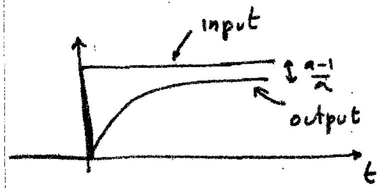
Chapter 6: System Time-Response Characteristics

- Review of time-responses of continuous time systems:
- First order plant: $R(s) \rightarrow \boxed{\frac{1}{s+a}} \rightarrow C(s)$

if $R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{1}{s(s+a)} \overset{\mathcal{L}^{-1}}{\Longleftrightarrow} c(t) = \frac{1}{a}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{1}{a}\left(1 - e^{-at}\right)$



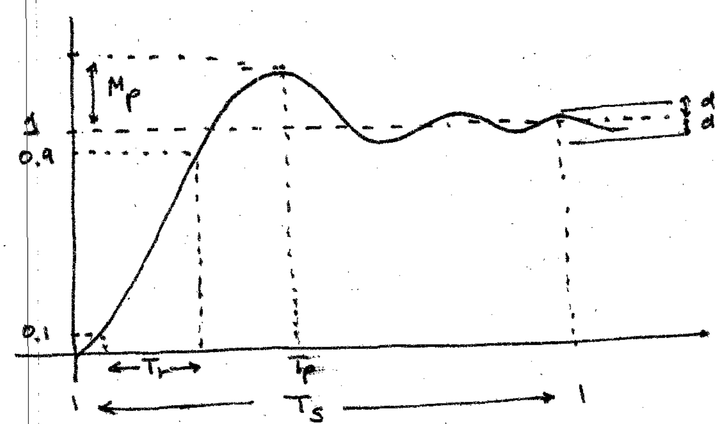
$\Rightarrow$ has non-zero steady state error

Time constant $\tau = \frac{1}{a}$

Steady state is reached in about 4τ

- 2^{nd} order plant: $R(s) \rightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \rightarrow C(s)$

ζ: damping ratio
$0 < \zeta < 1$: underdamped
$\zeta = 1$ critically damped
$\zeta > 1$ overdamped
ω_n: natural frequency



$d \sim 1\%, 5\%$

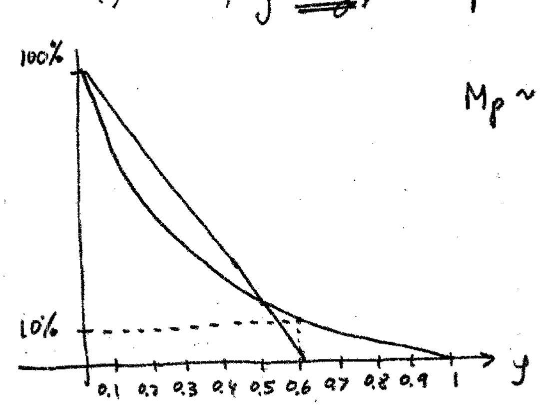
T_r: rise time (function of ζ, ω_n) $\quad T_r \cong \frac{1.8}{\omega_n}$ for $\zeta = 0.5$

T_p: peak time (function of ζ, ω_n) $\quad T_p = \pi/(\omega_n\sqrt{1-\zeta^2})$

T_s: settling time (function of $\zeta \cdot \omega_n$) $\quad T_s = 4/\zeta\omega_n$

M_p: % overshoot (function of ζ only) $\quad M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$

Approximation:



$M_p \sim \left(1 - \frac{\zeta}{0.6}\right) \quad 0 \le \zeta \le 0.6$

(10% max error @ $\zeta = 0.6$)

ζ is also related to the phase margin: $PM \sim 100 \cdot \zeta$

- Now we need to find the equivalent responses for discrete time systems.

