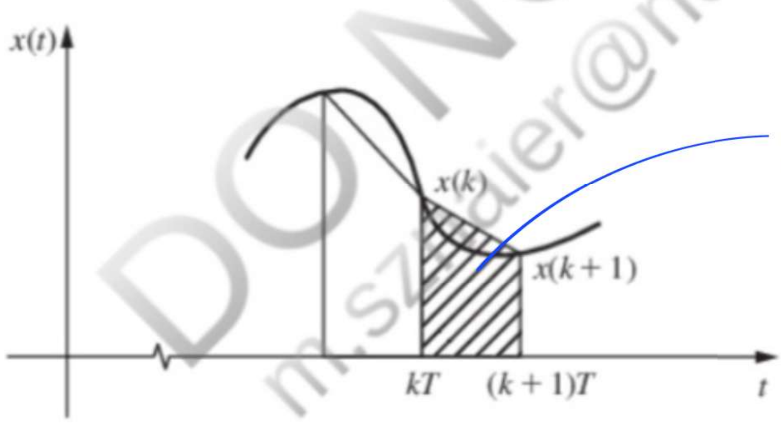
Problem 2.2.2

2.2-2. The trapezoidal rule (modified Euler method) for numerical integration approximates the integral of a function $x(t)$ by summing trapezoid areas as shown in **Fig. P2.2-2**. Let $y(t)$ be the integral of $x(t)$.



Area (////)

$$T \cdot \left(\frac{x(kT) + x((k+1)T)}{2} \right)$$

Figure P2.2-2 Trapezoidal rule for numerical integration.

Figure P2.2-2 Full Alternative Text

a. Write the difference equation relating $y[(k+1)T]$, $y(kT)$, $x[(k+1)T]$, and $x(kT)$ for this rule.

b. Show that the transfer function for this integrator is given by

$$\frac{(\;)}{(\;)} = \frac{(\;/2)(\;+1)}{\;-1}$$

(a)

$$y[(k+1)T] = y(kT) + \int_{kT}^{(k+1)T} x(t)\,dt \quad \approx \frac{T}{2}[x(kT) + x((k+1)T)]$$

D.F.

$$= y(kT) + \frac{T}{2}[x(kT) + x((x+1)T)] \quad \#$$

or

$$y[k+1] = y[k] + \frac{T}{2}(x[k] + x[k+1]) \quad \#$$

(b)

(z advance)

$$\mathcal{Z}\{y((k+1)T)\} = z \cdot Y(z)$$

$$\mathcal{Z}\{y(kT)\} = Y(z)$$

$$\mathcal{Z}\{x((k+1)T)\} = z X(z)$$

$$\mathcal{Z}\{x(kT)\} = X(z)$$

$$\therefore \quad zY(z) = Y(z) + \frac{T}{2}\left(X(z) + zX(z)\right)$$

$$(z-1)Y(z) = \frac{T}{2}(1+z)X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{T}{2}\frac{(z+1)}{(z-1)} \quad \#$$

Problem 2.2.3

- 2.2-3.

a. The transfer function for the right-side rectangular-rule integrator was found in **Problem 2.2-1** to be $Y(z)/X(z) = Tz/(z-1)$. We would suspect that the reciprocal of this transfer function should yield an approximation to a differentiator. That is, if $w(kT)$ is a numerical derivative of $x(t)$ at $t = kT$,

$\frac{X(z)}{Y(z)} = \frac{(z-1)}{Tz}$

$$\frac{W(z)}{X(z)} = \frac{z-1}{Tz}$$

Write the $\boxed{\text{difference equation}}$ describing this differentiator.

b. Draw a figure similar to those in **Fig. P2.2-1** illustrating the approximate differentiation.

- c. Repeat part (a) for the left-side rule, where $W(z)/X(z) = T/(z-1)$.

- d. Repeat part (b) for the differentiator of part (c).

