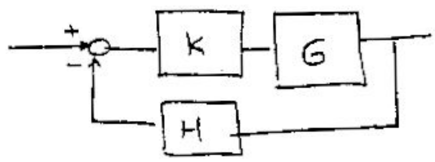
ROOT LOCUS (Chapter 7)

Def: The root locus of a system is a plot of the roots of the system's characteristic equation as some parameter (usually a gain) of the system changes



Characteristic equation: $1 + KG(s)H(s) = 0$

$\Rightarrow$ A point s_1 belongs to the Root Locus iff $1 + KG(s_1)H(s_1) = 0$ for some value of K.

Usually (but not always) K is taken to be positive, i.e. $0 \le K < \infty$

Q: How do we know if a point $s_1 \in RL$?

A: if $s_1 \in RL$, then for some $K_1 \geq 0$, $1 + KG(s_1)H(s_1) = 0$

$\Leftrightarrow KG(s_1)H(s_1) = -1 \Leftrightarrow |G(s_1)H(s_1)| = \frac{1}{K}$ ← magnitude criterion

$\angle G(s_1)H(s_1) = 180^\circ$ ← angle criterion

Note that the angle criterion is independent of K.

Example: $G = \frac{1}{s(s+2)}$, $H = 1$

$s_1 = -1 + j$ $\quad G(s_1) = \frac{1}{(-1+j)(1+j)} = -\frac{1}{2} \Rightarrow \angle G(s_1) = 180$

$s_1 \in RL$

$K = \frac{1}{|G(s_1)|} = 2$

$s_2 = -4$ $\quad G(s_2) = \frac{1}{(-4)(2-4)} = \frac{1}{(-4)(-2)} = \frac{1}{8} \Rightarrow \angle G(s_2) = 0$

$s_2 \notin RL$. There is no value of $K > 0$ such that $s_2 = -4$ can be made a pole of the system (i.e. $1 + KG(-4) = 0$)

Geometric interpretation of the angle criterion:

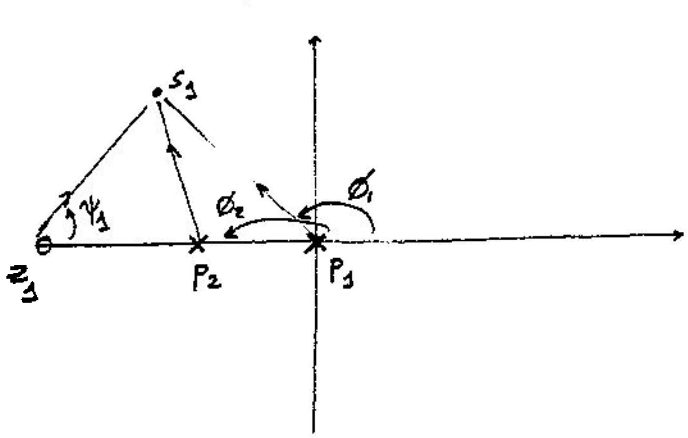
Suppose that the open loop function $(G(s)H(s))$ has the following form:

$$G(s)\,H(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Then: $\angle GH = \angle s-z_1 + \angle s-z_2 + \cdots \angle s-z_m - \angle s-p_1 - \angle s-p_2 \cdots - \angle s-p_n$

Hence a point $s_1 \in RL$ iff: $\sum \begin{pmatrix}\text{all angles} \\ \text{from zeros}\end{pmatrix} - \sum \begin{pmatrix}\text{angles} \\ \text{from} \\ \text{poles}\end{pmatrix} = (2r+1)180$

$r = 0, \pm 1, \pm 2$



In this case $s_1 \in RL$ iff

$$\psi_1 - \phi_1 - \phi_2 = (2r+1)\,180$$

- We want to be able to sketch a reasonably accurate root locus without having to actually solve the characteristic equation for each value of K. We will do so using <u>only</u> information about the <u>open loop</u> system

Rules for sketching the Root Locus

1) The R.L. is symmetrical with respect to the real axis (recall that if the char. eq has a complex root it <u>must</u> have also its complex conjugate)

   It has n branches (the char. eq. has exactly n roots)

