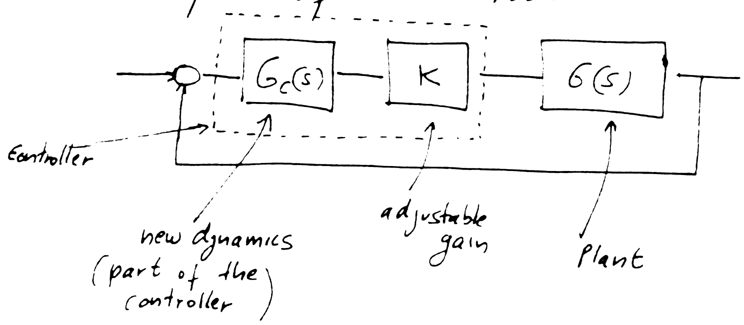
• Design using Root Locus techniques (section 7.6)

Sometimes gain adjustments alone cannot yield a satisfactory design (example: the 2 mass - spring system)

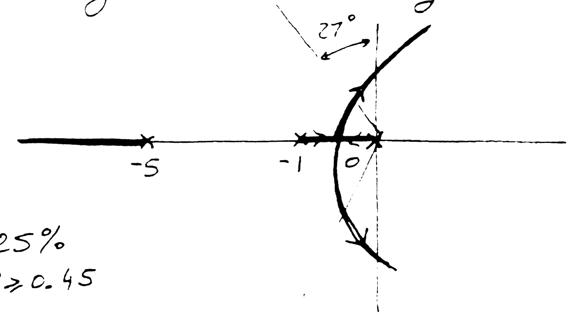
In these cases we need to change the dynamics of the system, i.e. add poles & zeros so we can change the "shape" of the root locus.



Examples: 1) Transient o.k. but steady-state error too large

Solution: increase the gain without reducing stability

$$G(s) = \frac{1}{s(s+1)(s+5)}$$



Suppose we want overshoot $\leq 25\%$

$\Rightarrow$ for the dominant poles $\zeta \geq 0.45$

Graphically (or from Matlab) we get roots at $s = -0.404 \pm j0.802$ corresponding to $K = 4.188$

Then $K_v = \lim_{s \to 0} s\,G(s) = \frac{4.188}{5} = 0.838$

$\Rightarrow$ The final design has the following properties:

$\zeta \cong 0.45$ (overshoot $\sim 25\%$)

$\omega_n = 0.898$

$t_r \cong \frac{1.8}{\omega_n} \cong 2$ sec

$T_s = \frac{4}{\zeta\omega_n} \cong 10$ sec

$K_v = 0.838 \Rightarrow$ error to unit ramp $\cong 1.19$

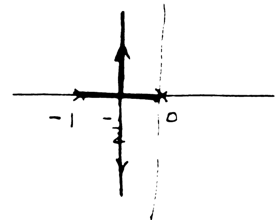
Suppose that we want to decrease the steady state error to a ramp

$\Rightarrow$ We need to increase k, but doing so will move the closed-loop poles towards the $j\omega$-axis and decrease ζ $\Rightarrow$

smaller stability margin, more overshoot $\Rightarrow$ <u>not acceptable</u>

2) <u>Transient not satisfactory</u> $\Rightarrow$ Root Locus must be moved towards the left.

Example: $KG = \dfrac{k}{s(s+1)}$



In this case the best settling time T_s we can get is $T_s \cong 8$ (why?)

Additionally, suppose we want $\omega_n \geq 2$ (from rise time considerations)

The characteristic equation is: $s^2 + s + k = 0 \Rightarrow \omega_n = \sqrt{K}$

$\omega_n = 2 \Rightarrow K = 4$

but, if $K = 4 \Rightarrow 2\zeta\omega_n = 1 \;//\; \zeta = \frac{1}{4}$

$\Rightarrow$ large overshoot!

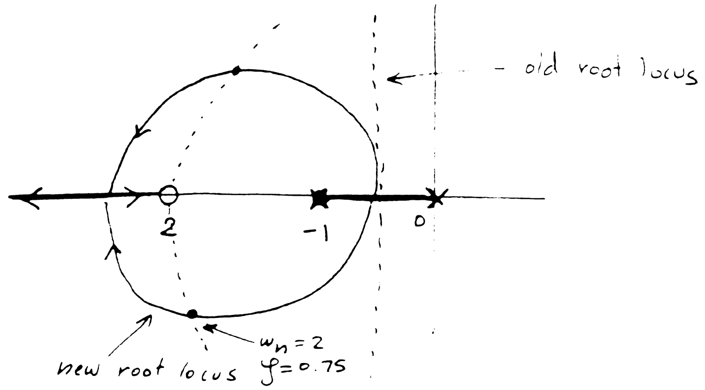
3) Transient <u>and</u> steady-state not satisfactory $\Rightarrow$ need to both increase the gain <u>and</u> move the R.L. towards the left

- 2 types of compensators are widely used:
  - Lead compensator: faster system, better transient (similar to adding a zero)
  - Lag compensator: better steady-state (however: less stable)

Back to example 2: $KG = \dfrac{1}{s(s+1)}$

Suppose we add a zero at $s = -2 \Rightarrow KG = \dfrac{K(s+2)}{s(s+1)}$

The new Root Locus is:



$G(s) = \dfrac{s+2}{s(s+1)}$ $\Rightarrow$ a) $m=1$, $n=2 \Rightarrow n-m=1$, only asymptote real axis

b) char eq: $s^2+(1+k)s+2k$ → no $j\omega$ axis intersect

c) $\dfrac{d}{ds}\left(\dfrac{1}{GH}\right)=0 \Leftrightarrow \dfrac{(2s+1)(s+2)-s(s+1)}{(s+2)^2}=0 \Leftrightarrow s^2+4s+2=0$ $\begin{cases} s=-0.58 \\ s=-3.41 \end{cases}$

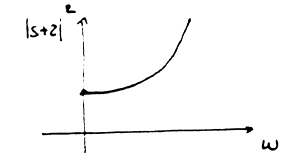
The effect of this additional zero is to move the R.L. to the left.
Now we can have closed-loop poles at $w_n=2$, $\zeta=0.75$

Problem with this: 1) not physically realizable, since the compensator is

$G_c(s) = k(s+2)$ $\Rightarrow$ more zeros than poles

2) If we look at the gain of the compensator as a function of the frequency we get:

$|s+2|^2 = |2+j\omega|^2 = 4+\omega^2$

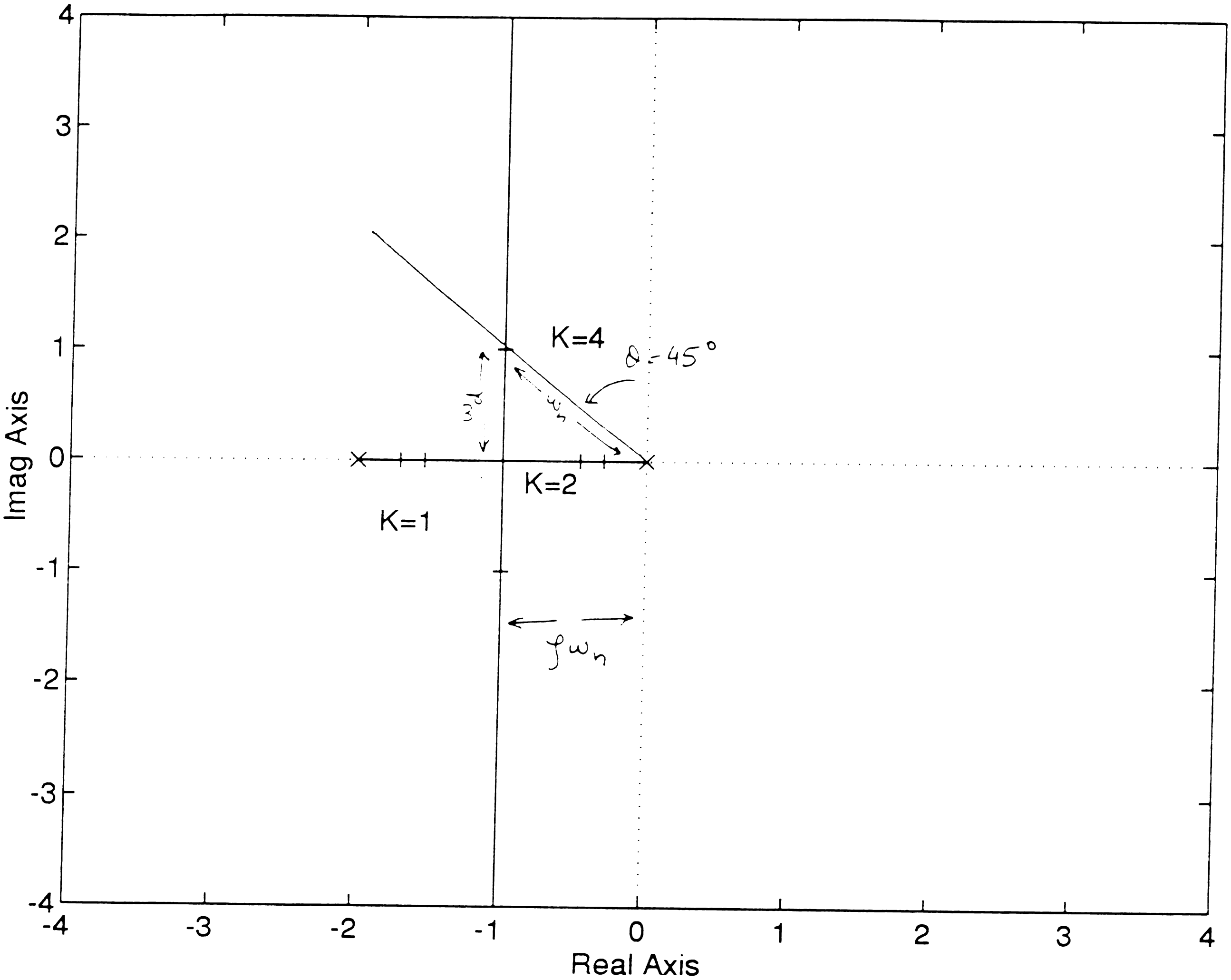


$\Rightarrow$ The compensator will amplify high frequency noise

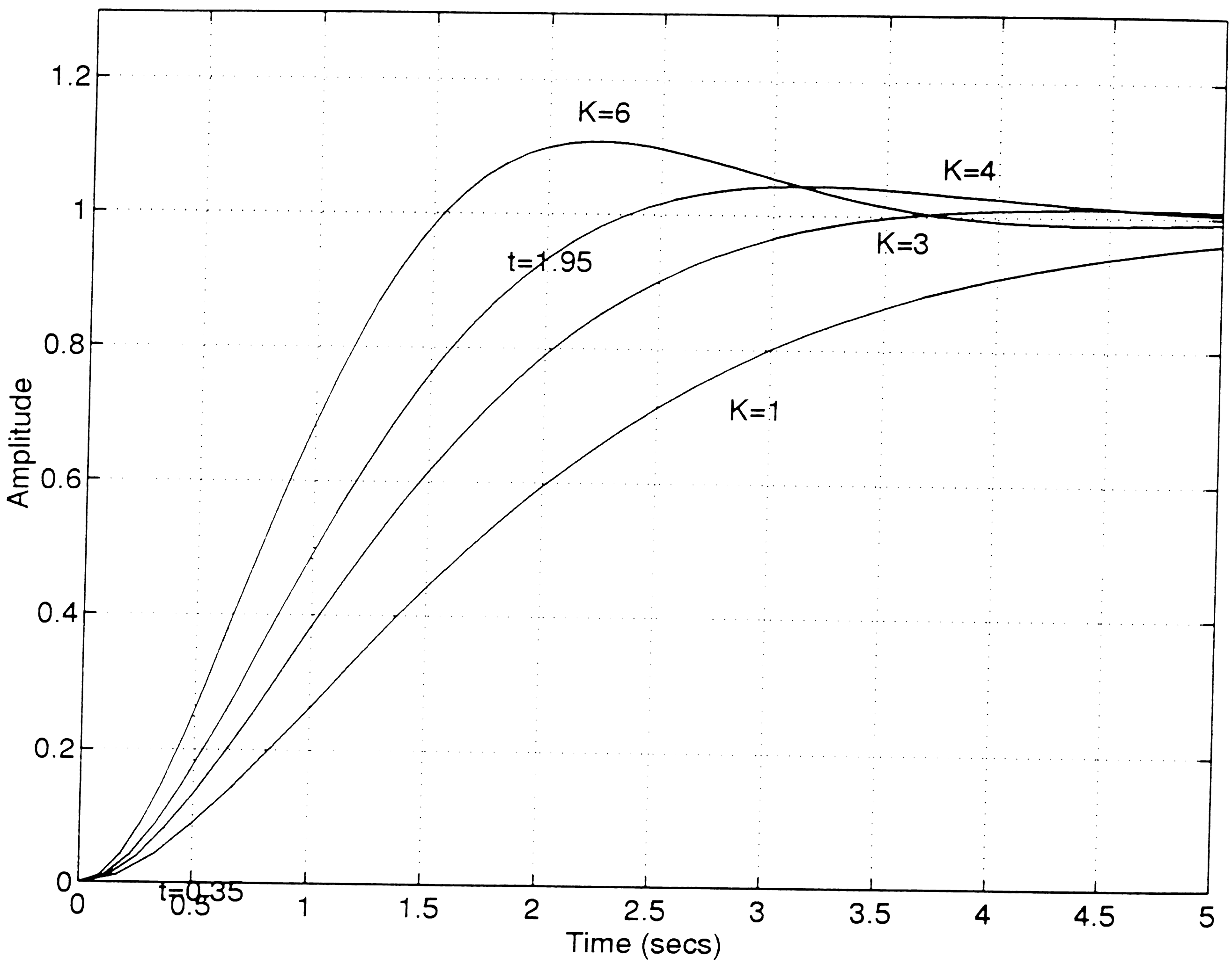
Solution: Since the effect of the zero is more important near $w_n=2$, the effect of the compensator will not be reduced if we add a high frequency pole (for instance at $s=-20$)

The effect of this pole is twofold:

1) The compensator is now physically realizable $\left(G_c = \dfrac{k(s+2)}{(s+20)}\right)$
2) Limits the high frequency gain.