I use Matlab

(a) $\frac{W(z)}{X(z)} = \frac{(z-1)}{Tz} = \frac{(1-z^{-1})}{T}$

$\Rightarrow W(z)\cdot T = X(z) - z^{-1}X(z)$

$\downarrow$ inverse Z

$\Rightarrow w(kT)\cdot T = x(kT) - x((k-1)T)$

$\Rightarrow w(kT) = [x(kT) - x((k-1)T)]/T$ #

(c) $\frac{W(z)}{X(z)} = \frac{T}{(z-1)} = \frac{Tz^{-1}}{(1-z^{-1})}$

$\Rightarrow T\cdot z^{-1}X(z) = W(z) - z^{-1}W(z)$

$\downarrow$ inverse Z

$\Rightarrow T\cdot x[(k-1)T] = w[kT] - w[(k-1)T]$

But, it's the integrator. I can't do part (d). ($\because$ problem requests a differentiator)

Awww, I think the question should be like using $\frac{Y(z)}{X(z)} = \frac{T}{(z-1)}$

to describe the integrator, then $\frac{W(z)}{X(z)} = \frac{(z-1)}{T}$ is the differentiator.

let do it according to the question above.

$\frac{W(z)}{X(z)} = \frac{(z-1)}{T} \Rightarrow T\cdot W(z) = X(z)(z-1) = zX(z) - z$

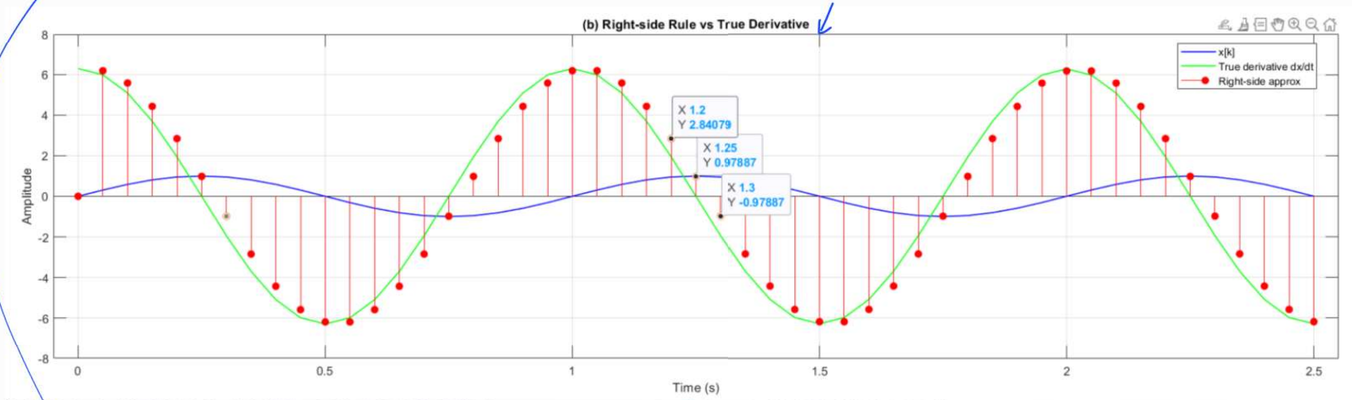
$\downarrow z^{-1}$

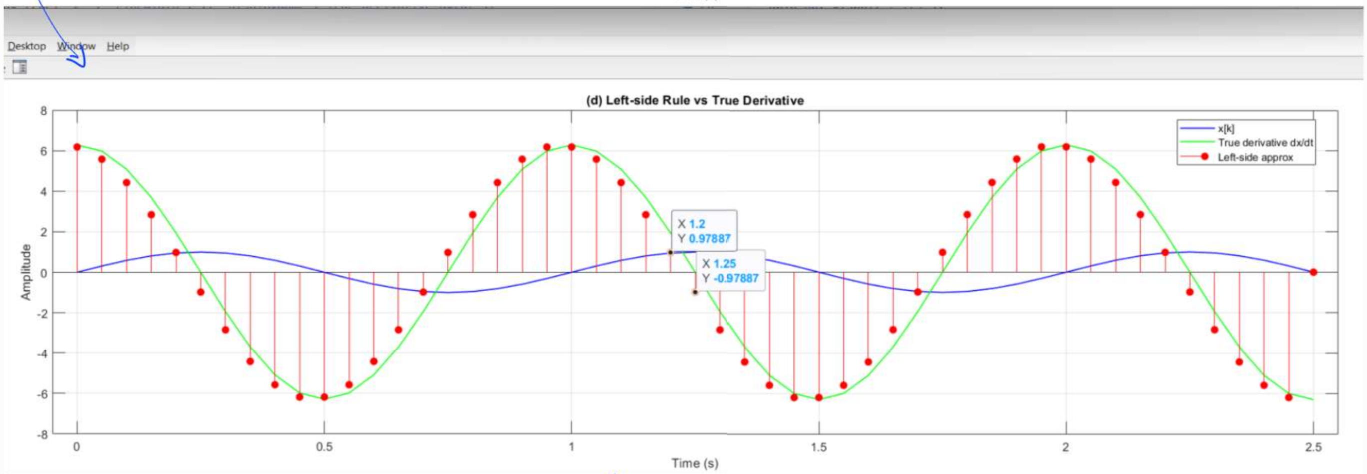
$Tw(kT) = x((k+1)T) - x(kT)$

$\Rightarrow w(kT) = [x((k+1)T) - x(kT)]/T$ #

(b)(d)

