Jury's Stability Test (Section 7.5 Book)

Jury's test is similar to Routh–Hurwitz in the sense that it counts the number of unstable roots of the **discrete-time** characteristic equation.

You form an array using the coefficients of the polynomial, starting with two rows of length n. From these you compute a successor row of length $n-1$, then another of length $n-2$, and so on, until we get a row of length 1.

Stability is related to the entries of the first column, as follows:

Assume that the characteristic polynomial is given by:

$$\phi(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n > 0$$

Step 1: Form Jury's Array

$$
\begin{array}{cccccccc}
z^0 & z^1 & z^2 & z^3 & \cdots & z^{n-1} & z^{n-1} & z^n \\
\\
a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_{n-1} & a_n \\
a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & a_0 \\
\\
b_0 & b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} & \\
b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_{n-1} & b_0 & \\
\\
c_0 & c_1 & c_2 & c_3 & \cdots & c_{n-2} & & \\
c_{n-2} & c_{n-3} & c_{n-4} & c_{n-5} & \cdots & c_0 & & \\
\vdots & & & & \vdots & & & \\
\ell_0 & \ell_1 & \ell_2 & \ell_3 & & & & \\
\ell_3 & \ell_2 & \ell_1 & \ell_0 & & & & \\
\\
m_0 & m_1 & m_2 & & & & &
\end{array}
$$

Remark: The elements of the *even-numbered rows* are the elements of the preceding row in reverse order.

$$
b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \; \cdots
$$

Step 2: Check the following conditions

(necessary and sufficient for having all roots in $|z| < 1$)

1. $\phi(1) > 0$

2. $(-1)^n \phi(-1) > 0$

3. $|a_0| > a_n$

   $|b_0| > |b_{n-1}|$

   $|c_0| > |c_{n-2}|$

   $|d_0| > |d_{n-3}|, \; \cdots$

   $|m_0| > |m_2|$

Remark: Check first that $\phi(1) > 0$, $(-1)^n \phi(-1) > 0$, and $a_n > |a_0|$.

If any of these conditions fails, the system is unstable and DON't need to proceed any further.

Example 1 (Linear example with $T = 0.1$):

$$\phi(z) = 1 + kG(z) = z^2 + (0.0085k - 1.5752)z + (0.0072k + 0.6065)$$

Conditions:

$$\phi(1) = 1 + (0.0085k - 1.5752) + (0.0072k + 0.6065) = 0.0314 + 0.0157k > 0$$

$$(-1)^2 \phi(-1) = 3.1817 - 0.0013k > 0$$

$$a_2 > |a_0| \quad \Rightarrow \quad |0.0072k + 0.6065| < 1$$

$$\Leftrightarrow -1 < 0.0072k + 0.6065 < 1$$

From these conditions, we have:

$$k > -2$$

$$k < 2.4475 \times 10^3$$

$$k < 54.713$$

$$k > -223.39$$

$$\Rightarrow \boxed{-2 < k < 54.713}$$

(Same conditions as before)

Example 2: A Robust Stability Example

Consider the following second-order system: $\phi(z) = z^2 + \alpha z + \beta$, where α and β are parameters.

The Jury Array:

z^0	z^1	z^2
β	α	1
1	α	β
$\beta^2 - 1$	$\alpha(\beta - 1)$	

Conditions:

$$\phi(1) > 0 \qquad \Rightarrow \alpha + \beta + 1 > 0$$

$$(-1)^2\phi(-1) > 0 \qquad \Rightarrow \beta - 1 - \alpha > 0$$

$$|a_0| < a_n \qquad \Rightarrow |\beta| < 1 \qquad \Leftrightarrow -1 < \beta < 1$$

$$|b_0| > |b_{n-1}| \qquad \Rightarrow |\beta^2 - 1| > |\alpha(\beta - 1)| \quad \Rightarrow |\beta + 1| > |\alpha|$$

Simplified:$|\beta + 1| > |\alpha|$ or equivalently:$1 + \beta > \alpha, \quad 1 + \beta > -\alpha$

Graphical Interpretation:

The stability region in the (α, β) plane is given by:

$$\text{stable parameter values: } \begin{cases} 1+\beta > \alpha \\ 1+\beta > -\alpha \\ |\beta| < 1 \end{cases}$$

Example 3: A Third-Order System

$$\phi(z) = z^3 - 1.82z^2 + 1.05z - 0.20$$

Checking conditions: $\phi(1) = 1 - 1.8 + 1.05 - 0.20 + 0.05 > 0$

$$(-1)^3\phi(-1) = (-1)(-1 - 1.8 - 1.05 - 0.2) = 4.05 > 0$$

$$|a_0| = 0.2 < a_3 = 1$$

Jury Array:

z^0	z^1	z^2	z^3
-0.2	1.05	-1.8	1
1	-1.8	1.05	-0.2
-0.96	1.59	-0.69	

$$|b_0| = 0.96 > |b_2| = 0.69$$

All conditions hold $\Rightarrow$ **System is stable.**

$$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = (-0.2) \times (-0.2) - 1 \times 1 = -0.96,$$

$$b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = (-0.2) \times 1.05 - 1 \times (-1.8) = 1.59,$$

$$b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix} = (-0.2) \times (-1.8) - 1 \times 1.05 = -0.69.$$

Singular Cases

Q: What do we do if we get an equality (rather than a strict inequality) in any of the conditions?