2 alternatives — a) derive everything again as in 580 Tedious!!

b) Find a relationship between locations in the s & z planes and use directly the results obtained in 580

We will pursue approach (b):

Start by recalling the relationship between the poles of $E(s)$, $E^*(s)$, $E(z)$

$$\underset{E(s)}{e(t)} \;\text{—/—}\; \underset{E^*(s)}{e^*(t)}$$

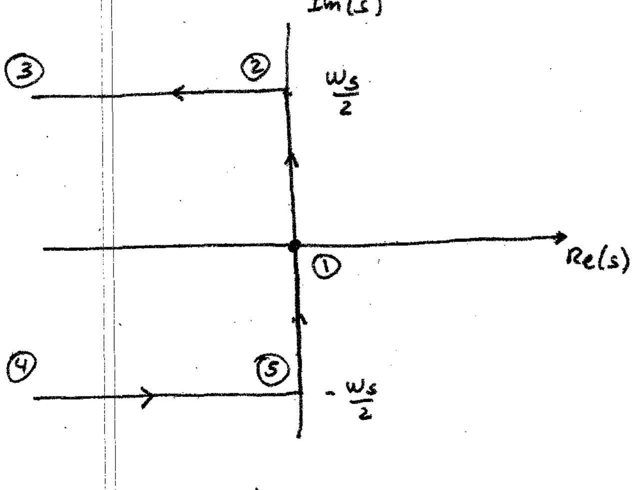
Remember: $$E(z) = \sum_{\substack{\text{at poles} \\ E(\lambda)}} \left[\text{residues } E(\lambda) \frac{z}{z - e^{\lambda T}} \right]$$

$\Rightarrow$ $E(s)$ pole at $s = p$ $\Leftrightarrow$ $E(z)$ pole at $z = e^{pT}$

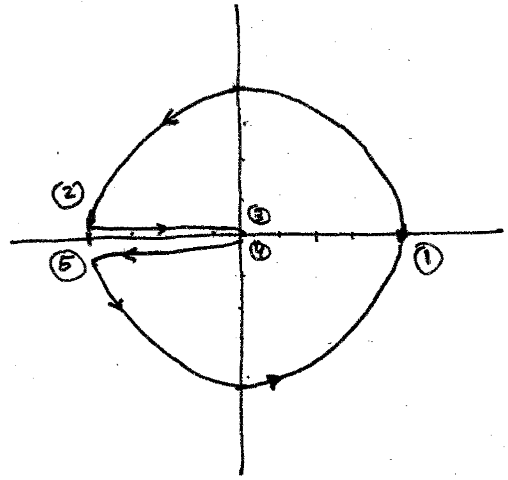
For instance if $E(s) = \dfrac{1}{s+a}$ $\Rightarrow$ $E^* = \dfrac{e^{sT}}{e^{sT} - e^{-aT}}$

$$E(z) = \frac{z}{z - e^{-aT}}$$

$\Rightarrow$ A pole at $z = z_1$ results in the transient response characteristics (at the sampling instants) of the continuous time function that has a pole at $s = s_1$ where $z_1 = e^{s_1 T}$. Using this relationship we have:



s - plane



z - plane (after sampling

(Note: since $e^{sT} = e^{s\frac{2\pi}{\omega}}$ is periodic with period $j\omega_s$, we need to consider only one such period, for instance $-\frac{\omega_s}{2} < \omega \leq \frac{\omega_s}{2}$)

- The $j\omega$ axis gets mapped to the unit circle:

  i.e if $s = j\omega \Rightarrow z = e^{j\omega T} = 1 \angle \omega T$

  Therefore, a pole at $z_1 = 1 \angle \omega_1 T$ generates oscillations with freq ω_1

- The line $s = \sigma + j\frac{\omega_s}{2}$ gets mapped to: $z = e^{\sigma T + j\frac{\omega_s}{2}T} = e^{\sigma T} \cdot e^{\pi} = -e^{\sigma T}$

  so if $\sigma < 0$, we get the portion of the real axis $-1 \leq z < 0$

- Similarly, the line $s = \sigma - j\frac{\omega_s}{2}$ gets mapped to the same place