2) <u>Origin and ending points:</u> (what happens as $k \to 0$ or $k \to \infty$)

$$1 + KGH = 0 \iff 1 + k\,\frac{\Pi(s-z_i)}{\Pi(s-p_i)} = 0 \iff \Pi(s-p_i) + k\,\Pi(s-z_i) = 0$$

but, if $k \to \infty$, this equation can be satisfied only if $s \to z_i$

(a zero of the open loop function). Note that we have to take into account both finite and infinite zeros.

if $GH = \frac{Q}{P} = \frac{b_m s^m + \cdots - b_0}{s^n + \cdots \quad a_0}$, with $n > m$

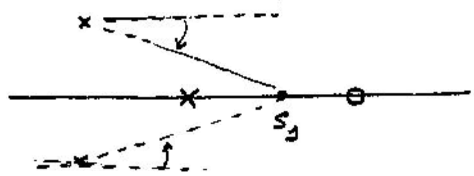
then, as $s \to \infty$ $\quad GH \to 0$ $\Rightarrow$ GH is said to have $n-m$ zeros at infinite (in addition to the m finite zeros)

When $k \to 0$, $\quad \Pi(s-p_i) + k\,\Pi(s-z_i) = 0 \iff s \to p_i$ (open loop poles)

$$\boxed{\text{Root Locus originates } (k=0) \text{ at the poles of } GH \text{ and terminates at the zeros of } GH \ (k \to \infty)}$$

3) Root Locus on the real axis

Consider a test point on the real axis:



Recall the angle criterion: $\quad s_1 \in RL \iff \Sigma \begin{smallmatrix}\text{angles}\\ \text{from}\\ \text{zeros}\end{smallmatrix} - \Sigma \begin{smallmatrix}\text{angles}\\ \text{from}\\ \text{poles}\end{smallmatrix} = (2l+1)180$

If the test point is on the axis, we have the following contributions

a) zeros or poles on the axis to the left of s_1: $\quad$ angle $= 0$

⇒ b) zeros or poles on the axis to the right of s_1: $\quad$ angle $= 180$

c) zeros/poles off the axis: they come in pairs so their contribution cancells out $\quad$ angle $= 0$

We see that the only non-zero angle is from poles/zeros on the real axis to the right of the test point.

If we have an even number of these $\Rightarrow$ angle $= 2 \cdot n \cdot 180 = n \cdot 360$ $\quad$ NOT on RL

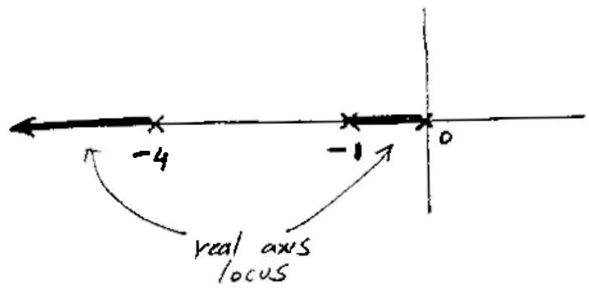
If we have an odd number of these $\Rightarrow$ angle $= (2n+1)180 \Rightarrow$ on RL

Root Locus on the real axis: points on the axis to the left of an odd number of poles + zeros

Example: $G(s) = \dfrac{K}{s(s+4)(s+1)}$

origin $0, -1, -4$

ends: 3 infinite zeros



4) Asymptotes for large k: (We want to find out the behavior for large k)

Recall that we have already shown that for $k \to \infty$, the RL $\to$ open loop zeros (finite or infinite). Here we want to sharpen this result and find out how the infinite zeros are reached.

$$KG(s)H(s) = \frac{K(b_m s^m + \cdots}{s^n + }$$

let $\alpha = n - m$

α = pole-zero excess = number of infinite zeros

as $s \to \infty$

$$1 + KGH \sim 1 + \frac{K b_m}{(s-\sigma)^\alpha}$$

(m of the zeros cancel-out m of the poles)