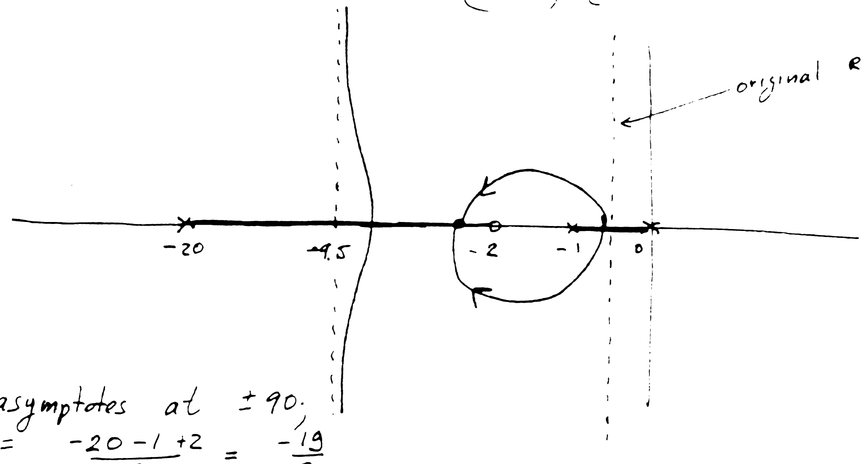


K=4
$\theta = 45°$
ω_d
ω_n
K=2
K=1
$\zeta\omega_n$
Imag Axis
Real Axis



K=6
K=4
K=3
K=1
t=1.95
t=0.35
Amplitude
Time (secs)
0
0.2
0.4
0.6
0.8
1
1.2
0.5
1.5
2
2.5
3
3.5
4
4.5
5

Root Locus of $G_c G = k \dfrac{s+2}{(s+20)s(s+1)}$



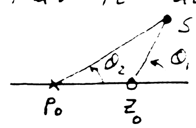
2 asymptotes at ± 90,

$\sigma = \dfrac{-20-1+2}{2} = \dfrac{-19}{2}$

Note that the root locus remains virtually unchanged in the neighborhood of $s=-2$.

- This type of compensation is called phase-lead

(This name comes from the fact that it adds a positive phase to the overall Transfer function)



total angle contribution:
$\phi = \phi_1 - \phi_2 > 0$)

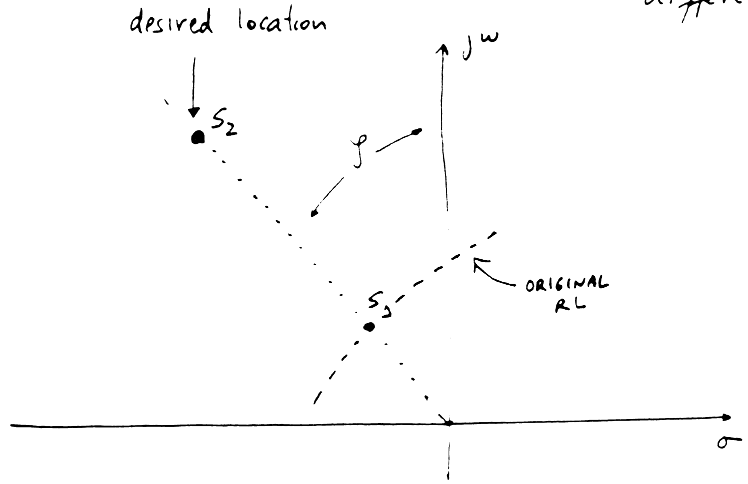
<u>Summary</u>: Phase lead compensation: $G_c(s) = k\dfrac{(s-z_0)}{(s-p_0)}$ where $|z_0| < |p_0|$

alternative description: $G_c(s) = K\dfrac{(s+1/T)}{(s+\frac{1}{\alpha}T)}$ where $\alpha \ll 1$ (common choice is $\alpha = 0.1$)

effect: adds a positive angle $\Rightarrow$ moves RL towards the left

Note: $\alpha \geq 0.1$. If α too small $\Rightarrow$ poor noise supression
α too large $\Rightarrow$ very little compensation

• <u>General Design Method</u>: Suppose that we have a root at $\underline{s_1}$ and we want to move it to a different location



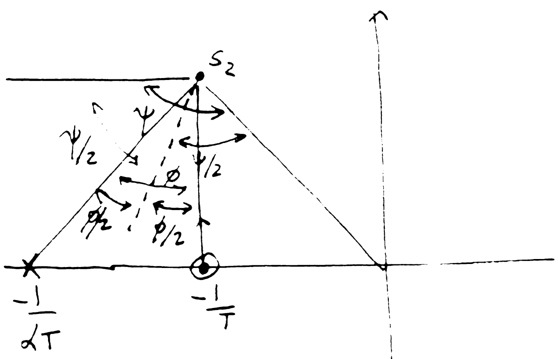
We have 3 parameters (α, T, k) and 2 equations $(1 + KGH = 0)$ $\Rightarrow$ 1 degree of freedom

To force s_2 to be part of the root locus, you need to select the location of the zero & the pole of the compensator so that the angle criterion is satisfied

<u>Fact 8</u> The following algorithm <u>yields</u> the maximum α (less additional gain)

Let the angle at s_2 due to the original poles and zeros be:

$\angle s_2 = 180 - \phi \Rightarrow$ we need to add an angle ϕ in order to make s_2 part of the root locus.

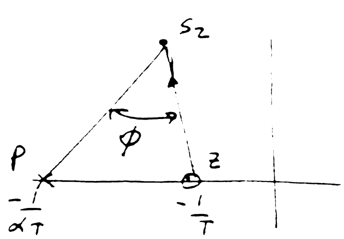


a) draw the line bisecting the angle ψ

b) Take $\frac{\phi}{2}$ at each side of this line

c) where these lines intersect the real axis are the locations of the pole & zero of the compensator

It is easy to show that indeed this construction adds a net angle of ϕ at the point $s = s_2$



$\angle s_2 - z = \angle s - p + \phi \Rightarrow$

$\angle s_2 - z - \angle s - p = \phi$

• They are all sorts of other recipes for locating the zero:

a) cancel a pole of the original system: Type 1: The rightmost (stable pole excluding the orig

Type 0: second largest stable pole

b) immediately below the desired closed-loop location (easiest)