A: Use the transformation

$$z = (1+\epsilon)\hat{z}, \quad (\epsilon > 0)$$

This transformation amounts to expanding the unit circle.

i.e. If we have a root at $z = z_0$, then $\phi(\hat{z})$ will have a root at $\hat{z} = \dfrac{z_0}{1+\epsilon}$.

Let $\phi(z) = \sum a_i z^i \quad \Rightarrow \quad \phi(\hat{z}) = \sum_i a_i (1+\epsilon)^i \hat{z}^i$

Since for small ϵ, $(1+\epsilon)^i \approx (1+i\epsilon)$, the transformation is equivalent to multiplying the coefficients of the original polynomial by $(1+k\epsilon)$, i.e.,

$$a_k \to (1+k\epsilon)a_k$$

Raible's Tabular Form of Jury's Criterion

Ref: H. Raible, *"A Simplification of Jury's Tabular Form," IEEE Trans. Autom. Control,* AC-19, June 1974, pp. 248–250.

Jury's Array:

$$
\begin{array}{cccccc}
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\
a_n k_a & a_{n-1} k_a & a_{n-2} k_a & \cdots & a_1 k_a & a_0 k_a \\
\\
b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \\
b_{n-1} k_b & b_{n-2} k_b & & \cdots & b_1 k_b & \\
\\
c_0 & c_1 & & \cdots & c_{n-2} & \\
\vdots & \vdots & & & \vdots & \\
\\
q_0 & q_1 & & & & \\
q_0 k_q & q_1 k_q & & & & \\
\\
r_0 & & & & &
\end{array}
$$

$$
k_a = \frac{a_0}{a_n}, \quad k_b = \frac{b_{n-1}}{b_0}, \quad k_c = \frac{c_{n-2}}{c_0}, \quad k_q = \frac{q_1}{q_0}.
$$

where: $b_i = a_{n-i} - k_a a_i$, i.e., you obtain the rows marked with arrows by subtracting the previous row from the one immediately above it.

Then: Interpretation of Raible's Tabular Form

Number of positive elements in the first column of the calculated rows (the ones marked with an arrow)

$\Rightarrow$ Number of roots inside the unit disk, i.e.

System stable $\iff$ all the elements in the first column of the marked rows are positive.

Example 3 Revisited

$$\phi(z) = z^3 - 1.82z^2 + 1.05z - 0.20$$

$$\begin{array}{ccccccc}
 & 1 & -1.8 & 1.05 & -0.2 & & k_a = -0.2 \\
 & 0.04 & -0.21 & 0.36 & & & \\
\Rightarrow & 0.96 & -1.59 & 0.69 & & & k_b = 0.7188 \\
 & 0.4959 & -1.1428 & & & & \\
\Rightarrow & 0.4641 & -0.4472 & & & & k_c = -0.9636 \\
 & 0.4309 & & & & & \\
\Rightarrow & 0.0152 & & & & &
\end{array}$$

$$b_0,\ c_0,\ d_0 > 0 \quad \Rightarrow \quad \textbf{System Stable}$$

Example 2 Revisited

$$\phi(z) = z^2 + \alpha z + \beta$$

	1	α	β	$k_a = \beta$
	β^2	$\alpha\beta$		
$\Rightarrow$	$1-\beta^2$	$\alpha(1-\beta)$		$k_p = \dfrac{\alpha(1-\beta)}{(1-\beta^2)} = \dfrac{\alpha}{1+\beta}$
	$\alpha^2\dfrac{(1-\beta)}{(1+\beta)}$			
$\Rightarrow$	$(1-\beta^2) - \alpha^2\dfrac{(1-\beta)}{(1+\beta)}$			

Stability Conditions

$$1-\beta^2 > 0 \quad \Rightarrow \beta^2 < 1 \qquad\qquad \Rightarrow |\beta| < 1$$

$$(1-\beta^2) - \alpha^2\frac{(1-\beta)}{1+\beta} > 0 \quad \Leftrightarrow (1+\beta)^2 > \alpha^2$$

$$\Leftrightarrow 1+\beta > \alpha, \quad 1+\beta > -\alpha$$

Same Conditions as Before.

Note that these conditions are redundant and can be obtained from:

$$\phi(1) > 0 \qquad \Rightarrow \beta + 1 > -\alpha$$

$$\phi(-1) > 0 \qquad \Rightarrow \beta + 1 > \alpha$$

$$|a_0| < a_n \quad \Rightarrow |\beta| < 1$$