4)} \quad \text{unstable!}$$

$\Rightarrow$  Input/output stability does not guarantee that all possible transfer functions are stable (due to pole-zero cancellations)

We need the concept of internal stability

Definition: The feedback loop shown above is internally stable if and only if the 4 transfer functions are stable

Claim: The feedback system is internally stable iff there are no closed-loop poles in  $\text{Re}(s) > 0$

Proof: write  $G = \frac{N_G}{M_G}$ ,  $C = \frac{N_C}{M_C}$   
 where the polynomials  $(N_G, M_G)$  and  $(N_C, M_C)$  are coprime  
 (i.e. no common factors)

Then the transfer functions are given by:

$$\begin{aligned} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} &= \frac{1}{1+GC} \begin{pmatrix} 1 & -C \\ G & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \frac{1}{N_G N_C + M_G M_C} \begin{pmatrix} M_G M_C & -N_G M_C \\ M_G N_C & M_G M_C \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

Sufficiency: if  $N_G N_C + M_G M_C$  doesn't have zeros in  $\text{Re}(s) > 0$ , all 4 transfer functions are stable

Necessity: We need to show that if the 4 transfer functions are stable, then  $N_G N_C + M_G M_C$  does not have roots in  $\text{Re}(s) > 0$ .  
 To show this we need to show that they are no pole-zero cancellation in  $\text{Re}(s) > 0$

(otherwise the transfer functions could be stable and yet  $N_G N_C + M_G M_C$  can have zeros in  $\text{Re}(s) > 0$ , if these zeros cancel out in all 4 T.F.)

However, it can be easily shown that there cannot be any pole-zero cancellations.

Assume that for some  $s_0$  we have:

$$\left. \begin{aligned} (N_G N_C + M_G M_C)|_{s_0} &= 0 \\ (1) \quad M_G M_C|_{s_0} &= 0 \\ (2) \quad N_G M_C|_{s_0} &= 0 \\ (3) \quad M_G N_C|_{s_0} &= 0 \end{aligned} \right\} \Rightarrow (4) \quad N_G N_C|_{s_0} = 0$$

Assume that  $N_G(s_0) = 0$ . Since  $M_G, N_G$  are coprime  $\Rightarrow M_G(s_0) \neq 0$

but (1) implies that  $M_c(s_0) = 0 \Rightarrow N_c(s_0) \neq 0 \Rightarrow (3) M_c(s_0) = 0$   
against the assumption that  $N_6, M_6$  coprime.

- Alternative characterization: The feedback loop is internally stable iff the following 2 conditions hold:

and 1)  $\frac{1}{1+GC}$  is stable (i.e. no poles in the RHP)

2) there are no RHP pole/zero cancellations in  $G(s)C(s)$

Proof ( $\Rightarrow$ ) if the system is internally stable  $\Rightarrow \frac{1}{1+GC}$  is stable

writing  $G = \frac{N_6}{M_6}$   $C = \frac{N_c}{M_c}$  we have that the

characteristic polynomial is given by:  $M_6(s)M_c(s) + N_6(s)N_c(s) = \varphi(s)$

Since  $\varphi(s)$  has no zeros in  $\text{Re}(s) \geq 0$  ( $M_c, N_6$ ) can't have any  
 $\Rightarrow$  no unstable pole-zero cancellations ( $M_6, N_c$ )

( $\Leftarrow$ ) Let  $s_0$  be a root of  $\varphi(s)$ . We need to show that  $\text{Re}(s_0) < 0$

Assume to the contrary that  $\text{Re}(s_0) \geq 0$ . If  $M_6(s_0)M_c(s_0) = 0$   
 $\Rightarrow N_6(s_0)N_c(s_0) = 0$ , but this violates (2)  $\Rightarrow$

$M_6(s_0)M_c(s_0) \neq 0$  and we can divide  $\varphi(s)$  by it to get:

$$\frac{\varphi(s_0)}{M_6(s_0)M_c(s_0)} = 1 + \frac{N_6(s_0)N_c(s_0)}{M_6(s_0)M_c(s_0)} = 0 \Leftrightarrow 1 + GC = 0$$

which violates (1) #

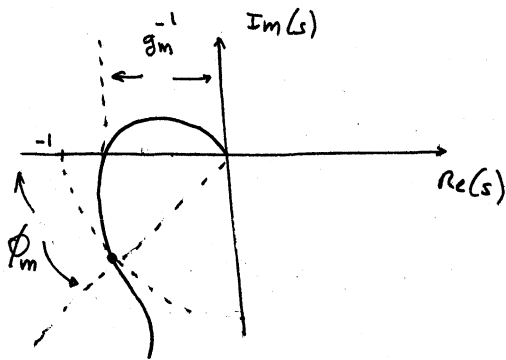
Warning: Internal stability rules-out unstable pole/zero cancellations. However, there still could be stable cancellations. Although this does not affect stability, it could have adverse effects upon performance.

## Robust Stability (section 2.4)

Loosely speaking: "robustness"  $\leftrightarrow$  ability of the closed loop system to have "acceptable" performance, even when the "true" plant does not match the plant used in the design process.

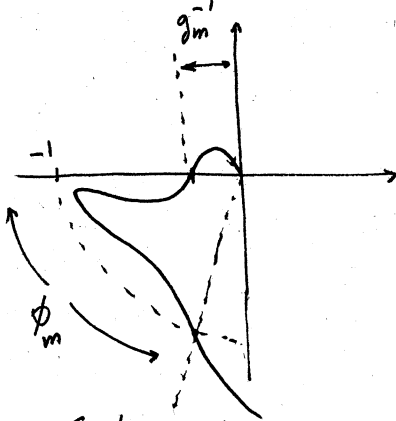
Classical control approach:

Assess robustness using gain & phase margins: the "distance" in angle or gain from the Nyquist plot to the critical point  $z = -1$

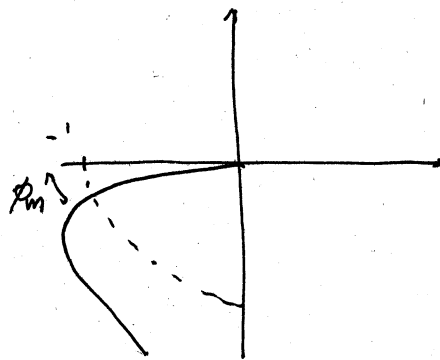


However, these measures are effective only if uncertainty is confined to either phase or gain. They do not guarantee stability against (even very small) simultaneous gain and phase perturbations.

For instance:



System "A"



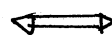
System "B"

System "A" has large  $g_m$  and  $\phi_m$ . However, a small combined perturbation can destabilize it.

System "B" has a small  $\phi_m$  (but  $\infty g_m$ ). This can be acceptable if we have a-priori information (reliable) that uncertainty appears only in the gain.

