

• Nyquist criterion, general case

In general  $G(s)$  may have  $P$  poles in the RHP

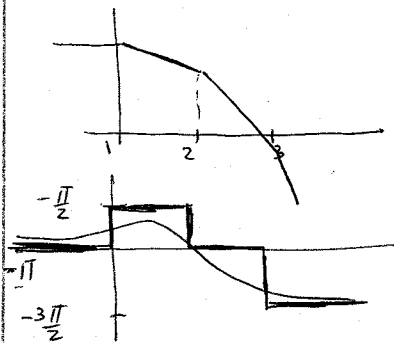
$\Rightarrow$  for the closed loop system to be stable we need  $Z=0$ . Hence

$$N = \frac{Z}{-P} \quad // \quad \boxed{N = -P}$$

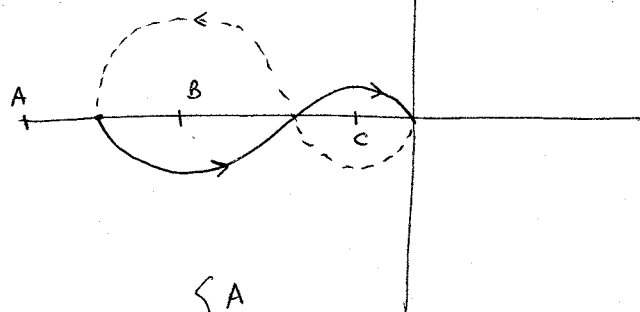
Example where  $P \neq 0$  (open loop unstable)

$$GH = \frac{k}{(s-1)(s+2)(s+3)} = \frac{k}{s^3 + 4s^2 + s - 6}$$

Bode Plots:



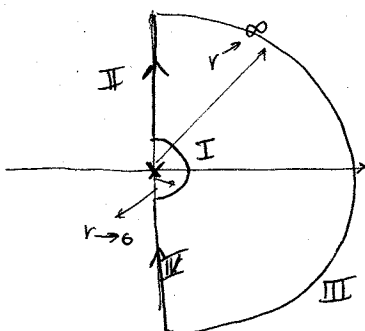
Polar plot:



$P=1$ : 3 possible cases:  $-1 = \begin{cases} A \\ B \\ C \end{cases}$

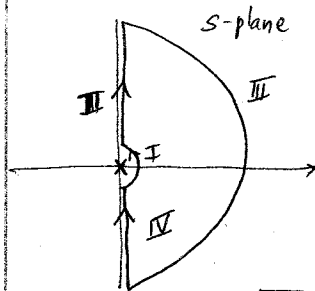
In A)  $N=0$   $Z=1 \Rightarrow$  unstable  
 B)  $N=-1$   $Z=N+P=0 \Rightarrow$  stable  
 C)  $N=1$   $Z=N+P=2 \Rightarrow$  unstable

• Note that the argument's principle is valid only if  $C$  does not go through any pole of  $F(s)$ . Hence if  $F(s)$  has poles on the  $j\omega$  axis (such as at  $s=0$ ) we need to change  $C$  (to avoid these poles). For instance, assume a pole at the origin. Then we can avoid it by taking a small semicircle (with radius  $r \rightarrow 0$ ) around the pole



Now, as  $r \rightarrow 0$  the region I will map into  $\infty$  in the  $w$  plane.

Example:  $G(s) = \frac{k}{s(s+1)}$

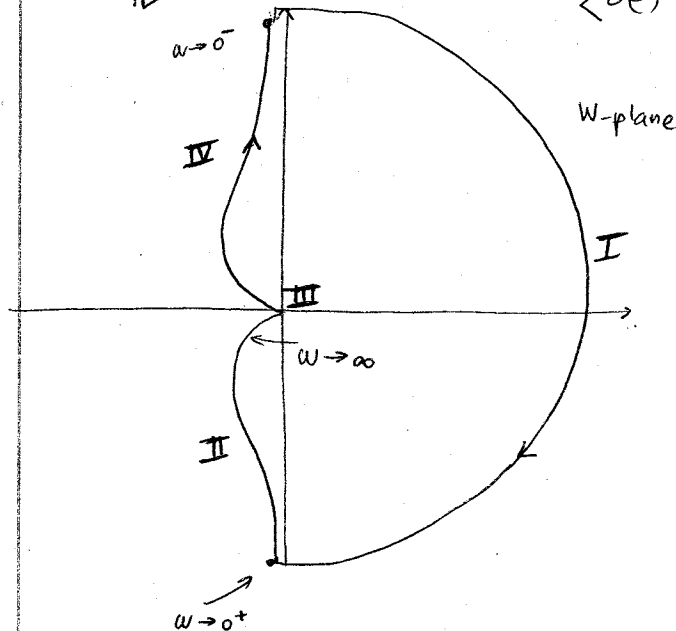


In region I,  $s = re^{j\theta}$  with  $r \rightarrow 0$

Hence  $G(s) \approx \frac{k}{re^{j\theta}} = \frac{k}{r} e^{-j\theta}$

when  $\theta$  goes from  $-\frac{\pi}{2}$  to  $+\frac{\pi}{2}$  counterclockwise