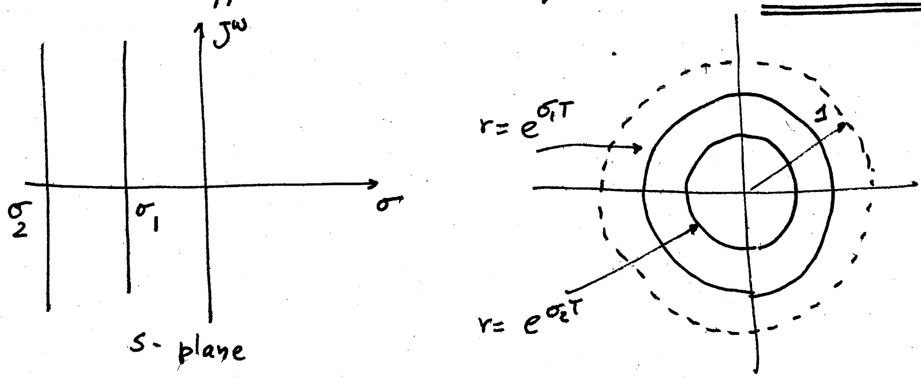
- A pole at $z = z_1 = r\angle\theta \longleftrightarrow s = s_1 = \sigma + j\omega$ with $r = e^{\sigma T}$, $\angle\theta = \omega T$

$$c_k = e^{k\sigma T}(\cos(k\omega T + \phi)) \longleftarrow c(t) = e^{\sigma t}\cos(\omega t + \phi)$$

$$c_k = (r)^k \cos[k\theta + \phi]$$

(consistent with what we knew, if $r < 1 \Rightarrow c_k \to 0$ as $k \to \infty$; $r > 1$ $c_k \to \infty$, unstable)

- What happens to the loci of constant <u>time constant</u>? (i.e. constant real part)

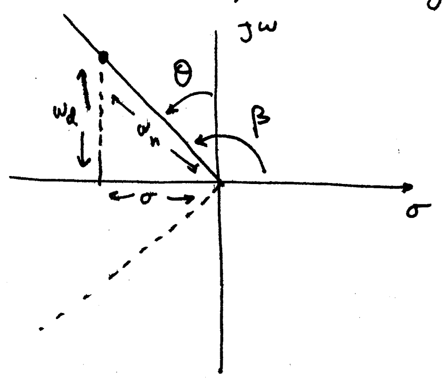


$$s = \sigma_1 + j\omega \Rightarrow z = e^{sT} = e^{\sigma_1 T} e^{j\omega T} = e^{\sigma_1 T} \angle \omega T$$ circle of radius $e^{\sigma_1 T}$

In the discrete time case: $\boxed{\text{time constant related to } |z|}$

- What about constant damping ratio?

Recall that $\zeta = \sin\theta \Rightarrow$ in the continuous time case the locus of constant ζ is a ray through the origin



s-plane

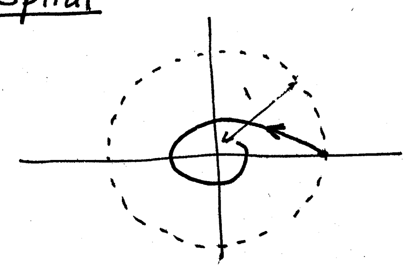
$\zeta = \sin\theta$

$\frac{\omega_d}{\sigma} = \tan\beta$

$$\Longrightarrow \quad z = e^{sT} = e^{\sigma T} \angle \omega T = e^{\sigma T} \angle \sigma T \cdot \tan(\beta)$$

$$\Rightarrow \beta \text{ constant:} \quad z = e^{\sigma T} \angle \sigma T \tan(\beta)$$

