

- Nyquist criterion, general case

In general we may have P poles in the RHP

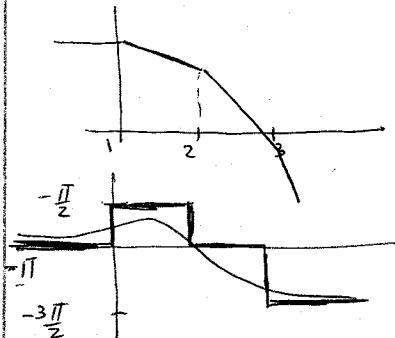
⇒ for the closed loop system to be stable we need $Z=0$. Hence

$$N = Z - P \quad // \quad \boxed{N = -P}$$

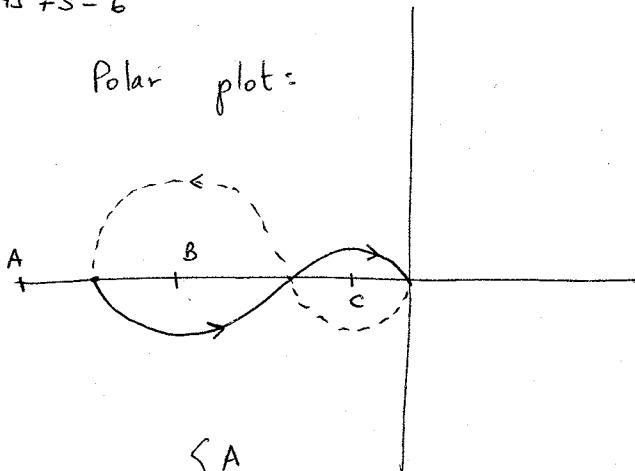
Example where $P \neq 0$ (open loop unstable)

$$G(s) = \frac{k}{(s-1)(s+2)(s+3)} = \frac{k}{s^3 + 4s^2 + s - 6}$$

Bode Plots:



Polar plot:



$$P=1 :$$

$$3 \text{ possible cases: } -1 = \begin{cases} A \\ B \\ C \end{cases}$$

1h

$$A) \quad N=0$$

$$Z=1 \Rightarrow \text{unstable}$$

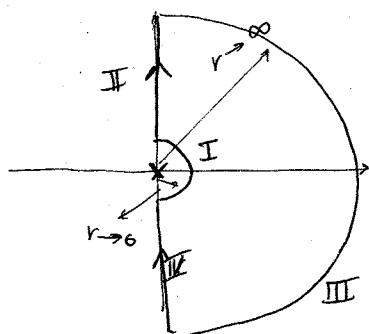
$$B) \quad N=-1$$

$$Z=N+P=0 \Rightarrow \text{stable}$$

$$C) \quad N=1$$

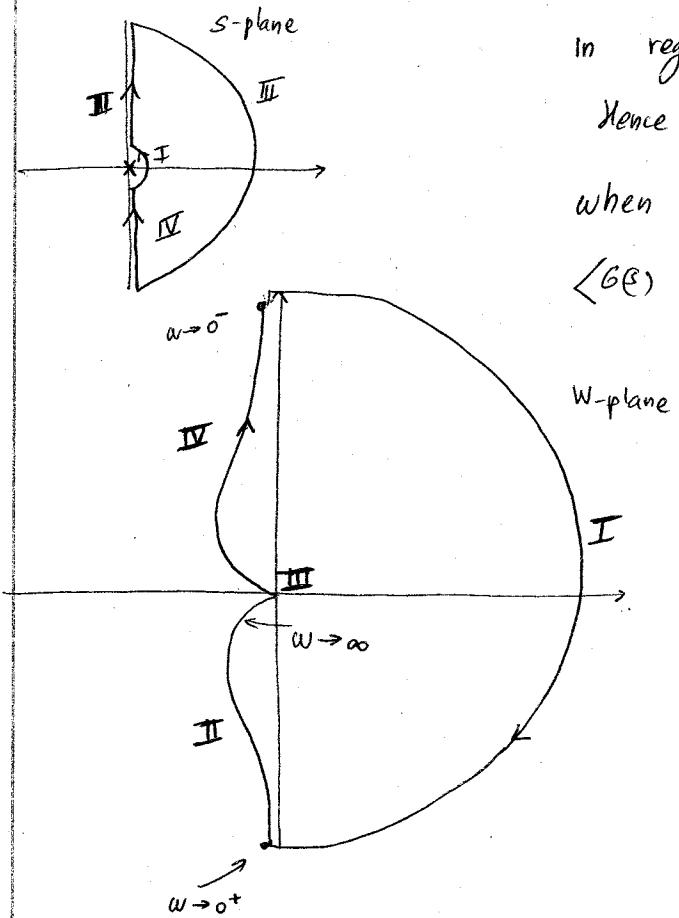
$$Z=N+P=2 \Rightarrow \text{unstable}$$

- Note that the argument's principle is valid only if C does not go through any pole of $F(s)$. Hence if $F(s)$ has poles on the $j\omega$ axis (such as at $s=0$) we need to change C (to avoid these poles). For instance, assume a pole at the origin. Then we can avoid it by taking a small semicircle (with radius $r \rightarrow 0$) around the pole