Main point

stability margin



Type of uncertainty

- A given stability margin is a good indicator of the robust stability properties of the system only when it is associated with a specific type of uncertainty

## • Uncertainty descriptions:

From the discussion above, the controller has to be tailored to the uncertainty that we expect to have in the system:

For instance, a system subject to either phase or gain uncertainty can be represented by a family of models of the form:

$$g(s) = g_0 \cdot c \cdot e^{j\phi} \quad \text{with } c \text{ real, } c \in [c_{\min}, c_{\max}] \\ \phi \text{ real, } \phi \in [\phi_{\min}, \phi_{\max}]$$

This is a special case of multiplicative global dynamic uncertainty

- Global dynamic uncertainty:
  - 1) related to the dynamics (rather than some parameters)
  - 2) covers globally the plant

When is it useful?

When the order (or structure) of the model (i.e. the differential equations) is unknown.

It arises in practice in several different situations:

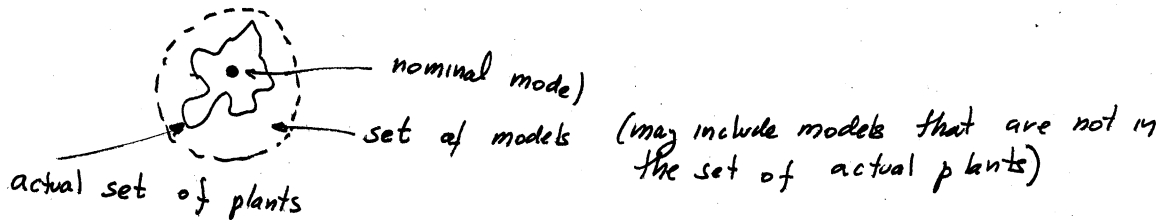
- (a) Certain phenomena (such as flexibility) is not included in the model in order to get tractable problems
- (b) A high order model is approximated by a lower order one  
Example: on-line control of fast processes. The computation time constraint requires using simple models
- (c) Linearization of a non-linear model around different operating points. For instance an aircraft model linearized around different Mach numbers and angles of attack, or an active vision system linearized around different focal lengths.
- (d) Approximation of an infinite dimensional model (such as a time delay or a distributed parameters system) by a FDLTI model

Our approach: Describe the plant using a family of models rather than a single one

$$\text{Family of models } \mathcal{F} = \{ \text{nominal plant} + \text{bounded uncertainty} \}$$

- Goal of robust control: Design a single controller that works (in the sense of achieving stability and a guaranteed level of performance) for all members of the family.

Note: the tighter the uncertainty description, the less conservative the design



- Uncertainty description:  $\text{uncertainty} = \underbrace{\text{fixed weighting function}}_{W(s)} + \underbrace{\Delta(s)}_{\text{bounded uncertainty}}$   
represents all that is known or fixed about the uncertainty

- Simplest case of multiplicative uncertainty:

$$g(s) = g_0(s) (1 + \delta W(s))$$

$g_0$ : nominal model  
 $W(s)$ : fixed weight

$\delta$ : unknown complex number with  $|\delta| \leq 1$

Example: A plant with uncertainty in the location of the zeros

$$g(s) = \frac{s+a}{s+2} \quad \text{with } a \in [0, 2]$$

Take  $a_0 = 1$  (nominal plant  $\frac{s+1}{s+2}$ )  $\Rightarrow$  need  $\delta W$  to cover the set

$$\frac{g - g_0}{g_0} = \frac{\frac{s+a}{s+2} - \frac{s+1}{s+2}}{\frac{s+1}{s+2}} = \frac{a-1}{s+1} \Rightarrow a-1 = \delta, \quad |\delta| \leq 1$$

$$W = \frac{1}{s+1}$$

The family  $\mathcal{F} = \left\{ \frac{s+1}{s+2} (1 + W\delta) \right\}, \quad |\delta| \leq 1, \quad W = \frac{1}{s+1}$   
covers the set of possible plants

Q: Is the description tight?

A: No, only if we make  $\delta$  real, rather than complex (real versus complex uncertainty)

Trade-off: "tightness" of the description — tractability of the resulting problem

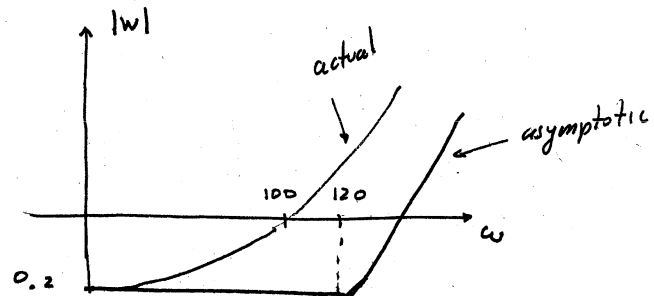
Taking  $\delta$  complex is conservative but leads to a tractable  $H_\infty$  control problem  
 Taking  $\delta$  real is tight but the resulting problem may be NP complete.

Example 2: Plant with uncertainty in the high order dynamics:

$$\mathcal{G} = \left\{ g: g = \frac{3 \left( 1 + \frac{s}{5} + \frac{s \cdot \delta}{100} \right)}{(s+1)(s+3)} \right\} \quad |\delta| \leq 1$$

$$g_0 = \frac{3}{(s+1)(s+3)}; \quad W\delta \text{ needs to cover: } \frac{g-g_0}{g_0} = \frac{\delta(s+20)}{100}$$

Bode plot of the uncertainty:



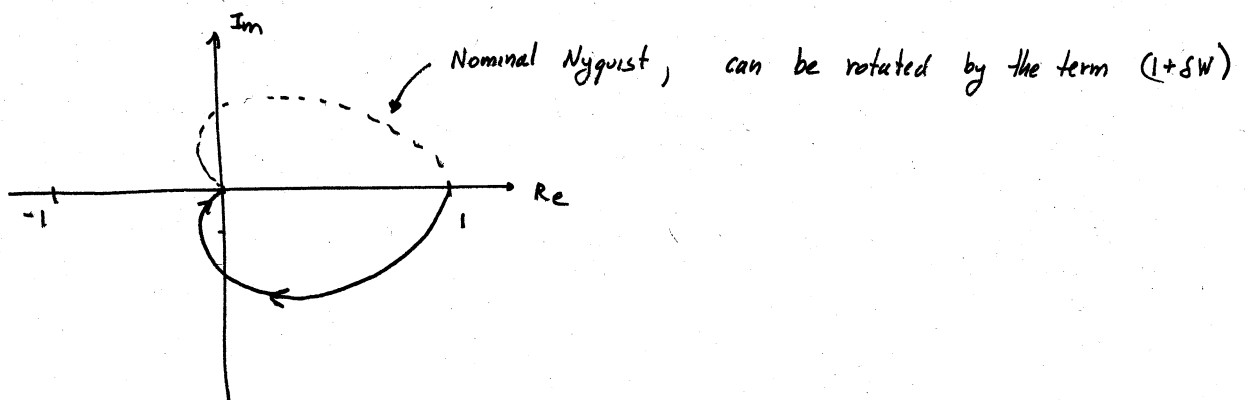
Break frequency:  $f_0 = 20 \text{ Hz}$ ,  $\omega_b \approx 2\pi f_0 \approx 120 \text{ rad/sec}$ . Note that at frequencies above  $100 \text{ rad/sec}$ ,  $|W| > 1 \Rightarrow$