$\angle G(s)$  goes from  $+\frac{\pi}{2}$  to  $-\frac{\pi}{2}$  clockwise



$P=0, N=0$

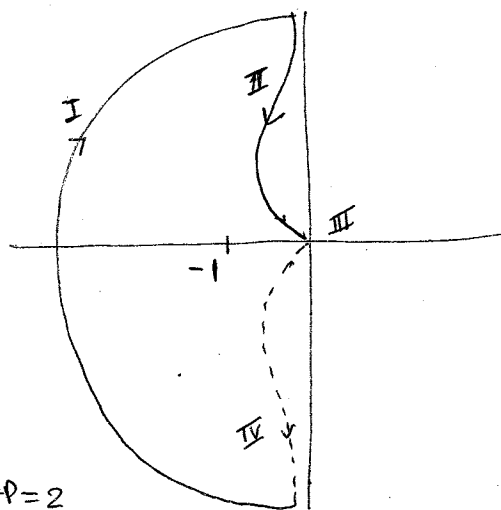
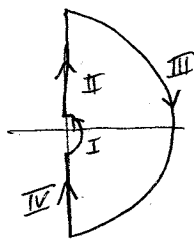
$\Rightarrow Z=0$

stable for all  $k > 0$

• Example: (non minimum phase system:)

$G(s) = \frac{k}{s(s-1)}$

$\Rightarrow P=1$



for  $s = pe^{j\theta}$   
 $p \rightarrow 0$

$G(s) \sim -\frac{k}{pe^{j\theta}}$

$= \frac{k}{p} \angle 180 - \theta$

$\left. \begin{matrix} P=1 \\ N=1 \end{matrix} \right\} Z = N+P=2$

unstable for all  $k > 0$ , with 2 unstable roots.

## • Relative Stability (Concepts of Phase & Gain margin)

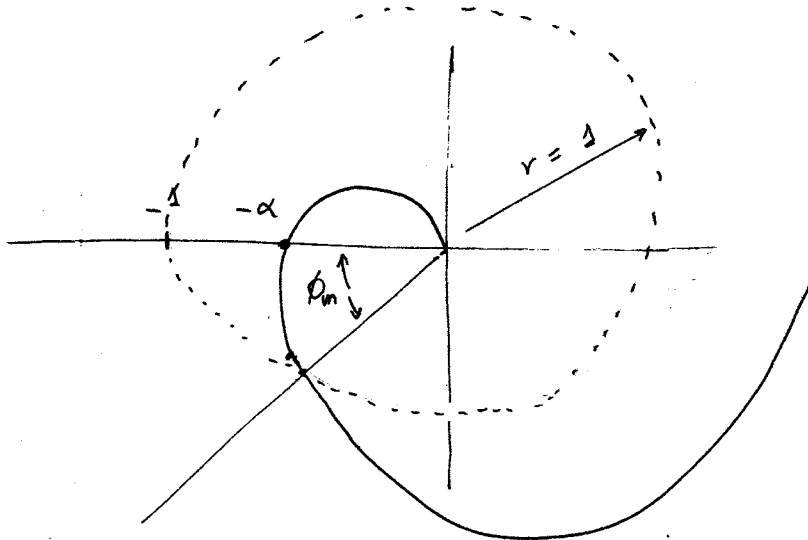
• Concept of "relative stability": How close is a given system to being unstable?

(ie: how much can we change  $G(s)$  and still have a stable system?) This gives a measure of the "robustness" of the system against model uncertainty

Classically 2 measures:

a) Gain margin: How much gain can we add to our system and still keep it stable?  
(equivalently: how much gain do I have to add to the system to make it unstable)

b) Phase margin: How much phase do I have to add to the system to make it marginally stable?