Now, as $r \rightarrow 0$ the region I will map into ∞ in the w plane.

Example: $G(s) = \frac{k}{s(s+1)}$



In region I, $s = re^{j\theta}$ with $r \rightarrow 0$

$$\text{Hence } G(s) \approx \frac{k}{re^{j\theta}} = \frac{k}{r} e^{-j\theta}$$

when θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ counterclockwise

$\angle G(s)$ goes from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$ clockwise

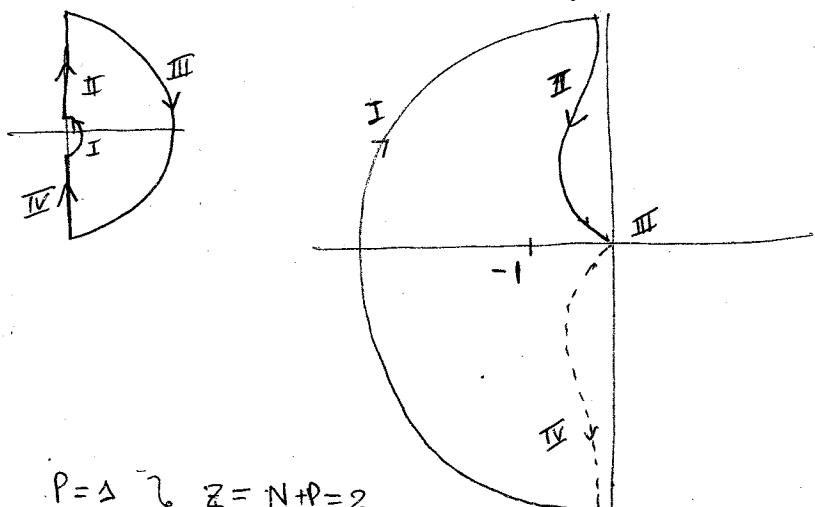
$$P=0, N=0$$

$$\Rightarrow Z=0$$

stable for all $k > 0$

Example: (non minimum phase system:)

$$G(s) = \frac{k}{s(s-1)} \Rightarrow P=1$$



$$\begin{aligned} \text{for } s &= pe^{j\theta} \\ p &\rightarrow 0 \\ G(s) &\sim -\frac{k}{pe^{j\theta}} \\ &= \frac{k}{p} e^{-j\theta} \angle 180 - \theta \end{aligned}$$

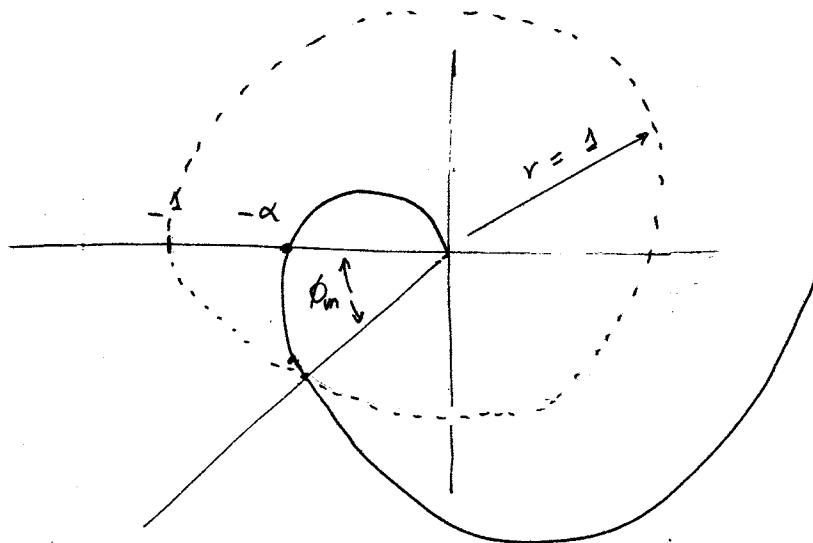
$$\left. \begin{array}{l} P=1 \\ N=1 \end{array} \right\} Z=N+P=2$$

unstable for all $k > 0$, with 2 unstable roots.