$(1 + \delta W)$  can have either positive or negative real part

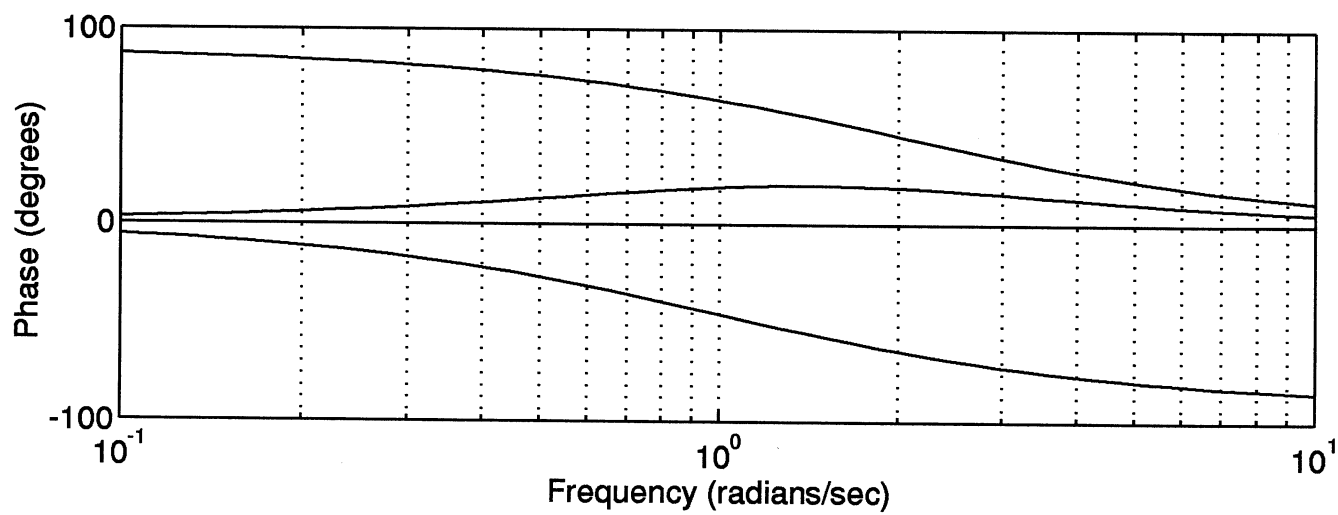
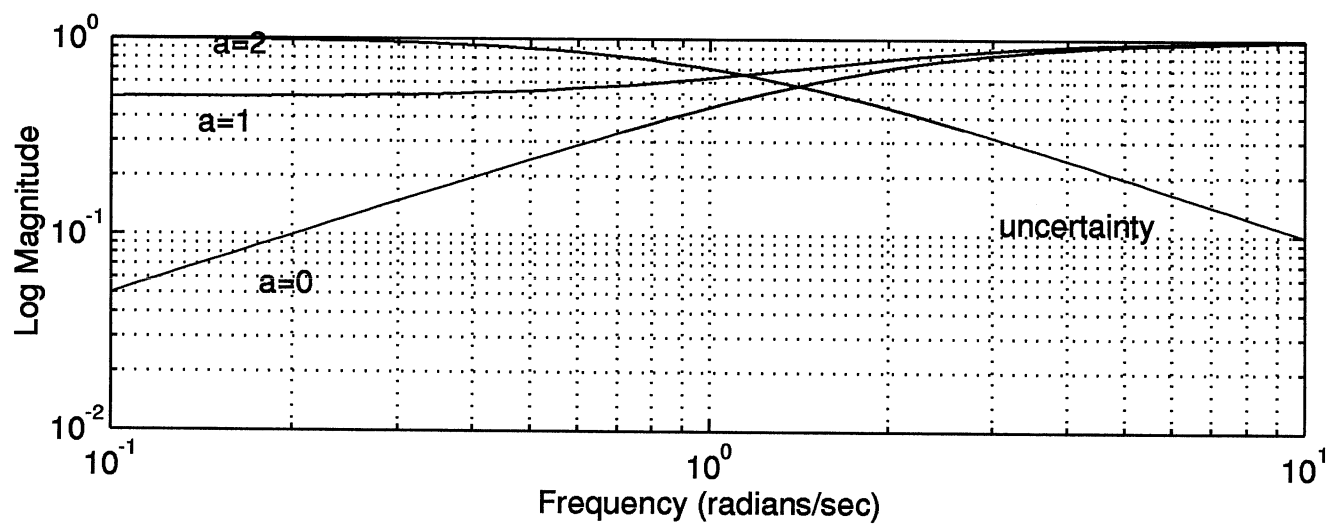
$\Rightarrow$  Only way we can guarantee stability is to design  $C(s)$  such that

