

## • STABILITY ANALYSIS (Chapter 6)

Recall: A system is BIBO stable iff the output remains bounded for every bounded input

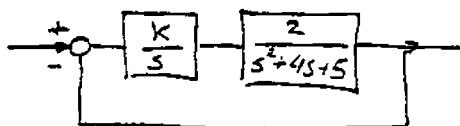
$$\int_0^{\infty} |g(t)| dt < M \quad (\text{a testable condition})$$

All poles of  $G(s)$  are in the left half plane

This last condition is far easier to test than computing the integral. it entails finding all the roots of a polynomial, which for polynomials of degree  $> 4$  must be done numerically.

Key observation: If we are interested in <sup>stability</sup> we do not need to find all the roots, we just need to find out whether or not there are roots in the  $j\omega$ -axis or the right half plane.

Example: laser position control with integral action:



We have seen that in order to reduce steady-state error to a ramp we'd like to take  $K$  as large as possible. However, large  $K$  (in this case  $K \geq 10$ ) render the system unstable.

Characteristic equation:  $s^3 + 4s^2 + 5s + 2K = 0$

$\Rightarrow$  Rather than solving the equation for each value of  $K$ , we'd like to be able to tell whether or not we have roots in the RHP (hopefully an easier task)

Q: is finding-out the number of roots in the RHP easier than solving the equation?

A: Yes! We are going to see a procedure that accomplishes just that. This procedure is based upon the relationship between the roots and the coefficients of a polynomial

• Relationship between the roots and the coefficients of a polynomial

• Consider a second order polynomial:  $P_2(s) = s^2 + a_1 s + a_0 = (s - p_1)(s - p_2)$

$$= s^2 - (p_1 + p_2)s + p_1 p_2$$

Equating power of  $s$  yields:  $a_1 = -(p_1 + p_2)$  (sum of the roots)  
 $a_0 = p_1 \cdot p_2$  (product of roots)

• For a 3<sup>rd</sup> order polynomial we have:

$$P_3(s) = s^3 + a_2 s^2 + a_1 s + a_0 = (s - p_1)(s - p_2)(s - p_3) = s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_2 p_3 + p_1 p_3)s - p_1 p_2 p_3$$

Equating powers of  $s$ :  $a_2 = -(p_1 + p_2 + p_3)$   
 $a_1 = (p_1 p_2 + p_1 p_3 + p_2 p_3)$   
 $a_0 = -p_1 \cdot p_2 \cdot p_3$

• In general we have:  $P_n(s) = s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0 = (s - p_1)(s - p_2) \dots (s - p_n)$

$$a_{n-1} = -\sum p_i \quad \text{-(sum of the roots)}$$

$$a_{n-2} = \sum_{i \neq j} p_i p_j \quad \text{(sum of products)}$$

$$a_0 = (-1)^n p_1 p_2 \dots p_n$$

What can we conclude from these relationships?

If all the roots have negative real parts then  
 all  $a_i > 0$ . Conversely if there is at least  
 one  $a_i \leq 0$  then there are roots on the jw-axis or RHP

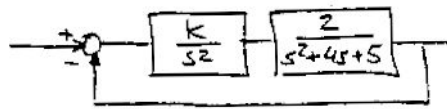
Examples: 1)  $p_1 = -1 + j \Rightarrow (s - p_1)(s - p_2) = (s + 1 - j)(s + 1 + j) = s^2 + 2s + 2$   
 $p_2 = -1 - j$   
 $a_1 = 2 = -(p_1 + p_2)$   
 $a_0 = 2 = p_1 p_2$

Notice that the imaginary parts cancel out

2)  $p_1 = (-1 + j) \Rightarrow (s - p_1)(s - p_2)(s - p_3) = s^3 + s^2 - 2$   
 $p_2 = (-1 - j)$   
 $p_3 = 1$

Here we have:  $a_2 = 1$   
 $a_1 = 0$   
 $a_0 = -2$

3) Laser position control with double integral action:



char. equation:

$$s^4 + 4s^3 + 5s^2 + 2K = 0$$

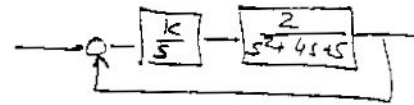
missing the term in  $s^1 \Rightarrow$   
always has roots in RHP

$\Rightarrow$  unstable for all  $K$

Q: Is this a reliable way of telling?

A: yes and no!

