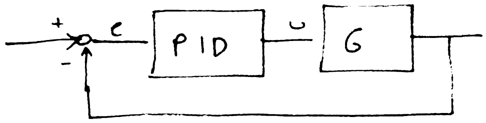
- **PID Controller Design:**

PID's are among the most commonly used compensators.

The output of a PID controller is defined by the equation:

$$u(t) = \underbrace{K_p\, e(t)}_{\text{proportional term}} + \underbrace{K_I \int_0^t e(\tau)\, d\tau}_{\text{integral term}} + \underbrace{K_D \frac{de(t)}{dt}}_{\text{derivative term}}$$



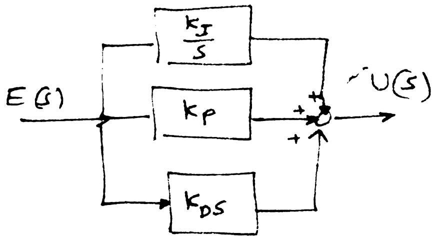
The combination provides an acceptable degree of error reduction simultaneously with acceptable stability and damping.

Commercially available process control systems typically are PID's where the control engineer has to adjust 3 constants

In the Laplace domain we have:

$$U(s) = \left(k_p + \frac{k_I}{s} + k_D s\right) E(s)$$

$$G_c(s) = k_p + \frac{k_I}{s} + k_D s$$



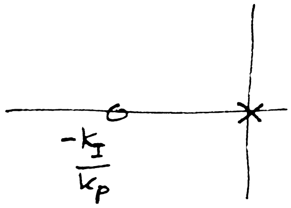
- **Analysis of each term:**

P controllers (proportional) The controller is just a pure gain, that you adjust using Root Locus techniques

PI control $\quad G_c(s) = k_p + \frac{k_I}{s} = \frac{k_p s + k_I}{s} = k_p\left(\frac{s + {}^{k_I}/_{k_p}}{s}\right)$

This controller increases the type of the system by 1 (ie: 0 → 1)
Hence it improves the steady state response

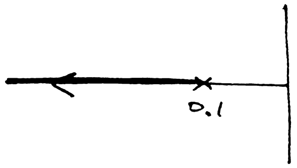
Note that is a phase lag (limiting case of phase lag when the pole → 0)



⇒ Improves steady state

However: can render the system slower and less stable.

Example : $KG = \frac{0.25k}{s+0.1}$



$$K_p = \lim_{s\to 0} k\,G(s) = 2.5k$$

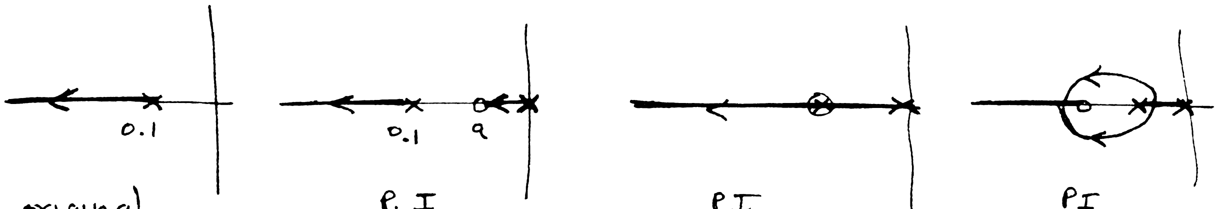
$$e_{ss}^{step} = \frac{1}{1+k_p} = \frac{1}{1+2.5k}$$

If we want 0 steady-state error to a step, we need a Type 1 system

$\Rightarrow$ use PI control: $\quad G_c(s) = \frac{k_p(s + {k_s}/{k_p})}{s} = \left(\frac{s+a}{s}\right)k_p$

$$GG_c(s) = K' \; 0.25\frac{(s+a)}{s(s+0.1)} \qquad \text{where } k' = k k_p$$

$\Rightarrow$



original

P.I (a<0.1)

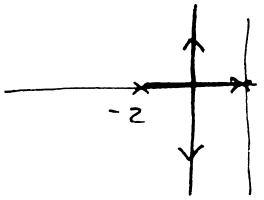
PI a=0.1

PI a>0.1

- PD control: $\quad G_c(s) = k_p + k_D s = k_D\left(s + \frac{k_p}{k_D}\right) \Rightarrow$ introduces a single zero

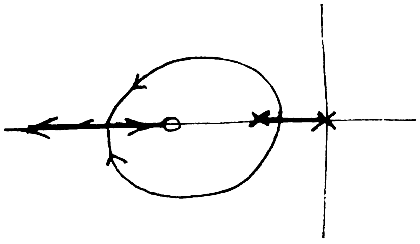
$\Rightarrow$ PD behaves as an "ideal" phase lead : moves RL to the left, improves transient