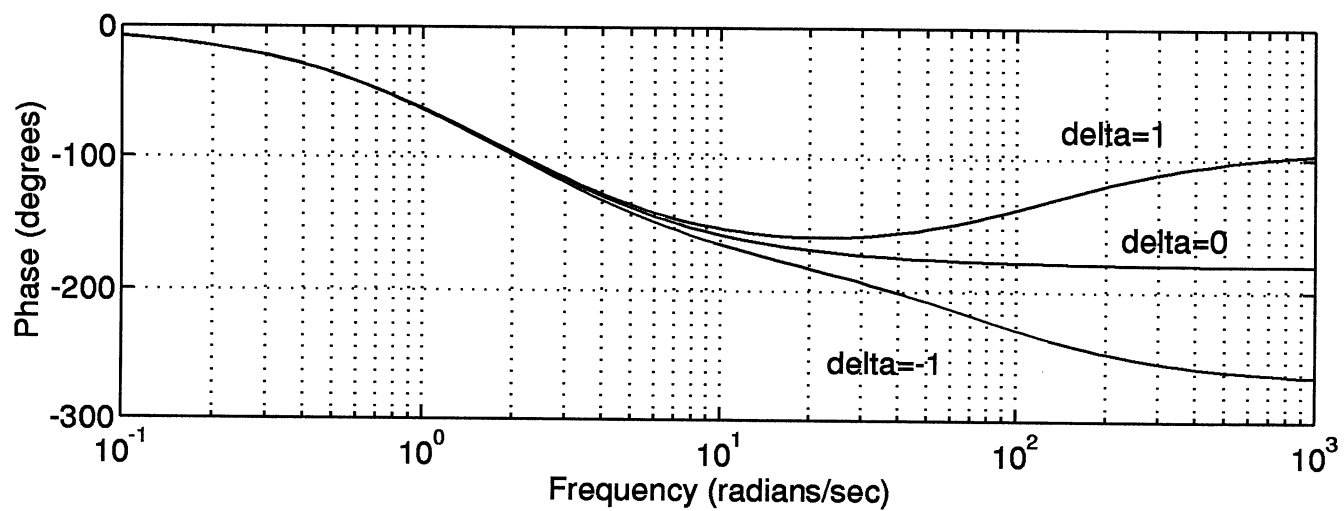
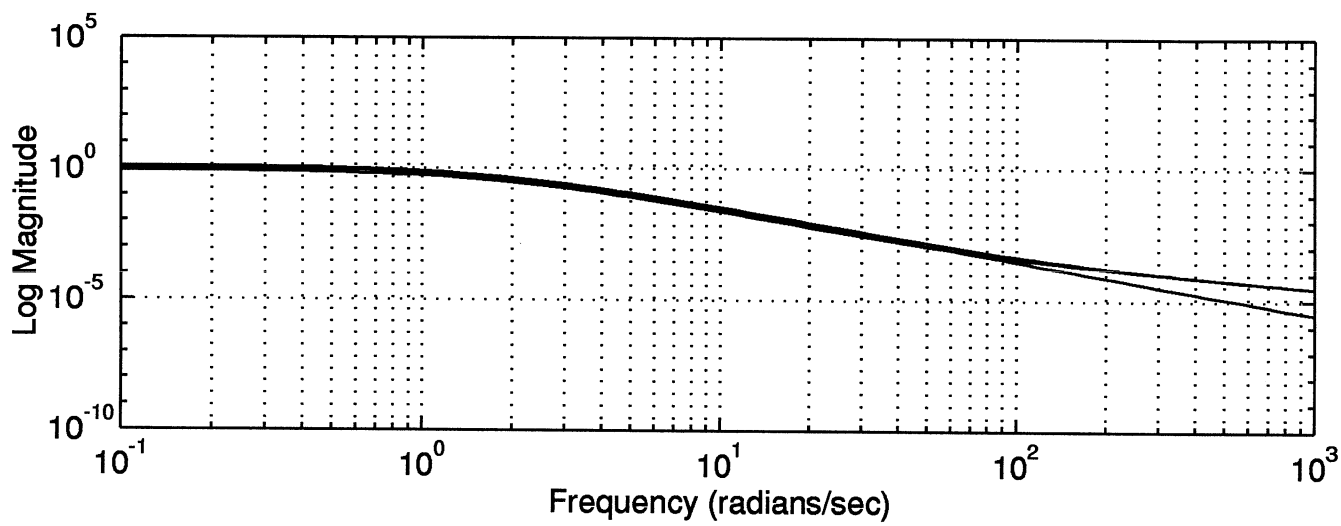
$$|C(j\omega)G(j\omega)| < 1 \text{ for all frequencies above } \omega = 100 \text{ rad/sec}$$

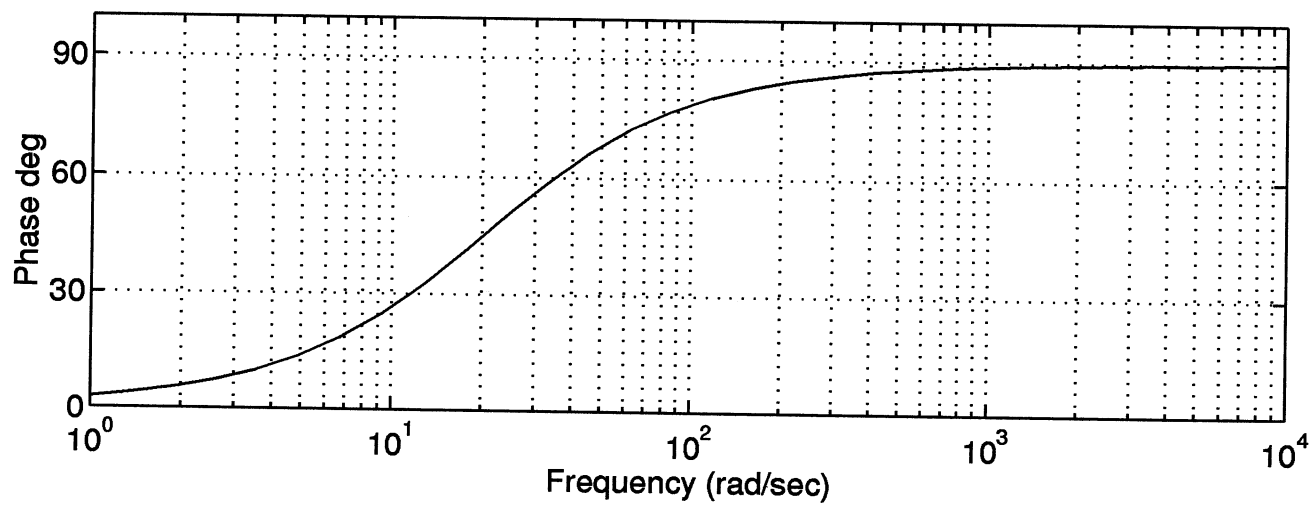
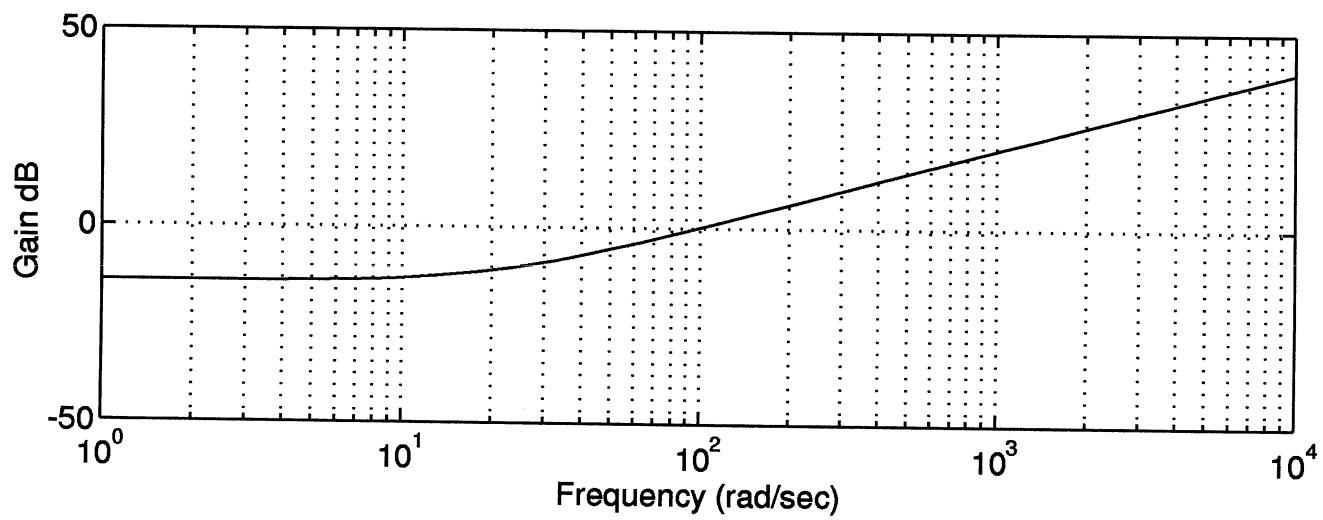
(to make sure that the Nyquist plot does not encircle the -1 point)



$\Rightarrow$  The presence of uncertainty poses a fundamental limitation on the bandwidth of the closed-loop system.







