

• Other uncertainty descriptions:

So far we have used multiplicative dynamic uncertainty. However, this is not the only description available.

Other types of uncertainty descriptions are:

$$g_a = \left\{ g_0(s) + \delta W_a(s), \quad |\delta| \leq 1 \right\} \quad \text{additive uncertainty}$$

$$g_q = \left\{ \frac{g_0}{1 + \delta W_q(s)} \quad |\delta| \leq 1 \right\} \quad \text{quotient uncertainty}$$

$$g_i = \left\{ \frac{g_0}{(1 + \delta W_i g_0)} \quad |\delta| \leq 1 \right\} \quad \text{inverse dynamics uncertainty}$$

Q: which one should we use?

A: depends on the specific application.

Example 1: approximation of a high order model by a lower order one

$$g(s) \cong g_0(s) + g_{h.o.}(s) = g_0(s) + \delta W_a(s)$$

↓
 reduced order model take this as uncertainty

⇒ in this case the "natural" description is additive uncertainty

Example 2: system with pole location uncertainty:

$$g(s) = \frac{1}{(s+3)(s+6)} \quad |\delta| \leq 1$$

If we use quotient uncertainty we get

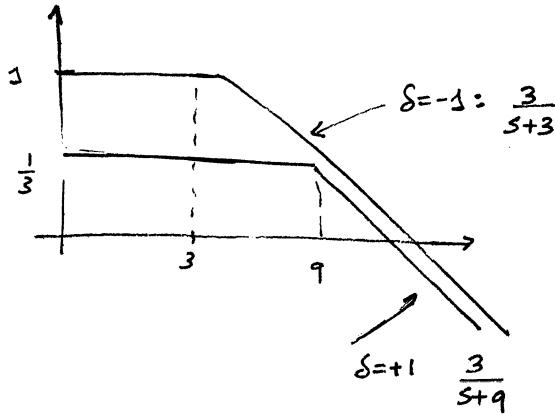
$$g_0(s) = g(s) \Big|_{s=0} = \frac{1}{s+6}$$

$$g(s) = \frac{1}{(s+6)} \left(\frac{1}{1 + \delta \frac{3}{s+6}} \right) = g_0 \frac{1}{1 + \delta W_q} \quad \text{with } W_q = \frac{3}{s+6}$$

What happens if we try to use multiplicative uncertainty

$$\text{ie: } g(s) = g_0(s) \left(1 + \delta_m W_m(s)\right), \quad |\delta_m| \leq \gamma_m$$

We need to cover the set: $\frac{g(s) - g_0(s)}{g_0(s)} = \frac{-\beta s}{s+3(2+\beta)}, \quad |\beta| \leq 1$



In this case we can use $W_m = \frac{3}{s+3}, \quad |\delta_m| \leq 1$

$$G_m = \left\{ g(s) : \frac{1}{(s+6)} \left(1 + \frac{\delta_m 3}{s+3}\right), \quad |\delta_m| \leq 1 \right\}$$

However, this includes second order systems in the family of models. For instance, for $\delta_m = 0.5$ we get

$$g = \frac{1}{(s+6)(s+3)} (s+4.5)$$

But the original set had only first order systems \Rightarrow
Since G_m has more plants, any design satisfying some specs for all elements of G_m is necessarily more conservative than one that takes into account only the original set.

- In some cases, the "best" choice is a blend of several descriptions:

$$\begin{aligned} G &= \left\{ \frac{s+2(1+0.3\delta_Z)}{s+3(1+0.5\delta_P)} \right\} = \frac{s+2}{s+3} \left[\frac{1 + \frac{0.3}{s+2} \delta_Z}{1 + \frac{0.5}{s+3} \delta_P} \right] \\ &= g_0 \left(\frac{1 + \delta_Z W_Z}{1 + \delta_P W_P} \right) \quad \text{multiplicative and quotient} \end{aligned}$$

Note that now we have 2 different δ 's. This is related to the concept of structured uncertainty

- Depending on the uncertainty description we get different conditions for Robust Stability

Type of uncertainty	Set of models	Robust Stability condition
multiplicative	$g_0(1 + \delta_w w_m)$	$\ w_m T(s)\ _\infty < 1$
additive	$g_0 + \delta_a w_a$	$\ w_a k(s) S(s)\ _\infty < 1$
quotient	$g_0 (1 + \delta_q w_q)^{-1}$	$\ w_q S\ _\infty < 1$
inverse	$g_0 (1 + \delta_w g_0)^{-1}$	$\ w_i \cdot g(s) S(s)\ _\infty < 1$