Example:



Take  $K=10 \Rightarrow$  char equation:

all  $a_i > 0$ , yet:

$$s^3 + 4s^2 + 5s + 20 = 0$$

$$p_1 = -4$$

$$p_2 = +2.236j$$

$$p_3 = -2.236j$$

on  $j\omega$ -axis  
marginally stable

What happens is that we got only a necessary condition for stability (not sufficient),

i.e if at least one  $a_i \leq 0 \Rightarrow$  NOT BIBO stable  
if all  $a_i > 0$  CAN'T TELL

So we need to keep working to get a necessary and sufficient condition

### Routh-Hurwitz Stability Criterion

simple proof: A. Chappellat, M. Mansour & S.P. Bhattacharyya, "Elementary Proof of Some Classical Stability Criteria," IEEE Trans on Education, 33, 3, 1990

It is an analytical procedure to determine if all roots of a polynomial have negative real parts. The test gives the number of roots with positive real part, without actually having to solve for these roots.

Consider the polynomial:  $P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$

1) Step one: Form the Routh array:

$s^n$	$a_n$	$a_{n-2}$	$a_{n-4}$	$a_{n-6}$
$s^{n-1}$	$a_{n-1}$	$a_{n-3}$	$a_{n-5}$	$a_{n-7}$
$s^{n-2}$	$b_1$	$b_2$	$b_3$	$b_4$
$s^{n-3}$	$c_1$	$c_2$	$c_3$	$c_4$
$s^2$	$k_1$	$k_2$		
$s^1$	$l_1$			
$s^0$	$m_1$			

Each row, except for the first 2, is calculated from the 2 rows directly above it as follows:

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}; \quad b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}; \quad c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

For the  $i^{th}$  coefficient of a given row you take  $(-1)^i$  determinant formed by the first and  $(i+1)$  columns of the 2 preceding rows and divide it by the first element of the preceding row.

Example: 1)  $P_1(s) = s^3 + s^2 + 2s + 8$  (roots at  $-2, \frac{1}{2} \pm j\frac{\sqrt{15}}{2}$ )

$s^3$	1	2
$s^2$	1	8
$s^1$	$\frac{2-8}{1} = -6$	0
$s^0$	8	

2)  $P_2(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$

$s^6$	1	3	1	4
$s^5$	$\frac{4}{2}$	$\frac{2}{1}$	$\frac{4}{2}$	
$s^4$	$\frac{5}{2}$	0	4	
$s^3$	1	$-\frac{6}{5}$	0	
$s^2$	3	4		
$s^1$	$-\frac{38}{15}$			
$s^0$	4			

→ note: you can divide (or multiply) a row by a number. Then you don't have to deal with large numbers (or fractions)

roots at:  $-3.26, -0.605 \pm j0.99$   
 $0.698 \pm j0.15$   
 $-0.386$

## • Routh Hurwitz Criterion:

The number of roots in the right half plane is equal to the number of sign changes in the first column of the array.

(Original work: Routh 1877, Hurwitz 1845)

Simple proof: K. Chappellat, M. Mansour & S. P. Bhattacharyya: "Elementary proof of some classical stability criteria, IEEE Trans. on Education, Vol 33, No 3, (1990)

Examples: for  $P_1(s)$  2 sign changes  $\Rightarrow$  2 roots in the RHP

$P_2(s)$  2 sign changes  $\Rightarrow$  2 roots in the RHP

Consequence: System is stable if and only if all elements of the first column of the array are positive.

• What happens if one of the elements of the first column is zero?  
For example:

$$P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

$$\text{roots at: } \begin{aligned} &0.8950 \pm j1.456 \\ &-1.2407 \pm j1.0375 \\ &-1.3087 \end{aligned}$$

$s^5$	1	2	11
$s^4$	2	4	10
$s^3$	0	6	
$s^2$	:		

← in principle we can't proceed any further, since to get the elements on the row  $s^2$  we would have to divide by 0!