• A time delay example:

$$g_0 = \frac{1}{s^2} ; \quad g(s) = \frac{1}{s^2} e^{-sz} \text{ with } 0 \leq z \leq 0.1$$

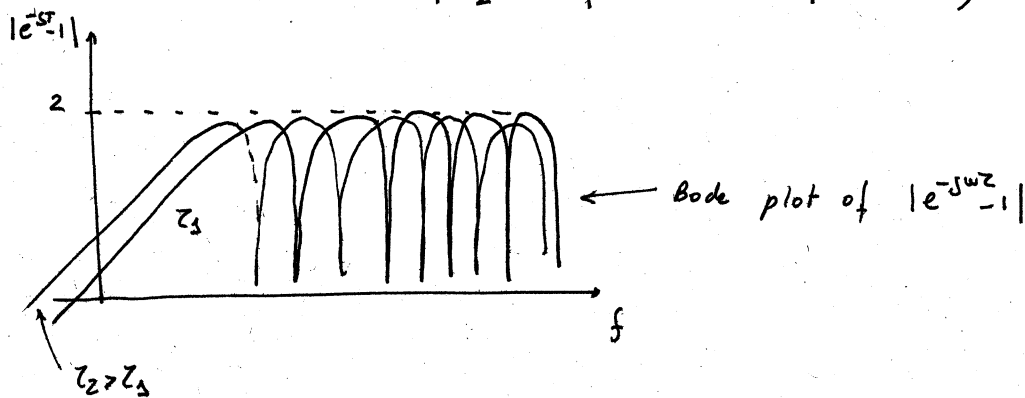
This can also be treated as multiplicative uncertainty, but it will require a  $\delta(s)$  (a transfer function) rather than just a number

Model  $g(s)$  as:  $g(s) = g_0(s)(1 + W_2(s)\delta(s))$ ,  $\|\delta\|_\infty \leq 1$

We need to select  $W_2(s)$  so that  $W_2 \cdot \delta$  "covers" all possible plants:

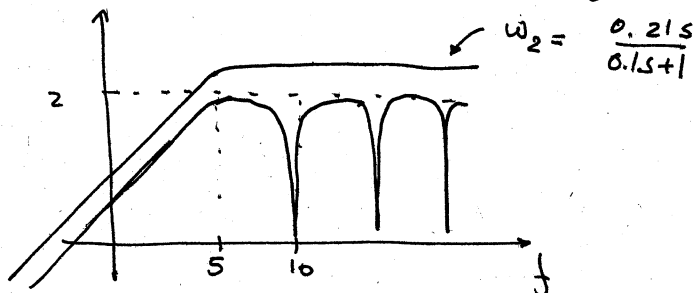
$$W_2 \delta = \frac{g - g_0}{g_0} = \frac{\frac{1}{s^2}(e^{-sz} - 1)}{\frac{1}{s^2}} \Rightarrow W_2 \delta \text{ should "cover" } e^{-sz} - 1$$

i.e. we need  $|W_2(s)\delta(s)| \geq |1 - e^{-j\omega z}| \quad \forall \omega, \text{ all } 0 \leq z \leq 0.1$



$\Rightarrow$  Worst possible case (as expected) corresponds to the largest value of  $z$  ( $z=0.1$ )

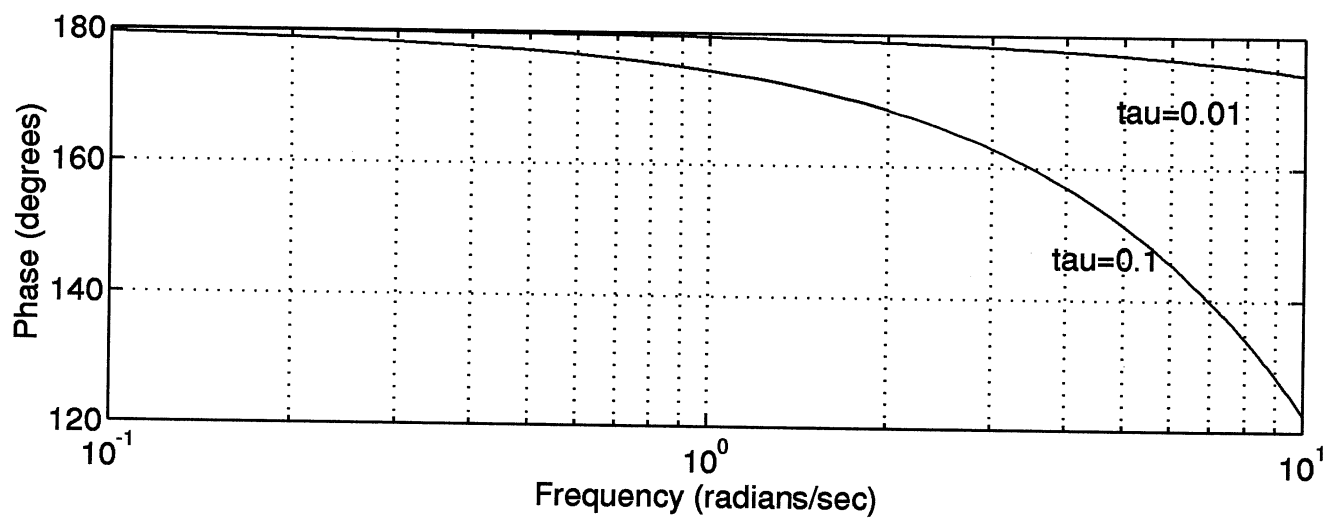
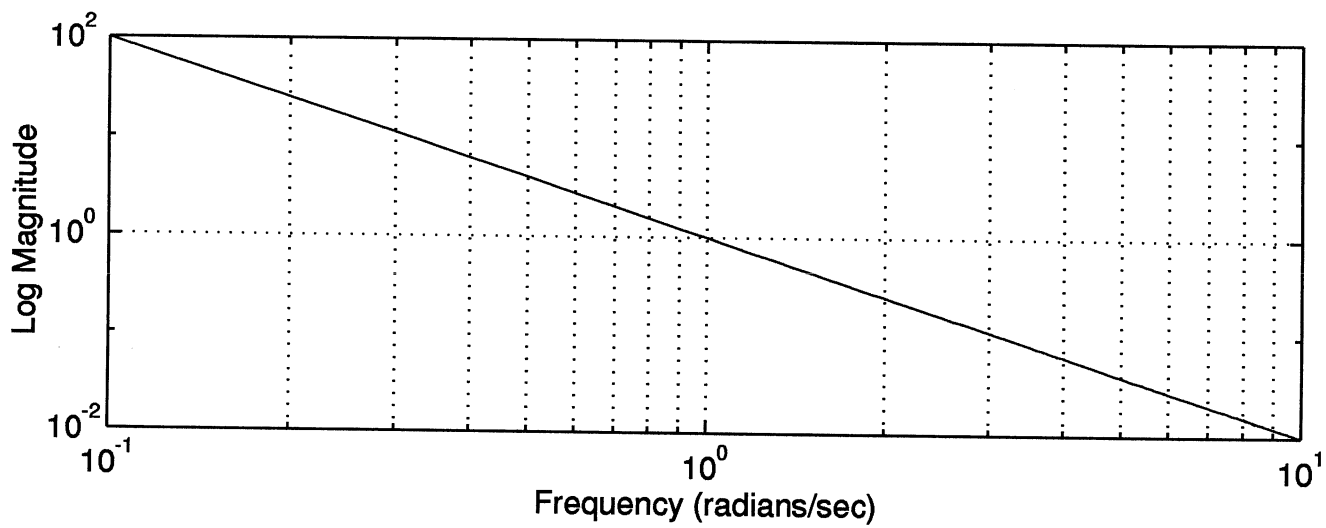
$\Rightarrow$  Need  $W_2$  to cover the (magnitude) Bode plot of  $|1 - e^{-j\omega 0.1}|$