Example of proof: $g = g_0 + \delta_a w_a$

$$1 + kg \neq 0 \quad \text{all } |\delta_a| \leq 1, \text{ all } s, \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow 1 + k g_0 + k \delta_a w_a \neq 0 \quad \text{all } s, \operatorname{Re}(s) \geq 0 \text{ all } \delta$$

$$\Leftrightarrow 1 + \frac{k \delta_a w_a}{1 + k g_0} \neq 0$$

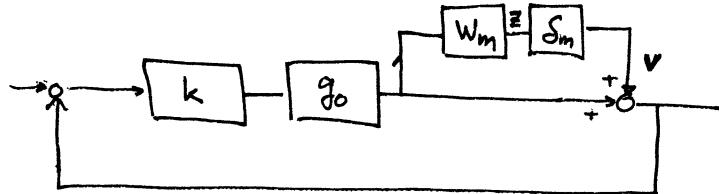
$$\Leftrightarrow \left| \frac{k w_a}{1 + k g_0} \right| < 1 \quad \text{all } s, \operatorname{Re}(s) \geq 0$$

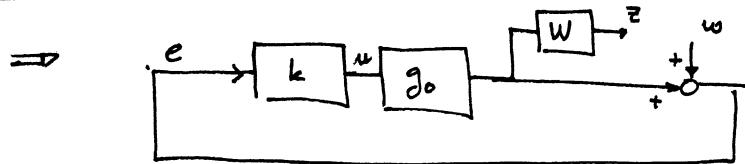
$$\Leftrightarrow \|k w_a S\|_\infty < 1$$

What about unstable pole/zero cancellations? It can be shown that they are precluded by the condition above

Rest of the proofs left as an exercise

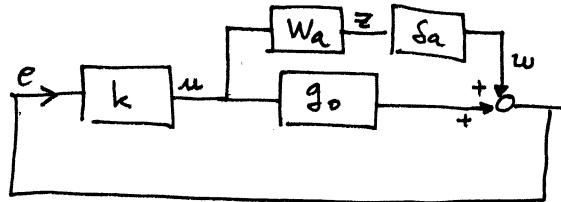
Note that for the multiplicative uncertainty case we have





$$T_{zw} = \frac{W k g_o}{1 + k g_o} \Leftrightarrow \|WT\|_\infty < 1 \Leftrightarrow \|T_{zw}\|_\infty < 1$$

for the additive uncertainty case we have:

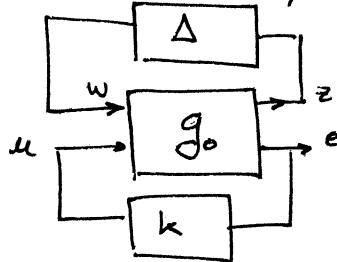


$$T_{zw} = \frac{W_a k}{1 + k g_o}$$

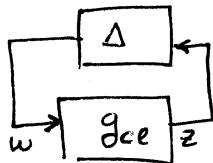
$$T_{zw} = W_a k S$$

$$\|T_{zw}\|_\infty = \|W_a k S\|_\infty$$

All of these cases are special cases of:



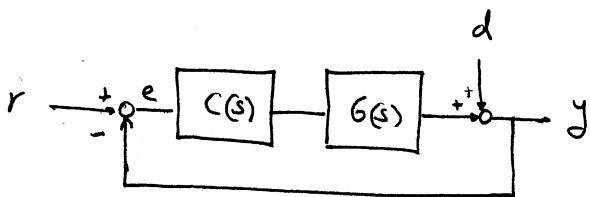
and the robust stability conditions become special cases of the "small gain" theorem:



The interconnection is stable for all Δ , $\|\Delta\|_\infty \leq 1 \Leftrightarrow \|T_{zw}\|_\infty \leq 1$

- Exercise: derive the condition for the inverse dynamics type uncertainty using this approach.

Nominal performance:



Classical notion of performance:

- 1) ability to track a known signal $r(t)$
- 2) ability to reject disturbances of a given type (sinusoidal, steps, etc)

These problems are equivalent. For instance if r is a step function perfect (steady state) tracking is achieved iff:

$$\lim_{t \rightarrow \infty} e(t) = 0 \Leftrightarrow \lim_{s \rightarrow 0} \frac{1}{1 + G(s)(s)} = 0 \Leftrightarrow S(0) = 0 \Leftrightarrow PC(0) = \infty$$

i.e. either P or C have a pole at the origin (integrator)

i.e. to track a step we need at least a type 1 system
(as we know from undergrad control)

On the other hand to reject a step disturbance d we need:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s y(s) = \lim_{s \rightarrow 0} \frac{s}{1 + GC} \frac{d}{s} = S(0)d = 0 \Leftrightarrow S(0) = 0$$

i.e. Type 1 or higher system.

This is a special case of the "internal model principle": to achieve perfect tracking the loop function $L = G(s)C(s)$ must contain a model of the unstable poles of the input.

