

Name: _____ Signature: _____

Instructions:

1. This is a closed-book test but **one** $8\frac{1}{2} \times 11$ single-sided cheat-sheet is allowed.
2. Work as many problems as you can. Try not to spend too much time working on a single problem. If you get stuck, try working on a different question.
3. Show all your work, but try to be as concise as possible.
4. • **DO NOT LOOK** at the problems until told to do so.
5. • **STOP** working after the “time’s up” announcement.
6. • **GOOD LUCK !**

Problem 1: (50 points)

1.1- (10 pts) The (continuous-time) transfer function of a *first hold* is given by:

$$G_{foh}(s) = \frac{1 + Ts}{T} \left[\frac{1 - e^{-Ts}}{s} \right]^2 \quad (1)$$

Find an analytic expression (in the time domain) for its impulse response and sketch it.

1.2- (20 pts) Consider the block diagram shown in Figure 1 Show that this diagram is equivalent to a first order hold. Hint: put an appropriate input, sketch the corresponding output $c(t)$ and compare it with the output of a first order hold.

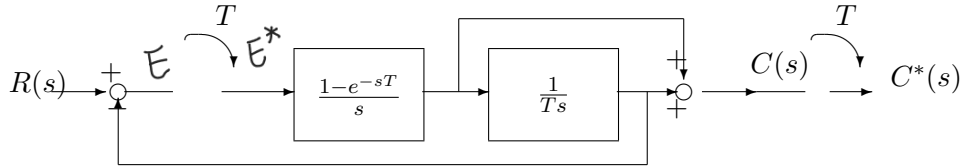


Figure 1: Block Diagram for Problem 1

1.3- (20 pts)

- (a) Show that for frequencies $\frac{\omega}{\omega_s} \ll 1$ the phase characteristic of the first order hold (equation (1) is given by

$$\text{angle}\{G_{foh}(j\omega)\} \simeq -\frac{2\pi\omega}{\omega_s} \quad (2)$$

where $\omega_s = \frac{2\pi}{T}$ is the sampling frequency in radians per second.

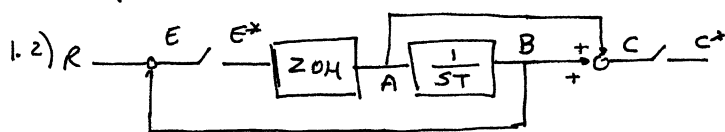
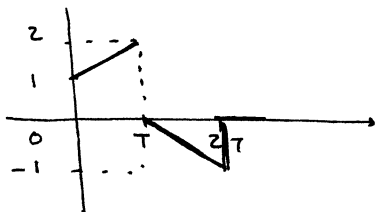
- (b) Assume that this hold will be used in a closed-loop system where the plant has bandwidth 10 Hz and can tolerate up to $\frac{\pi}{6}$ radians of phase lag. Find the minimum sampling interval so that this condition is satisfied. For this part you may use the approximation (2).

$$1.1) G_{fch} = \left(\frac{1+Ts}{T}\right) \left(\frac{1-e^{-sT}}{s}\right)^2 = \left(\frac{1}{s} + \frac{1}{Ts}\right) (1 + e^{-2sT} - 2e^{-sT})$$

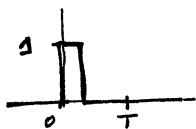
$$g_{fch}(t) = \mathcal{L}^{-1}[G_{fch}] = \frac{1}{T} [t v(t) - 2(t-T) v(t-T) + (t-2T) v(t-2T)] + [v(t) - 2v(t-T) + v(t-2T)]$$

collecting terms yields: $\left(1 + \frac{t}{T}\right) v(t) - 2\frac{t}{T} v(t-T) + \left(\frac{t}{T} - 1\right) v(t-2T)$ or

$$\left[\left(1 + \frac{t}{T}\right) [v(t) - v(t-T)] + \left(1 - \frac{t}{T}\right) [v(t-T) - v(t-2T)] \right]$$



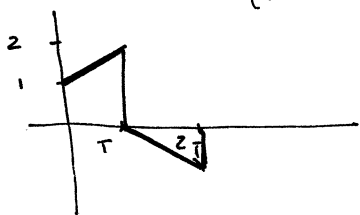
Take as input R



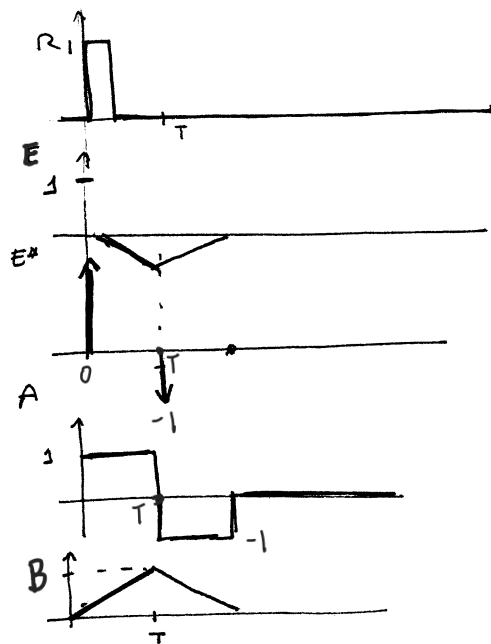
$$E^* = \sum_0^{\infty} e(k) \delta(t-kT)$$