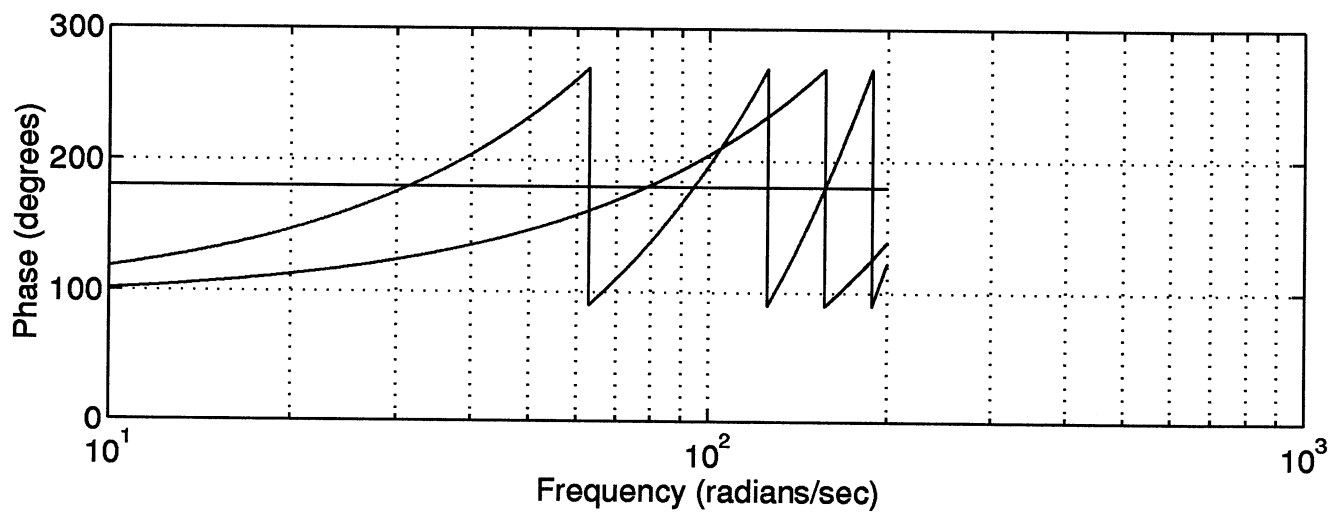
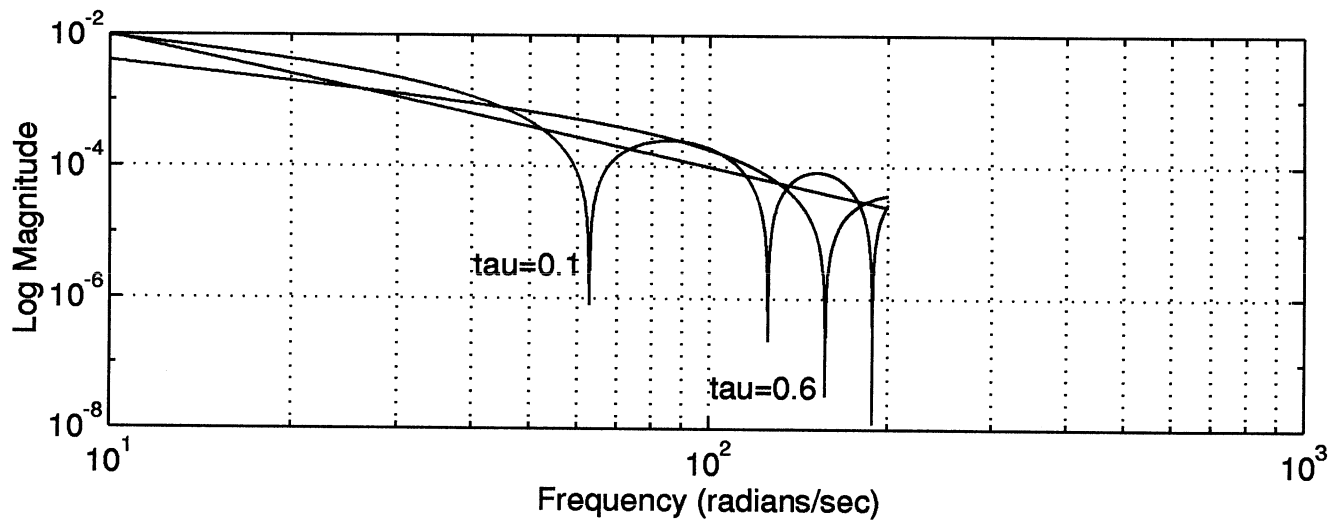


