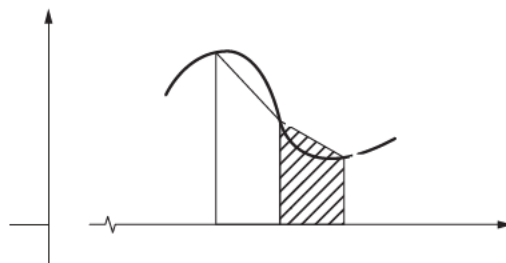**2.2-2.** The trapezoidal rule (modified Euler method) for numerical

$$\frac{Y(z)}{X(z)} = \frac{(T/2)(z+1)}{z-1}$$

Solution:

(a) $y(k+1) = y(k) + T\dfrac{x(k)+x(k+1)}{2}$

(b) $zY(z) = Y(z) + \dfrac{T}{2}[X(z) + zX(z)] \Rightarrow Y(z) = \dfrac{T}{2}\dfrac{z+1}{z-1}X(z)$

2.2-3. (a) The transfer function for the right-side rectangular-rule integrator was found in Problem 2.2-1 to be $Y(z)/X(z) = Tz/(z-1)$. We would suspect that the reciprocal of this transfer function should yield an approximation to a differentiator. That is, if $w(kT)$ is a numerical derivative of $x(t)$ at $t = kT$,

$$\frac{W(z)}{X(z)} = \frac{z-1}{Tz}$$

Write the difference equation describing this differentiator.

(b) Draw a figure similar to those in Fig. P2.2-1 illustrating the approximate differentiation.

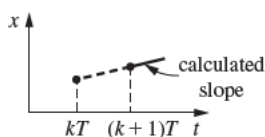(c) Repeat part (a) for the left-side rule, where $W(z)/X(z) = T/(z-1)$.

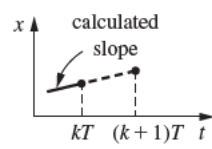(d) Repeat part (b) for the differentiator of part (c).

Solution:

(a) $Tz\,W(z) = zX(z) - X(z)$

$$w(k+1) = \frac{1}{T}[x(k+1) - x(k)]$$

(b)

(c) $TW(z) = zX(z) - X(z)$

$$w(k) = \frac{1}{T}\left[x(k+1) - x(k)\right]$$

2.6-3. Given the difference equation

$$x(k) - x(k-1) + x(k-2) = e(k)$$

where $e(k) = 1$ for $k \geq 0$.

(a) Solve for $x(k)$ as a function of k, using the z-transform. Give the values of $x(0), x(1)$, and $x(2)$.

(b) Verify the values $x(0)$, $x(1)$, and $x(2)$, using the power-series method.

(c) Verify the values $x(0)$, $x(1)$, and $x(2)$ by solving the difference equation directly.

(d) Will the final-value property give the correct value for $x(\infty)$?

Solution:

(a) $[1 - z^{-1} + z^{-2}]X(z) = E(z) = \dfrac{z}{z-1}$

$$X(z) = \frac{z^3}{(z-1)(z^2-z+1)}, \qquad \text{poles: } z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1\angle \pm 60°$$

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{k_1}{z-p_1} + \frac{k_1^*}{z-p_1^*} \quad \text{with } p = 1\angle 60°$$

$$k_1 = \left.\frac{z^2}{(z-1)(z-1\angle-60°)}\right|_{z=1\angle 60°} = \frac{1\angle 120°}{(.5+j.866-1)(.5+j.866-.5+j.866)}$$

$$= \frac{1\angle 120°}{1\angle 120°\,[j2(0.866)]} = 0.5774\angle-90°$$

$$\therefore aT = \ln\left(|p_1|\right) = 0; \quad bT = \arg p_1 = \frac{\pi}{3}$$

$$A = 2|k_1| = 1.155; \quad \theta = \arg k_1 = -90°$$

$$\therefore x(k) = 1 + 1.155 \cos\left(\frac{\pi}{3}k - 90°\right) = 1 + 1.155 \sin\left(\frac{\pi}{3}k\right)$$

$$x(0) = 1, \quad x(1) = 2, \quad x(2) = 2$$

(b)

$$z^3 - 2z^2 + 2z - 1\,\overline{\big)\,z^3} \qquad \underset{}{\dfrac{1 + 2z^{-1} + 2z^{-2} + \cdots}{}}$$

$$\underline{z^3 - 2z^2 + 2z - 1}$$
$$2z^2 - 2z + 1$$
$$\underline{2z^2 - 4z + 4 - 2z^{-1}}$$
$$2z + \cdots$$

$\therefore x(0) = 1$
$x(1) = 2$
$x(2) = 2$

(c) $x(k) = 1 + x(k-1) - x(k-2)$

$x(0) = 1 + 0 - 0 = 1$

$x(1) = 1 + 1 - 0 = 2$

$x(2) = 1 + 2 - 1 = 2$

(d) *No*, 3 poles for $X(z)$ on the unit circle.