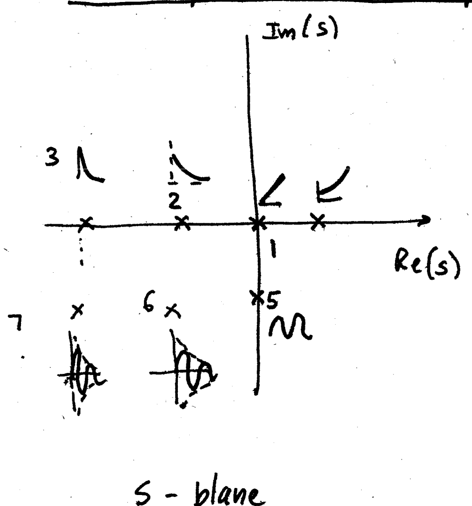
this is the equation of a <u>logarithmic spiral</u>



z - plane

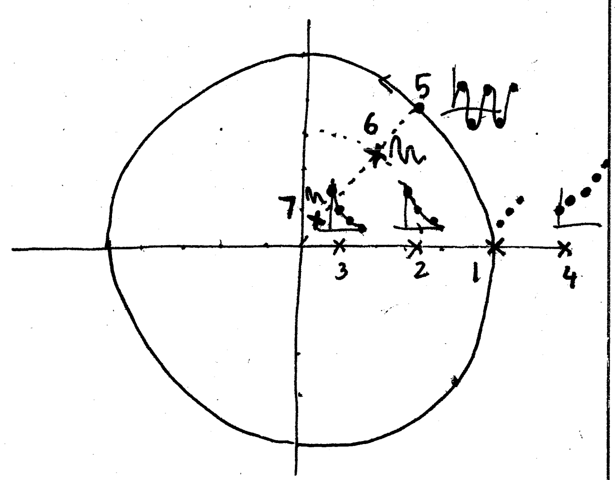
<u>Recap</u>: $z = e^{sT}$

<u>Time responses in the s-plane</u>