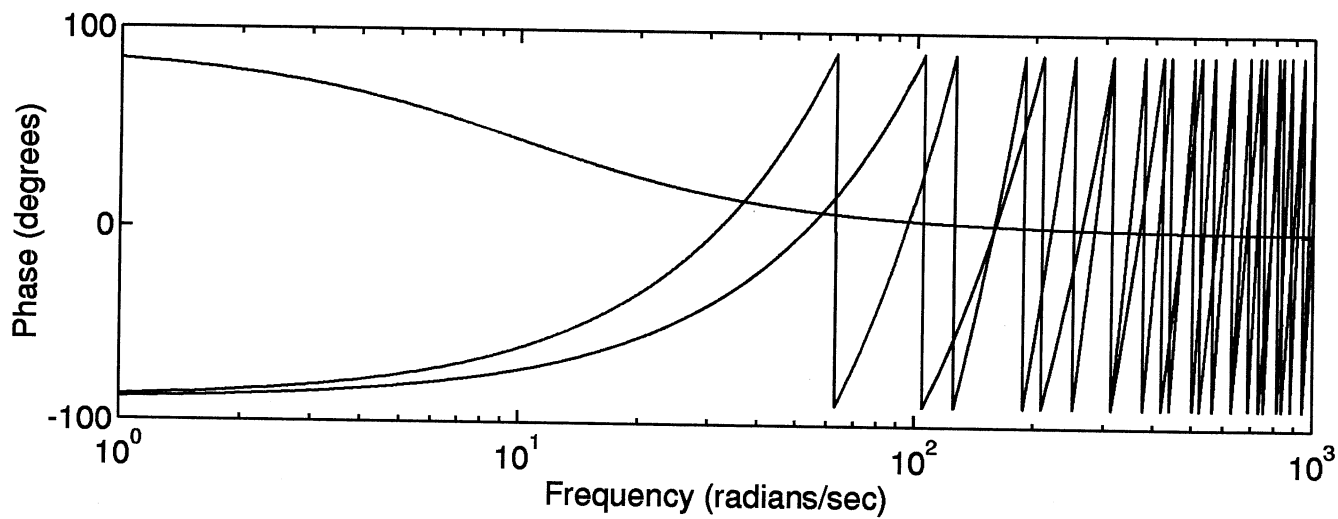
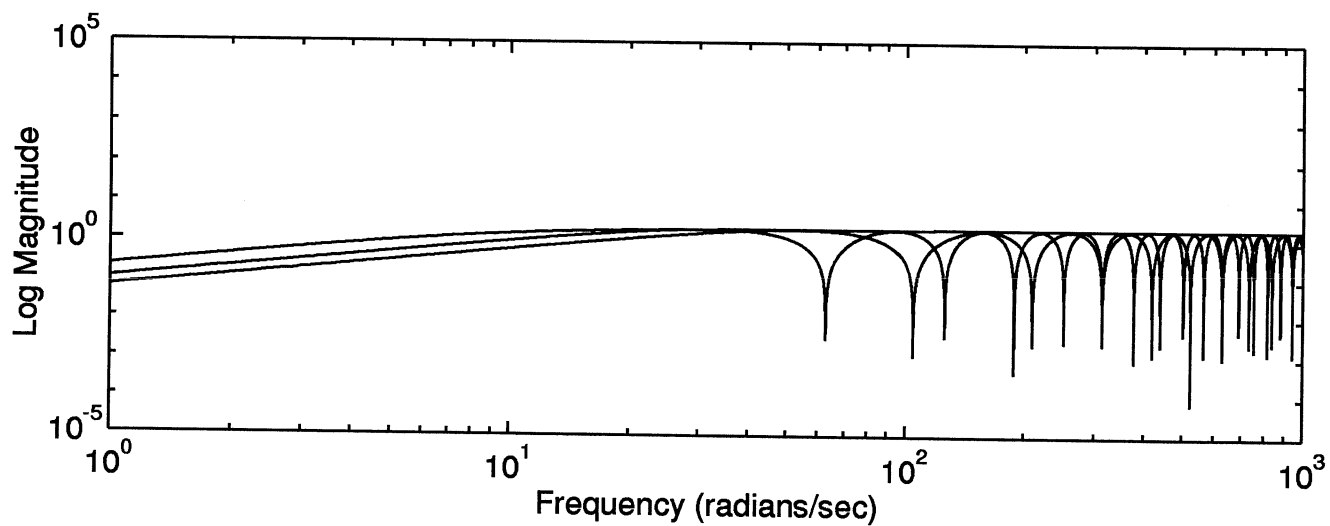
The Bode plot of  $|1 - e^{-j\omega 0.1}|$  can be covered by  $W_2 = \frac{0.21s}{0.1s + 1}$

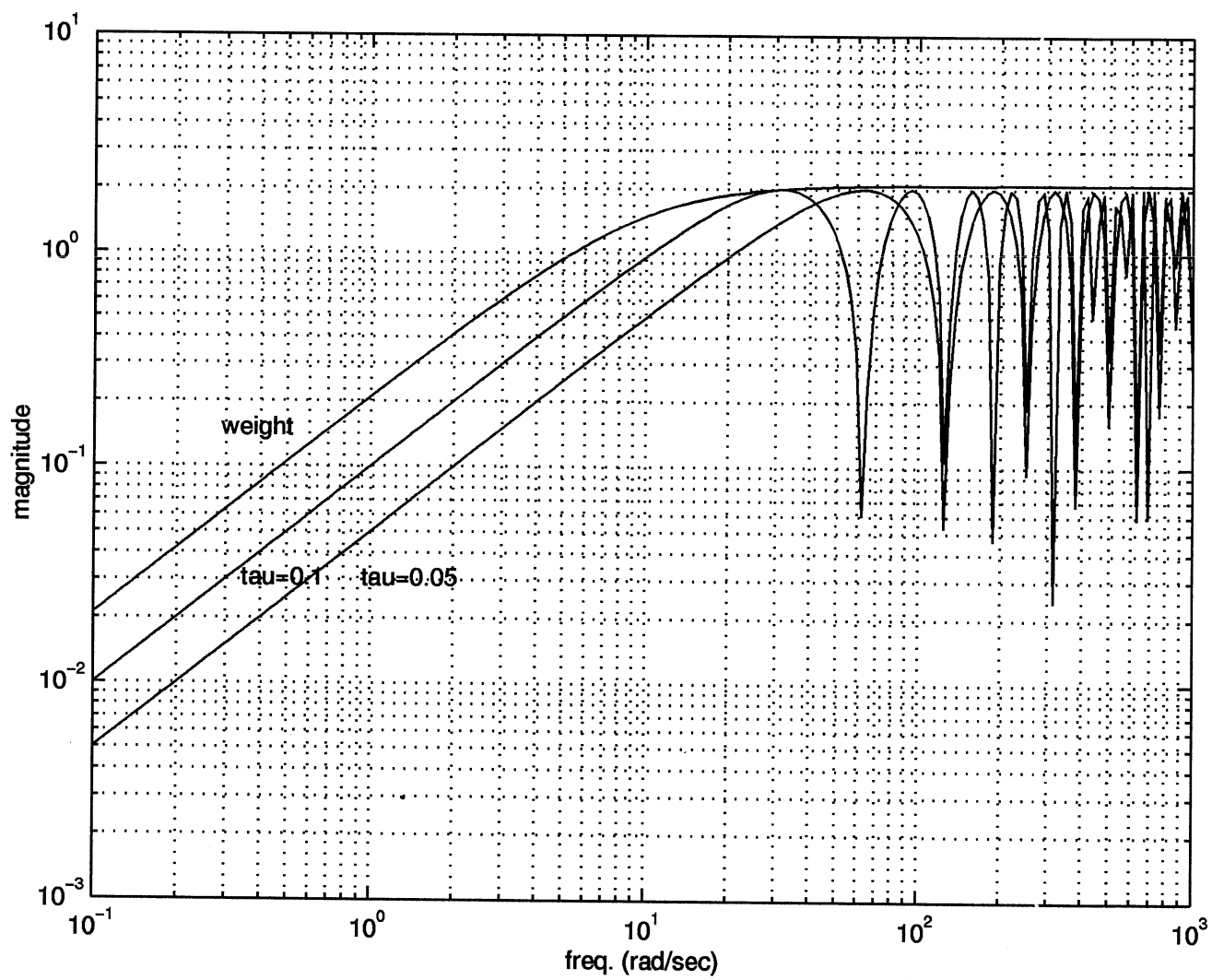
$$\Rightarrow g = \frac{1}{s^2} (1 + W_2 \delta) \quad W_2 = \frac{0.21s}{0.1s + 1}, \quad \|\delta\|_\infty \leq 1$$