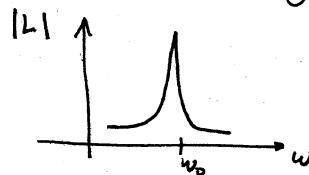
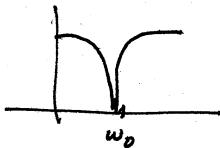
Example: $d = e^{j\omega_0 t}$

$$\lim_{t \rightarrow \infty} y(t) = \frac{1}{1 + G(j\omega_0)C(j\omega_0)} e^{j\omega_0 t} = S(j\omega_0) e^{j\omega_0 t}$$

$\Rightarrow \lim_{t \rightarrow \infty} y(t) = 0 \Leftrightarrow S(j\omega_0) = 0 \Leftrightarrow GC$ has poles at $s = \pm j\omega_0$

Graphically we want $L(j\omega)$ to look like

so that $S(j\omega)$ looks like



"Modern" Control Theory (i.e LQG) uses the same concept of performance, with the difference that the concept of optimality is introduced:

We define a performance index such as $J = \int_0^{\infty} (e^2 + u^2) dt$

and find the controller that minimizes this index

J provides a trade-off among different objectives (such as tracking error versus control action)

In both cases (classical and modern) the designer is assumed to have knowledge of the perturbation (or the reference signal) and this knowledge gets incorporated into the controller.

(In other words, the controller is "tuned" to this specific type of signal)

- Q: What happens if there is a mismatch between this assumption and reality?
- A: Performance degrades dramatically: For instance designing a controller to minimize the effect of a disturbance with frequency ω_0 can amplify the effect of a disturbance at ω_1 .
- Q: Is there a way out of this?
- A: "Robust" approach: Try to guarantee performance for a whole family of inputs, rather than for a single signal.

Exemple 1: A robotic manipulator must be able to track different trajectories, as the workspace changes in time.

Example 2: We want to reject sinusoidal disturbances when the only information available is that their frequency is contained in a given band.

Extreme case of the example above: reject a sinusoidal wave of unknown frequency and unity amplitude.

Suppose that the plant is minimum phase (i.e all zeros in the LHP).

We want $|S(j\omega)|$ to be ~ 0 for all frequencies:

Trivial solution: Take $C(s) = k \rightarrow \infty$ (since plant is minimum phase a simple root locus argument shows that as $k \rightarrow \infty$ the closed-loop system remains stable)

$$\text{Then } S = \frac{1}{1+kG} \rightarrow 0$$

However: we know that $T+S=1 \Rightarrow$ if $S \rightarrow 0$ $T = \frac{kG}{1+kG} \rightarrow 1 \Rightarrow$ robust stability is compromised.

- A more realistic approach: Make S "small" only over a certain frequency range

Example: position control of large mechanical systems: need to make sure that S is small at the resonant frequencies (low frequencies)

In general the frequency range of interest can be represented by a weighting function: $W_1(s)$ (without loss of generality W_1 can be taken to be stable, minimum phase)

The condition for nominal performance becomes now "small" $W_1(s)S(s)$

All we need to do now is to define precisely the meaning of small W_1S . As before, we will measure sizes of operators by their induced norm

$$\|W_1S\| = \sup_{\|d\|_1 \leq 1} \|W_1Sd\| = \sup_{\|d\|_1 \leq 1} \|W_1y\|$$

2 cases: (a) Use the (time domain) ℓ_∞ norm. In this case performance is measured in terms of the induced peak-to-peak norm of $\|W_1S\|_{\ell_\infty \rightarrow \ell_\infty} = \|W_1S\|_{\ell_\infty}$

This theory was first formulated around 1987 (Vidyasagar, Bakhsh and Pearson). The main result shows that this problem reduces to Linear Programming and can be solved (with arbitrary precision) using L.P.

(b) Use the ℓ_2 norm (energy) \Rightarrow Performance is measured in terms of $\|W_1S\|_{\ell_2 \rightarrow \ell_2} = \sup_{\|d\|_2 \leq 1} \|W_1Sd\|_2 = \|W_1S\|_{\ell_2}$

In this context, performance is optimized by minimizing $\|W_1S\|_{\ell_2}$
This is the H_∞ control problem.

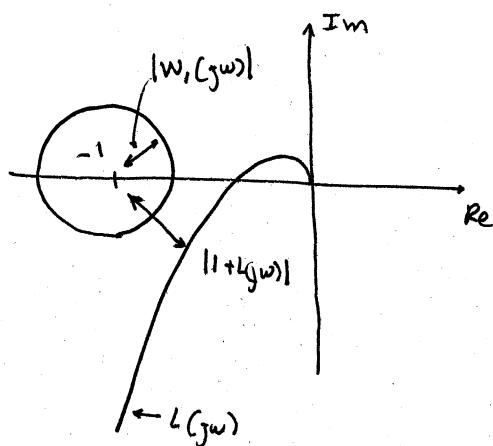
- Graphical interpretation of nominal performance:

- Def: The closed loop system achieves nominal performance iff

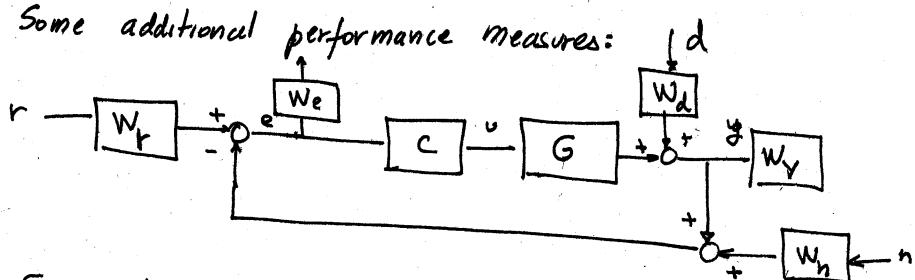
$$\|W_1 S\|_\infty = \sup_w |W_1(j\omega) S(j\omega)| \leq 1$$

Note that $|W_1(j\omega) S(j\omega)| \leq 1 \Leftrightarrow |W_1(j\omega)| \leq \underbrace{|1 + G(j\omega) C(j\omega)|}_{\text{distance from } L(j\omega) \text{ to the } (-1, 0) \text{ point}}$

- Nominal performance is achieved \Leftrightarrow the disk centered at -1 with radius $|W_1(j\omega)|$ does not intersect the Nyquist plot of $G \cdot C$



- Some additional performance measures:



For instance, if we are concerned about the control effort we can define nominal performance as

$$\text{nominal performance} \Leftrightarrow \sup_{\|r\|_2 \leq 1} \|u\|_2 \leq 1 \Leftrightarrow \left\| \frac{C W_r}{1 + GC} \right\|_\infty \leq 1$$

If we want sensor noise attenuation we can define:

$$\text{nominal performance} \Leftrightarrow \sup_{\|n\|_2 \leq 1} \|W_y y\|_2 \leq 1 \Leftrightarrow \left\| W_y \frac{C G}{1 + G C} W_n \right\|_\infty = \|W_y^T W_n\|_\infty \leq 1$$

Note that in this case performance is related to T rather than S .

This last condition has a nice graphical interpretation in terms of the inverse Nyquist plot:

$$\|WT\|_{\infty} \leq 1 \Leftrightarrow |W(j\omega)T(j\omega)| \leq 1 \text{ for all } \omega$$

$$\Leftrightarrow \left| \frac{W(j\omega)L(j\omega)}{1+L(j\omega)} \right| \leq 1 \Leftrightarrow \left| \frac{W(j\omega)}{1 + \frac{1}{L(j\omega)}} \right| \leq 1 \Leftrightarrow$$

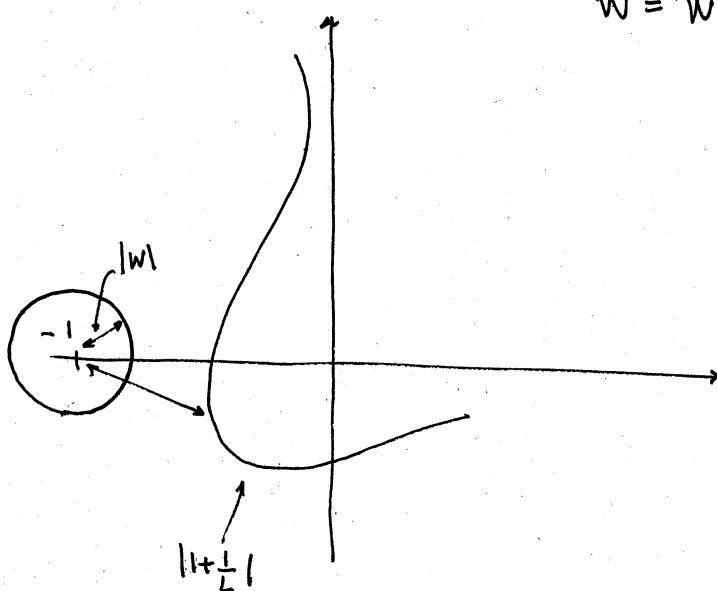
$$|W(j\omega)| \leq \underbrace{\left| 1 + \frac{1}{L(j\omega)} \right|}_{\text{distance from the } -1 \text{ point to the}} \text{ for all } \omega$$

inverse Nyquist plot

Nominal performance
(sensor noise)
attenuation

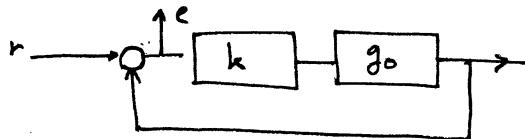
Inverse Nyquist plot does not intersect the disk centered at -1 with radius $|W(j\omega)|$ where

$$W = W_y \cdot W_n$$



- Robust performance

Recall that we achieve nominal performance if:



$$\|W_1 e\|_2 < 1 \text{ for all signals}$$

$$r, \|r\|_2 \leq 1 \Leftrightarrow \|W_1 S\|_\infty < 1$$

On the other hand, if rather than the nominal plant g_0 we have the family $g = g_0(1 + W_2 S)$, $|S| \leq 1$
we achieve robust stability if $\|W_2 T\|_\infty < 1$

⇒ Necessary and sufficient condition for nominal performance and robust stability is:

$$\|\max \{ |W_1 S|, |W_2 T| \}\|_\infty < 1$$

However, in most cases robust stability & nominal performance is not enough since the performance for off-nominal plants can be unacceptable \Rightarrow

We'd like to guarantee a given level of performance for all plants in the family (not just g_0)

This is the robust performance problem.