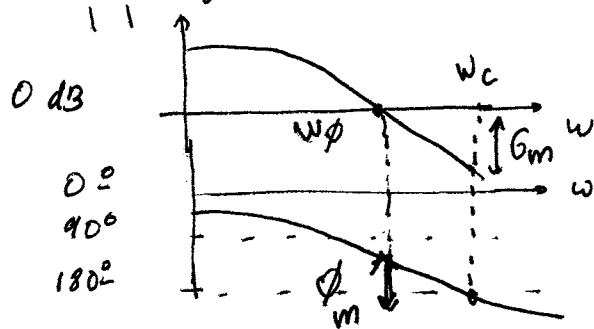


• Gain margin: Let  $\alpha$  be the magnitude of  $G(j\omega)$  at the  $180^\circ$  cross-over (ie  $G(j\omega) = -\alpha$ )  $\Rightarrow$  Then, the gain margin is  $1/\alpha$   
(usually expressed in dB)

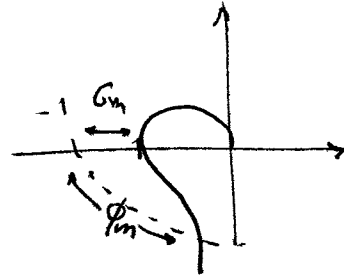
• Phase margin: magnitude of the minimum angle by which the Nyquist diagram must be rotated in order to go through the  $-1$  point  
 $\Rightarrow$  in the diagram:  $\phi_m$

Gain & phase margins can be read (in most cases) from the Bode diagrams

If the system is open loop stable, then



Example:  $G(s) = \frac{2500}{s(s+5)(s+50)}$



To find  $G_m$ , first find  $w_c$ :  $G(jw) = \frac{2500}{jw[(250-w^2)+j55w]}$

$\Rightarrow w_c^2 = 250 \Rightarrow w_c = 15.8 \text{ rad/sec}$

$|G(jw_c)| = \frac{1}{5.5} \Rightarrow \boxed{G_m = 5.5 = 14.8 \text{ dB}}$

To find  $\phi_m$ , we need to find the frequency where  $|G(jw)| = 1$

$\Rightarrow |G(jw)|^2 = 1 \Rightarrow \frac{(2500)^2}{w^2 [(250-w^2)^2 + (55)^2 w^2]} = 1$

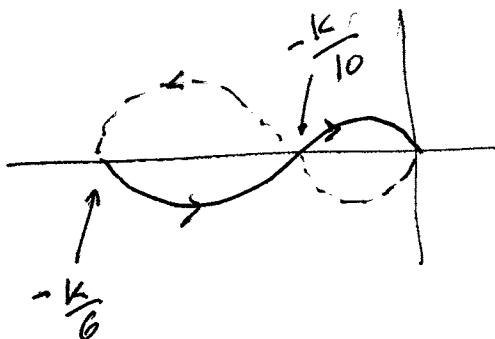
Let  $w^2 = x \Rightarrow x(x^2 - 500x + (55)^2 x + (250)^2) - (2500)^2 = 0$

$\Rightarrow w_\phi = \sqrt{x} = 6.2 \text{ rad/sec}$

$\angle G(jw_\phi) = -148.3^\circ \Rightarrow \phi = 180 + \angle G(jw) = \boxed{31.7^\circ}$

However, need to be careful if the plant is open-loop unstable

Back to the example  $G(s) = \frac{k}{(s-1)(s+2)(s+3)} = \frac{k}{s^3 + 4s^2 + s - 6}$



stable for  $6 < k < 10$   
and it can be made unstable  
by both increasing and  
decreasing the gain.

For instance, for  $K_0=8$ ,  $-1$  is encircled once, counterclockwise  $\Rightarrow$   
 $N=-1 \Rightarrow Z=N+P=0$  (stable)

As  $K$  increases, the system becomes unstable when the point at  $-\frac{K}{10}$  lands on top of  $-1 \Rightarrow K=10$

$$\Rightarrow G_m = \frac{10}{8} = 1.25 = \boxed{1.94 \text{ dB}} \quad (\text{upward gain margin})$$

But, the system can also be made unstable by decreasing  $K$  so that the point  $-\frac{K}{6}$  moves on top of  $-1$

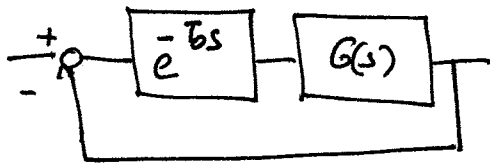
$$G_m = \frac{6}{8} \approx -2.5 \text{ dB} \# \quad (\text{downward gain margin})$$

