

Name: \_\_\_\_\_ Signature: \_\_\_\_\_

**Instructions:**

1. This is a closed-book test but **one**  $8\frac{1}{2} \times 11$  single-sided cheat-sheet is allowed.
2. Work as many problems as you can. Try not to spend too much time working on a single problem. If you get stuck, try working on a different question.
3. Show all your work, but try to be as concise as possible.
4. • **DO NOT LOOK** at the problems until told to do so.
5. • **STOP** working after the “time’s up” announcement.
6. • **GOOD LUCK !**

## Problem 1: (50 points)

1.1- (10 pts) The (continuous-time) transfer function of a *first hold* is given by:

$$G_{fob}(s) = \frac{1+Ts}{T} \left[ \frac{1-e^{-Ts}}{s} \right]^2 \quad (1)$$

Find an analytic expression (in the time domain) for its impulse response and sketch it.

1.2- (20 pts) Consider the block diagram shown in Figure 1. Show that this diagram is equivalent to a first order hold. Hint: put an appropriate input, sketch the corresponding output  $c(t)$  and compare it with the output of a first order hold.

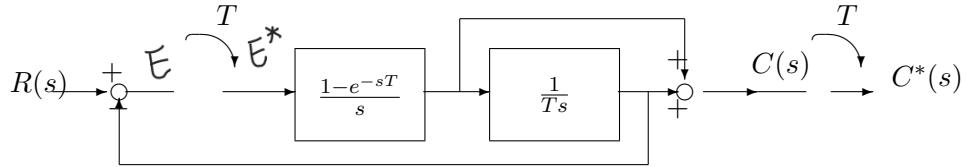


Figure 1: Block Diagram for Problem 1

1.3- (20 pts)

- (a) Show that for frequencies  $\frac{\omega}{\omega_s} \ll 1$  the phase characteristic of the first order hold (equation (1)) is given by

$$\text{angle}\{G_{fob}(j\omega)\} \simeq -\frac{2\pi\omega}{\omega_s} \quad (2)$$

where  $\omega_s = \frac{2\pi}{T}$  is the sampling frequency in radians per second.

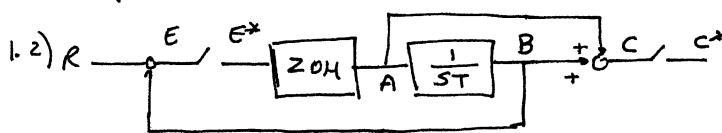
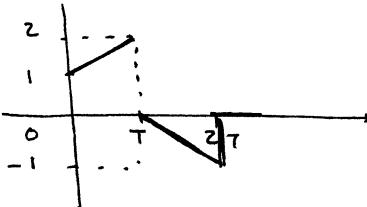
- (b) Assume that this hold will be used in a closed-loop system where the plant has bandwith 10 Hz and can tolerate up to  $\frac{\pi}{6}$  radians of phase lag. Find the minimum sampling interval so that this condition is satisfied. For this part you may use the approximation (2).

$$b) G_{fob} = \left(\frac{1+sT}{T}\right) \left(1 - e^{-sT}\right)^2 = \left(\frac{1}{s} + \frac{1}{s^2 T^2}\right) \left(1 + e^{-2sT} - 2e^{-sT}\right)$$

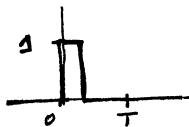
$$g_{fob}(t) = \mathcal{L}^{-1}[G_{fob}] = \frac{1}{T} \left[ t u(t) - 2(t-T) u(t-T) + (t-2T) u(t-2T) \right] + [u(t) - 2u(t-T) + u(t-2T)]$$

collecting terms yields:  $\left(1 + \frac{t}{T}\right) u(t) - 2\frac{t}{T} u(t-T) + \left(\frac{t}{T} - 1\right) u(t-2T)$  or

$$\boxed{\left(1 + \frac{t}{T}\right) [u(t) - u(t-T)] + \left(1 - \frac{t}{T}\right) [u(t-T) - u(t-2T)]}$$



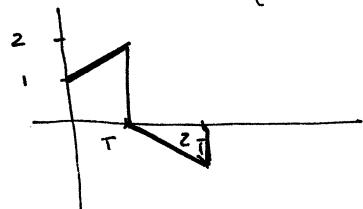
Take as input R



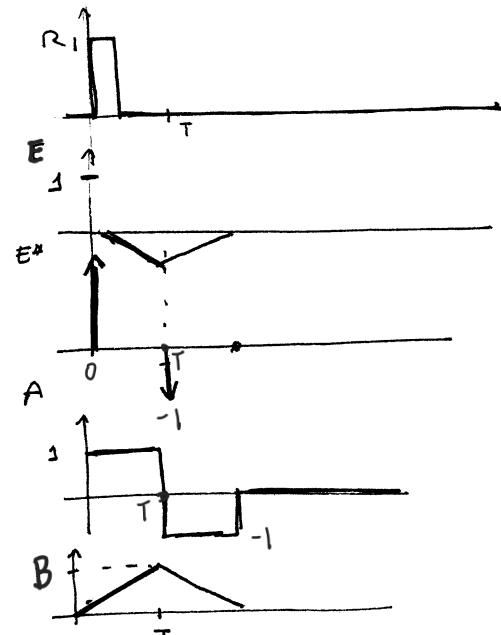
$$E^* = \sum_0^\infty e(k) \delta(t-kT)$$