x[k] = x[k] - x[k-1] #





x[k] = x[k+1] - x[k] #

Problem 2.6.3

2.6-3. Given the difference equation

$$x(k) - x(k-1) + x(k-2) = e(k)$$

where $e(k) = 1$ for $k \geq 0$.

a. Solve for $x(k)$ as a function of k, using the z-transform. Give the values of $x(0)$, $x(1)$, and $x(2)$.

b. Verify the values $x(0)$, $x(1)$, and $x(2)$, using the power-series method.

c. Verify the values $x(0)$, $x(1)$, and $x(2)$ by solving the difference equation directly.

d. Will the final-value property give the correct value for $x(\infty)$?

(a)

$x(k) - x(k-1) + x(k-2) = e(k) = 1$ (unit step)

$$X(z) - z^{-1}X(z) + z^{-2}X(z) = E(z) = \frac{z}{z-1}$$

$$X(z)(1 - z^{-1} + z^{-2}) = \frac{z}{z-1}$$

$$X(z) = \frac{z^3}{(z-1)(z^2-z+1)} \quad \left(\text{pole} = 1,\ \frac{1 \pm \sqrt{3}j}{2}\right)$$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)(z^2-z+1)} = \frac{A}{(z-1)} + \frac{Bz + C}{(z^2-z+1)}$$

$$= \frac{A(z^2-z+1) + Bz(z-1) + C(z-1)}{(z-1)(z^2-z+1)} \quad \begin{cases} A + B = 1 \\ -A - B + C = 0 \\ A - C = 0 \end{cases} \Rightarrow \begin{cases} C = 1 \\ A = C \end{cases} \Rightarrow B = 0$$

$$\Rightarrow X(z) = \frac{z}{(z-1)} + \frac{z}{z^2-z+1} \xrightarrow{z^{-1}} x(k) = 1 + 2\sqrt{3}^{-1}\sin\left(\frac{\pi k}{3}\right) \quad \begin{cases} x(0) = 1 \\ x(1) = 1 + \cancel{\frac{2}{\sqrt{3}}}\cancel{\frac{\sqrt{3}}{2}} = 2 \\ x(2) = 1 + \cancel{\frac{2}{\sqrt{3}}}\cancel{\frac{\sqrt{3}}{2}} = 2 \end{cases}$$

#

(b) power series method:

$$X(z) = \frac{z^3}{(z-1)(z^2-z+1)} = \frac{z^3}{z^3-2z^2+2z-1}$$

$$\begin{array}{r|l}
 & 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{1}{z^3} + \cdots \\
\hline
z^3-2z^2+2z-1 & z^3 \\
 & z^3-2z^2+2z-1 \\
\hline
 & 2z^2-2z+1 \\
 & 2z^2-4z+4-\frac{2}{z} \\
\hline
 & 2z-3+\frac{2}{z} \\
 & 2z-4+\frac{4}{z}-\frac{2}{z^2} \\
\hline
 & 1-\frac{2}{z}+\frac{2}{z^2} \\
 & 1-\frac{2}{z}+\frac{2}{z^2}-\frac{1}{z^3} \\
\hline
 & \frac{1}{z^3}
\end{array}$$

$\therefore$ first term of the sequence are: $\{1, 2, 2, 1, \cdots$

$\Rightarrow x(0)=1,\ x(1)=2,\ x(2)=2,\ \cdots$ #

(c) solve directly

$$x(k) = x(k-1) - x(k-2) + 1$$

let initial conditions: $x(-1) = x(-2) = 0$

then $\begin{cases} x(0) = 1 \\ x(1) = x(0) - x(-1) + 1 = 2 \\ x(2) = x(1) - x(0) + 1 = 2 \end{cases}$ #

(d) Final Value Theorem is:

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}(z-1)\cdot\frac{z^3}{(z-1)(z^2-z+1)}$$

$$= \frac{z^3}{(z^2-z+1)} = 1. \quad \text{has poles} = e^{\frac{j\pi}{3}}, e^{\frac{-j\pi}{3}}$$

but $\left(\left|e^{\frac{j\pi}{3}}\right|^2 + \left|e^{\frac{-j\pi}{3}}\right|^2\right)^{\frac{1}{2}}$ = magnitude = 1 (it's on unit circle, not inside)

also, $\lim_{k\to\infty} x[k]$ does NOT exist since the sequence

$x[k] = (1, 2, 2, 0, 0 \cdots) \Rightarrow$ it's **NOT CONVERGENCE** #

to sum up: $\boxed{\textbf{NO}}$ the FVT is NOT Applicable.