Example : $KG = \frac{k}{s(s+2)}$



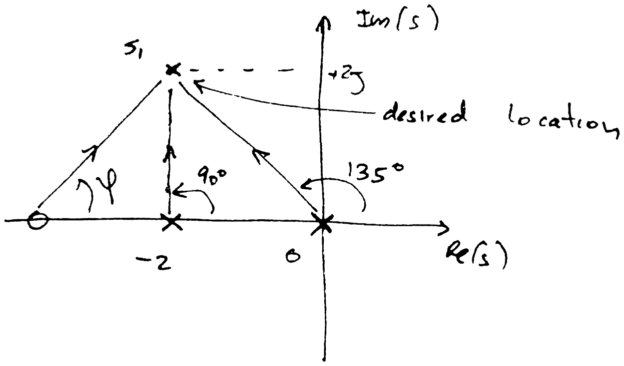
uncompensated

if we use PD:



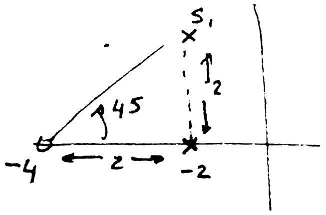
Suppose we want a specific location for our closed loop poles (say $s_{1,2} = -2 \pm j2$). How do we select the parameters?

2 options : 1) use the angle criterion:



$s_1 \in RL \Leftrightarrow \varphi - 90 - 135 = (2\ell+1)180$ // $\varphi = 45$ (for $\ell=-1$)

$\Rightarrow$ if $\varphi = 45$, then the zero of the controller should be at $s = -4$



Finally, we can get k from the magnitude condition: $k\frac{|s+4|}{|s||s+2|} = 1$

$$K \frac{|2+2j|}{|-2+2j||2j|} = K\frac{|1+j|}{|-1+j||2|}$$

$$\Rightarrow \boxed{k=2} \quad \#$$

Alternative way of solving:

with the compensator in place the closed-loop characteristic polynomial is given by:

$$s(s+2) + k(s+a) = s^2 + (2+k)s + ak$$

Since we want a root at $s_1 = -2 + j2$ we must have also a root at $\bar{s}_1 = -2-2j$. Therefore, the char polynomial should be of the form:

$$(s+2+2j)(s+2-2j) = (s+2)^2 + 4 = s^2 + 4s + 8$$

Equating both expressions we get: $\begin{matrix} 2+k = 4 \\ ak = 8 \end{matrix} \Rightarrow \boxed{\begin{matrix} k=2 \\ a=4 \end{matrix}}$

- **PID control**: $G_c(s) = k_P + \frac{k_I}{s} + k_D s = \frac{k_D s^2 + k_P s + k_I}{s}$ 2 zeros 1 pole

you can use the PI part to improve steady state and the PD part to improve the transient.

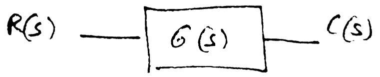
Frequency Response Analysis

- Most classical control design tools are based upon the use of frequency domain information (i.e. the way the system responds to signals of various frequencies)

From 1960-1980 most of the theoretical work was based upon "time-domain" techniques (so called "Modern Control"). However, practicing engineers kept using freq. domain

Now we use a mixture (some freq. domain motivated methods, but implemented using "state space" techniques : $\mathcal{H}_\infty$, robust control, μ)

- Start by considering the output of a system excited by a sinusoidal input (later we will see how to use this information for analysis)



G(s) stable, LTI system

Assume r(t) is a sinusoidal wave : $r(t) = A\cos\omega_0 t = A\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$

$$\Rightarrow R(s) = \frac{A}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$\Rightarrow C(s) = G(s)R(s) = \frac{A}{2}G(s)\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$\underset{\substack{\uparrow \\ \text{partial} \\ \text{fraction expansion}}}{=} \frac{k_1}{s-j\omega_0} + \frac{k_2}{s+j\omega_0} + \underset{\substack{\uparrow \\ \text{due to the} \\ \text{poles of } G}}{C_g(s)}$$

where $k_1 = \frac{A}{2}G(j\omega_0)$

$k_2 = \frac{A}{2}G(-j\omega_0) = k_1^*$

Taking the inverse Laplace transform we get:

$$c(t) = k_1 e^{j\omega_0 t} + k_2 e^{j\omega_0 t} + c_g(t)$$

where $c_g(t) \to 0$ as $t \to \infty$ since $G(s)$ is stable

Therefore in steady-state we get: $c(t) = k_1 e^{j\omega_0 t} + k_1^* e^{-j\omega_0 t}$ where $k_1 = \frac{A}{2}G(j\omega_0)$

write now $G(j\omega_0) = |G(j\omega_0)|e^{j\theta} \Rightarrow$ $k_1 = \frac{A}{2}|G(j\omega_0)|e^{j\theta}$

$k_1^* = \frac{A}{2}|G(j\omega_0)|e^{-j\theta}$

$$\Rightarrow c(t) = A|G(j\omega_0)|\,\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2} = |G(j\omega_0)|\,A\cos(\omega_0 t+\theta) \quad \#$$

The output is another sinusoidal wave of the same frequency, but the magnitude has been multiplied by $|G(j\omega_0)|$ and the phase shifted by $\theta = \angle G(j\omega_0)$

Gain & phase shift are frequency dependent

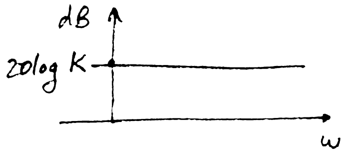
$G(j\omega)$ is called the <u>frequency response</u>. It is usually shown in the form of a plot $G(j\omega)$ vs ω.