(B is the integral of A)

$$C = A + B$$



same as in part (1)



$$1.3) G_{fob}(j\omega) = \left(\frac{1+j\omega T}{T}\right) \left(1 - e^{-j\omega T}\right)^2 = \left(\frac{1+j\omega T}{T}\right) e^{-j\omega T} \left(T \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2 j\omega T/2}\right)^2 = \\ G(j\omega) = \left(\frac{1+sT}{T}\right) \left(1 - e^{-sT}\right)^2 = (1 + j\omega T) e^{-j\omega T} T \underbrace{\sin^2 \frac{\omega T}{2}}_{\text{always positive}}$$

$$\angle G = -\omega T + \angle(1+j\omega T) \sim -\omega T \text{ if } \omega T \ll 1$$

$$\angle G \approx -2\pi \frac{\omega}{\omega_s} = \omega T \text{ where } T = \frac{2\pi}{\omega_s}$$

want  $-2\pi \frac{\omega}{\omega_s} < \frac{\pi}{6}$  for all  $\omega \in [0, 10 \text{ Hz}]$   $\Rightarrow \omega_s > 12 \omega_B \Rightarrow$

- $f_s > 120 \text{ Hz}$
- $T < 8.3 \text{ msec}$

for statement (question)

bandwidth frequency.

## Problem 2: (50 points)

2.1- (20 pts) Consider the model of a first order hold shown in Figure 1 ([previous page](#)).

- (a) Find the discrete transfer function  $E(z)/R(z)$ .
- (b) Find the discrete transfer function  $C(z)/R(z)$ . Does your result make sense (recall that this block diagram is supposed to implement a first order hold)

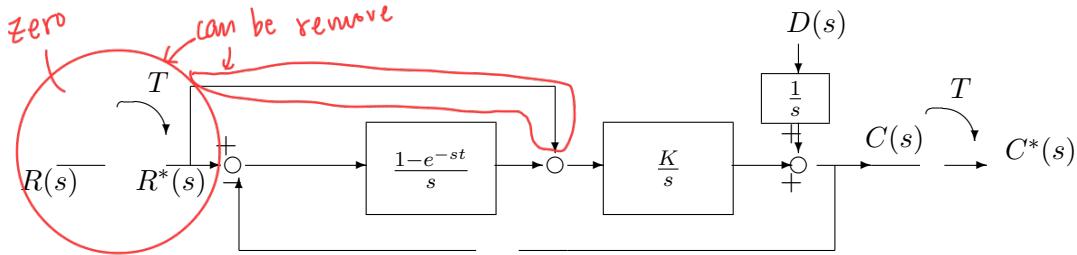
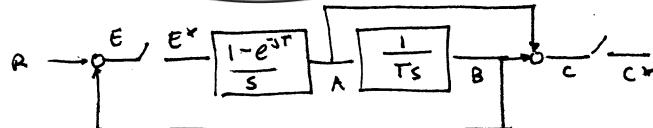
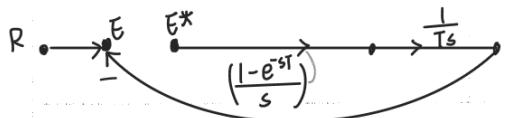


Figure 2: Block Diagram for Parts 2.2 and 2.3

2.2- (15 pts) Consider the block diagram shown in Figure 2 above. Assume that  $R = 0$ . Does the discrete transfer function from  $D$  to  $C$  exist? If so, find it. If not, fully justify your answer and find  $C(z)$  for the case where  $D(s) = 1$  (i.e.  $d(t) = \delta(t)$ ).