2.7-1. (a) Find $e(0)$, $e(1)$, and $e(10)$ for

$$E(z) = \frac{0.1}{z(z - 0.9)}$$

using the inversion formula.

(b) Check the value of $e(0)$ using the initial-value property.

(c) Check the values calculated in part (a) using partial fractions.

(d) Find $e(k)$ for $k = 0, 1, 2, 3,$ and 4 if $\mathscr{z}[e(k)]$ is given by

$$E(z) = \frac{1.98z}{\left(z^2 - 0.9z + 0.9\right)\left(z - 0.8\right)\left(z^2 - 1.2z + 0.27\right)}$$

(e) Find a function $e(t)$ which, when sampled at a rate of 10 Hz $(T = 0.1s)$, results in the transform $E(z) = 2z/(z - 0.8)$.

(f) Repeat part (e) for $E(z) = 2z/(z + 0.8)$.

(g) From parts (e) and (f), what is the effect on the inverse z-transform of changing the sign on a real pole?

Solution:

(a) $e(k) = \sum\limits_{\text{residues}} \dfrac{0.1z^{k-1}}{z(z-0.9)} = \sum\limits_{\text{residues}} \dfrac{0.1z^{k-2}}{z - 0.9}$

$k = 0 : \text{fcn} = \dfrac{0.1}{z^2(z-0.9)}, \quad \therefore \text{residue}\big|_{z=0\,9} = \dfrac{0.1}{(0.9)^2} = 0.1235$

$\text{residue}\big|_{z=0} = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \dfrac{-0.1(1)}{(z-0.9)^2}\bigg|_{z=0} = \dfrac{-0.1}{(0.9)^2} = -0.1235$

$\therefore e(0) = 0$

$k = 1 : e(1) = \dfrac{0.1}{z-0.9}\bigg|_{z=0} + \dfrac{0.1}{z}\bigg|_{z=0\,9} = 0$

$k = 10 : e(10) = 0.1(0.9)^8$

(b) $e(0) = \lim\limits_{z \to \infty} E(z) = \lim\limits_{z \to \infty} \dfrac{0.1}{z(z-0.9)} = 0$

(c) $\dfrac{E(z)}{z} = \dfrac{0.1}{z^2(z-0.9)} = \dfrac{k_1}{z^2} + \dfrac{k_2}{z} + \dfrac{k_3}{z-0.9}$

$k_1 = \dfrac{-0.1}{0.9} = -\dfrac{1}{9}; \quad k_3 = \dfrac{0.1}{(0.9)^2} = \dfrac{1}{8.1}$

$k_2 = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \dfrac{-1}{8.1}, \text{ from (a)}$

$$\therefore e(k) = \frac{-1}{8.1}\delta(k) - \frac{1}{9}\delta(k-1) + \frac{1}{8.1}(0.9)^k$$

$$x(0) = -\frac{1}{8.1} + 0 + \frac{1}{8.1} = 0; \quad x(1) = -0 - \frac{1}{9} + \frac{0.9}{8.1} = 0$$

$$x(10) = -0 - 0 + \frac{0.1}{(0.9)^2}(0.9)^{10} = 0.1(0.9)^8$$

(d) $\quad E(z) = \dfrac{1.98z}{z^5 + \cdots} = 1.98z^{-4} + (\cdot)z^{-5} + (\cdot)z^{-6} + \cdots$

$\therefore e(0) = e(1) = e(2) = e(3) = 0; \quad e(4) = 1.98$

(e) $\quad E(z) = \dfrac{2z}{z - 0.8} = \dfrac{2z}{z - \varepsilon^{-aT}} \quad \therefore \varepsilon^{-aT} = 0.8 \Rightarrow aT = 0.2231$

$\therefore a = \dfrac{0.2231}{0.1} = 2.231, \quad \therefore e(t) = 2\varepsilon^{-2.231t}u(t)$

(f) $\quad E(z) = \dfrac{2z}{z - (-0.8)}; \quad \therefore \varepsilon^{-aT}\varepsilon^{j\pi} = -0.8 \Rightarrow aT = 2.231$

$\therefore e(t) = 2e^{-2.231t}\cos 10\pi t \quad \text{where} \quad \dfrac{\omega_s}{2} = 10\pi$

(g) (e) $e(k) = (0.8)^k$; \quad (f) $e(k) = (-0.8)^k$

$\therefore$ sign alternates on $e(k)$.