- Relative Stability (Concepts of Phase & Gain margin)
- Concept of "relative stability": How close is a given system to being unstable?
(ie: how much can we change $G(j\omega)$ and still have a stable system?) This gives a measure of the "robustness" of the system against model uncertainty

Classically 2 measures:

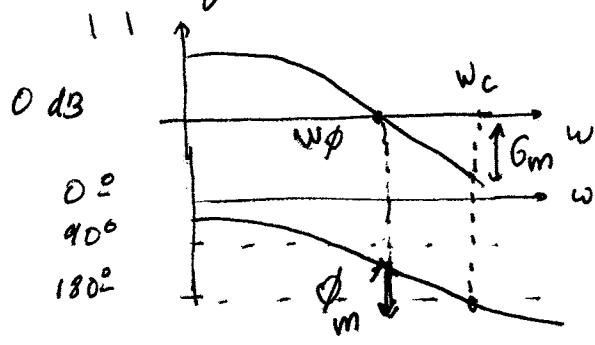
- a) Gain margin: How much gain can we add to our system and still keep it stable?
(equivalently: how much gain ~~add~~ I have to add to the system to make it unstable)
- b) Phase margin: How much phase do I have to add to the system to make it marginally stable?



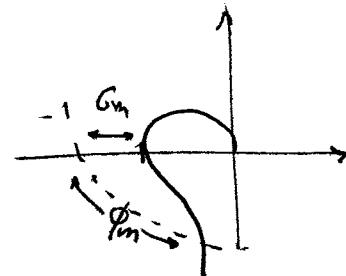
- Gain margin: Let α be the magnitude of $G(j\omega)$ at the 180° cross-over (ie $\theta(j\omega) = -\alpha$) \Rightarrow Then, the gain margin is $1/\alpha$
(usually expressed in dB)
- Phase margin: magnitude of the minimum angle by which the Nyquist diagram must be rotated in order to go through the -1 point
 \Rightarrow in the diagram: ϕ_m

Gain & phase margins can be read (in most cases) from the Bode diagrams

If the system is open loop stable, then



Example: $G(s) = \frac{2500}{s(s+5)(s+50)}$



To find G_m , first find w_c : $G(jw) = \frac{2500}{jw[(250-w^2)+j55w]}$

$$\Rightarrow w_c^2 = 250 \Rightarrow w_c = 15.8 \text{ rad/sec}$$

$$|G(jw_c)| = \frac{1}{5.5} \Rightarrow G_m = 5.5 = 14.8 \text{ dB}$$

To find ϕ_m , we need to find the frequency where $|G(jw)|=1$

$$\Rightarrow |G(jw)|^2 = 1 \Rightarrow (2500)^2$$

$$\frac{w^2[(250-w^2)^2 + (55)^2 w^2]}{[(250-w^2)^2 + (55)^2 w^2]} = 1$$

$$\text{Let } w^2 = x \Rightarrow x(x^2 - 500x + (55)^2 x + (250)^2) - (2500)^2 = 0$$

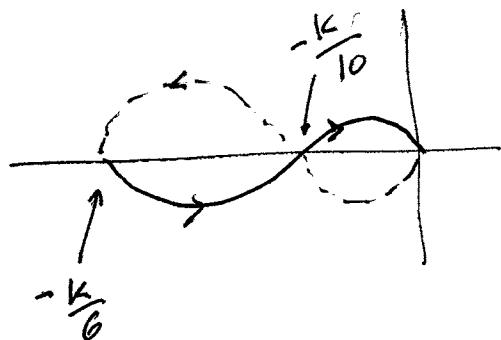
$$\Rightarrow w = \sqrt{x} = 6.2 \text{ rad/sec}$$

$$\angle G(jw_\phi) = -148.3^\circ \Rightarrow \phi = 180 + \angle G(jw)$$

$$= 31.7^\circ$$

However, need to be careful if the plant is open-loop unstable

Back to the example $G(s) = \frac{K}{(s-1)(s+2)(s+3)} = \frac{K}{s^3 + 4s^2 + s - 6}$



stable for $6 < K < 10$

and it can be made unstable by both increasing and decreasing the gain.