We want to find σ and the angle α

Let's take $s_0 = Re^{j\phi}$ (with R large and fixed) and apply the angle criterion to:

$$1 + kGH \sim 1 + \frac{k b_m}{(s-\sigma)^\alpha} = 1 + \frac{k b_m}{(Re^{j\phi} - \sigma)^\alpha}$$

$$\frac{k b_m}{(Re^{j\phi} - \sigma)^\alpha} = -1 \iff \angle (Re^{j\phi} - \sigma)^\alpha = 180 + \ell \cdot 360 \quad \ell = 0, 1 \ldots$$

$$\iff \alpha \angle Re^{j\phi} - \sigma = 180 + \ell 360$$

$$\Rightarrow \angle Re^{j\phi} - \sigma = \frac{180 + \ell 360}{\alpha} \quad \ell = 0, 1, \ldots$$

$\Rightarrow$ As $s \to \infty$, the angle of the RL approaches $\dfrac{180 + \ell 360}{\alpha}$ $\ell = 0, 1 \ldots \alpha - 1$

$$\boxed{\text{Let } \alpha = n-m \Rightarrow \text{There are } \alpha \text{ asymptotes that approach } \infty \text{ at angles } \quad \theta_a = \frac{180 + \ell\cdot 360}{\alpha},\ \ell = 0, 1 \cdots \alpha-1}$$

Now we need to determine σ (the intersection of the asymptotes)

Recall that for a polynomial of the form: $p = s^n + a_{n-1}s^{n-1} + \cdots$

$$a_{n-1} = -\Sigma p_i = -\left(\text{sum of the roots}\right)$$

In our case the characteristic equation is:

$$s^n + a_1 s^{n-1} + \cdots + a_n + k\left(s^m b_1 + \cdots\right) = 0 \quad \Rightarrow \quad \text{if } m < n-1$$

(i.e. 2 or more asymptotes)

Then $\Sigma\left(\begin{array}{c}\text{closed-loop}\\\text{poles}\end{array}\right) = -a_1 = \Sigma\left(\begin{array}{c}\text{open-loop}\\\text{poles}\end{array}\right)$

For large K, m of the closed-loop poles move on top of m of the open-loop zeros. The remaining m-n (the asymptotes) come from: $\dfrac{1}{(s-\sigma)^\alpha}$

$$\Rightarrow \Sigma \text{ roots} = \underbrace{\alpha\cdot\sigma}_{\text{asymptotes}} + \underbrace{\Sigma z_i}_{\substack{\text{closed loop poles}\\ \to \text{open loop zeros}}} = \overbrace{\Sigma p_i}^{\text{open loop poles}} \quad // \quad \boxed{\sigma = \frac{\Sigma p_i - \Sigma z_i}{n-m}}$$

- Laser example: $\quad G(s) = \dfrac{K}{s(s+4)(s+1)} \qquad \begin{array}{l} m=0\\ n=3\end{array} \Rightarrow \alpha = 3$ asymptotes

$$\Sigma z_i = 0, \qquad \Sigma p_i = 0 - 4 - 1 = -5$$

$$\Rightarrow \sigma = -\frac{5}{3} \qquad \theta = \frac{180 + \ell\cdot 360}{3} = 60,\ 180,\ 300$$

(Note: the point at $s = -\sigma$ NOT neccesarily $\in$ R.L.)

5) <u>Break-away (break-in) points</u>: points where 2 or more branches of the root locus intersect

At a break-away (or break-in) point the characteristic equation has multiple roots

Char. equation: $1 + KGH = 1 + k\frac{N(s)}{D(s)} = 0$

Let $P(s) = D(s) + kN(s)$ (characteristic polynomial)

If $P(s)$ has a multiple root at $s = s_o$, then it can be factored as:

$P(s) = (s - s_o)^{\gamma} P_1(s) \qquad \gamma > 1$

$\Rightarrow \quad P'(s) = \gamma (s - s_o)^{\gamma - 1} P_1(s) + (s - s_o)^{\gamma} P_1'(s) \quad \Rightarrow P'(s_o) = 0$

If s_o is a root of $P(s)$ with multiplicity γ, then it is also a root of $P'(s)$ with multiplicity $\gamma - 1$.

$$\left.\begin{array}{l} P'(s) = D'(s) + KN'(s) \;\Rightarrow\; D'(s_o) + kN'(s_o) = 0 \\ \text{also, since } D(s_o) + KN(s_o) = 0 \Rightarrow K = -\dfrac{D'(s_o)}{N'(s_o)} \end{array}\right\} \Rightarrow D'(s_o)N(s_o) - D(s_o)N'(s_o) = 0$$