For this example Matlab's  $[G_m, \phi_m] = \text{margin}(\text{sys})$  yields:

$$G_m = 1.94 \text{ dB} \quad \leftarrow \text{only one of the values}$$

$$\phi_m = 2.54^\circ$$

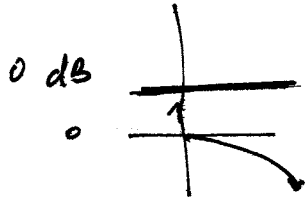
# • Effects of time delay on stability



Want to analyze stability of  $1 + G(s)e^{-Ts} = 0$

(can't use Routh Hurwitz since this is not a polynomial)

Recall that the Bode plot of a time delay is

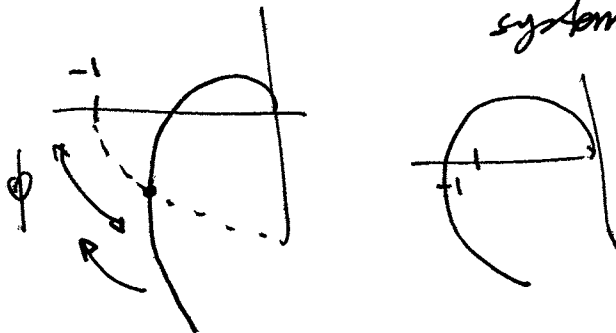


$$|e^{-sT_0}| = 1$$

$$\angle e^{-j\omega_p T_0} = -\omega_p T_0$$

$\Rightarrow$  time delay adds phase to the Loop function

$\Rightarrow$  Nyquist plot gets rotated (clockwise) by an amount  $\omega_p T_0$  radians at frequency  $\Rightarrow$  even if the original system was stable, the added rotation



can cause the plot to encircle the -1 point and destabilize the system

Q: how much delay can we tolerate?

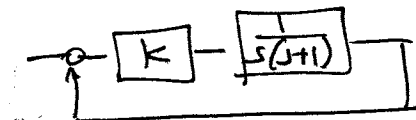
A: System becomes unstable when Nyquist plot passes through -1

$\Rightarrow$  we need to add  $\phi$  radians, but  $\phi$  is precisely the phase margin

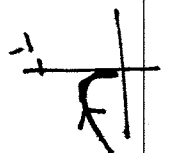
$$\Rightarrow \phi_m = \omega_p T_0 \Rightarrow T_0 = \frac{\phi_m \leftarrow \text{radians}}{\omega_p \leftarrow \text{rad/sec}} = \frac{\phi_m^{\text{deg}} \pi}{180 \omega_0}$$

Example:

$$G(s) = \frac{K}{s(s+1)}$$



From RH, stable for all K, gain margin  $\infty$



However,  $\phi_m = 51.83^\circ$  at  $\omega_\phi = 0.7862 \text{ rad/sec}$

$\Rightarrow$  system can tolerate at delay of at most  
 $T = 1.15 \text{ seconds}$

What if  $k=10$ ?  $\Rightarrow$  Then  $\phi_m = 17.96$  and  $\omega_\phi = 3.08$   
 $\Rightarrow T_0 = 0.1 \text{ sec}$  #  $\leftarrow$  very small

Note that for the closed loop system we have (without the time delay)

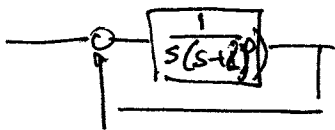
$$G_{cl} = \frac{k}{s^2 + s + k} \Rightarrow \omega_n = \sqrt{k} \quad 2\zeta\omega_n = 1 \Rightarrow \zeta = \frac{1}{2\omega_n} = \frac{1}{2\sqrt{k}}$$

$$\Rightarrow \text{in this case } \zeta = 0.158 \Rightarrow M_p \approx 60\%$$

Seems that there is a connection here between  
low damping and small phase margins  $\leftrightarrow$  small tolerance  
to time delays

This is actually the case!