However: this description is conservative!



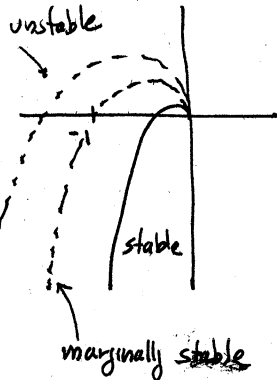






- Before we can derive conditions for robust stability, we need to examine with more detail the properties of  $W(s)$  and  $S(s)$

Suppose that we have a family of models that depend on a parameter? How do we detect when the family becomes unstable? When the Nyquist plot crosses over the critical  $(-1)$  point  $\Rightarrow$  The number of encirclements of  $-1$  changes



Suppose now that there are members of the family  $\mathcal{F}$  with different number of unstable open loop poles  $\Rightarrow$  Even if the closed loop stays stable, the corresponding Nyquist plots go around  $-1$  a different number of points

$\Rightarrow$  The Nyquist band already includes the  $-1$  point  $\Rightarrow$  very difficult to tell whether or not the system becomes unstable as the parameter changes.

To rule this out, we will require all members of the family  $\mathcal{F}$  to have the same number of unstable poles. Equivalently:

(a)  $W(s)$  stable

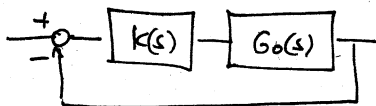
(b)  $\delta(s)$  such that no unstable p/z cancellations when forming  $g_o(1+W\delta)$

} admissible uncertainty

If (a) & (b) hold all TF of the form  $g(s) = g_o(1+W\delta)$  have the same number of unstable poles.

- Now we can find a condition guaranteeing robust stability:

Assume that we have a controller  $K(s)$  that internally stabilizes the nominal plant  $G_o(s)$



Under what conditions will the controller stabilize the whole family  $G(s) = G_o(1+W\delta)$

Recall that the closed loop is internally stable if:

(a)  $\frac{1}{1+GK}$  stable

(b) No RHP pole/zero cancellations

⇒ We need to impose these conditions for all members of the family  $\mathcal{F}$

Let's look first at condition (a):

$\frac{1}{1+GK}$  stable for all  $G = G_0(1+WS) \iff$

$1+G_0K(1+WS) \neq 0$  for all  $s, \operatorname{Re}(s) \geq 0, |S(s)| \leq 1 \iff$

$1+G_0K \neq G_0KW S \iff$

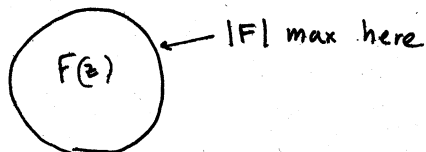
$S^{-1} \neq \frac{G_0KW}{1+G_0K} \quad \forall S, |S| \leq 1, \operatorname{Re}(s) \geq 0$

$\iff \left| \frac{G_0KW}{1+G_0K} \right| < 1 \quad \forall s, \operatorname{Re}(s) \geq 0$

$\iff |T(s)W(s)| < 1 \quad \forall s, \operatorname{Re}(s) \geq 0$ , where  $T$  is the complementary sensitivity function for the nominal plant.

Now we are going to make use of the maximum modulus theorem:

Recall that if a function is analytic in an open, simply connected region  $\mathcal{D}$  then  $|F(s)|$  achieves its maximum on the boundary of  $\mathcal{D}$



In our case since  $W(s)$  and  $T(s)$  are stable ( $W$  by choice,  $T$  since  $K$  stabilizes  $G_0$ ) then  $WT$  is analytic in  $\operatorname{Re}(s) > 0 \implies$

$|WT|$  achieves its maximum on the boundary of the RHP, i.e. the  $j\omega$  axis

$|T(s)W(s)| < 1$  for all  $s, \operatorname{Re}(s) \geq 0 \iff |T(j\omega)W(j\omega)| < 1$  all  $\omega$

$\iff \boxed{\|T(s)W(s)\|_{\infty} < 1}$

What about condition (b)? Since  $K$  internally stabilizes  $G_0 \implies$  no unstable p/z cancellations between  $G_0$  &  $K$

⇒ For any open loop RHP pole  $p_*$  of  $K(s)G_o(s)$  we have:

$$T(p_*) = \frac{KG_o}{1+KG_o} = 1 \Rightarrow \text{if } \|WT\|_\infty < 1 \text{ then } |W(p_*)| < 1$$

but this implies that  $[1 + W(s)S(s)] \neq 0$  for all  $|s| \leq 1$   
and all RHP poles of  $KG_o$

⇒  $KG_o(1 + SW(s))$  can't have unstable p/z cancellations

• Recap:

$$\left. \begin{array}{l} \text{(a) } (1 + GK)^{-1} \text{ stable} \\ \text{(b) No RHP p/z cancellations} \end{array} \right\} \Rightarrow \|TW\|_\infty < 1$$

necessary & sufficient  
condition for robust stability

• Interpretation in terms of the Nyquist plot:

$$\|TW\|_\infty < 1 \Leftrightarrow |G_o(j\omega)K(j\omega)W_2(j\omega)| < |1 + G_o(j\omega)K(j\omega)|$$

⇒ the disk centered at  $G_o(j\omega)K(j\omega)$  with radius  $|G_oK W_2|$  does not enclose the critical point