Def: Robust performance is achieved if:

$$\|W_1 e\|_2 < 1 \text{ for all signals } r, \|r\|_2 \leq 1 \text{ and}$$

all plants $g(s) = g_0(1 + W_2 s)$

Next we are going to derive a necessary and sufficient condition for robust stability

• Claim: Robust performance is achieved iff

$$\sup_w \{ |W_1 S| + |W_2 T| \} = \|\|W_1 S| + |W_2 T|\|_\infty < 1$$

Proof: Robust performance is achieved if

$$\|W_1 e\|_2 \leq 1 \quad \text{for all } r, \|r\|_2 \leq 1, \quad g = g_0(1 + \delta w_2)$$

$$\Leftrightarrow |W_1 \cdot \frac{1}{1+gk}| \leq 1 \quad \text{all } w, S$$

$$\Leftrightarrow \left| W_1 \frac{\frac{1}{1+kg_0(1+\delta w_2)}}{(1+kg_0)} \right| = \left| \frac{W_1}{(1+kg_0)} \cdot \frac{\frac{1}{1+w_2 S}}{\frac{kg_0}{(1+kg_0)}} \right| \leq 1 \quad \text{all } w, S, |\delta| \leq 1$$

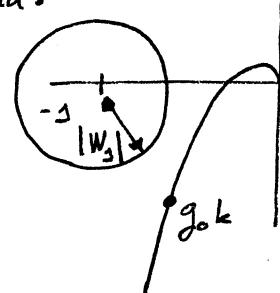
$$\Leftrightarrow |W_1 \cdot S| \leq |1 + \delta w_2 T| \quad \text{all } w, S, |\delta| \leq 1$$

$$\Leftrightarrow |W_1 S| \leq 1 - |W_2 T| \quad \text{all } w$$

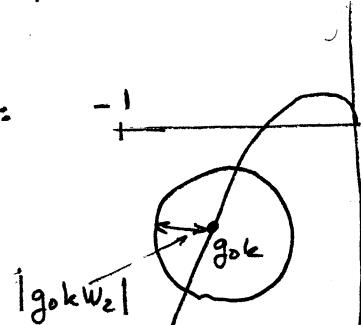
$$\Leftrightarrow |W_1 S| + |W_2 T| < 1 \quad \text{all } w$$

$$\Leftrightarrow \boxed{\|W_1 S| + |W_2 T|\|_\infty < 1}$$

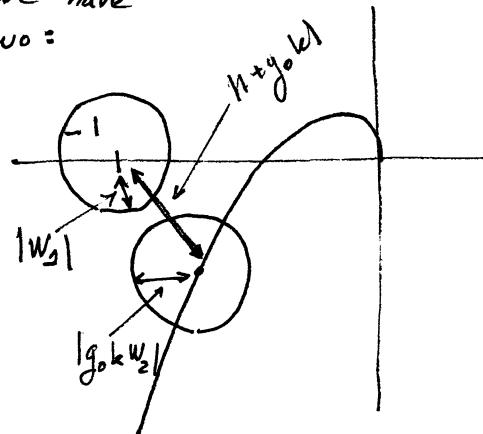
Graphical interpretation: Recall that for nominal performance we had:



The graphical condition for robust stability is:



For robust performance we have a combination of these two:



Now the disks must be disjoint

Note that the discs are disjoint iff

$$|1+g_{ok}| > |w_1| + |g_{ok} w_2| \quad \text{all } w$$

$$\Leftrightarrow 1 > \left| \frac{w_1}{1+g_{ok}} \right| + \left| w_2 \frac{g_{ok}}{1+g_{ok}} \right| \quad \text{all } w$$

$$\Leftrightarrow 1 > \| |w_1 s| + |w_2 T| \|_\infty$$

As expected, the condition for robust performance is stronger than just nominal performance and robust stability. However, these conditions are not too far apart in the following sense:

$$\max \{ |w_1 s|, |w_2 T| \} \leq |w_1 s| + |w_2 T| \leq 2 \max \{ |w_1 s|, |w_2 T| \}$$

So if we have nominal performance and robust stability, with a "safety factor" of 2 (i.e. $\|w_1 s\|_\infty < \frac{1}{2}$, $\|w_2 T\|_\infty < \frac{1}{2}$) we

are guaranteed robust performance.

The condition $\| |w_1 s| + |w_2 T| \|_\infty < 1$ gives a necessary and sufficient condition for robust stability. However, it is very difficult to use for synthesis (i.e. to find a controller such that the condition is satisfied), so it is useful to look for alternatives.