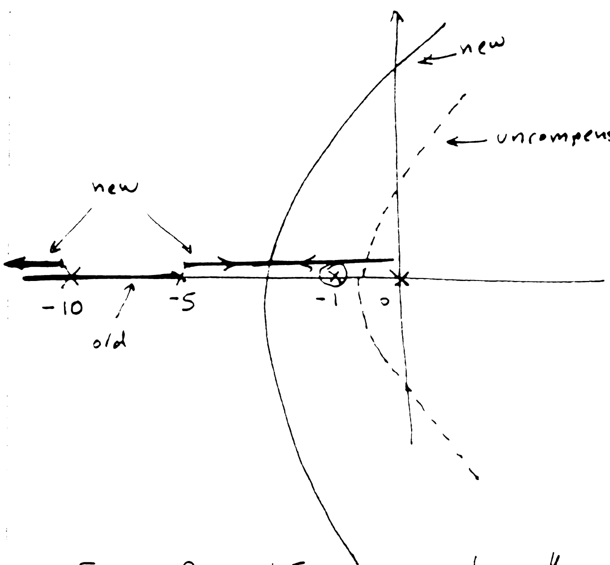
c) slightly to the left

- **Example of lead compensation:** $G = \dfrac{K}{s(s+1)(s+5)}$

choose $\alpha = 0.1$

$\Rightarrow G_c = k_c \dfrac{s + 1/T}{s + 10/T}$

$$G_s = G_c G = \frac{K k_c \,(s + 1/T)}{s(s+1)(s+5)\left(s + \frac{1}{\alpha T}\right)}$$



Let $k' = K k_c$

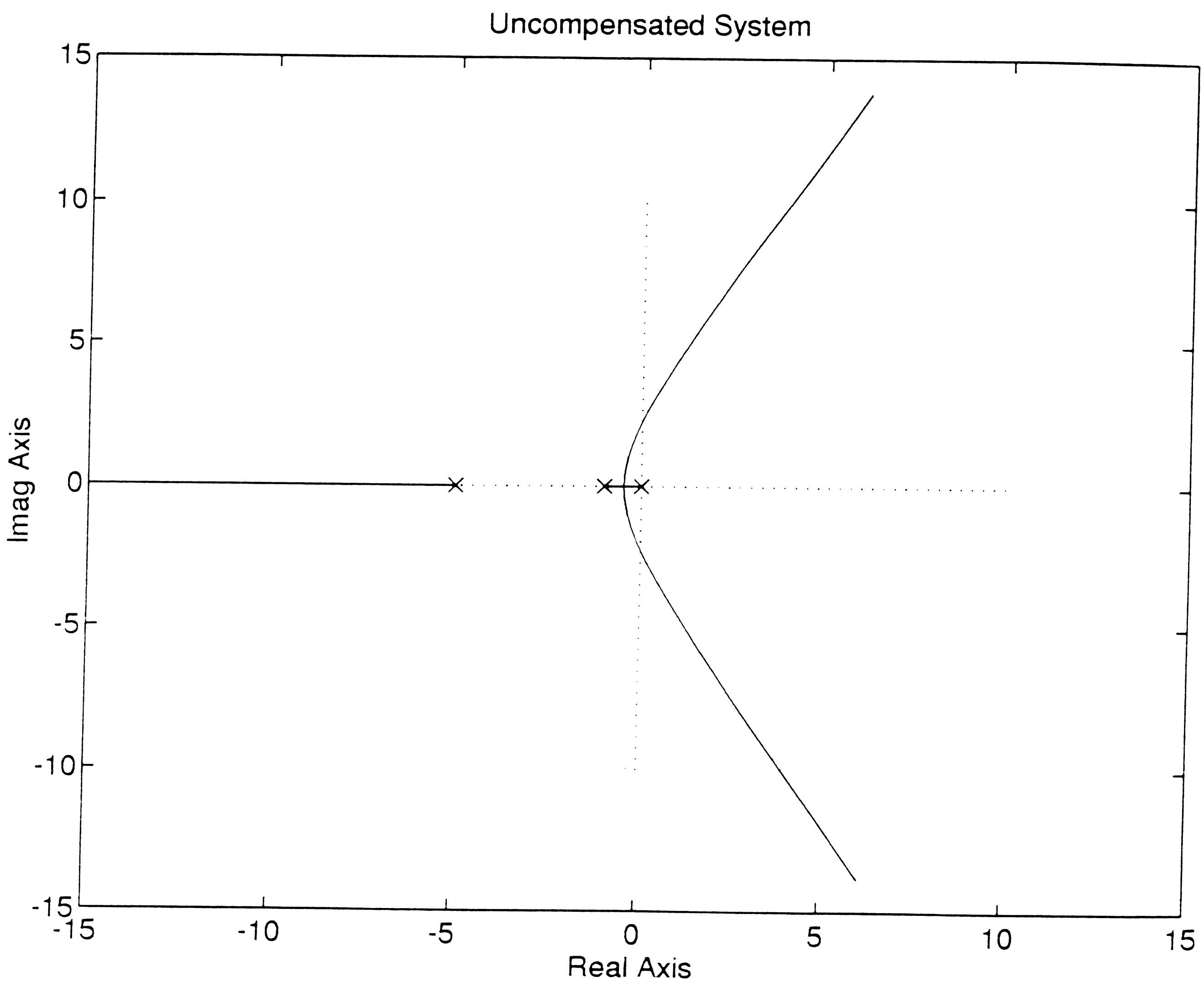
Then $k' = \dfrac{|s|\,|s+1|\,|s+5|\,|s + \frac{1}{\alpha T}|}{|s + 1/T|}$

(from magnitude criterion)

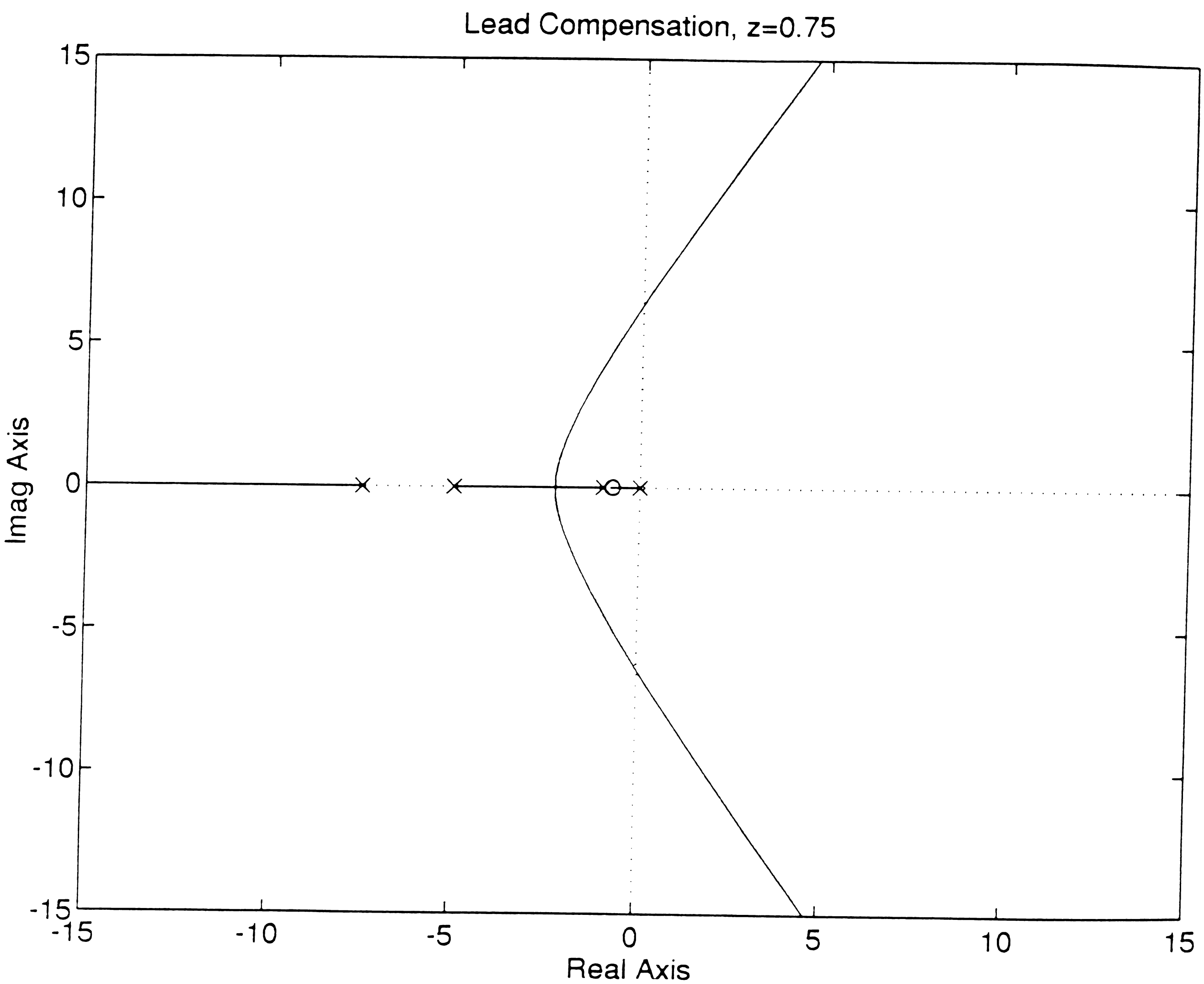
$$k_v = \lim_{s\to 0} sG(s) = \frac{k'\alpha}{5}$$

