Open Loop Discrete Time Systems (chapter 4)

So far we have seen how to deal with ideal sampling (via $*$ transform) and ideal data reconstruction.

Now we will finally put these tools together to get <u>discrete-time transfer functions.</u>

The first step is to examine:

- <u>Relationship between $E(z)$ and $E^*(s)$:</u>

As we will see next $E^*(s)$ provides the link between $E(s)$ (continuous time) and $E(z)$ (discrete time). Let's compare definitions:

$$\left.\begin{aligned} E^*(s) &= \sum_{k=0}^{\infty} e(kT)\, e^{-kTs} \\ E(z) &= \sum_{k=0}^{\infty} e(kT)\, z^{-k} \end{aligned}\right\} \Rightarrow \boxed{E^*(s)\big|_{z=e^{+sT}} = E(z)}$$

(this relationship allows for considering the z-transform as a special case of the Laplace transform)

Suppose we sample a continuous time function and want to get its z-transform:

<u>Old way:</u>

1) expand $E(s)$ in partial fractions
2) find $e(t) = \mathcal{L}^{-1}\left[E(s)\right]$
3) find $e(kT)$
4) find $E(z) = \sum_{0}^{\infty} e(kT)\, e^{-kTs}$

<u>Alternative</u>

1) find $E^*(s)$
2) $E(z) = E^*(s)\big|_{z=e^{Ts}}$

Obviously the alternative works better <u>only</u> if we have an efficient way of computing $E^*(s)$ from $E(s)$. Here is where the <u>residues formula</u> becomes useful.

Recall that $$E^*(s) = \sum_{\substack{\text{poles}\\ E(\lambda)}} \text{Res}\left[E(\lambda)\, \frac{1}{1-e^{-T(s-\lambda)}}\right]$$

$$\Rightarrow \boxed{E(z) = \sum_{\substack{\text{poles}\\ E(\lambda)}} \left[\text{residues of } E(\lambda)\, \frac{1}{1-\frac{1}{z}e^{\lambda T}}\right]}$$

Example: $E(s) = \dfrac{1}{(s+1)(s+2)}$

Old way:

1) $E(s) = \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)}$

2) $e(t) = e^{-t} - e^{-2t}$

3) $e(kT) = e^{-kT} - e^{-2kT}$

4) $E(z) = \sum_{0}^{\infty} e^{-kT} \cdot z^{-k} - \sum_{0}^{\infty} e^{-2kT} \cdot z^{-k}$

$$= \frac{1}{1 - \frac{e^{-T}}{z}} - \frac{1}{1 - \frac{e^{-2T}}{z}}$$

$=$

"New" way

$$E(z) = \sum_{\substack{\lambda=-1\\ \lambda=-2}} \text{Res}\left\{ \frac{1}{(\lambda+1)(\lambda+2)} \frac{1}{1 - e^{\lambda T} \cdot \frac{1}{z}} \right\}$$

$$= \frac{1}{1 - \frac{e^{-T}}{z}} - \frac{1}{1 - \frac{e^{-2T}}{z}}$$

Q: We just stated that we can view the z-transform as a special case of the Laplace Transform (by using the * transform). So, why do we need it? What's wrong with using E^*?

A: Let's compare $E^*(s)$ versus $E(z)$. For the example above we have:

$$E^*(s) = \frac{e^{sT}(e^{-T} - e^{-2T})}{(e^{sT} - e^{-T})(e^{sT} - e^{-2T})} \qquad E(z) = \frac{z(e^{-T} - e^{-2T})}{(z - e^{-T})(z - e^{-2T})}$$

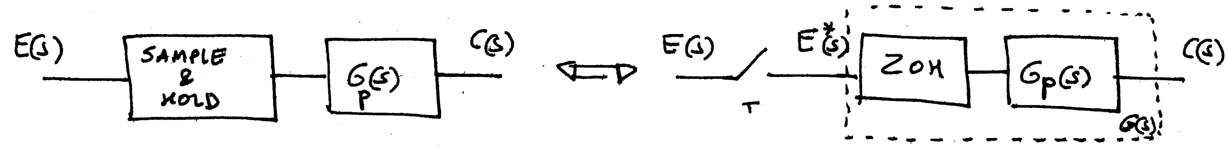
∞ poles at $s = -1 \pm j n \omega_s$, $s = -2 \pm j n \omega_s$

2 poles: $z = e^{-T}$, $z = e^{-2T}$

$\Rightarrow$ Any analysis based on pole/zeros will be far easier to carry out in the z-plane. We will see more advantages latter on.