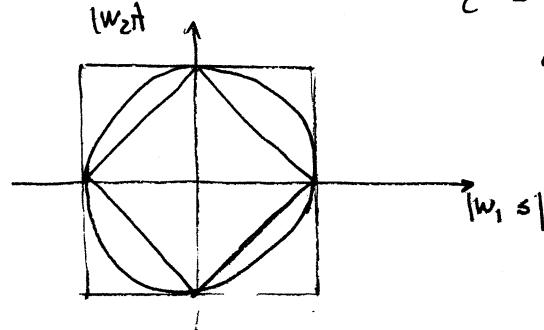
A compromise between nominal performance + robust stability and robust performance is given by:

$$\| (|w_1 s|^2 + |w_2 T|^2)^{1/2} \|_\infty < 1$$

This condition arises in the context of the so called mixed sensitivity problem.

Note that:

$$\max \{ |w_1 s|, |w_2 T| \} \leq (|w_1 s|^2 + |w_2 T|^2)^{1/2} \leq |w_1 s| + |w_2 T|$$



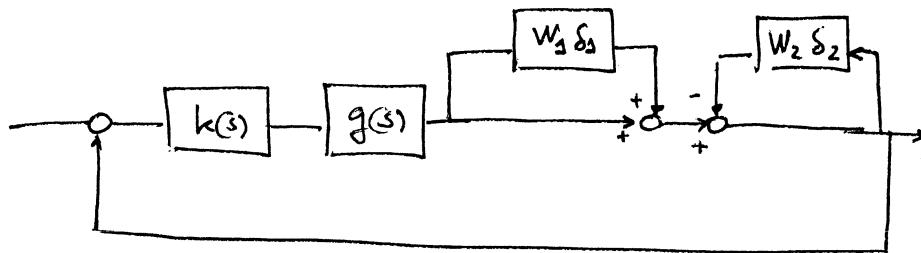
and that:

$$\begin{aligned} \frac{1}{\sqrt{2}} \{ |w_1 s| + |w_2 T| \} &\leq (|w_1 s|^2 + |w_2 T|^2)^{1/2} \\ &\leq \sqrt{2} \max \{ |w_1 s|, |w_2 T| \} \end{aligned}$$

\Rightarrow The condition $\|(W_1 S)^2 + (W_2 T)^2\|_\infty^{1/2} < 1$
 is never off by more than a factor of $\sqrt{2}$

- Connexion between Robust Stability and Robust Performance

Consider the following family (subject to 2 perturbations)



$$g(s) = \left\{ g_0 \frac{(1 + W_1 \delta_1)}{(1 + W_2 \delta_2)} \right\}$$

In this case it is easy to show that Robust stability is achieved iff:

$$1 + g(s)k(s) \neq 0 \quad \text{all } |\delta_i| \leq 1, \quad \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow 1 + g_0 k \left(\frac{1 + W_1 \delta_1}{1 + W_2 \delta_2} \right) \neq 0 \quad \text{all } |\delta_i| \leq 1, \quad \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow (1 + g_0 k + W_2 \delta_2 + g_0 k W_1 \delta_1) \neq 0 \quad \text{all } |\delta_i| \leq 1, \quad \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow (1 + g_0 k) \left(1 + \left(\frac{W_2}{1 + g_0 k} \right) \delta_2 + \frac{W_1 g_0 k}{1 + g_0 k} \delta_1 \right) \neq 0 \quad \text{all } |\delta_i| \leq 1, \quad \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow 1 - |W_2 S| - |W_1 T| \neq 0 \quad \text{all } \operatorname{Re}(s) \geq 0$$