Answer: The fact that we get 0 implies that the system is not BIBO stable. However, we may want to proceed with the array in order to be able to count the number of unstable roots. If we want to proceed we have the following alternatives:

Method 1: Replace the "zero" by a small number  $\epsilon$ , continue the calculation of the array and then let  $\epsilon \rightarrow 0$

$s^5$	1	2	11
$s^4$	1	2	10
$s^3$	$\epsilon$	6	
$s^2$	$\frac{2\epsilon - 6}{\epsilon}$	5	
$s^1$	$6 + o(\epsilon)$		
$s^0$	5		

↑ infinitesimal of order  $\epsilon$

let  $\epsilon \rightarrow 0$ , since  $\epsilon \ll -6$   
 $\Rightarrow$  2 sign changes in the first column

(regardless of the sign of  $\epsilon$ )

$\Rightarrow$  2 roots on the RHP

Example 2:  $P(s) = s^3 + s^2 + 2s + 2$

$$\begin{array}{rcl} s^3 & 1 & 2 \\ s^2 & 1 & 2 \\ s^1 & \phi & E \\ s^0 & 2 & \end{array}$$

roots at  $s_1 = -1$   
 $s_{2,3} = \pm j\sqrt{2}$

note that: if  $E > 0$  no sign changes  $\Rightarrow$  no roots RHP  
 $E < 0$  2 sign changes  $\Rightarrow$  2 roots RHP

$\Rightarrow$  continuity implies that in this case for  $E=0$  we have 2 roots on the  $j\omega$  axis.

Method 2: Use the change of variable  $x = 1/s$

Note that  $\phi(s) = 0 \Leftrightarrow \phi(1/x) \stackrel{\text{def}}{=} \phi_1(x) = 0$

Since the roots of  $\phi(s)$  and  $\phi_1(x)$  are the reciprocal of each other, stability is not affected (i.e. if  $\phi$  has a root in the RHP, so does  $\phi_1$ )

Example:  $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

$$P(1/x) = \frac{1}{x^5} + 2\frac{1}{x^4} + \frac{2}{x^3} + \frac{4}{x^2} + \frac{11}{x} + 10 = \frac{(1 + 2x + 2x^2 + 4x^3 + 11x^4 + 10x^5)}{x^5}$$

$$\Rightarrow P_3(x) = 10x^5 + 11x^4 + 4x^3 + 2x^2 + 2x + 1$$

(note that the order of the coefficients is reversed)

$$\begin{array}{rcl} x^5 & 10 & 4^2 & 2^1 \\ x^4 & 11 & 2 & 1 \\ x^3 & \frac{12}{11} & \frac{6}{11} & \\ x^2 & -7 & 1 & \\ x^1 & +9/7 & & \\ x^0 & 1 & & \end{array}$$

2 sign changes  $\Rightarrow$  2 roots RHP

- Note that to be able to apply these methods you need at least one element in the row to be non-zero
- Case where all elements of a row in the Routh array are zero

Example :  $P(s) = s^3 + s^2 + 2s + 2 = (s+1)(s^2+2) \Rightarrow$  roots at  $-1 \pm j\sqrt{2}$

$s^3$	1	2
$s^2$	1	2
$s^1$	0	
$s^0$	2	

Note : here the "E" trick will work, the  $1/x$  won't.

Note that in this case the polynomial  $P(s)$  contains an even polynomial as a factor. This is no accident : if we get a row of "zeros" in the array, then the polynomial contains an even (or odd) polynomial as a factor. This polynomial is called the auxiliary polynomial, and its coefficient appear in the row immediately above the zeros. (In this case the auxiliary polynomial is  $P_2(s) = s^2 + 2$ )

To continue with the Routh array, the auxiliary polynomial is differentiated with respect to  $s$ , and the coefficients of the resulting polynomial replace the zero row.

$$\begin{aligned} \phi(s) &= s^2 + 2 \\ \phi'(s) &= 2s \end{aligned}$$

$s^3$	1	2
$s^2$	1	2
$s^1$	2	0
$s^0$	2	

No sign changes

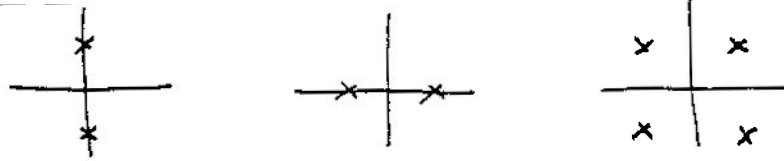
$\Rightarrow$  Number of roots in the RHP = number of sign changes in the modified array

+  
# roots of the auxiliary polynomial in the RHP.