s - plane

<u>Time responses in the z-plane</u>



Note 2 & 6 are at the same distance from $z=0$

3 & 7 are at the same distance from $z=0$

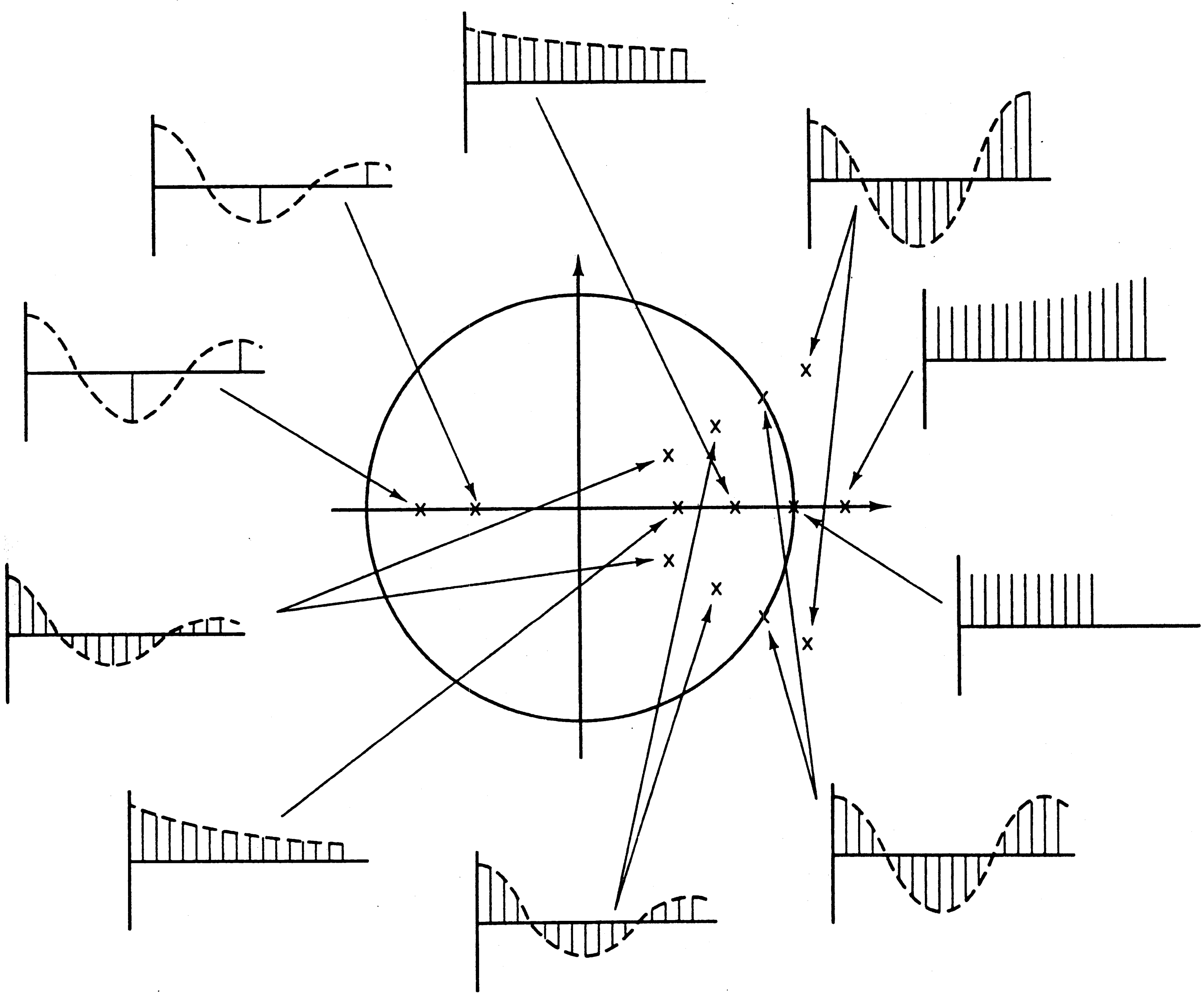
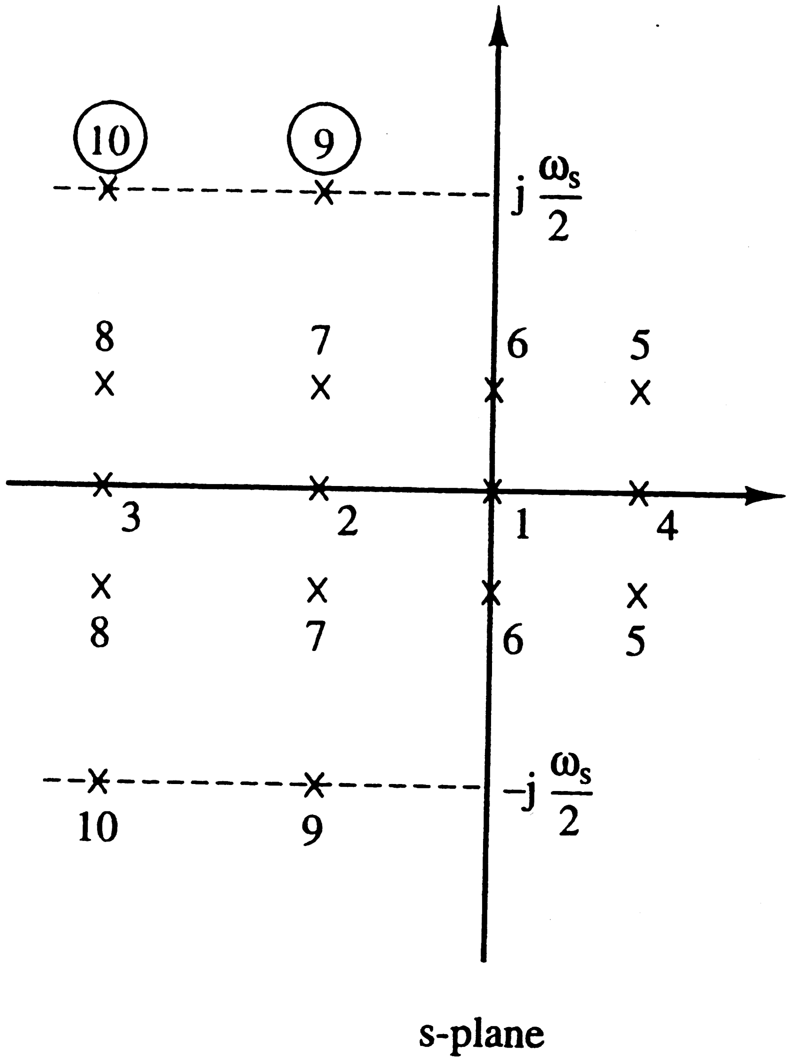