$$\Leftrightarrow |W_2 S| + |W_1 T| < 1 \quad \text{all } \omega$$

$$\Rightarrow \| |W_2 S| + |W_1 T| \|_\infty < 1$$

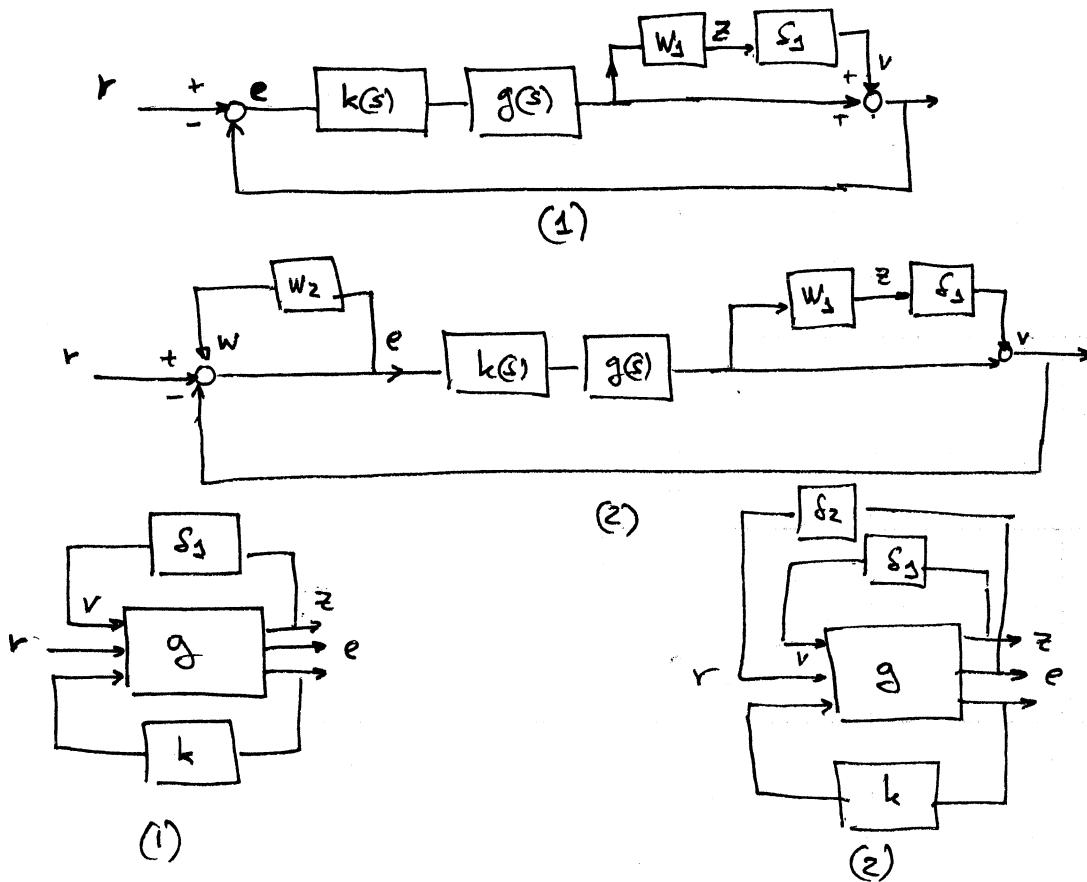
But this is exactly the condition for robust performance

$$\Rightarrow \boxed{\begin{array}{l} \text{Robust performance} = \text{Robust stability} \\ \text{with one perturbation} \quad \text{with 2 perturbations} \end{array}}$$

The Robust Stability and the Robust Performance problems are equivalent provided that

- 1) Performance is also assessed using the $\| \cdot \|_\infty$ norm
- 2) The fictitious perturbation is connected at the appropriate "points"

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS



Robust performance of (1) = Robust stability of (2)

δ_1 is called the "performance" block

The difficulty is that now we have to deal with a "structured" perturbation (the perturbation has now the structure)

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

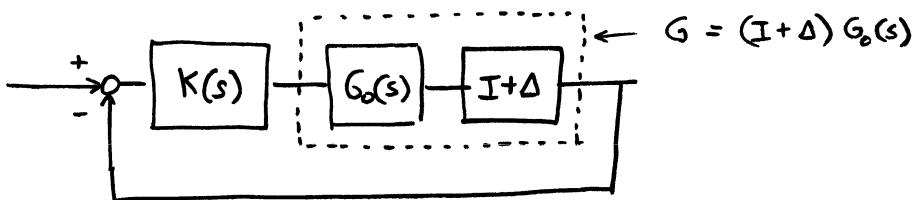
and with a MIMO system.

Question: How do we deal with MIMO systems?

We need to generalize our robust stability test to MIMO systems. However, this may not be straight forward.

Naive approach: Try to reduce the problem to a collection of SISO problems by looking at one loop at a time

Example:



Suppose that the nominal plant is given by:

$$G_0(s) = \begin{bmatrix} \frac{s+2}{s} & \frac{-1}{2(s+1)} \\ \frac{(s+1)(s+2)}{s^2} & \frac{1}{2s} \end{bmatrix} \quad (2 \text{ inputs } \& 2 \text{ outputs})$$

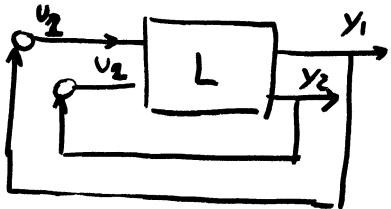
$$K(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{s}{2(s+1)(s+2)} \\ 0 & 1 \end{bmatrix} \quad (2 \text{ inputs } \& 2 \text{ outputs})$$

The nominal loop and complementary sensitivity are given by:

$$L(s) = GK = \frac{1}{s} \begin{bmatrix} 1 & 0 \\ \frac{s+1}{s} & 1 \end{bmatrix}$$

$$T(s) = L(I + L)^{-1} = \frac{1}{(s+1)} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

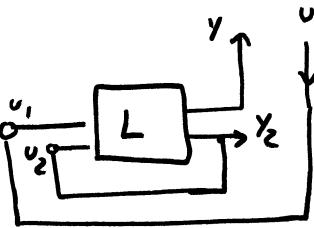
If we want to perform a classical gain/phase margin analysis (one loop at a time) we have