(B is the \int of A)

$$C = A + B$$



same as in part (1)



$$1.3) G_{fch}(j\omega) = \left(\frac{1+j\omega T}{T}\right) \left(\frac{1-e^{-j\omega T}}{j\omega}\right)^2 = \left(\frac{1+j\omega T}{T}\right) e^{-j\omega T} \left(T \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j\omega T/2} \right)^2 =$$

$$= (1+j\omega T) e^{-j\omega T} T \underbrace{\text{sinc}^2 \frac{\omega T}{2}}_{\text{always positive}}$$

$$\angle G = -\omega T + \angle(1+j\omega T) \sim -\omega T \text{ if } \omega T \ll 1$$

$$\angle G \approx -2\pi \frac{\omega}{\omega_s} = -\omega T \text{ where } T = \frac{2\pi}{\omega_s}$$

$$\text{want } -2\pi \frac{\omega}{\omega_s} < \frac{\pi}{6} \text{ for all } \omega \in [0, 10 \text{ kHz}] \Rightarrow \omega_s > 12 \omega_b \Rightarrow f_s > 120 \text{ Hz}$$

$$T < 8.3 \text{ msec}$$

bandwidth frequency.
for statement (question)

Problem 2: (50 points)

2.1- (20 pts) Consider the model of a first order hold shown in Figure 1 (**previous page**).

- Find the discrete transfer function $E(z)/R(z)$.
- Find the discrete transfer function $C(z)/R(z)$. Does your result make sense (recall that this block diagram is supposed to implement a first order hold)

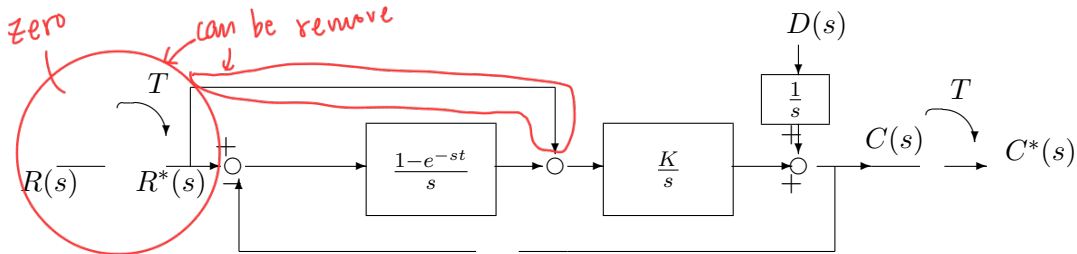
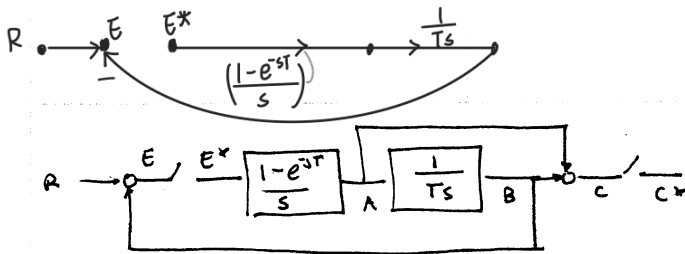


Figure 2: Block Diagram for Parts 2.2 and 2.3

- (15 pts) Consider the block diagram shown in Figure 2 above. Assume that $R = 0$. Does the discrete transfer function from D to C exist? If so, find it. If not, fully justify your answer and find $C(z)$ for the case where $D(s) = 1$ (i.e. $d(t) = \delta(t)$).
- (15 pts) Find a pair of values for K and T such that, when $R = 0$ and $D(s) = 1$, the steady state value of $c_{ss} = \lim_{k \rightarrow \infty} c_k$ satisfies $|c_{ss}| < 0.1$. Compare the answer with the one that you would have gotten in the case of a continuous time system (ie same loop without sampling and holding action). Explain both similarities and differences.