Figure 6-11 Transient response characteristics of the z-plane pole locations.



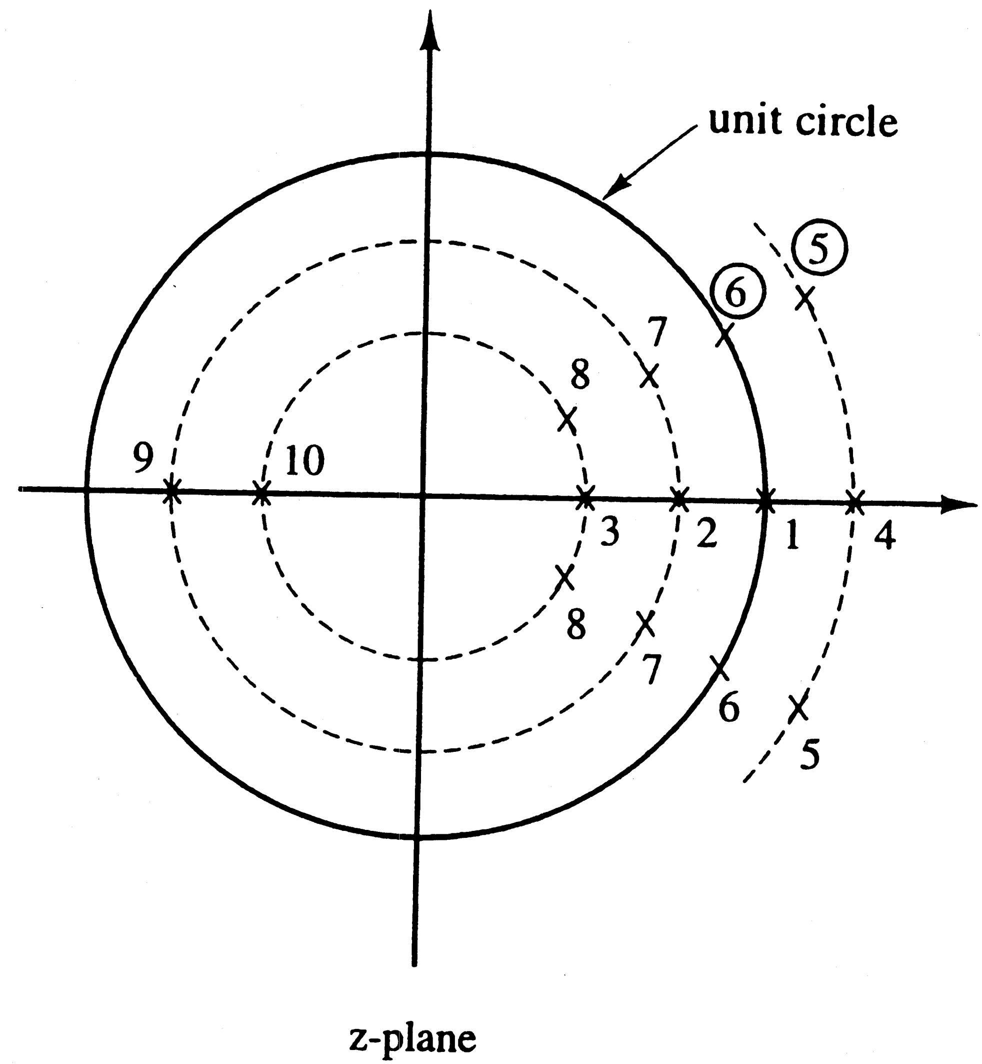


Figure 6-10 Corresponding pole locations between the s-plane and the z-plane.

- If we have a second order system (in the s-plane):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The equivalent z-plane poles are located at: $z = e^{sT}\Big|_{s_1,s_2} = e^{-\zeta\omega_n T} \angle \omega_n T\sqrt{1-\zeta^2} = r\angle \pm\theta$

where (1) $r = e^{-\zeta\omega_n T} \Rightarrow \zeta\omega_n T = -\ln r$

(2) $\theta = \omega_n T\sqrt{1-\zeta^2}$

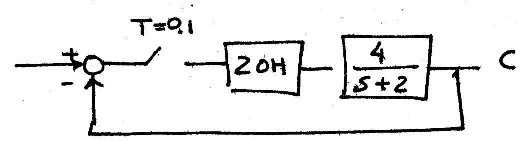
From (1), (2): $\frac{-\ln r}{\theta} = \frac{\zeta}{\sqrt{1-\zeta^2}} \Rightarrow$

$$\boxed{\zeta = \frac{-\ln r}{\sqrt{\theta^2 + \ln^2 r}} \qquad \omega_n = \frac{1}{T}\sqrt{\theta^2 + \ln^2 r}}$$

Finally, the time constant is given by: $\boxed{\tau = \frac{1}{\zeta\omega_n} = \frac{-T}{\ln r}}$