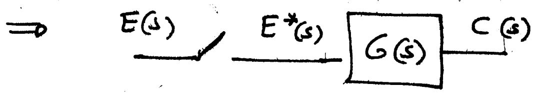
• Pulse Transfer Function:

Remember that our goal was to get a transfer function representation for:



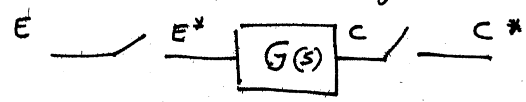
The idea is to group the plant & ZOH and consider it a single block $G(s)$

$\Rightarrow$



and $C(s) = G(s)E^*(s)$

Assume now that we only look at the output at discrete time instants: kT



We want to find out and expression for $C^*(s)$ (as an intermediate step to finding $c(kT)$)

$$C^*(s) = \frac{1}{T}\sum_{-\infty}^{+\infty} C(s+jn\omega_s) = \frac{1}{T}\sum_{-\infty}^{+\infty} G(s+jn\omega_s)E^*(s+jn\omega_s)$$

But E^* is periodic $\Rightarrow$ $E^*(s+jn\omega_s) = E^*(s)$

$$\Rightarrow C^*(s) = \frac{1}{T}\sum_{-\infty}^{+\infty} G(s+jn\omega_s)E^*(s) = E^*(s)\left[\frac{1}{T}\sum_{-\infty}^{+\infty} G(s+jn\omega_s)\right] = E^*(s)G^*(s)$$

$\Rightarrow$ We have just shown that:

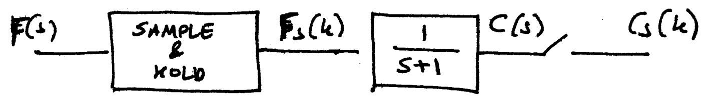
$$\boxed{C^*(s) = \left[G(s)E^*(s)\right]^* = G^*(s)E^*(s)} \quad \#$$

or, using z-transforms: $\boxed{C(z) = G(z)E(z)}$

We will call $G(z)$ the <u>pulse transfer function</u>

(Roughly speaking: TF between sampled input & sampled output at the sampling instants)

Suppose that we want to find the TF of a simple plant:



Brute force approach: $C(s) = \frac{1}{(s+1)} F_s \overset{\mathcal{L}^{-1}}{\Rightarrow} \dot{c}(t) + c(t) = f_s(t)$

Solving this first order differential equation yields:

$$c(t) = A e^{-(t-kT)} + f(kT) \qquad \text{(recall that } f_s(t) = f(kT) \quad kT \le t < (k+1)T)$$

Now we need to impose the initial condition $c(t)\big|_{t=kT} = c(kT)$ and solve for A:

$$A = c(kT) - f(kT) \Rightarrow$$

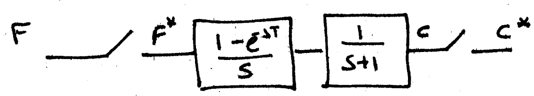
$$c(t) = c(kT)\, e^{-(t-kT)} + \left[1 - e^{-(t-kT)}\right] f(kT) \qquad kT \le t < (k+1)T$$

$$\Rightarrow c((k+1)T) = c(kT)\, e^{-T} + (1 - e^{-T}) f(kT)$$

$\Downarrow$ z transform

$$(z - e^{-T})\, C(z) = (1 - e^{-T}) F(z) \Rightarrow \boxed{\frac{C(z)}{F(z)} = \frac{1 - e^{-T}}{z - e^{-T}}}$$

Lot's of work, even for a very simple plant! Let's try now our tools:



$$G(s) = \frac{1 - e^{-sT}}{s(s+1)} = \frac{C(s)}{F^*}$$

$$\Rightarrow \frac{C(z)}{F(z)} = Z\left\{ \frac{1 - e^{-sT}}{s(s+1)} \right\} = \left[\frac{1 - e^{-sT}}{s(s+1)}\right]^*\Bigg|_{z = e^{sT}}$$