In our cases we have

0 sign changes on the modified array + 2 roots of the aux. polynomial  
 $\Rightarrow$  total 2 RHP roots

The roots of an even polynomial are symmetric with respect to the origin (quadrupolar symmetry)



Example 6.4 text:

$$P(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

roots at:  $\pm j\sqrt{2}$   
 $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$s^4$	1	3	2
$s^3$	1	2	
$s^2$	1	2	
$s^1$	0		
$s^0$			

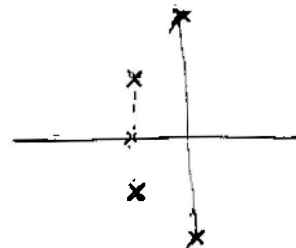
$\Rightarrow$  auxiliary equation:  $\Phi(s) = s^2 + 2 = 0$   
 $\Phi'(s) = 2s$

$\Rightarrow$  modified array

$s^4$	1	3	2
$s^3$	1	2	
$s^2$	1	2	
$s^1$	$\emptyset$	2	
$s^0$	2		

no sign changes here  
 however:  $s^2 + 2$  2 roots at  $\pm j\sqrt{2}$   
 $\Rightarrow$  2 roots in RHP +  $j\omega$  axis

$$P(s) = (s^2 + s + 1)(s^2 + 2)$$

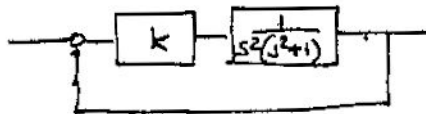


Suppose that we want to design a controller for satellite attitude control:

Performance specs: zero steady state error  
settling time  $\sim 10$  to  $15$  sec  
reasonable overshoot

Plant:  $G(s) = \frac{1}{s^2(s^2+1)}$   
rigid body mode  $\rightarrow$  first bending mode (freq 1 rad/sec)

Type 2 plant  $\Rightarrow e_{ss} = 0$  if stable. Try a simple controller  $G_c(s) = k$



Characteristic equation:  $s^4 + s^2 + k = 0 \Rightarrow$  missing  $s^3, s^1$  terms  $\Rightarrow$  unstable for all  $k$

$\Rightarrow$  Need a more sophisticated controller

$$\text{Try } G_c(s) = \frac{(s+0.5)((s+0.01)^2+1)}{(0.1s+1)^3}$$

(Later on we will learn how to choose appropriate controllers)

The corresponding closed-loop characteristic equation is:

$$\Delta(s) = 0 \Leftrightarrow 10^{-3}s^7 + 3 \cdot 10^{-2}s^6 + 0.301s^5 + 1.03s^4 + 1.3s^3 + 1.52s^2 + 1.0101s + 0.5005$$

From Routh-Hurwitz (or matlab) you can show that it is stable:

$$\text{roots } e: \begin{aligned} &-12.41 \pm j 3.45 \\ &-3.972 \\ &-0.0061 \pm j 1.0117 \\ &-0.5953 \pm j 0.6225 \end{aligned}$$

• Suppose now that we only know the mass of the satellite to within 10%

$\Rightarrow$  The transfer function of the plant is now:  $G_p(s) = \frac{1}{s^2(s^2+p)}$

with  $p \in [0.9, 1.1]$

and for  $p = 0.9$  the system is unstable!

The characteristic equation (as a function of  $p$ ) is:

$$\Delta(s, p) = 10^{-3}s^7 + 3 \cdot 10^{-2}s^6 + (3 \cdot 10^{-1} + 10^{-3}p)s^5 + (1 + 3 \cdot 10^{-2}p)s^4 + (3 \cdot 10^{-1}p + 1)s^3 + (p + 0.52)s^2 + 1.0101s + 0.5005$$

To establish stability, in principle we could use Routh Hurwitz. The array will be now a function of " $p$ " and we need the first column to be positive for all  $p \in [0.9, 1.1]$

The array looks like:

$s^7$	$10^{-3}$	$0.3 + 10^{-3}p$	$1 + 0.3p$	$1.0101$
$s^6$	$3 \cdot 10^{-2}$	$1 + 0.03p$	$0.52 + p$	$0.5005$
$s^5$	$f_1(p)$	$f_2(p)$	$f_3(p)$	
$s^4$	$g_1(p)$	$g_2(p)$	$0.5005$	
$\vdots$				
$\vdots$				
$\vdots$				

To have stability we need  $f_1(p) > 0$ ,  $g_1(p) > 0$   $\therefore$

Even with a single parameter the analysis gets pretty involved. If on top of this we assume that the controller is also subject to uncertainty; say:

$$G(s) = \frac{(s+0.5)(s+0.01)^2 + z}{(0.1s+1)^3} \quad \text{with } z \in [z_{\min}, z_{\max}]$$

things get untractable pretty fast.