For $\zeta = 0.45$ we get the following data:

	G_c	roots	ω_n	k_v	Additional root	t_s
	$\frac{s+0.75}{s+7.5}$	$-0.404 \pm j0.802$	0.898	0.84	-5.192	9.48
	$\frac{s+0.75}{s+7.5}$	$-1.522 \pm j3.021$	3.38	2.05	$-0.689, -9.77$	2.78
BEST TRANSIENT	$\frac{s+1}{s+10}$	$-1.588 \pm j3.152$	3.52	2.95	-11.83	2.45
BEST SS ("but lower")	$\frac{s+1.5}{s+15}$	$-1.531 \pm j3.039$	3.40	4.4	$-1.76, -16.18$	2.50

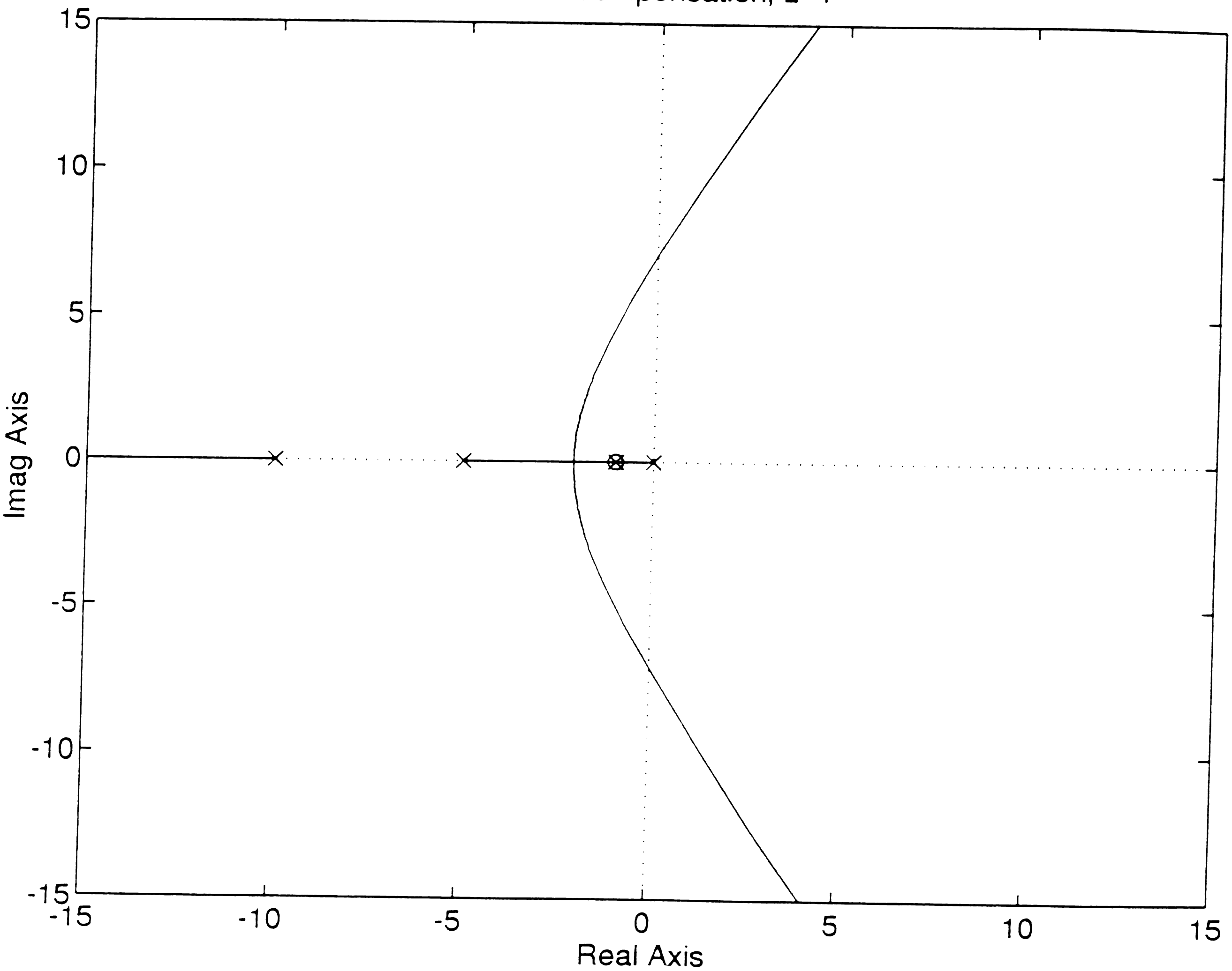


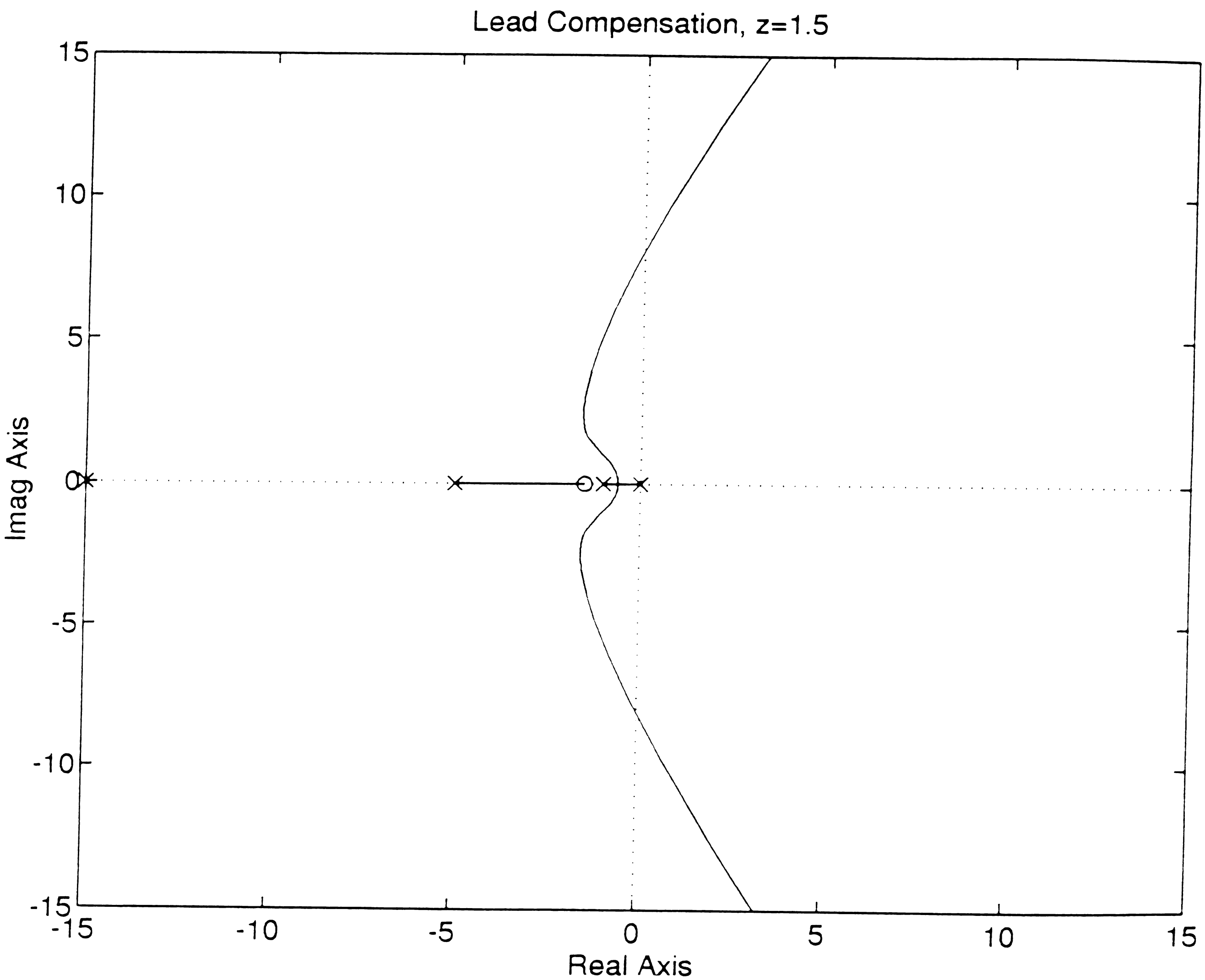
Uncompensated System
15
10
5
0
-5
-10
-15
Imag Axis
-15
-10
-5
0
5
10
15
Real Axis

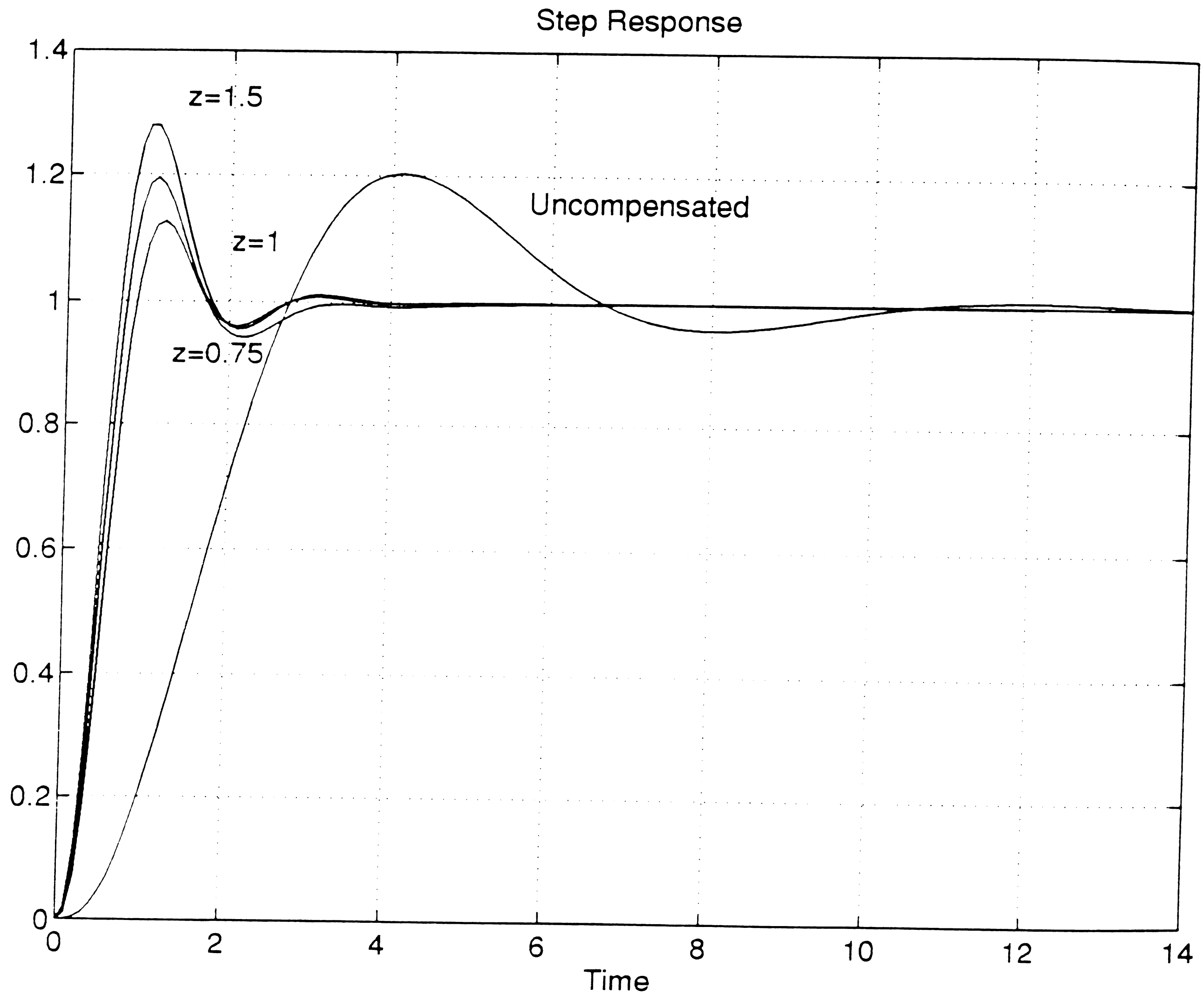

Lead Compensation, z=0.75
Imag Axis
Real Axis
15
10
5
0
-5
-10
-15
-15
-10
-5
0
5
10
15