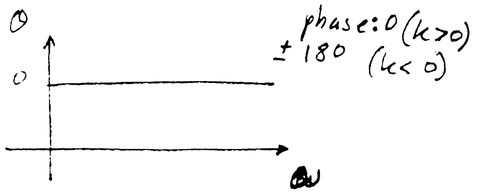
- <u>Bode diagrams</u>: Plot of the magnitude versus frequency <u>and</u> phase versus frequency

The freq & magnitude axes have a logarithmic scale.

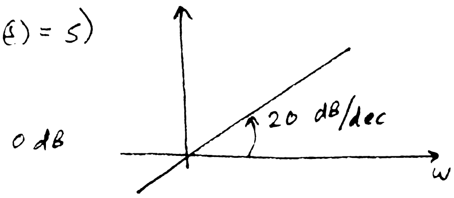
<u>Summary of Bode plots</u>

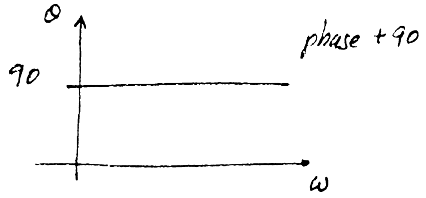
- constant gain:



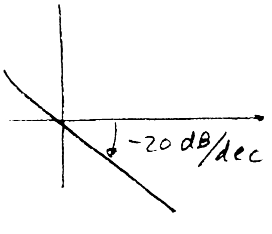


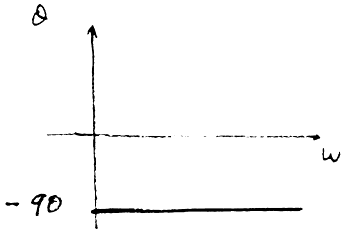
- zero at the origin $(G(s) = s)$



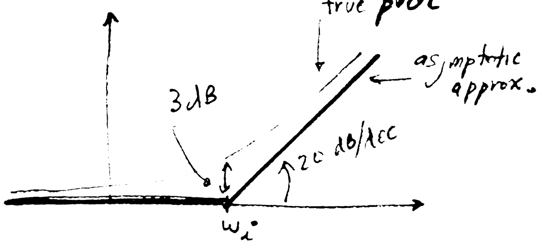


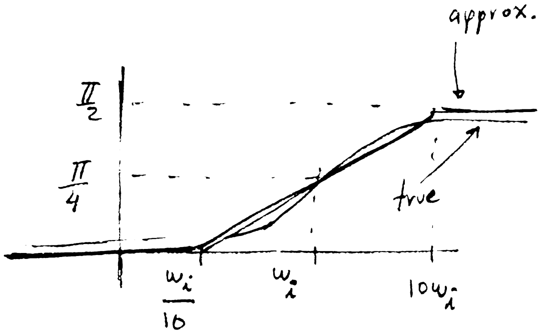
- pole at the origin $(G(s) = 1/s)$



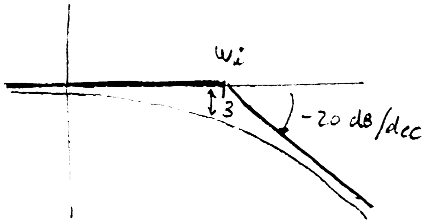


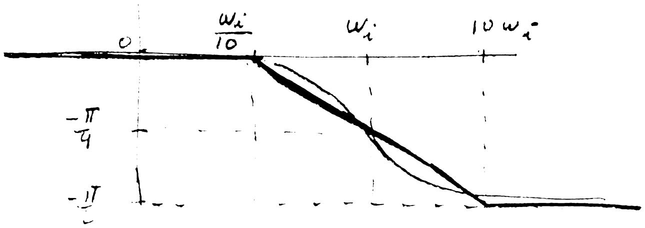
- real zero: $G(s) = 1 + \frac{s}{\omega_i}$





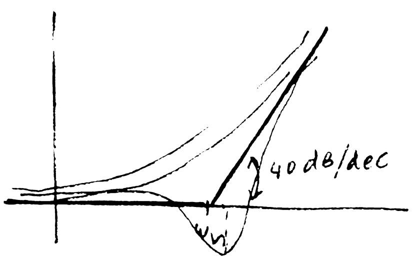
- real pole : $G(s) = \frac{1}{1+s/\omega_i}$