Problem 2.7.1

2.7-1. (a) Find $e(0)$, $e(1)$, and $e(10)$ for

$$E(z) = \frac{0.1}{z(z-0.9)}$$

$e_k = \frac{1}{2\pi j}\oint E(z) z^{k-1} dz = \Sigma \operatorname{Res}\{E(z) z^{k-1}\}$

using the inversion formula.

(b) Check the value of $e(0)$ using the initial-value property. $\lim_{k \to 0} x[k] = \lim_{z\to\infty} X(z)$

(c) Check the values calculated in part (a) using partial fractions. $E(z) = z \cdot \frac{P(z)}{Q(z)}$

(d) Find $e(k)$ for $k = 0, 1, 2, 3,$ and 4 if $\mathcal{z}[e(k)]$ is given by

$$E(z) = \frac{1.98z}{(z^2 - 0.9z + 0.9)(z-0.8)\underbrace{(z^2 - 1.2z + 0.27)}_{(z-0.9)(z-0.3)}}$$

(e) Find a function $e(t)$ which, when sampled at a rate of 10 Hz $(T = 0.1s)$, results in the transform $E(z) = 2z/(z-0.8)$.

(f) Repeat part (e) for $E(z) = 2z/(z+0.8)$. $\Rightarrow A = -\frac{1}{9}, B = \frac{1}{9}$

(g) From parts (e) and (f), what is the effect on the inverse z-transform of changing the sign on a real pole?

(a) $E(z) = (0.1) \cdot \frac{1}{z(z-0.9)} \Rightarrow E(z) z^{k-1} = 0.1 \cdot \frac{z^{k-1}}{z(z-0.9)} = 0.1 \cdot \frac{z^k}{z^2(z-0.9)}$

it has pole $z = 0.9$ and repeated poles at $z = 0$ (order 2)

$$e[k] = \frac{1}{2\pi j}\oint E(z) z^{k-1} dz = \Sigma \operatorname{Res}\{E(z) z^{k-1}\}$$

$k=0$ ($e[0]$)

* At $z = 0.9$: (singularity of order 1)

$$\operatorname{Res}\{E(z) z^{k-1}\} = \lim_{z\to z_i}[(z - z_i) E(z) z^{k-1}]$$
$$= \lim_{z\to 0.9} \cancel{(z-0.9)} \frac{0.1 z^0}{z^2 \cancel{(z-0.9)}} = 0.1 \cdot (0.9)^{-2} \quad (a)$$

$\therefore e[0] = \Sigma \operatorname{Res}\{E(z) z^{k-1}\} = (a) + (b) = 0$ #

* At $z = 0$: ($n = 2$ (order))

$$\operatorname{Res}\{E(z) z^{k-1}\} = \lim_{z\to 0} \frac{1}{(2-1)!} \cdot \frac{d^{2-1}}{dz^{2-1}}\left[\cancel{(z-0)^2} \cdot 0.1 \frac{z^0}{\cancel{z^2}(z-0.9)}\right]$$
$$= \lim_{z\to 0} \frac{1}{1} \cdot \frac{d}{dz}\left[\frac{z^0}{(z-0.9)} \cdot (0.1)\right] = \lim_{z\to 0}\left(\frac{-0.1}{(z-0.9)^2}\right) = -0.1 \cdot (0.9)^{-2} \quad (b)$$

$(k=1)\ e[1]:\ E(z)\cdot z^{k-1} \overset{}{=} \dfrac{0.1}{z(z-0.9)}\cdot 1$

$z=0.9: \lim_{z\to 0.9} \cancel{(z-0.9)}\cdot\dfrac{0.1}{z\cancel{(z-0.9)}} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$ ⓐ