$\Rightarrow$  This leads to the study of the stability of interval polynomials, i.e.

polynomials of the form:

$$p(s) = a_0 + a_1s + a_2s^2 + \dots + a_ns^n$$

where the coefficients  $a_i$  are only known to belong to a "box", i.e.

$$a_i \in [a_i^-, a_i^+]$$

• Surprising result (Kharitonov theorem, 1978)

If the coefficients are independent, then the family is stable if and only if 4 vertex polynomials are stable:

$$p_1(s) = a_0^+ + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + a_5^- s^5 + \dots$$

$$p_2(s) = a_0^- + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + a_5^+ s^5 + \dots$$

$$p_3(s) = a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + a_5^- s^5 + \dots$$

$$p_4(s) = a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4 + a_5^+ s^5 + \dots$$

If the coefficients are not independent this provides only a sufficient (perhaps conservative) condition.

Stability of interval plants is still a very active research area

(We have now available results for cases where the coefficients are linearly related and for some forms of polynomial dependence)

In our case (satellite) the coefficients are not independent, so Kharitonov is sufficient (but not necessary)

A quick check shows that the 4 polynomials

$$p_1 = 0.3150 + 0.6401s + 1.62s^2 + 1.33s^3 + 1.027s^4 + 0.3003s^5 + 0.03s^6 + 0.001s^7$$

$$p_2 = 0.3150 + 0.7801s + 1.62s^2 + 1.27s^3 + 1.027s^4 + 0.3011s^5 + 0.03s^6 + 0.001s^7$$

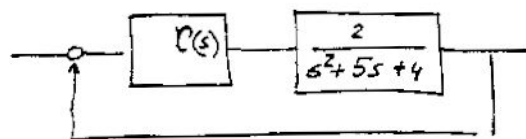
$$p_3 = 0.3851 + 0.6401s + 1.42s^2 + 1.33s^3 + 1.033s^4 + 0.3003s^5 + 0.03s^6 + 0.001s^7$$

$$p_4 = 0.3851 + 0.7801s + 1.42s^2 + 1.27s^3 + 1.033s^4 + 0.3011s^5 + 0.03s^6 + 0.001s^7$$

are all stable  $\Rightarrow$  all possible combinations of plant & controller for  $p \in [0.9, 1.1]$  and  $z \in [0.9, 1.1]$  are stable

- Use of the Routh Hurwitz criterion as a design tool:

Remember the position control of a laser:



Suppose the specs call for 0 steady state error to a step  
0.1 steady-state error to a ramp

$\Rightarrow$  Take  $C(s) = \frac{K}{s} \Rightarrow$  Type 1 system:

$$e_{ss}^{step} = 0$$

$$e_{ss}^{ramp} = \frac{1}{K_v} = \frac{1}{\lim_{s \rightarrow 0} s G(s)} = \frac{2}{K} \Rightarrow K \geq 20$$

However, this analysis is valid ONLY if the closed-loop system is stable

Let's check the stability limits:

Characteristic equation:  $s^3 + 5s^2 + 4s + 2K = 0$   
Routh Hurwitz yields:

$$\begin{array}{ccc} s^3 & 1 & 4 \\ s^2 & 5 & 2K \\ s^1 & 20-2K & \\ s^0 & 2K & \end{array}$$

$\Rightarrow$  stable for  $0 < K < 10$

$\Rightarrow$  Maximum value of  $K_v = \frac{K_{max}}{2} = 5 \Rightarrow e_{min}^{ss} = \frac{1}{5} = 0.2$

$\Rightarrow$  Specification NOT achievable with a controller of the form  $C(s) = \frac{K}{s}$

For  $K=10$  we get:

$$\begin{array}{ccc} s^3 & 1 & 4 \\ s^2 & 5 & 10 \\ s^1 & 0 & \\ s^0 & 20 & \end{array}$$

$E > 0$  no changes  
 $E < 0$  2 changes  $\Rightarrow$

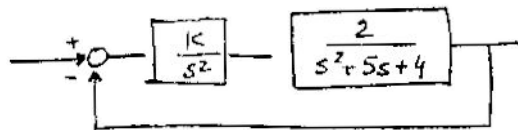
2 roots on the jw-axis

System marginally stable  $\Rightarrow$  oscillation

We can also get the frequency of oscillation from the auxiliary equation as follows:

Aux eq:  $5s^2 + 10 = 0 \parallel s^2 = -2 \quad \boxed{s = \pm j\sqrt{2}}$

Suppose that we want to make it a type 2 system:



Characteristic equation:  $s^4 + 5s^3 + 4s^2 + 2k = 0$   $\Rightarrow$  missing  $s^1$  power  $\Rightarrow$  unstable for all  $K$

$\frac{k}{s^2}$  will never work. If we are determined to make it a type 2 system we need something like:  $G_c(s) = \frac{k(s+a)}{s^2}$  (where  $a$  is to be determined)

The characteristic equation is now:  $s^4 + 5s^3 + 4s^2 + 2ks + 2ka = 0$   
(note that we got back a term in  $s^1$ , so this may work)

Routh Hurwitz yields:

$s^4$	1	4	$2ka$
$s^3$	5	$2k$	
$s^2$	$\frac{20-2k}{5}$	$2ka$	
	$10-k$	$5ka$	
$s^1$	$20k-2k^2-25ka$		
$s^0$	$5ka$		

Stable if  $\left. \begin{array}{l} 10-k > 0 \quad // \quad k < 10 \\ ka > 0 \\ (20-25a-2k)k > 0 \end{array} \right\} \Rightarrow \begin{array}{l} a = 0.2 \\ 0 < k < 7.5 \end{array}$

Assume  $a = 0.2 \Rightarrow k(15-2k) > 0 \quad // \quad k < 7.5$

- Returning to the simpler case of  $G_c(s) = \frac{k}{s}$ , recall that the closed-loop is stable for  $0 < k < 10$ . Suppose that we add some more specifications to the problem, to make it realistic

Specs:

a) closed-loop stable

b) zero steady state error to a step

c) "reasonable" settling time & overshoot

} can be taken care of by Routh Hurwitz

To take care of (c) we need the actual roots (just to find out whether or not they are in the LHP is not enough)

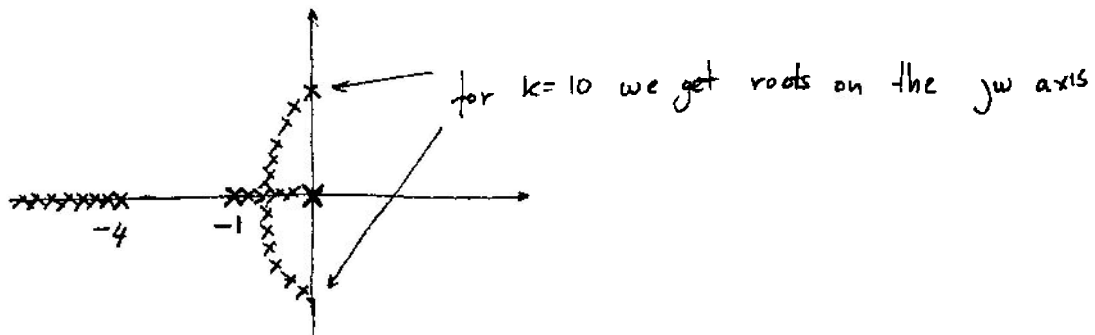
The characteristic equation is  $s^3 + 5s^2 + 4s + 2K = 0$

Suppose we solve this equation for all  $K \in [0, 10]$ , say with increments of 0.1, and plot the roots:

A sample Matlab code to achieve this is:

```
r=[];  
for k=0:0.1:10  
    r=[r; roots([1 5 4 2*k])];  
end;  
plot(real(r), imag(r), 'x')
```

This will give a plot that looks like:



Suppose we want to get  $\zeta$  (damping ratio)  $\geq \frac{\sqrt{2}}{2}$

$\Rightarrow$  we could intersect the plot with the ray through the origin at  $45^\circ$  and find the corresponding value of  $K$ .

In this case this yields:

$$K \approx 0.8$$

roots at:

$$-0.438 \pm j0.443 \leftarrow \text{dominant}$$

$$-4.12$$

$$\Rightarrow \zeta = 0.704$$

$$T_s \approx \frac{4}{\sigma} \approx 10 \text{ sec.}$$

$$M_p \approx 5\%$$

This plot of the roots of the characteristic equation as a function of the gain  $K$  is called the root-locus