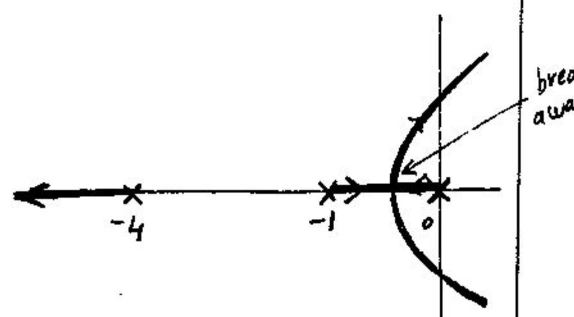
which is equivalent to: $\frac{d}{ds}\left(\frac{D}{N}\right) = \boxed{\frac{d}{ds}\left(\frac{1}{GH}\right) = 0}$

or $\boxed{\frac{d}{ds}(GH) = 0}$ (note that $GH(s_o) \neq 0$ since $1 + G(s_o)H(s_o) = 0$)

Back to the example: $GH = \frac{2}{s(s+1)(s+4)} \Rightarrow \frac{1}{GH} = \frac{s^3 + 5s^2 + 4s}{2}$

$$\frac{d}{ds}\left(\frac{1}{GH}\right) = \frac{1}{2}(3s^2 + 10s + 4) = 0 \qquad s_{1,2} = -\frac{5}{3} \pm \frac{\sqrt{13}}{3} \quad \begin{cases} -\frac{5}{3} + \frac{\sqrt{13}}{3} & \text{in RL} \\ -\frac{5}{3} - \frac{\sqrt{13}}{3} & \text{not in RL} \end{cases}$$

<u>Note:</u> once you get the roots of $d\left(\frac{1}{GH}\right)$ you have to keep only those that belong to the RL



6) jω-axis intersection: The value of K such that the roots are on the jω-axis can be found using Routh-Hurwitz

Example: char equation: $s^3 + 5s^2 + 4s + 2K$

s^3	1	4
s^2	5	$2K$
s^1	$\frac{20-2K}{5}$	
s^0	$2K$	

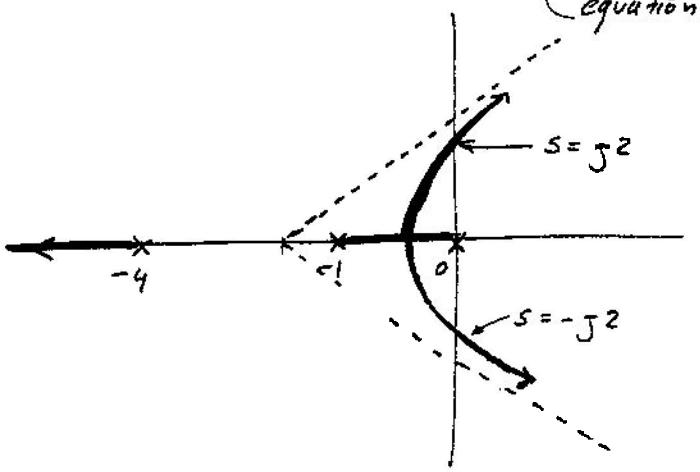
$\Rightarrow$ marginally stable for $K = 10$

jω axis intersection as:

(auxiliary equation) $5s^2 + 20 = 0 \Rightarrow s^2 = -4$

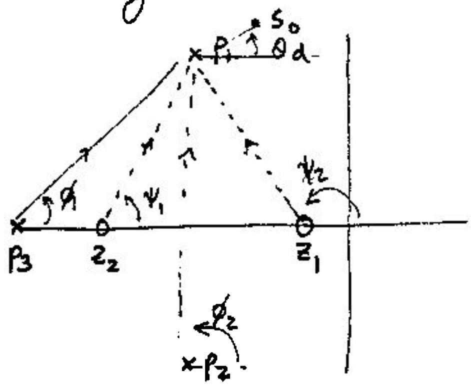
$s = \pm j2$

jω axis intersect



7) Angle of departure (arrival):

The angle at which the Root Locus leaves a pole (or approaches a zero)