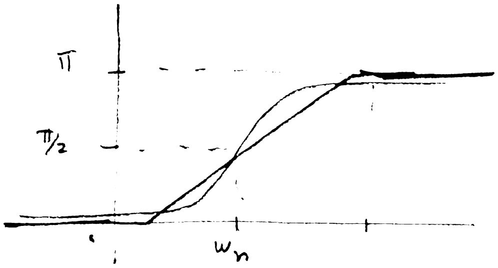




- complex conjugate zeros

$$G(s) = \frac{s^2}{\omega_n^2} + 2\frac{\rho s}{\omega_n} + 1$$



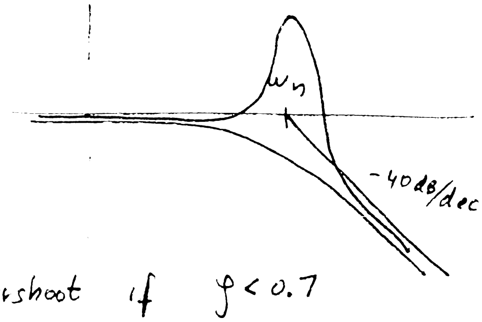


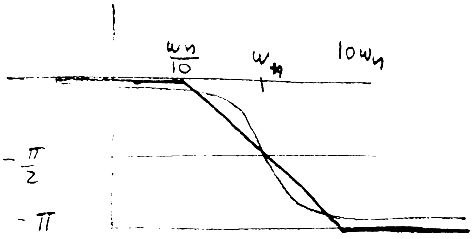
overshoot for $\rho < 0.7$; $|G| = \frac{1}{2\rho\sqrt{1-\rho^2}}$

$\omega_n = \omega_n\sqrt{1-2\rho^2}$

- Complex conjugate poles :

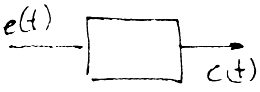
$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\frac{\rho s}{\omega_n} + 1}$$





Overshoot if $\rho < 0.7$

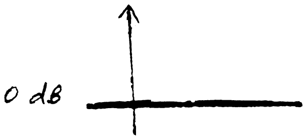
- Ideal time delay:

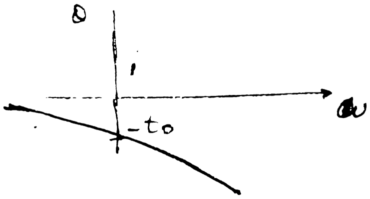


$c(t) = e(t-t_0)\,u(t-t_0)$

$\Rightarrow C(s) = E(s)\,e^{-st_0} \Rightarrow G = \frac{C}{E} = e^{-st_0} \Rightarrow G(j\omega) = e^{-j\omega t_0}$

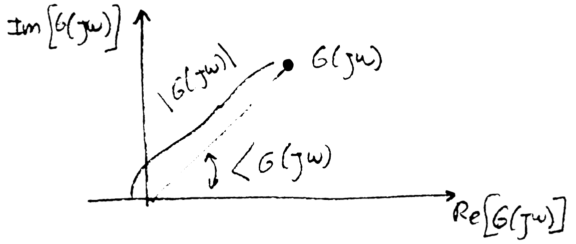
$|G(j\omega)| = 1$, $\angle G(j\omega) = -\omega t_0$





- Polar plots :

Plot of $G(j\omega)$ in the complex plane for $0 \leq \omega < \infty$
(ω is a parameter that does not appear explicitly in the plot)

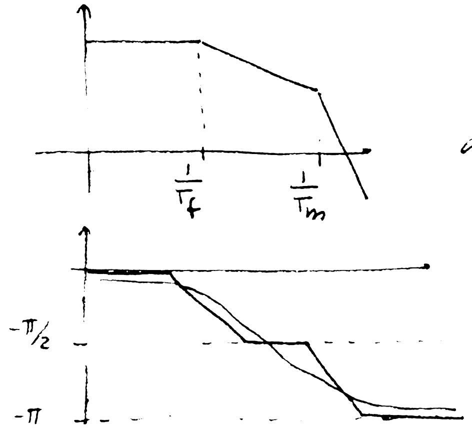


The data for drawing (an approximate) polar plot can be obtained from the Bode diagram

- Example : Type 0 system (for instance a servo motor)

$$G(j\omega) = \frac{k_0}{(1+j\omega T_f)(1+j\omega T_m)} \Rightarrow G(j\omega) \to \begin{cases} K_0 & \omega \to 0^+ \\ 0\angle{-186} & \omega \to \infty \end{cases}$$