$z=0: \lim_{z\to 0} \dfrac{1}{(2-1)!}\cdot\dfrac{d^{2-1}}{dz^{2-1}}\left[(z-0)^{\cancel{2}}\cdot\dfrac{0.1}{\cancel{z}(z-0.9)}\right] = \dfrac{-1}{9}$ ⓑ

$\therefore e[1] = \Sigma \text{Res}\, E(z) = ⓐ + ⓑ = 0$ #

$(k=10)\ e[10]:\ E(z)\cdot z^{k-1} = \dfrac{0.1\cdot z^9}{\cancel{z}(z-0.9)} = \dfrac{0.1\cdot z^8}{(z-0.9)}$

$z=0.9: \lim_{z\to 0.9} \cancel{(z-0.9)}\cdot\dfrac{0.1\cdot z^8}{\cancel{(z-0.9)}} = 0.1\cdot 0.9^8$

$z=0: \lim_{z\to 0} \dfrac{1}{(2-1)!}\cdot\dfrac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2\cdot\dfrac{0.1\cdot z^8}{(z-0.9)}\right] = 0.$

$\therefore e[10] = \Sigma \text{Res}\{E(z) z^9\} = 0.1\,(0.9)^8$ #

(b)

IVT: $e(0) = \lim_{z\to\infty} E(z) = \lim_{z\to\infty}\left(\dfrac{0.1}{z(z-0.9)}\right) = 0 \Rightarrow$ verified ✓ #

(c)

$E(z) = \dfrac{0.1}{z(z-0.9)} = \dfrac{A}{z} + \dfrac{B}{z-0.9}$

$= A(z-0.9) + Bz$

$\begin{cases} A+B=0 \\ -0.9A = 0.1 \end{cases} \Rightarrow A = \dfrac{-0.1}{0.9} \Rightarrow A = -\dfrac{1}{9},\ B = \dfrac{1}{9}$

then: $E(z) = \dfrac{-1}{9} z^{-1} + \dfrac{1}{9}\cdot\dfrac{1}{(z-0.9)}$

z^{-1}

$e[k] = \dfrac{1}{9}\left(0.9^{k-1} u[k-1] - \delta[k-1]\right)$

verify:

$k=0: e[0] = \frac{1}{9}(0.9^{-1}u[-1] - \delta[-1]) = 0 + 0 = 0$ ✓

$k=1: e[1] = \frac{1}{9}(0.9^0 u[0] - \delta[0]) = \frac{1}{9}(1-1) = 0$ ✓

$k=10: e[10] = \frac{1}{9}(0.9^9 u[9] - \delta[9]) = \frac{1}{9}(0 + 0.9^9) = (0.1)\cdot 0.9^8$ ✓ #

(d) since the denominator is order 5, and try to divide the numerator of order 1. As a result, the coefficient of first three terms should be zero by power series method $\Rightarrow e(0) = e(1) = e(2) = e(3) = 0$

then, $e(4)$ is equal to 1.98 because we need to minus the numerator $\Rightarrow \begin{cases} e(0) = e(1) = e(2) = e(3) = 0 \\ e(4) = 1.98 \end{cases}$

(the first nonzero term start from k

where k = degree(Denominator) − degree(Nominator) = 4) #

(e) find $e(t)$, with $T = 10\ Hz = 0.1\ sec$ and $E(z) = \frac{2z}{(z-0.8)}$

$$\Rightarrow E(z) = 2 \cdot \frac{z}{(z-0.8)} \xrightarrow{z^{-1}} e[k] = 2(0.8)^k,\ k = 0, 1, \cdots$$

$$0.8 = e^{10s} \Rightarrow \log_e 0.8 = 10s \Rightarrow s = 10^{-1} \log_e(0.8)$$

$$\therefore e(t) = e[kT] = 2(0.8)^k = 2 \cdot e^{(10s)t},\ \text{where } s = 10^{-1} \cdot \log_e(0.8)\ \#$$

(f) again, find $e(t)$, with $T = 10\ Hz = 0.1\ sec$ but $E(z) = \frac{2z}{(z+0.8)}$

$$\Rightarrow E(z) = 2 \cdot \frac{z}{(z+0.8)} \xrightarrow{z^{-1}} e[k] = 2(0.8)^k(-1)^k,\ k = 0, 1, \cdots$$