Take a test point s_0 close to p_1

We'd like to find out the angle θ_d (angle of the segment $\overline{s_0 \cdot p_1}$)

From the angle condition we have:

$$\Sigma \angle s_0 - z_i - \Sigma \angle s_0 - p_i = (2r+1)\cdot 180$$

If s_0 is very close to p_1 then: $\angle s - z_i \rightarrow \psi_i$

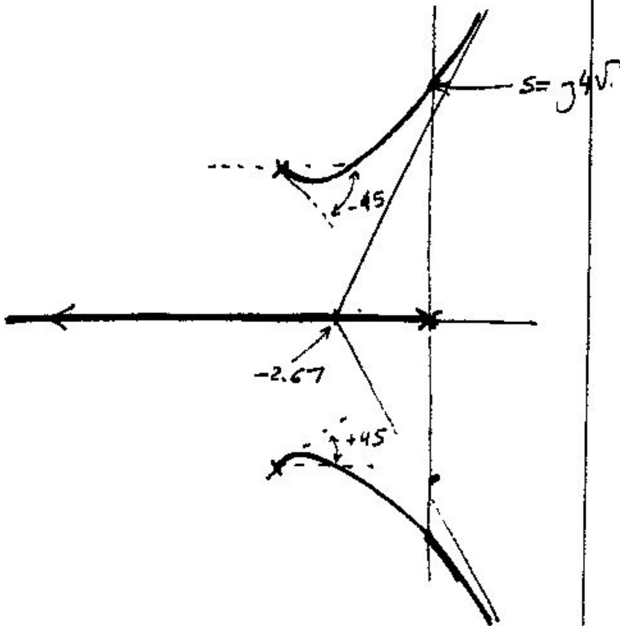
$\angle s - p_i \rightarrow \phi_i$

$\angle s - p_1 \rightarrow \theta_d$ (angle of departure)

$$\boxed{\theta_d = \Sigma \psi_i - \Sigma \phi_i - (2r+1)\cdot 180}$$

Similarly: $\boxed{\theta_a = \sum \phi_i - \sum \psi_i + (2\ell+1)180}$

Example: $G = \dfrac{1}{s\left[(s+4)^2+16\right]}$



a) open loop poles at $s=0$, $s=-4\pm 4j$
open loop zeros: none finite
3 at ∞

b) RL axis: (to the left of $s=0$)

c) 3 asymptotes. Intersect at:
$$\sigma = \frac{-4+4j-4-4j}{3} = \frac{-8}{3} = -2.67$$
angles: 60, 180, 300

d) break-away / break-in

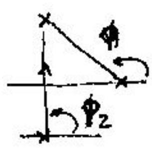
$$\frac{1}{GH} = s^3+8s^2+32s$$
$$\frac{d}{ds}\left(\frac{1}{GH}\right) = 3s^2+16s+32=0 \Rightarrow s = -2.67 \pm j1.189$$ do <u>not</u> belong to R.L.

e) $j\omega$-axis intersect: char eq. $s^3+8s^2+32s+K$

s^3	1	32
s^2	8	K
s^1	$\frac{256-K}{8}$	
s^0	K	

$\Rightarrow K=256$
$8s^2+256=0$ // $s = \pm j4\sqrt{2} = \pm j5.66$

f) Departure angle from $-4+4j$:



$\phi_1 = 135°$
$\phi_2 = 90$

$$\theta_d = -135 - 90 - (2\ell+1)180 = -225+180 = -45$$
$(\ell = -1)$

Example (example 7.8 book):

$$KGH = \frac{K}{s(s+20)(s+2)(s+0.1)} = \frac{K}{s^4 + 22.1 s^3 + 42.2 s^2 + 4s}$$

a) open loop poles: $0, -0.1, -2, -20$: $n = 4$
open loop zeros: all @ ∞ : $m = 0$