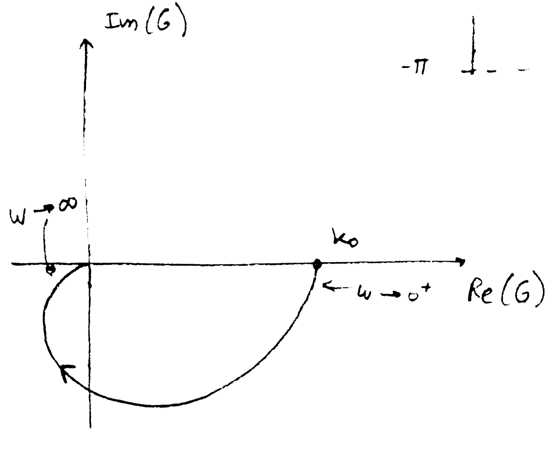
Rough Bode diagram:



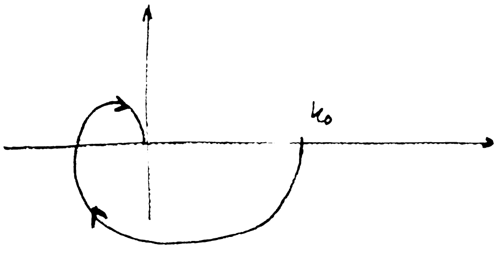
Magnitude always decreasing

Phase starts at 0, ends at -180 and changes continuously (goes thru -90)

$\Rightarrow$ Plot is continuously decreasing going clockwise from 0 to -180



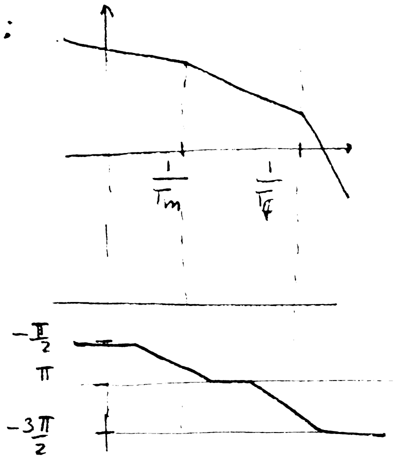
If we add an additional pole, it will add an extra $-\pi/2$ to the phase (i.e : if $G(s) = \frac{K_0}{(1+sT_1)(1+sT_2)(1+sT_3)}$)



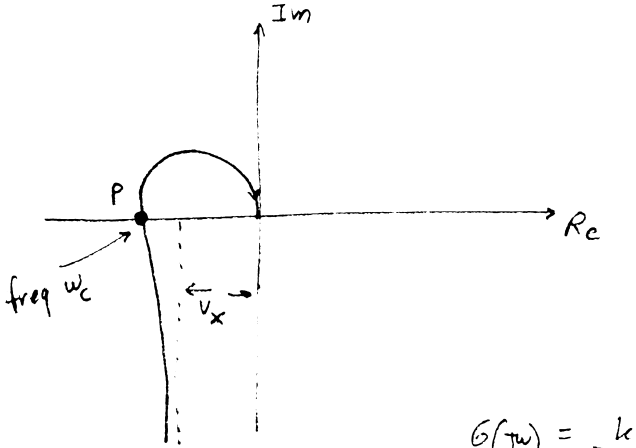
Example 2 : Type 1 system : $G(s) = \dfrac{k_1}{s\,(1+sT_m)(1+sT_f)}$

$$G(jw) \rightarrow \begin{cases} \infty \angle{-90} & w \rightarrow 0^+ \\ 0 \angle{-270} & w \rightarrow \infty \end{cases}$$

Bode plot:



Phase starts at -90, ends at -270
Magnitude starts at ∞ and decreases as the system goes thru the third and second quadrant



$$V_x = \lim_{w \to 0} Re\,(G(jw))$$

w_c frequency at which the plot crosses the real axis

$$G(jw) = \frac{k_1}{jw\,(1+jwT_m)(1+jwT_f)}$$

$$= \frac{k_1(-j\cancel{w})(1-jwT_m)(1-jwT_f)}{w^{\cancel{2}}(1+T_m^2w^2)(1+T_f^2w_f^2)} \Rightarrow \lim_{w\to 0} Re\,[G(jw)] = k_1(T_m+T_f) = V_x$$

To find P and w_c, need to solve $Im\{G(jw_c)\} = 0$

In this case: $G(jw) = -j\,k_1 \dfrac{(1 - w^2 T_m T_f - jw(T_m+T_f))}{w\,(1+w^2T_m^2)(1+w^2T_f^2)}$

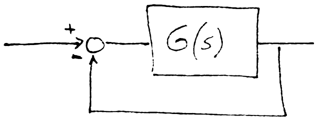
$Im\{G(jw_c)\} = 0 \Rightarrow -k_1(1 - w_c^2 T_m T_f) = 0$ // $w_c = \dfrac{1}{\sqrt{T_m T_f}}$ #

$$P = Re\{G(jw_c)\} = (-j)^2 \frac{k_1}{\cancel{w_c}} \frac{\cancel{w_c}(T_m+T_f)}{(1+w_c^2T_m^2)(1+w_c^2T_f^2)} = -\frac{k_1(T_m+T_f)}{\left(1+\frac{T_m^{\cancel{2}}}{\cancel{T_m}T_f}\right)\left(1+\frac{T_f^{\cancel{2}}}{T_m\cancel{T_f}}\right)} =$$

$$= -k_1 \frac{T_m T_f}{(T_m+T_f)} = -k_1 \frac{1}{\left(\frac{1}{T_m}+\frac{1}{T_f}\right)}$$