Note that we have a delay $\Rightarrow$ (3-10) <u>does not</u> apply. However since the delay is an integer number of periods we can use the alternative formula:

$$\left[e^{-mTs} G_1(s)\right]^* = e^{-mTs} \left[G_1(s)\right]^* \Leftrightarrow Z\left[e^{-mTs} G_1(s)\right] = \frac{1}{z^m} G_1(z)$$

In this case we have: $G_1(s) = \frac{1}{s(s+1)} \Leftrightarrow G_1(z) = \sum_{\substack{\lambda=0 \\ \lambda=-1}} \text{Res}\left[\frac{1}{\lambda(\lambda+1)} \cdot \frac{1}{\left(1 - \frac{e^{\lambda T}}{z}\right)}\right]$

$$= \frac{1}{1 - \frac{1}{z}} - \frac{1}{1 - e^{-T}/z} = \frac{z(1 - e^{-T})}{(z-1)(z - e^{-T})}$$

and $G(z) = G_1(z) - \frac{1}{z} G_1(z) = \left(\frac{z-1}{z}\right) G_1(z) = \boxed{\frac{1 - e^{-T}}{z - e^{-T}}}$ same as before

(but we did not have to solve a differential eq.)

Assume now that the input is a step: $f(kT) = 1, 1, \ldots$

$$F(z) = \frac{z}{z-1}$$

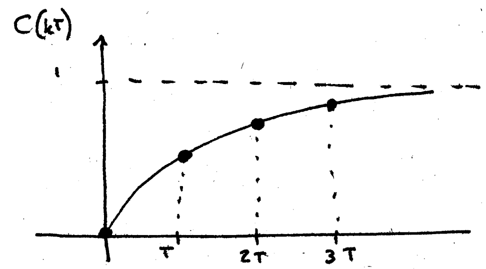
$$\Rightarrow C(z) = G(z)F(z) = \frac{z}{(z-1)}\frac{(1-e^{-T})}{(z-e^{-T})} = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

Going back to the time domain we get:

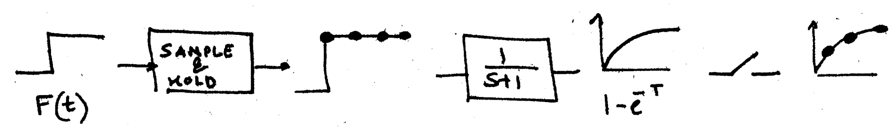
$$c(kT) = \mathcal{Z}^{-1}[C(z)] = \mathcal{Z}^{-1}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 1 - e^{-kT} \ \#$$

(Note: you can get the same result from residues, without having to do partial fractions:

$$c_k = \sum_{\substack{z=1 \\ z=e^{-T}}} \mathrm{Res}\left\{ z^{k-1} \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})} \right\} = \left(\frac{1-e^{-T}}{1-e^{-T}}\right) + e^{-kT}\left(\frac{1-e^{-T}}{e^{-T}-1}\right) = 1 - e^{-kT} \ \# \ )$$



Sanity check: Physically



Important: You can get $c(kT)$ from $c(t)$ but the converse does not work. You cannot recover $c(t)$ from $c(kT)$
(Obvious, since $c(kT)$ does not contain information about what happens in between samples)

- **Definition:** DC gain:

$$\frac{\lim_{k\to\infty} c_k}{\lim_{k\to\infty} F_k} = \frac{c_{ss}}{F_{ss}} \overset{\text{(Final value theorem)}}{=} \lim_{z\to 1} \frac{(z-1)C(z)}{F_{ss}}$$

$$= \lim_{z\to 1} (z-1)\frac{G(z)F(z)}{F_{ss}} = \lim_{z\to 1} \cancel{(z-1)} \frac{G(z)}{\cancel{F_{ss}}} \cancel{F_{ss}} \frac{z}{\cancel{(z-1)}}$$

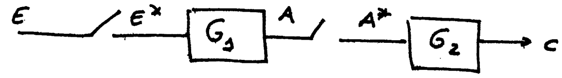
$$= G(1)$$

$$\Rightarrow \boxed{\text{DC gain: } G(1)}$$

(compare with continuous time: DC gain $= G(s)\big|_{s=0}$)

but then again: $z = e^{Ts} \Rightarrow s=0$ gets mapped to $z=1$

- Other configurations:

a) 