For instance, for $K_0=8$, -1 is encircled once, counterclockwise \Rightarrow
 $N=-1 \Rightarrow Z=N+P=0$ (stable)

As K increases, the system becomes unstable when the point at $-\frac{K}{10}$ lands on top of $-1 \Rightarrow K=10$

$$\Rightarrow G_m = \frac{10}{8} = 1.25 = \boxed{1.94 \text{ dB}} \quad (\text{upward gain margin})$$

But, the system can also be made unstable by decreasing K
so that the point $-\frac{K}{6}$ moves on top of -1

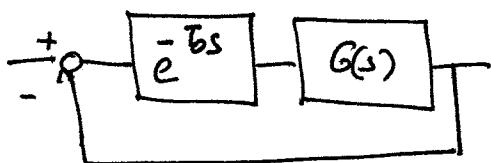
$$G_m = \frac{6}{8} \approx -2.5 \text{ dB} \# \quad (\text{downward gain margin})$$

For this example Matlab's $[G_m, P_m] = \text{margin}(\text{sys})$ yields:

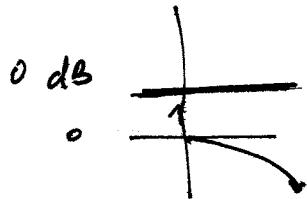
$$G_m = 1.94 \text{ dB} \quad \leftarrow \text{only one of the values}$$

$$\phi_m = 2.54^\circ$$

- Effects of time delay on stability



Recall that the Bode plot of a time delay is



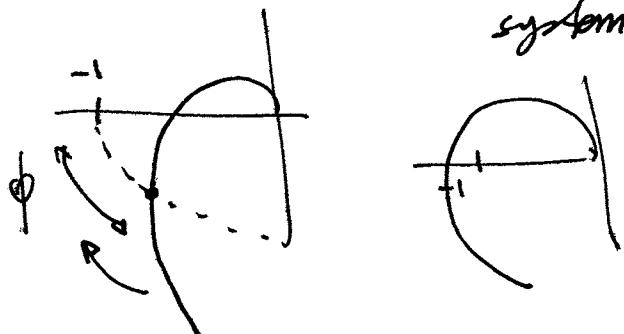
$$|e^{-sT_0}| = 1$$

$$\angle e^{-j\omega T_0} = -\omega T_0$$

Want to analyze stability of
 $1 + G(s)e^{-sT_0} = 0$

(can't use Routh Hurwitz since)
 this is not a polynomial

- Nyquist plot gets rotated (clockwise) by an amount ωT_0 radians at frequency ω \Rightarrow even if the original system was stable, the added rotation can cause the plot to encircle the -1 point and destabilize the system



Q: how much delay can we tolerate?

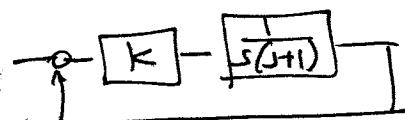
A: System becomes unstable when Nyquist plot passes through -1

\Rightarrow we need to add ϕ radians, but ϕ is precisely the phase margin

$$\phi_m = \omega \phi T_0 \Rightarrow T_0 = \frac{\phi_m}{\omega \phi} \text{ radians} = \frac{\phi_m \pi}{180 \omega_0} \text{ deg}$$

Example:

$$G(s) = \frac{K}{s(s+1)}$$



From RH, stable for all K , gain margin ∞



However, $\phi_m = 51.83^\circ$ at $w_p = 0.7862 \text{ rad/sec}$

\Rightarrow system can tolerate a delay of at most
 $T = 1.15 \text{ seconds}$

What if $K=10$? \Rightarrow Then $\phi_m = 11.96$ and $w_p = 3.08$
 $\Rightarrow T_0 = 0.1 \text{ sec} \# \leftarrow \text{very small}$

Note that for the closed loop system we have (without the time delay)

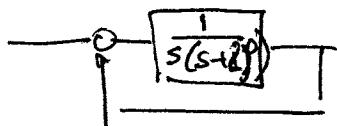
$$G_{CL} = \frac{K}{s^2 + s + K} \Rightarrow \omega_n = \sqrt{K} \quad 2\zeta\omega_n = 1 \Rightarrow \zeta = \frac{1}{2\omega_n} = \frac{1}{2\sqrt{K}}$$

\Rightarrow in this case $\zeta = 0.158 \Rightarrow M_p \approx 60\%$

Seems that there is a connection here between low damping and small phase margins \leftrightarrow small tolerance to time delays

This is actually the case!