$\Rightarrow$ From r, θ we can compute ζ, ω_n, τ and from there $T_s, M_p, T_r, \ldots$

- Example 1 (A first order plant)

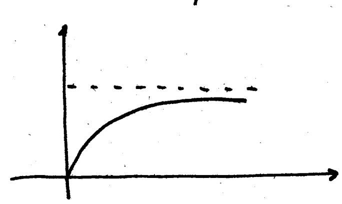


$$C(z) = \frac{G(z)}{1+G(z)} R(z) \quad \text{where} \quad G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\cdot\frac{4}{s+2}\right] = \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{4}{s(s+2)}\right] = \frac{0.3625}{z-0.8187}$$

$$C(z) = \frac{0.3625}{z-0.4562} R(z)$$

Since we have a pole at $z = 0.4562 = e^{-0.7848} = e^{-7.848\cdot(0.1)} = e^{-7.848T}$

we should expect a time response of the form



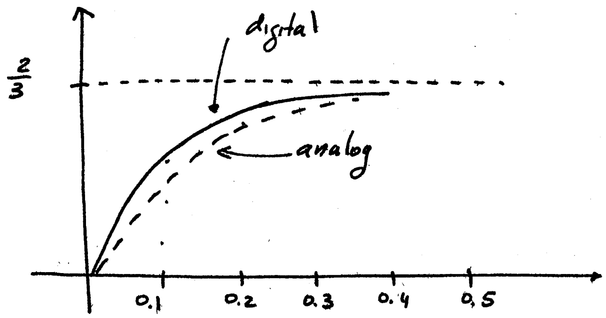
with a time constant $\tau \sim 0.127$ sec. Specifically,

$$R(z) = \frac{z}{z-1}, \quad C(z) = \frac{0.3625}{z-0.4562}\cdot\frac{z}{z-1} = \frac{2}{3}\left[\frac{z}{z-1} - \frac{z}{z-0.4562}\right] \xrightarrow{\mathcal{Z}^{-1}} c(nT) = \frac{2}{3}\left[1 - (0.4562)^k\right]$$

$$c(nT) = \frac{2}{3}\left[1-(0.4562)^k\right] = \frac{2}{3}\left[1 - e^{-7.848kT}\right]$$