- Relationship between phase margin and  $\zeta$  (for second order systems)



$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

To find the  $\phi_m$ , we need to find the freq  $\omega_\phi$  such that  $|G(j\omega_\phi)| = 1$   
 $\Rightarrow s = j\omega$ , define  $u = \frac{\omega}{\omega_n} \Rightarrow \left| -\frac{1}{u^2 + 2j\zeta u} \right| = 1 \Rightarrow u^4 + 4\zeta^2 u^2 - 1 = 0$

$$u_o^2 = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

$$\angle G(ju_o) = -\angle -u_o^2 + j2\zeta u_o = +180^\circ - \tan^{-1}\left(\frac{2\zeta u_o}{u_o^2}\right)$$

$$\Rightarrow \phi_m = \tan^{-1} \left[ \frac{2\zeta}{(\sqrt{4\zeta^4 + 1} - 2\zeta^2)^{1/2}} \right] \approx 100\zeta$$

good approximation in the range  
 $0.1 \leq \zeta \leq 0.7$

$\Rightarrow$  if you want  $\phi_m \geq 30^\circ$  you need  $\zeta > 0.3$

$\phi_m$  is also related to settling time:

$$T_s = \frac{4}{\zeta\omega_n}$$

$$\phi_m = \tan^{-1} \left[ 2 \frac{\zeta\omega_0}{\omega_n} \right] = \tan^{-1} \left[ \frac{8}{T_s (\sqrt{4\zeta^4 + 1} - 2\zeta^2)^{1/2}} \right]$$

## • Robust Control

The idea is to design a system such that some desirable properties (such as stability) are preserved in the presence of (unknown) perturbations

### • Robust stability:

Suppose that rather than having a known plant  $G_0(s)$  we are dealing with a family of plants

$$G(s) = G_0(s)(1 + \delta(s)) \quad \text{where } \delta(s) \text{ is a (frequency dependent) perturbation (such as unmodeled) dynamics}$$

Assume that we only know that, at each frequency  $\omega$ ,  $\delta$  is bounded by a known function, e.g.:

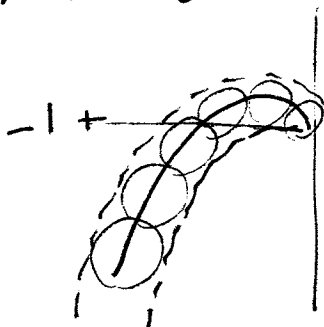
$$|\delta(j\omega)| < r(j\omega) \quad \text{where } r \text{ is known.}$$

We'd like to find out if the closed loop system is stable for all members of the family

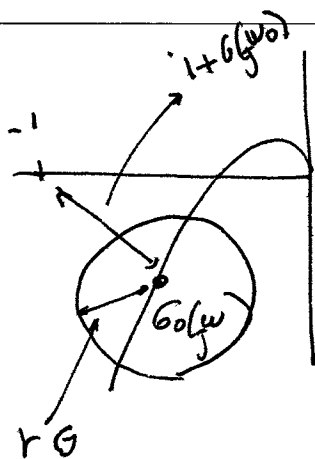
$\Rightarrow$  Let's apply Nyquist to the entire family

(assume that the entire family is open loop stable)  $\Rightarrow$

For a given frequency  $\omega_0$ , the Nyquist plot of the actual system can be anywhere inside the circle centered at  $G_0(j\omega_0)$  with radius  $r(\omega_0) |G_0(j\omega_0)| \Rightarrow$  rather than a single plot, we get a band



So, the family is stable iff the point  $(-1, 0)$  is excluded from the band, that is



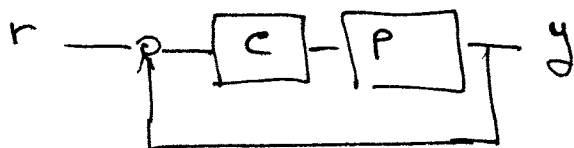
The condition for stability of the family becomes

$$|1 + G(jw_0)| > |rG(jw_0)|$$

distance from center of the disk to the  $(-1, 0)$  point

radius of the disk

$$\Rightarrow \text{family is stable} \iff 1 > \frac{r |G(jw_0)|}{|1 + G(jw_0)|}$$



$$\text{but } \frac{G(jw)}{1 + G(jw)} = T_{yr}$$

$\Rightarrow$  Robust stability requires that, for all frequencies

$$|T_{yr}(jw)| < \frac{1}{|r(jw)|}$$

$$\text{or, in compact form: } \|r(jw) \cdot T_{yr}(jw)\|_{\infty} < 1$$

$\uparrow$   
H-infinity norm

(for a function  $G(jw)$  we define its H-infinity norm as

$$\|G(jw)\|_{\infty} = \sup_w |G(jw)| \quad (\text{e.g. the peak value in the Bode plot})$$

Note that the conditions above indicate that

frequencies where  $|T(jw)|$  is large can tolerate

little uncertainty  $\Rightarrow$  need to make  $|T|$  small

in places where  $|r|$  is large

(e.g. places where the model is not well known)