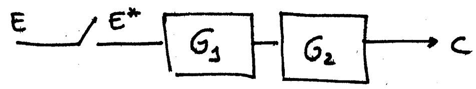
$$C(s) = G_2(s)\,A^*(s) \qquad A(s) = G_1(s)E^*(s) \Rightarrow A^* = (G_1 E^*)^* = G_1^* E^*$$

$$C^*(s) = G_2^* A^* = G_2^*(s)G_1^*(s)\,E^*(s)$$

$$\Downarrow$$

$$\boxed{C(z) = G_2(z)\,G_1(z)\,E(z)} \quad \text{or} \quad \boxed{G(z) = G_1(z)\cdot G_2(z)}$$

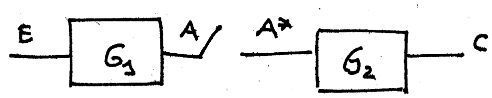
Key feature: we have an ideal sampler between G_1 & G_2

b) 

$$C(s) = G_2(s)\,G_1(s)\,E^*(s) \Rightarrow C^*(s) = [G_1\cdot G_2]^* E^*(s) \quad \text{or}$$

$$C(z) = \mathcal{Z}\{G_1\cdot G_2\}\cdot E(z) \Rightarrow \boxed{G(z) = \mathcal{Z}[G_1\cdot G_2]}$$

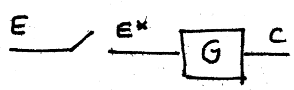
Note: $\mathcal{Z}[G_1 G_2] \neq G_1(z)\,G_2(z)$!!

c) 

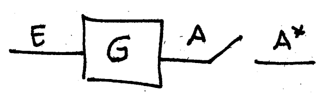
$$C = G_2 A^* \Rightarrow C^* = G_2^* [G_1 E]^*$$

For this system a transfer function cannot be written!
The reason is that you can't factor $E(z)$ out of $[G_1 E](z)$

Physical reason:

When you have , E^* contains information about E only at the sampling instants. ⇒ When you compute C^* you can factor E^* out.

On the other hand, if you have something like this:



$$A(s) = G(s)\,E(s)$$

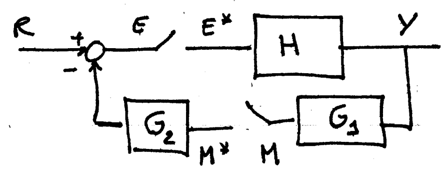
$$a(t) = \int_0^t g(t-\tau)\,e(\tau)\,d\tau \Rightarrow$$

to compute A^* (and latter C^*) you need information about $e(t)$ at all times, not just the sampling instants.

Important consequence: If you want to have a discrete T.F you have to select as your unknowns the inputs to the sampler

This variables will always "come free" after the equation sampling process and give a set of starred variables for which we can solve.

Example:



$$
\left.
\begin{aligned}
&E(s) = R - G_2 M^* \\
&E^* = R^* - [G_2 M^*]^* = R^* - G_2^* M^* \\
&M = (G_1 H) E^* \Rightarrow M^* = (G_1 H)^* E^*
\end{aligned}
\right\}
\quad E^* = R^* - G_2^* [G_1 H]^* E^*
$$

$$\Rightarrow \boxed{E^* = \frac{R^*}{1 + (G_1 H)^* G_2^*}}$$

or: $$\boxed{\frac{E(z)}{R(z)} = \frac{1}{1 + [G_1 H](z)\, G_2(z)}}$$

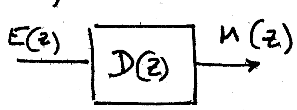
To obtain Y we can use the equation: $Y(s) = H E^* \Rightarrow Y^* = H^* E^*$

$$\Rightarrow \frac{Y^*}{R^*} = \frac{H^*}{1 + (G_1 H)^* G_2^*} \iff \boxed{\frac{Y(z)}{R(z)} = \frac{H(z)}{1 + (H G_1)(z)\, G_2(z)}}$$

Note that in this case we have a TF. This is because R goes right through a sampler before entering other blocks.