- <u>Nyquist Criterion</u>

We will use polar plots (i.e. frequency domain data) to assess the stability properties of the closed-loop system:



We will use the frequency response plot of the <u>open-loop</u> function ($G(j\omega)$) to determine whether or not the <u>closed-loop</u> system is stable

<u>Advantages</u>

- We don't need to know the T.F. We can get the required information from experimental data
- We also get information about the type of compensation required to stabilize the system and on how "robust" the system is. (i.e: how much uncertainty it can tolerate before becoming unstable)

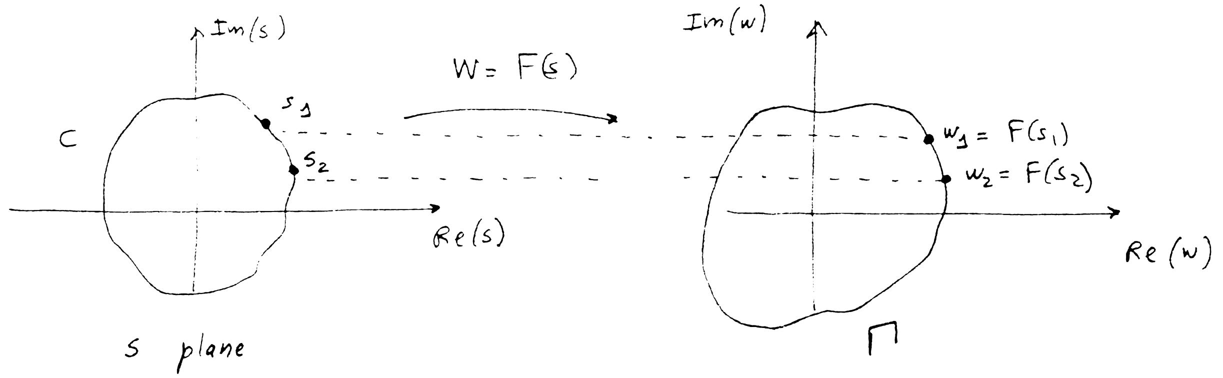
- <u>Basic tool</u>: <u>Cauchy's argument principle</u>

First we need the concept of mappings (or functions) in the complex plane

Suppose we have a function that maps the complex plane into itself i.e:

$$\underset{\text{complex number}}{W} = F(\underset{\text{complex number}}{s}) \qquad \left(\text{for instance: } W = (s-1)(s-2)\right)$$

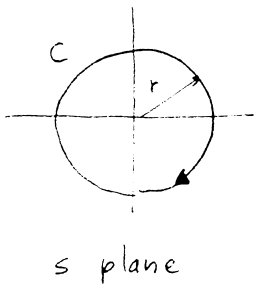
If we have a closed-curve C in the s plane, then F maps C into a closed-curve Γ in the w plane

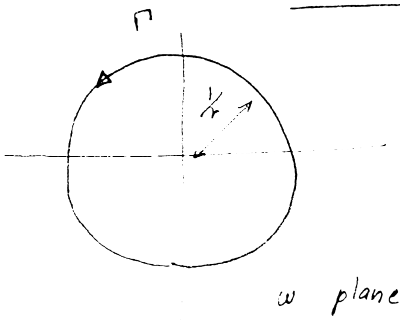


<u>Example 1</u> : $F(s) = \frac{1}{s}$ C: circle centered at $s=0$, radius r oriented clockwise

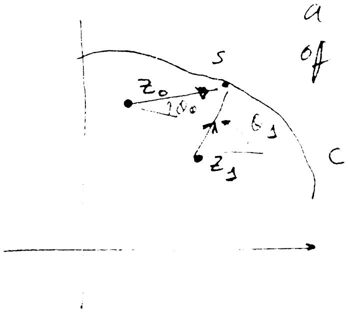
$\Rightarrow \Gamma$: circle centered at $s=0$, radius $1/r$ <u>oriented counterclockwise</u>

$\Rightarrow$ <u>Orientation matters</u>





<u>Example 2</u> : Consider now the function $F(s) = (s-z_0)(s-z_1)$
This function has 2 zeros $(s=z_0,\ s=z_1)$, and for a given s you get $F(s)$ by taking the product of the vectors $(s-z_0) \times (s-z_1)$