8

Lead Compensation, z=1




Lead Compensation, z=1.5
15
10
5
0
-5
-10
-15
Imag Axis
-15
-10
-5
0
5
10
15
Real Axis

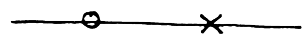


Step Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
z=1.5
z=1
z=0.75
Uncompensated
0
2
4
6
8
10
12
14
Time

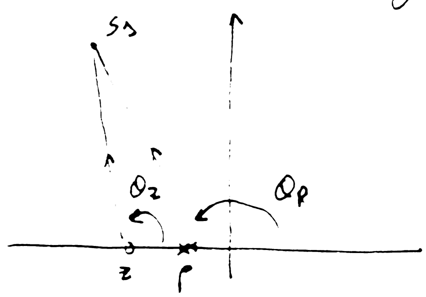
- <u>Phase-lag compensation</u>: Main purpose: improve steady-state by increasing the gain

In this case: $G_c(s) = K_c \dfrac{(s - z_0)}{(s - p_0)} = K_c \dfrac{(s + 1/T)}{(s + 1/\alpha T)}$ $\alpha > 1$

(i.e. $|p_0| < |z_0|$)



Therefore, it <u>adds a negative</u> angle to the angle criterion. Thus, it tends to move the RL towards the right (the unstable region!). This is obviously <u>undesirable</u>, so the idea is to keep the effect small by keeping the overall angle contribution <u>small</u>. This is achieved by placing the pole & zero very close <u>and</u> close to the origin.



Assume that the uncompensated system $G_p(s)$ is such that the R.L. goes through the point s_1.

Now we add the compensator and adjust the parameters so that the RL still goes through s_1 (approximately)

Assume that the original system was type 0 $\Rightarrow$ $K_p = \lim_{s \to 0} K_0 G = K_0 G(0)$

Since the new R.L. still goes through s_1 we have that

$$K K_c \frac{|s_1 + 1/T|}{|s_1 + \frac{1}{\alpha T}|} |G(s_1)| \sim K K_c |G(s_1)| = 1 \quad // \quad K K_c = \frac{1}{|G(s_1)|} = K_0$$

(assuming $1/T$, $\frac{1}{\alpha T}$ small)

but the new K_p is: $K'_p = \lim_{s \to 0} K K_c \dfrac{(s + 1/T)}{(s + 1/\alpha T)} G(s) = K K_c \alpha G(0) = \alpha K_p$

$\Rightarrow$ we increased K_p by a factor of α (say ~10) without (significantly) altering the transient.

To improve K_p we need α large, but in order to minimize the undesirable effects we need to get the pole-zero close to each other $\Rightarrow$

To get both α large and pole/zero close to each other, place <u>both</u> very close to the origin.

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Design example using phase-lag:

$$G = \frac{1}{s(s+1)(s+5)}$$ (same as before)

uncompensated system: for $\zeta = 0.45$, roots at $-0.404 \pm j0.802$

$K = 4.188$

$K_v = \lim_{s \to 0} s\,G(s) = \frac{4.188}{5} = 0.838$

($\Rightarrow$ steady state error to a ramp: $\cong 1.2$)

Suppose we want to bring this down by a factor of 10 without altering the transient $\Rightarrow$ use phase-lag with $\alpha = 10$

Try: $\alpha = 10$; $p = -0.005$
$z = \alpha p = -0.05$

$\Rightarrow$ Total angle contribution at the (old) closed-loop roots $= 2.6°$ $\Rightarrow$ very little change to the RL.

For the same ζ we had before, the new gain is now $k' = 4.01$ that yields closed-loop poles at:

$$s_{1,2} = -0.384 \pm j0.763$$

However, $K_v' = 8.02 \Rightarrow e_{ss}^{ramp} = 0.12$

Caveat: The compensated system has poles at $s = -5.183$
$s = -0.053$ *

$\Rightarrow$ There is a closed-loop root very close to the origin. This corresponds to a very slowly decaying transient, with small magnitude (due to the near pole-zero cancellation)

However, it might have serious influence upon the settling time $\Rightarrow$ a) put the pole-zero pair as far to the left as possible

b) do a "sanity check" after the design to see that you indeed meet the performance specifications.

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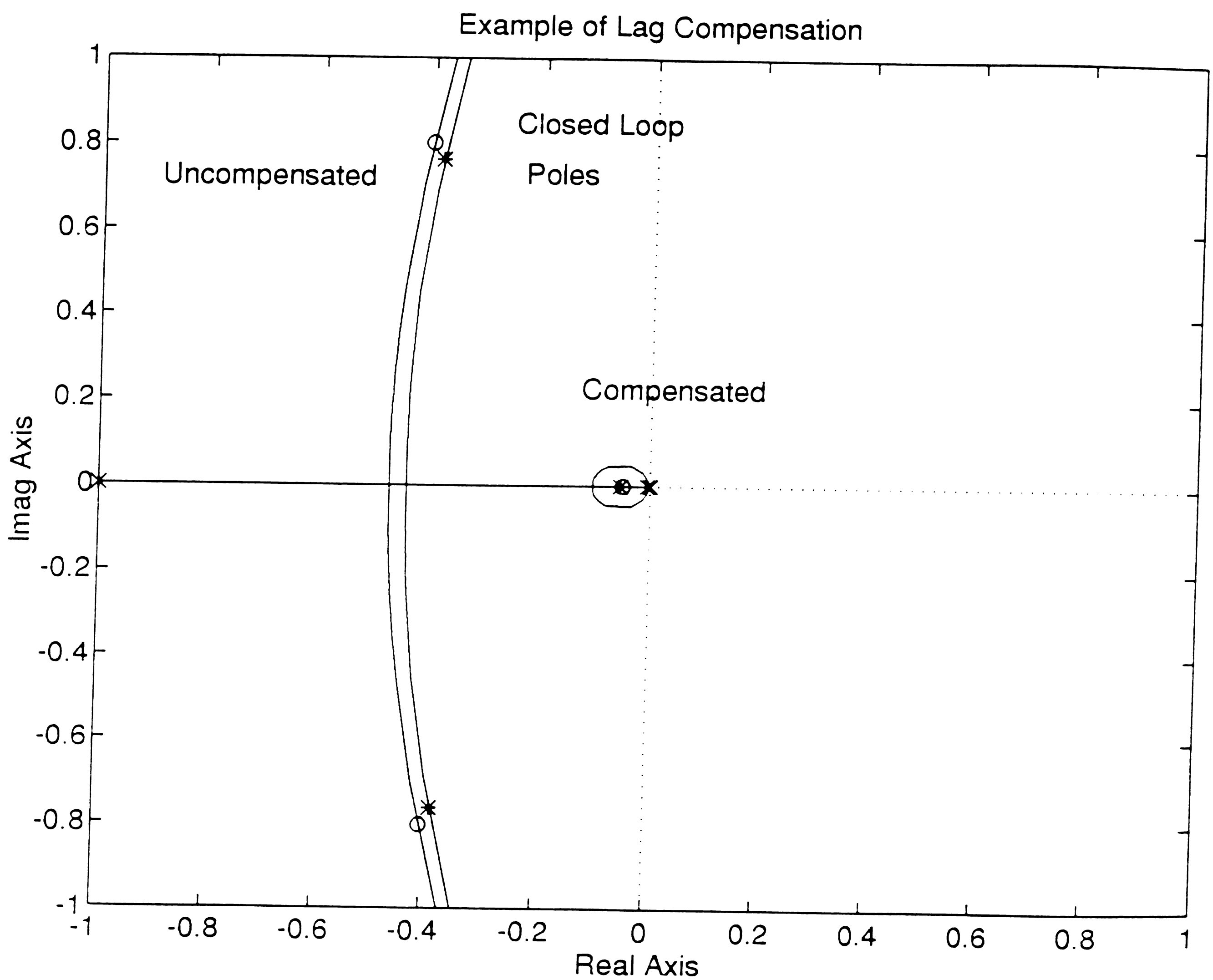
Example of Lag Compensation
Closed Loop
Poles
Uncompensated
Compensated
Imag Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