If we open the first loop we get

$$y_1 = -u; \quad y = L_{11}(s)u_1 + L_{12}(s)u_2 = -\frac{1}{s}u$$

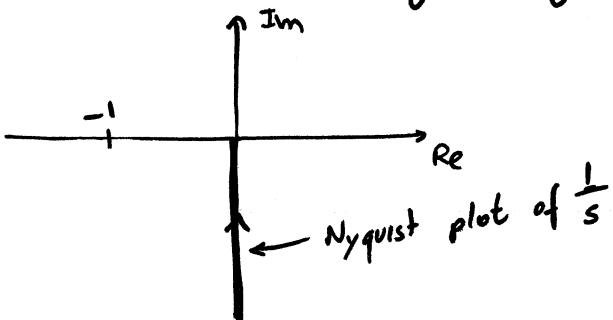
$$y_2 = \frac{s+1}{s^2}u_1 + \frac{1}{s}u_2 = -\left(\frac{s+1}{s^2}\right)u - \frac{1}{s}y_2 \quad \text{or} \quad y_2 \left(\frac{1+s}{s}\right) = -\left(\frac{1+s}{s^2}\right)u \Rightarrow y_2 = -\frac{1}{s}u$$



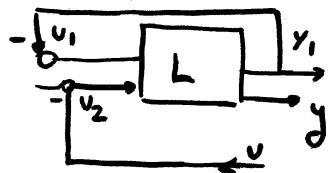
So the transfer function we need to consider for the margin analysis is

$$G = \frac{1}{s}$$

\Rightarrow Gain margin ∞
Phase margin 90°



If we open the second loop we get:



$$\begin{aligned} y &= L_{21}u_1 + L_{22}u_2 = L_{21}u_1 - L_{22}u \\ &= \left(\frac{s+1}{s^2}\right)u_1 - \frac{1}{s}u \end{aligned}$$

$$y_1 = L_{11}u_1 + L_{12}u_2 = \frac{1}{s}u_1 = -\frac{1}{s}y_1$$

$$\Rightarrow \left(1 + \frac{1}{s}\right)y_1 = 0 \quad \text{or} \quad y_1 = 0 \Rightarrow u_1 = 0$$

$$y = -\frac{1}{s}u$$

Again, the transfer function relevant to the stability analysis is:

$$G = \frac{1}{s} \Rightarrow \text{Gain margin } \infty \quad \text{Phase margin } 90^\circ$$

\Rightarrow According to the loop-at-a-time analysis, we have a fairly robust system

- Suppose now that we have multiplicative uncertainty

$$G = G_0(I + \Delta), \text{ with } \Delta = \begin{bmatrix} \delta_1 & \delta_2 \\ 0 & 0 \end{bmatrix}, |\delta_1| \leq 1, |\delta_2| \leq 1$$

In this case the loop function L is given by:

$$L = L_0(I + \Delta)$$

The characteristic polynomial is given by:

$$\begin{aligned} P(s) &= \det(I + L) = \det(I + L_0(I + \Delta)) = \det[(I + L_0)(I + (I + L_0)^{-1}L_0 \Delta)] \\ &= \det[(I + L_0) \cdot (I + T_0 \Delta)] = \det(I + L_0) \cdot \det(I + T_0 \Delta) \end{aligned}$$

where $T_0 = \frac{1}{s+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the nominal complementary sensitivity

$$\begin{aligned} \Rightarrow P(s) &= \begin{vmatrix} \frac{1+s}{s} & 0 \\ \frac{s+1}{s^2} & \frac{1+s}{s} \end{vmatrix} \cdot \begin{vmatrix} (1 & 0) + \left(\frac{1}{s+1} & 0\right) \left(\begin{smallmatrix} \delta_1 & \delta_2 \\ 0 & 0 \end{smallmatrix}\right) & \\ (0 & 1) + \left(\frac{1}{s+1} & \frac{1}{s+1}\right) \left(\begin{smallmatrix} \delta_1 & \delta_2 \\ 0 & 0 \end{smallmatrix}\right) & \end{vmatrix} \\ &= \frac{(s+1)^2}{s^2} \cdot \begin{vmatrix} 1 + \frac{\delta_1}{s+1} & \frac{\delta_2}{s+1} \\ \frac{\delta_1}{s+1} & 1 + \frac{\delta_2}{s+1} \end{vmatrix} = \frac{(s+1)^2}{s^2} \frac{(s+1+\delta_1)(s+1+\delta_2) - \delta_1 \delta_2}{(s+1)^2} \\ &= \frac{(s+1)(s+1+\delta_1+\delta_2)}{s^2} \end{aligned}$$

$$\Rightarrow \text{characteristic equation: } f(s) = \frac{(s+1)(s+1+\delta_1+\delta_2)}{s^2} = 0 \Rightarrow \text{poles at } \begin{cases} s = -1 \\ s = -1 - \delta_1 - \delta_2 \end{cases}$$

\Rightarrow The closed loop becomes unstable for $\delta_1 = -\frac{1}{2}$, $\delta_2 = -\frac{1}{2}$

\Rightarrow "Loop-at-a-time" analysis is overly optimistic