- Relationship between phase margin and ζ (for second order systems)



$$G(j\omega) = \frac{\omega_n^2}{s(s+2j\zeta\omega_n)}$$

To find the ϕ_m , we need to find the freq ω_p such that $|G(j\omega_p)| = 1$

$$\Rightarrow s=j\omega_p, \text{ define } v = \frac{\omega_p}{\omega_n} \Rightarrow \left| -\frac{1}{v^2 + 2j\zeta v} \right| = 1 \Rightarrow v^4 + 4\zeta^2 v^2 - 1 = 0$$

$$v^2 = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

$$\angle G(j\omega_p) = -\angle \left(-v^2 + j2\zeta v \right) = +180 - \tan^{-1} \left(2\zeta \frac{v^2}{v^2} \right)$$

$$\Rightarrow \phi_m = \tan^{-1} \left[\frac{2\varphi}{(\sqrt{4\varphi^4 + 1} - 2\varphi^2)^{1/2}} \right] \approx 100\varphi$$

good approximation in the range
 $0.1 \leq \varphi \leq 0.7$

\Rightarrow if you want $\phi_m \geq 30^\circ$ you need $\varphi > 0.3$

ϕ_m is also related to settling time:

$$T_s = \frac{4}{\varphi \omega_n}$$

$$\phi_m = \tan^{-1} \left[2 \frac{\varphi \omega_0}{\omega_n} \right] = \tan^{-1} \left[\frac{8}{T_s (\sqrt{4\varphi^4 + 1} - 2\varphi^2)^{1/2}} \right]$$

- Robust Control

The idea is to design a system such that some desirable properties (such as stability) are preserved in the presence of (unknown) perturbations

- Robust stability:

Suppose that rather than having a known plant $G_0(s)$ we are dealing with a family of plants

$$G(s) = G_0(s)(1 + \delta(s)) \quad \text{where } \delta(s) \text{ is a (frequency dependent) perturbation (such as unmodeled dynamics)}$$

Assume that we only know that, at each frequency ω , δ is bounded by a known function, e.g.:

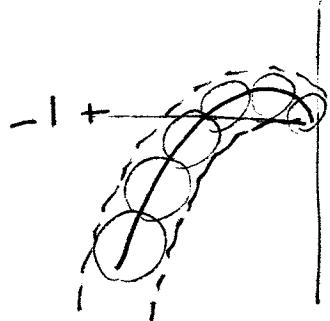
$$|\delta(j\omega)| < r(j\omega) \text{ where } r \text{ is known.}$$

We'd like to find out if the closed loop system is stable for all members of the family

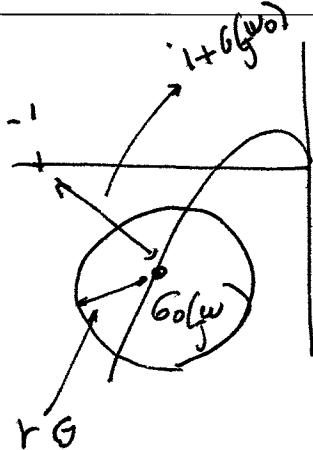
⇒ Let's apply Nyquist to the entire family

(assume that the entire family is open loop stable) ⇒

For a given frequency ω_0 , the Nyquist plot of the actual system can be anywhere inside the circle centered at $G_0(\omega_0)$ with radius $r(\omega_0)$. $G(\omega_0)$ = rather than a single plot, we get a band



So, the family is stable iff the point $(-1, 0)$ is excluded from the band, that is



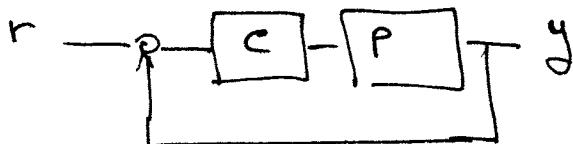
The condition for stability of the family becomes

$$|1 + G(j\omega_0)| > \underline{|r G(j\omega_0)|}$$

distance from center
of the disk to the $(-1, 0)$
point

radius of
the disk

$$\Rightarrow \text{family is stable} \Leftrightarrow 1 > \frac{r |G(j\omega_0)|}{|\Delta + G(j\omega_0)|}$$



$$\text{but } \frac{G(j\omega)}{1+G(j\omega)} = T_{yr}$$

\Rightarrow Robust stability requires that, for all frequencies

$$|T_{yr}(j\omega)| < \frac{1}{|r(j\omega)|}$$

$$\text{or, in compact form: } \|r(j\omega) \cdot T_{yr}(j\omega)\|_\infty < 1$$

\nearrow
H-infinity norm

(for a function $G(j\omega)$ we define its H-infinity norm as

$$\|G(j\omega)\|_\infty = \sup_\omega |G(j\omega)| \quad (\text{e.g. the peak value in the Bode plot})$$

Note that the conditions above indicate that frequencies where $|T(j\omega)|$ is large can tolerate little uncertainty \Rightarrow need to make $|T|$ small in places where $|r|$ is large (e.g. places where the model is not well known)