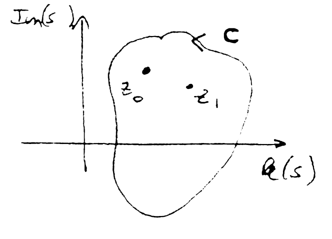


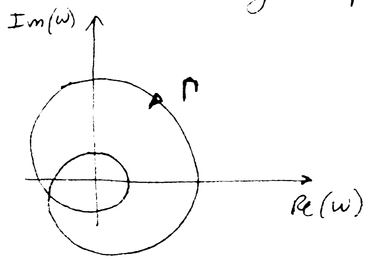
Suppose that we take a curve C that encircles <u>both</u> z_0 & z_1. Then, as **s** travels around C (clockwise):

$\theta \angle s - z_0$ changes by -360 $\Rightarrow$
$\theta \angle s - z_1$ changes by -360

$\theta \angle F(s) = \theta \angle s - z_0 + \theta \angle s - z_1$ changes by -720

which means that Γ <u>must</u> encircle the origin of the w plane <u>twice</u>, clockwise





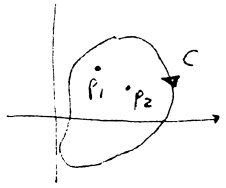
- <u>Example 3</u> : Suppose that rather than 2 zeros we have 2 poles, i.e:

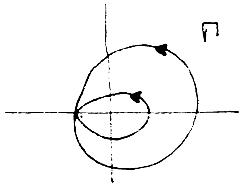
$F(s) = \frac{1}{(s-p_0)} \frac{1}{(s-p_1)}$ $\Rightarrow$ $F(s)$ formed by taking the product of the inverse of the vectors $(s-p_0)$ & $(s-p_1)$

$\Rightarrow \angle F(s) = \angle -(s-p_0) - \angle (s-p_1)$

In this case, as s travels around C, θ_1 changes from 0 to -360 and $\theta_2 = \angle s - p_1$ also changes from θ_1 to -360

$\Rightarrow \angle F(s)$ changes from 0 to $+720$ $\Rightarrow$ Γ encircles the origin in the w plane <u>twice</u> counter-clockwise





- All these examples are a special case of <u>Cauchy's argument principle</u>

This principle relates the number of poles & zeros of F(s) enclosed by C to the number of times that Γ encircles the origin of the w plane.

- <u>Theorem</u>: Consider a closed contour C in the s-plane and let F(s) be an analytic function (has continuous derivative) with a finite number <u>Z</u> of zeros and <u>P</u> of poles <u>inside</u> C and having neither zeros nor poles on C

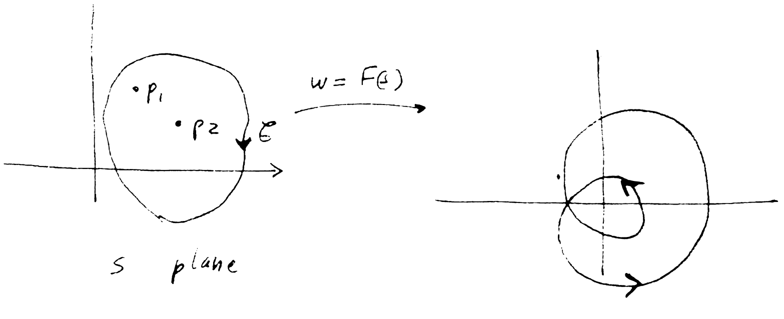
Then, the image Γ of C under the mapping w = F(s) encircles the origin of the w-plane a number N of times given by:

$$N = Z - P$$

where N: number of encirclements of the origin in the w-plane (taken with the same orientation as for C)

Z: number of zeros of F(s) enclosed by C
P: number of poles of F(s) enclosed by C

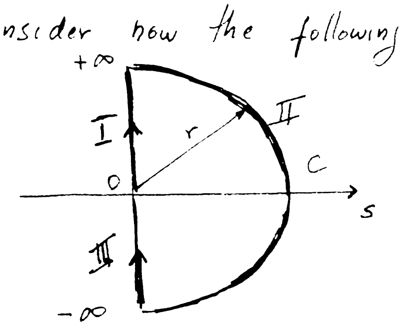
<u>Example</u>: $F(s) = \frac{k}{(s+p_1)(s+p_2)}$



$Z = 0, \quad P = 2$
$N = Z - P = -2$

(two times around, <u>counterclockwise</u>)