2.3- (15 pts) Find a pair of values for  $K$  and  $T$  such that, when  $R = 0$  and  $D(s) = 1$ , the steady state value of  $c_{ss} = \lim_{k \rightarrow \infty} c_k$  satisfies  $|c_{ss}| < 0.1$ . Compare the answer with the one that you would have gotten in the case of a continuous time system (ie same loop without sampling and holding action). Explain both similarities and differences.



choose  $E$  as output

input  $R, E^*$

output  $C, E$

$$a) E = R - \frac{A}{ST} = R - \left(\frac{1-e^{-sT}}{s}\right) \frac{1}{ST} E^* // E^* \left[1 + \left(\frac{1-e^{-sT}}{Ts^2}\right)^*\right] = R^*$$

$$E(z) = \frac{1}{1 + \left(\frac{z-1}{z}\right) \frac{1}{(z-1)^2}} R(z)$$

$$\text{From tables: } \mathcal{Z}\left[\frac{1}{Ts^2}\right] = \frac{z}{(z-1)^2} \Rightarrow E(z) = \frac{1}{1 + \left(\frac{z-1}{z}\right) \frac{1}{(z-1)^2}} R(z) = \left(\frac{z-1}{z}\right) R(z) \#$$

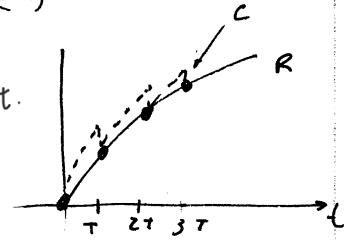
$$C = \left(1 + \frac{1}{Ts}\right) A = \left(1 + \frac{1}{Ts}\right) \left(1 - \frac{e^{-sT}}{s}\right) E^*$$

$$C^* = \left[\left(\frac{1-e^{-sT}}{s}\right)^* + \left(\frac{1-e^{-sT}}{Ts^2}\right)^*\right] E^* \Rightarrow C(z) = \left[\left(\frac{z-1}{z}\right) \left(\frac{z}{z-1}\right) + \left(\frac{z-1}{z}\right) \frac{z^2}{(z-1)^2}\right] E(z)$$

$$C(z) = (1 - e^{-sT}) \left[ \left(1 + \frac{1}{Ts}\right) \cdot \frac{1}{s} \right]^* E^* \Rightarrow C(z) = \frac{z-1}{z} \mathbb{Z}\left[\frac{1}{s} + \frac{1}{Ts^2}\right] E(z) \left[1 + \frac{1}{(z-1)}\right] E(z) = \frac{z}{(z-1)} E(z)$$

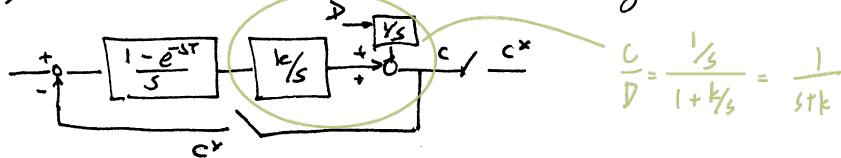
$$\frac{C(z)}{R(z)} = \frac{C}{E} \cdot \frac{E}{R} = \left(\frac{z}{z-1}\right) \cdot \left(\frac{z-1}{z}\right) = 1$$

sync of output should be equal to sync of input.



As expected. Since the block diagram implements a hold then  $c(kT) = r(kT)$  all  $k \Leftrightarrow C(z) = R(z)$

b) Since  $R=0 \Rightarrow$  eliminate it from diagram:



$$\frac{C}{D} = \frac{1/s}{1 + k/s} = \frac{1}{s + k}$$

TF from  $D$  to  $C$  ONE since  $D$  goes through the  $Y_s$  block before being sampled

pole  $\Rightarrow z = 1 - kT$

$$|z| < 1$$

$$\Rightarrow -1 < 1 - kT < 1$$

$$-2 < -kT < 0$$

$$\Rightarrow 2 > kT > 0$$

$$D = 1 \Rightarrow C(z) = \mathbb{Z}\left(\frac{1}{s}\right)$$

$$\frac{1}{1 + \left(\frac{z-1}{z}\right) \frac{1}{(z-1)^2} k} = \frac{z}{(z-1) \left[1 + \left(\frac{z-1}{z}\right) \frac{Tz}{(z-1)^2} k\right]} = \frac{z}{z-1+Tk}$$

$$c) FUT: C_{ss} = \lim_{z \rightarrow 1} (z-1) C(z) = \lim_{z \rightarrow 1} \frac{(z-1) z}{z-1+Tk} = 0 \quad \text{provided that is stable}$$

$$\Rightarrow -1 < -1 + Tk < 1 \quad / \quad 0 < Tk < 2 \quad \Rightarrow \text{pick } T, k \text{ so that}$$

$$0 < Tk < 2$$

d) Type I system, so it is expected to reject step disturbances. from continuous time:  $1 + \frac{k}{s} = 0$

$$s = -\frac{k}{s}$$

discrete time: unstable if  $Tk \geq 2$  due to extra phase lag

when  $k$  is positive  $\Rightarrow$  means STABLE

give discrete time poles

know settling time / like  $A^{-1}f$