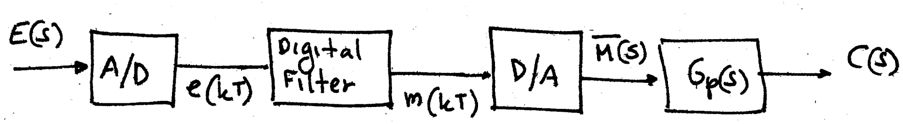
- **Open-loop Systems with Digital Filters**

We consider now the case where the sampled-data system contains a digital filter



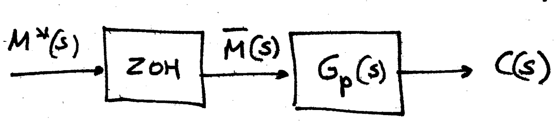
$$M(z) = D(z)\, E(z)$$
$$M^*(s) = D^*(s)\, E^*(s)$$

Physically:



We will assume that the D/A can be represented as a zero order hold

$\Rightarrow$



(Recall that $G_{h_0} = \dfrac{1-e^{-sT}}{s}$)

$$\Rightarrow \quad \begin{matrix} E(s) \\ e(t) \end{matrix} \;\; \diagup \;\; \xrightarrow[e(kT)]{E(z)} \boxed{D(z)} \xrightarrow{M(z)} \boxed{G(z)} \longrightarrow$$

$$C(s) = \underbrace{G_{h_0}(s) \cdot G_p(s)}_{G(s)} \, M^*(s) \quad \Rightarrow \quad C(z) = G(z) M(z) = G(z) D(z) E(z)$$

<u>Example</u>: Suppose that the digital filter is given by the following diff. eq.

$$m(kT) = 2e(kT) - e[(k-1)T]$$

$$M(z) = 2E(z) - \frac{E(z)}{z} = \frac{2z-1}{z} E(z)$$

$$\Rightarrow \quad D(z) = \frac{M}{E} = \boxed{\frac{2z-1}{z}}$$

$$G_p(s) = \frac{1}{(s+1)} \quad \Rightarrow \quad G(s) = G_{h_0} G_p = \frac{1-e^{-sT}}{s(s+1)}$$

$$G(z) = \left(1 - \frac{1}{z}\right) \mathcal{Z}\left[\frac{1}{s(s+1)}\right] = \left(1 - \frac{1}{z}\right)\left[\frac{1}{1-1/z} - \frac{1}{1-e^{-T}/z}\right]$$

$$= \frac{1-e^{-T}}{z-e^{-T}}$$

$$\xrightarrow{E(z)} \boxed{\frac{2z-1}{z}} \xrightarrow{M} \boxed{\frac{1-e^{-T}}{z-e^{-T}}} \xrightarrow{C}$$

Assume that $E(z)$ is a step: $E(z) = \dfrac{z}{z-1} \Rightarrow$

$$C(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right)\left(\frac{2z-1}{\cancel{z}}\right)\left(\frac{\cancel{z}}{z-1}\right)$$

From here we can get $c(kT)$ either by doing partal fraction expansion or using the residues formula.

Effect of T_s

Recall that $C^*(s)$ or $C(z)$ gives you information only on what happens at the sampling instants, but not in-between.

Q: Is this a problem?

A: Depends on how fast the dynamics of your plant are, compared to the sampling rate

Example: $G_p(s) = \dfrac{25}{s^2+2s+25} \Rightarrow \omega_n^2 = 25 \quad (\omega_n = 5)$

$\zeta = 0.2 \quad$ (under damped)

It can be shown that if we sample at $T=0.1$ and $T=1$ we get the following discrete time equivalents:

$$G(z) = \mathcal{Z}\left\{\left(\frac{1-e^{-sT}}{s}\right)\cdot G_p(s)\right\} = \left(1-\frac{1}{z}\right)\mathcal{Z}\left[\frac{25}{s(s^2+2s+25)}\right]$$

$$G_{T=1} = \frac{z-0.007}{z^2-0.1365z+0.1353}$$

$$G_{T=0.1} = \frac{0.115z-0.107}{z^2-1.6z+0.82}$$

Let's look at the step responses (next pages)

In one case the discrete output captures the relevant dynamics. In the second case there is significant intersample ripple