b) Real axis locus

c) $d = n - m = 4 \Rightarrow 4$ asymptotes
angles: 45, 135, 225, 315

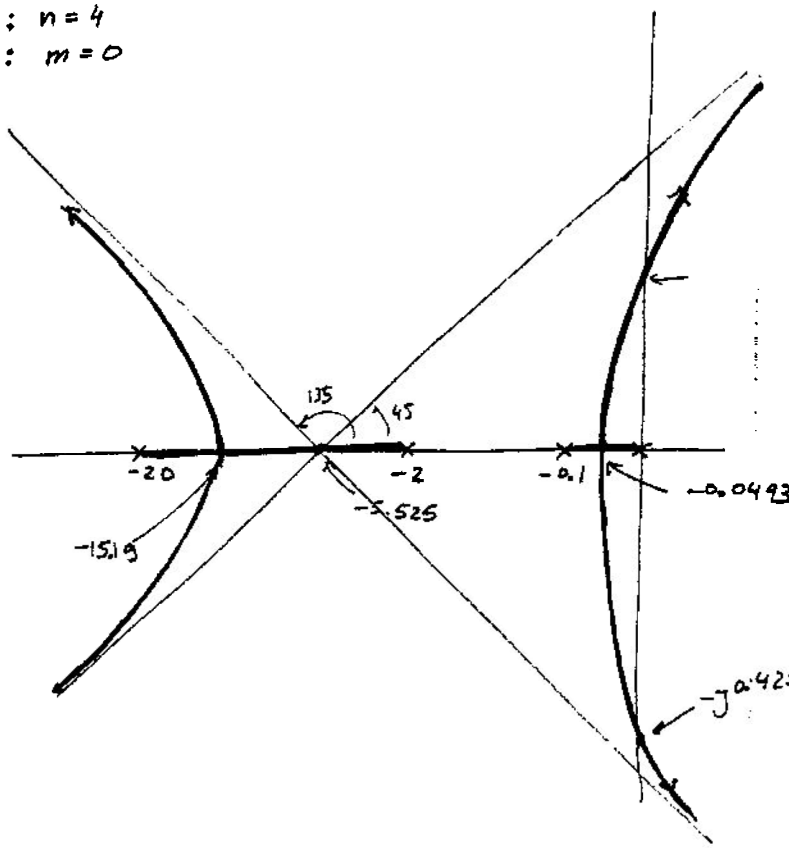
$$\sigma = \frac{-20-2-0.1}{4} = -5.525$$

d) Break-away:

$$\frac{d}{ds}\left(\frac{1}{GH}\right) = 0 \Rightarrow 4s^3 + 66.3s^2 + 84.4s + 4 = 0$$

$$s = -0.0493,\ \cancel{-1.335},\ -15.19$$

not in RL



e) imaginary axis intersect:

$$s^4 + 22.1 s^3 + 42.2 s^2 + 4s + K$$

s^4	1	42.2	K
s^3	22.1	4	
s^2	42.02	K	
s^1	$\frac{168.02 - 22.1K}{42.02}$		
s^0	K		

$\Rightarrow$ $K = 0$
$K = 7.6054$

for $K = 7.6054$, the auxiliary equation is:

$$42.02 s^2 + 7.6054 = 0 \Rightarrow s = \pm j\,0.4254$$

- **Summary** Guidelines for plotting the root locus

1) Mark the poles (x) and zeros (o) of the open-loop function

2) Draw the R.L on the real axis — points to the left of an odd number of real poles + zeros

3) Draw $n-m$ asymptotes at angles $\theta = (2\ell+1)\frac{180}{n-m}$; $\ell = 0, 1, \ldots n-m-1$

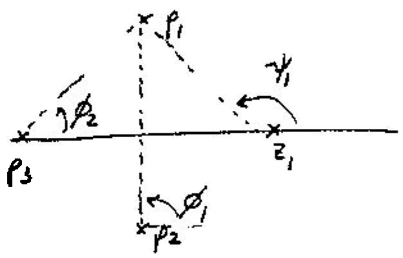
   The asymptotes intersect the real axis at the point $\sigma = \frac{\Sigma p - \Sigma z}{n-m}$

4) Break-away (break-in) points:

   Roots of: $\frac{d}{ds}\left(\frac{1}{GH}\right) = 0$ (only the roots that belong to the root-locus)

5) $j\omega$-axis intersection (Routh Hurwitz)

6) Departure (arrival) angle:



$$\theta_d = \Sigma \psi_i - \Sigma \phi_i - (2r+1)\,180$$

$$\theta_a = \Sigma \phi_i - \Sigma \psi_i + (2r+1)\,180$$