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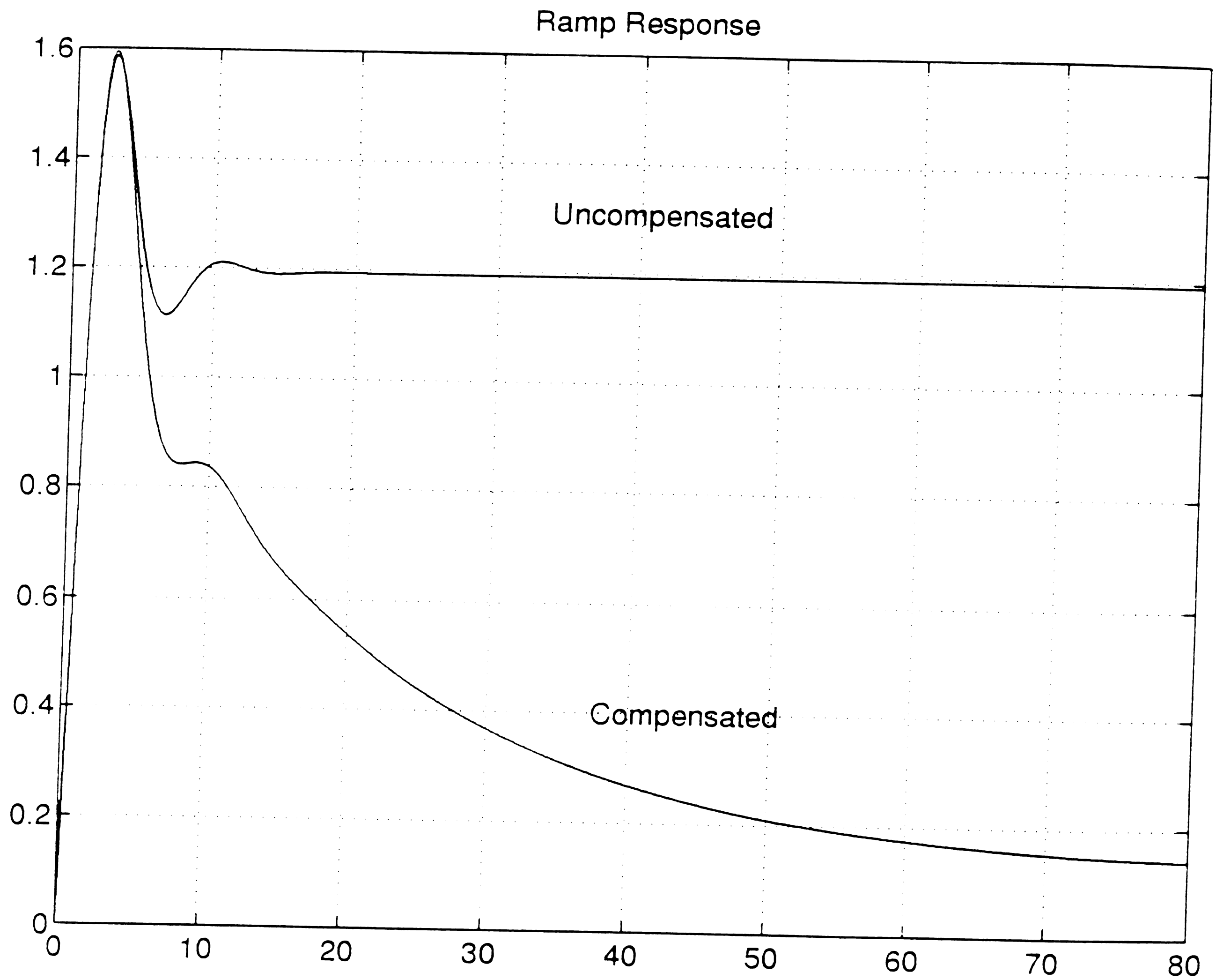


Ramp Response
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
Uncompensated
Compensated

Summary:

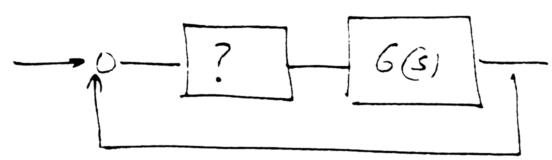
lead compensation:
- adds positive angle, shifts R.L towards the left ⇒ more stable, faster systems
- little effect upon steady state characteristics

lag compensation:
- adds negative angle, shifts R.L. towards the right (undesirable) ⇒ you use it in such a way as to change the R.L. as little as possible
- Increases error constants ⇒ improves steady state

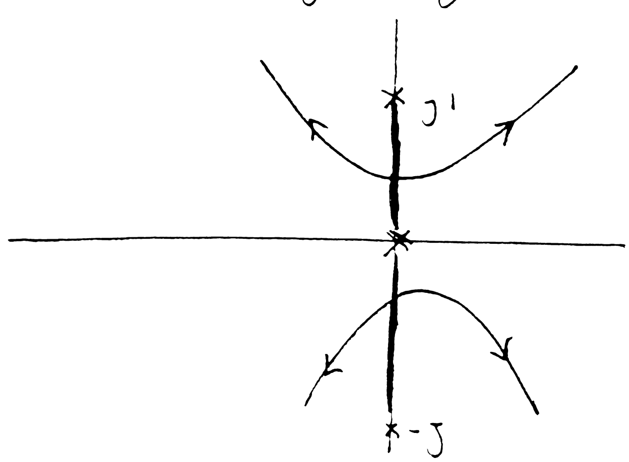
- **lead-lag compensation:** Combines the desirable characteristics of both. Disadvantage: more complex

- **Case study:** Design of a controller for a satellite attitude control

$$G(s) = \frac{1}{s^2(s^2+1)}$$



First, let's try using a simple gain:



Root locus shows that the "best" we can achieve using a simple gain is to get a marginally stable system ⇒ simple gain will not work.

Need to move R.L towards the left ⇒ lead compensation