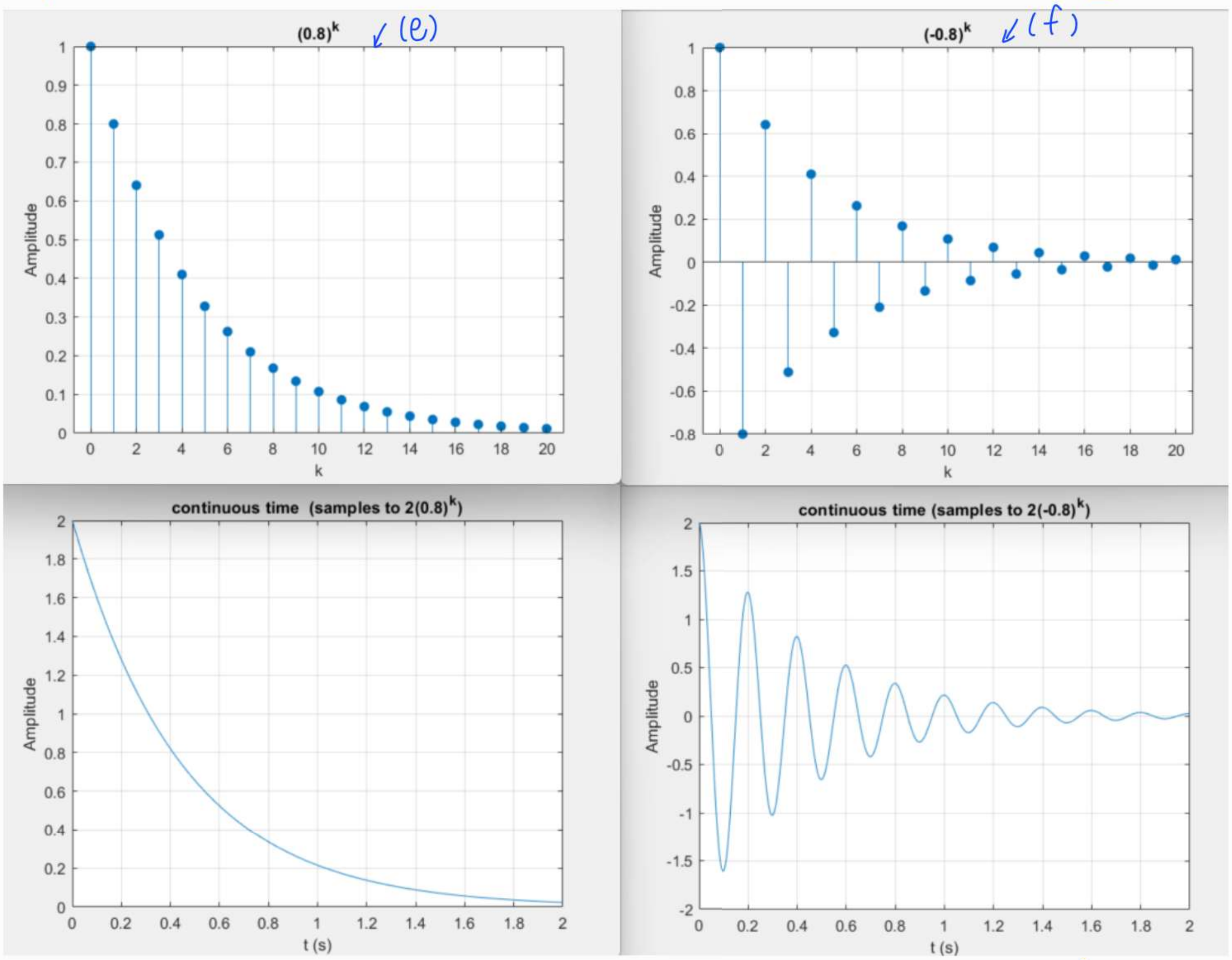
$$0.8 = e^{10s} \Rightarrow \log_e 0.8 = 10s \Rightarrow s = 10^{-1} \log_e(0.8)$$

$$(-1)^k = \cos(\pi k) = \cos!(\frac{\pi t}{0.1})$$

$$\therefore e(t) = e[kT] = 2(0.8)^k(-1)^k = 2 \cdot e^{(10s)t} \cdot \cos(10\pi t),\ \text{where } s = 10^{-1} \cdot \log_e(0.8)\ \#$$

(g) is in the next page!

(g) I use Matlab to plot (e) and (f) for understanding



Change a real pole from +0.8 to −0.8 causes the inverse z-transfer is multiplied by $(-1)^k$, which means it causes the corresponding time signal to oscillate at the highest freq. to be captured by the sampler. #