If we remove the sample & hold, we get:

$$T_c(s) = \frac{4}{s+6} \Rightarrow C(s) = \frac{4}{s(s+6)} = \frac{2}{3}\left[\frac{1}{s} - \frac{1}{s+6}\right] \Rightarrow c(t) = \frac{2}{3}\left[1 - e^{-6t}\right]u(t)$$



Note that the final value is the same for both systems.
This is a general result:

Since the ZOH does not have any effect in steady-state, the output of both systems should be the same (provided that both are stable)

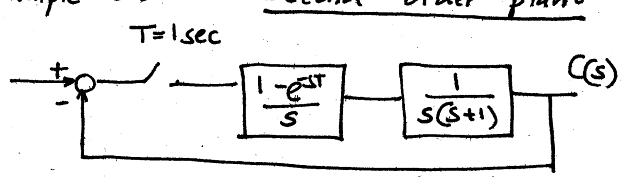
$\Rightarrow$ Same DC gain, i.e: $\lim_{s\to 0} \frac{G(s)}{1+G(s)} = \lim_{z\to 1} \frac{G(z)}{1+G(z)} = \frac{2}{3}$ #

If you want to find out what happens between sampling instants you can solve for

$$C(s) = G(s)\,E^* = G(s)\left[\frac{R}{1+G}\right]^*$$ and use the inverse Laplace transf

or use the modified z-transform.

- Example 2: Second order plant



$$G_{oe} = \mathcal{Z}\left[\frac{1-e^{-sT}}{s^2(s+1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{cl} = \frac{G_{oe}}{1+G_{oe}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Char equation: $z^2 - z + 0.632 = 0 \Rightarrow$ poles at $z = 0.7950\angle\pm 0.8906$

$$\zeta = \frac{-\ln(0.7950)}{\sqrt{[0.8906]^2 + (\ln[0.795])^2}} \cong 0.25$$

$$\omega_n = \frac{1}{1}\sqrt{\theta^2 + (\ln r)^2} \cong 0.92$$

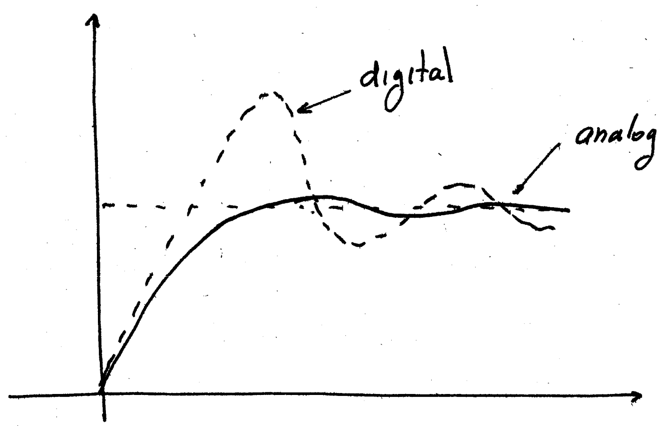
$$\tau = \frac{-1}{\ln(r)} \cong 4.36 \text{ sec}$$

Digital system ($T = 1$ sec)

$\zeta = 0.25 \Rightarrow M_p \cong 0.58$
$\omega_n \cong 0.92$ rad/sec
$\tau \cong 4.36$ sec

Analog system (no ZOH)

$\zeta = 0.5 \Rightarrow M_p \cong 16\%$
$\omega_n = 1$
$\tau = 2$ sec



You can see here that the effects of sampling are <u>destabilizing</u>

(The problem in this specific case is that the sampling rate is too low compared with the system's time constant: rule of thumb: want $T \ll \tau$)