- Consider now the following path in the s-plane

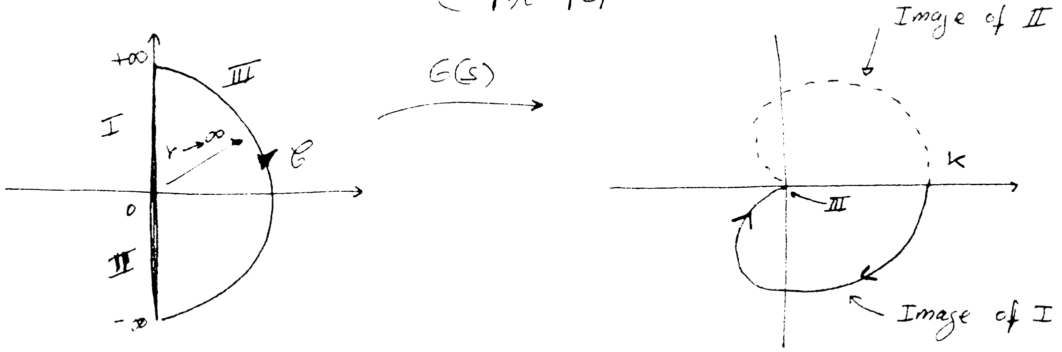


If we let $r \to \infty$, then C encloses all of the right half plane

The portion of C on the jw axis from 0 to +∞ gives <u>exactly</u> the polar plot of F

If F is a real rational function, the portion from $-\infty$ to 0 is the mirror image (around the real axis) of the polar plot.

- Example: $G(s) = \dfrac{k}{(s+p_1)(s+p_2)}$

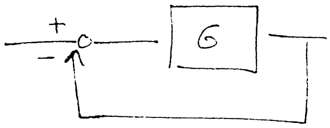


s - plane

w plane

- Application of the principle of the argument to control system design (Nyquist criterion)

The idea is to exploit this results to find out how many (if any) roots does the characteristic equation have in the right half plane



Let $F(s) = G(s) + 1$

$\Rightarrow$ closed loop stable $\Leftrightarrow$ F(s) does not have zeros in the RHP

If we take $\mathscr{C}$ has in the previous example (enclosing the RHP) we have that:

$$N = Z - P$$

number of zeros of $F = 1 + G(s)$ in the RHP

Assume for simplicity that G(s) does not have RHP poles $\Rightarrow P = 0$ and

$$N = Z \quad \Rightarrow \quad Z = 0 \Leftrightarrow N = 0$$

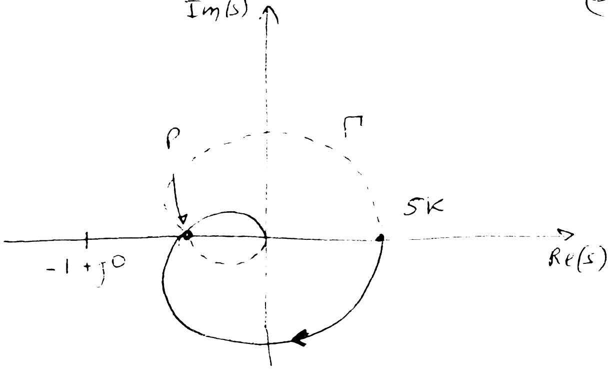
In other words, closed loop stable if and only if the polar plot of $F(s) = 1 + G(s)$ does not encircle the origin.

The closed loop system is stable <u>if and only if</u> the Nyquist plot of $GH(j\omega)$ does not encircle the origin

- <u>Example</u>: Let $G(s)H(s) = \frac{5K}{(s+1)^3}$

$\omega \to 0 \quad G(0)H(0) = 5K$

$\omega \to \infty \quad GH \to 0 \angle -270$

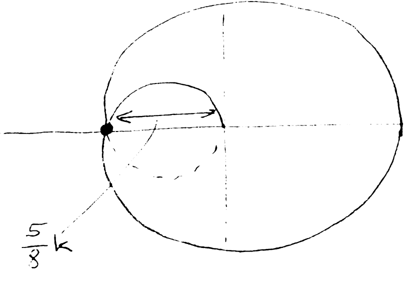


System stable iff Γ does not encircle $-1+j0$ ⇒ depends on the position of P

⇒ <u>Need to find P</u>: $\text{Im}\, GH(\omega)\Big|_P = 0 \Rightarrow$

$$\text{Im}\left\{\frac{5K}{1-3\omega^2 - j\omega(\omega^2-3)}\right\} = 0 \Leftrightarrow \omega = \sqrt{3}$$

for $\omega = \sqrt{3}$ we have $GH(j\sqrt{3}) = \frac{5K}{1-9} = -\frac{5}{8}K$



stable if $\frac{5}{8}K < 1 \Rightarrow -1$ outside plot

unstable if $\frac{5}{8}K > 1 \Rightarrow$ in this case $N=2$
⇒ 2 <u>unstable</u> roots

Sanity check using Routh Hurwitz:

Char. equation: $(s+1)^3 + 5K = 0 \qquad s^3 + 3s^2 + 3s + 1 + 5K = 0$

$$\begin{array}{lll} s^3 & 1 & 3 \\ s^2 & 3 & 1+5K \\ s & \frac{8-5K}{3} & \\ s^0 & 1+5k & \end{array}$$

stable if $8 - 5K > 0 \Rightarrow K < \frac{8}{5}$

if $k > \frac{8}{5} \Rightarrow$ 2 sign changes ⇒ 2 unstable roots.