choose E as output

input R, E*
output C, E

$$a) E = R - \frac{A}{sT} = R - \left(\frac{1-e^{-sT}}{s}\right) \frac{1}{sT} E^* \quad // \quad E^* \left(1 + \frac{1-e^{-sT}}{Ts^2}\right) = R^*$$

$$E(z) = \frac{1}{1 + \left(\frac{z-1}{z}\right)^2 \left[\frac{1}{Ts^2}\right]} R(z)$$

From tables: $\mathcal{Z}\left[\frac{1}{Ts^2}\right] = \frac{z}{(z-1)^2} \Rightarrow E(z) = \frac{1}{1 + \left(\frac{z-1}{z}\right)^2 \frac{z}{(z-1)^2}} R(z) = \left(\frac{z-1}{z}\right) R(z) \neq$

$$C = \left(1 + \frac{1}{Ts}\right) A = \left(1 + \frac{1}{Ts}\right) \left(\frac{1-e^{-sT}}{s}\right) E^*$$

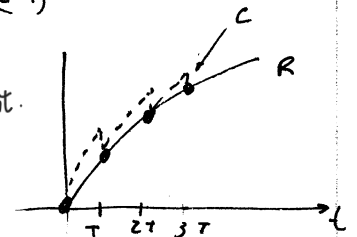
$$C^* = \left[\left(\frac{1-e^{-sT}}{s}\right)^* + \left(\frac{1-e^{-sT}}{Ts^2}\right)^*\right] E^* \Rightarrow C(z) = \left[\left(\frac{z-1}{z}\right) \left(\frac{z}{z-1}\right) + \left(\frac{z-1}{z}\right) \left(\frac{z}{(z-1)^2}\right)\right] E(z)$$

$$C(s) = (1-e^{-sT}) \left[\left(1 + \frac{1}{Ts}\right) \cdot \frac{1}{s}\right] E^* \Rightarrow C(z) = \frac{z-1}{z} \mathcal{Z}\left[\frac{1}{s} + \frac{1}{Ts}\right] E(z) \left[1 + \frac{1}{(z-1)}\right] E(z) = \frac{z}{(z-1)} E(z)$$

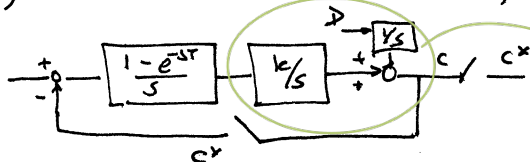
$$\frac{C(z)}{R(z)} = \frac{C}{E} \cdot \frac{E}{R} = \left(\frac{z}{z-1}\right) \cdot \left(\frac{z-1}{z}\right) = \boxed{1}$$

sync of output should be equal to sync of input.

As expected. Since the block diagram implements a hold then $c(kT) = r(kT)$ all $k \Leftrightarrow C(z) = R(z)$



b) Since $R=0 \Rightarrow$ eliminate it from diagram:



$$\frac{C}{D} = \frac{1/s}{1 + k/s} = \frac{1}{sTk}$$

TF from D to C ONE since D goes through the $1/s$ block before being sampled

$$C = \frac{D}{s} \Rightarrow \left[\left(1-e^{-sT}\right) \frac{k}{s^2}\right] C^* \Rightarrow C^* = \frac{\left(\frac{D}{s}\right)^*}{1 + \left(\frac{1-e^{-sT}}{s^2}\right)^* k}$$

$$D=1 \Rightarrow C(z) = \mathcal{Z}\left(\frac{1}{s}\right) \frac{1}{1 + \left(\frac{z-1}{z}\right)^2 \left(\frac{1}{s^2}\right) k} = \frac{z}{(z-1)^2} \left[1 + \left(\frac{z-1}{z}\right)^2 \frac{Tk}{(z-1)^2}\right] = \frac{z}{z-1+Tk}$$

$$c) \text{ FVT: } \lim_{z \rightarrow 1} (z-1) C(z) = \lim_{z \rightarrow 1} \frac{(z-1) z}{z-1+Tk} = \boxed{0} \quad \text{provided that is stable}$$

$$\Rightarrow -1 < -1+Tk < 1 \quad // \quad 0 < Tk < 2 \quad \Rightarrow \text{pick } T, k \text{ so that}$$

$$\boxed{0 < Tk < 2}$$

d) Type 1 system, so it is expected to reject step disturbances. However: continuous time = stable for all $k > 0$
discrete time = unstable if $Tk \geq 2$ due to extra phase lag

from continuous time: $1 + \frac{k}{s} = 0 \Rightarrow s = -k$
when k is positive \Rightarrow means STABLE

$$\text{pole} \Rightarrow z = 1 - kT$$

$$|z| < 1$$

$$\Rightarrow -1 < 1 - kT < 1$$

$$-2 < -kT < 0$$

$$\Rightarrow 2 > kT > 0$$

give discrete time poles

know settling time / like $\frac{4\pi}{\zeta}$