- **Steady state accuracy**
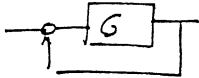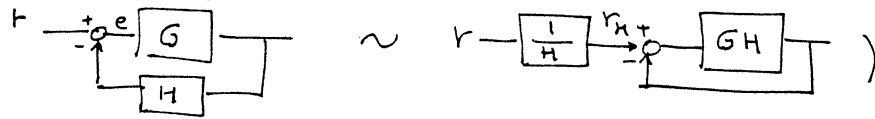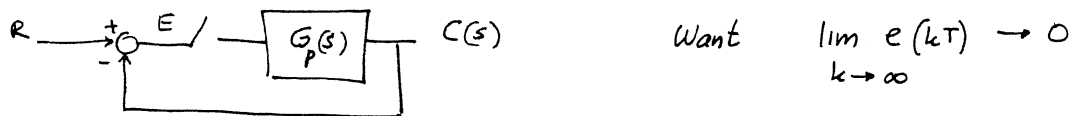
(For the time being we will assume unity feedback:

a non-unity feedback system can always be recast into this form with a prefilter:

)

The goal is to follow a reference trajectory (ideally without error)

Want $\lim\limits_{k \to \infty} e(kT) \to 0$

Assume that

$$G(z) = \frac{k \, \Pi(z - z_i)}{(z-1)^N \, \Pi(z - P_J)} = \frac{P(z)}{(z-1)^N \, Q_1(z)} \quad \text{where} \quad P(1) \neq 0$$
$$Q_J(1) \neq 0$$

As in the continuous time case, N is called the <u>system type</u>. As we will see next, this determines the ability of the system to track a reference input.

We are going to consider the following types of inputs : step
ramp
parabolic

Reasons
1) most commonly used inputs

2) any other input can be approximated by a combination of these three (to a large extent)

$$r(t) = \underbrace{r(0)}_{step} + \underbrace{\frac{dr}{dt}\Big|_0 t}_{ramp} + \underbrace{\frac{d^2r}{dt^2}\Big|_0 \frac{t^2}{2}}_{parabolic} + \cdots$$

<u>IF</u> the system is stable $\left(\text{standing assumption for the next two lectures}\right)$ then we can use the FVT

$$E(z) = \frac{1}{1 + G(z)} R(z) \implies e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (z-1) E(z) =$$
$$= \lim_{z \to 1} (z-1) \frac{R(z)}{1 + G(z)}$$

- **Step input:** $\quad R(z) = \dfrac{z}{z-1}$

$$\Rightarrow e_{ss} = \lim_{z \to 1} \frac{\cancel{(z-1)} \cdot z}{\cancel{(z-1)}} \frac{1}{1+G(z)} = \frac{1}{1 + \lim_{z \to 1} G(z)} = \frac{1}{1+k_p}$$

where $\boxed{k_p = \lim_{z \to 1} G(z)}$ (position error constant)

(a) Type 0: $\quad k_p = G(0) = DC$ gain $\Rightarrow e_{ss} = \dfrac{1}{1+k_p}$ __finite__ (non zero)

(b) Type 1 or higher: $\quad k_p \to \infty \qquad \Rightarrow e_{ss} = 0$

- **Ramp input:** $\quad r(t) = t \quad \Rightarrow R(z) = \dfrac{Tz}{(z-1)^2}$

$$e_{ss} = \lim_{z \to 1} \cancel{(z-1)} \frac{Tz}{(z-1)^2} \frac{1}{1+G(z)} = \lim_{z \to 1} \frac{T}{(z-1) G(z)} = \frac{1}{k_v}$$

where $\boxed{k_v = \lim_{z \to 1} \dfrac{(z-1) G(z)}{T}}$ (velocity error constant)

(a) Type 0: $\quad k_v = \lim_{z \to 1} (z-1) \dfrac{G(0)}{T} = 0 \Rightarrow e_{ss_{ramp}} = \infty$

(b) Type (1) $\quad k_v = \lim_{z \to 1} (z-1) \dfrac{G}{T} = $ finite value $\Rightarrow e_{ss_{ramp}} = \dfrac{1}{k_v}$

(c) Type (2) $\quad k_v = \lim_{z \to 1} (z-1) \dfrac{G}{T} = \infty \qquad \Rightarrow e_{ss_{ramp}} = 0$

- **Parabolic input:** $\quad r = \frac{1}{2} t^2 \Rightarrow R = \dfrac{T^2 z \,(z+1)}{2(z-1)^3} \Rightarrow E(z) = \lim_{z \to 1} \dfrac{T^2}{(z-1)^2 G(z)} = \dfrac{1}{k_a}$

where $k_a = \lim_{z \to 1} \dfrac{(z-1)^2 G(z)}{T^2}$

(a) Type 0 or 1: $\quad k_a = 0 \Rightarrow e_{ss} = \infty$

(b) Type 2 $\quad k_a$ finite $\Rightarrow e_{ss} = \dfrac{1}{k_a}$ (finite)

(c) Type 3 or higher $\quad k_a = \infty \qquad \Rightarrow e_{ss} = 0$

__General property__: A system of type N can follow without error an input of the form: $\dfrac{A}{(z-1)^k}$ with $k \le N$, and with finite error an input $\dfrac{1}{z^{N+1}}$

(for $k > N+1$, $e_{ss} \to \infty$)

Summary:

$$R(z) \qquad \frac{z}{z-1} \qquad \frac{Tz}{(z-1)^2} \qquad \frac{T^2 z(z+1)}{2(z-1)^3}$$

N	$\frac{z}{z-1}$	$\frac{Tz}{(z-1)^2}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
0	$\frac{1}{1+k_p}$	∞	∞
1	0	$\frac{1}{k_v}$	∞
2	0	0	$\frac{1}{k_a}$

where

$$k_p = \lim_{z \to 1} G(z)$$

$$K_v = \lim_{z \to 1} \frac{(z-1)\,G(z)}{T}$$

$$k_a = \lim_{z \to 1} (z-1)^2 \frac{G(z)}{T^2}$$

Trade-off: The higher the type, the more accurate the system. However, it is more difficult to stabilize.

Laser example :
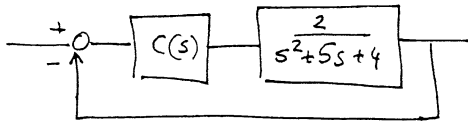
If $C(s) = k \implies k_p = \frac{k}{2}$, $e_{ss}^{step} = \frac{1}{1+k/2} = \frac{2}{2+k}$ $\underline{\text{provided}}$ that the closed loop is stable

Q: How do we assess stability?
A: Use Routh Hurwitz: Char eq: $s^2 + 5s + 4 + 2k = 0 \implies$

stable for all $\underline{k > 0}$

Now let's try to make it a type 1: $C(s) = \frac{k}{s}$

$\implies e_{ss}^{step} = 0$ $\underline{\underline{if}}$ stable.   Char eq: $s^3 + 5s^2 + 4s + 2k = 0$

Routh Hurwitz

s^3	1	4
s^2	5	2k
s^1	20-2k	
s^0	2k	

$\implies$ stable if $\begin{matrix} 0 < 2k \\ 20-2k < 0 \end{matrix} \implies$

$0 < k < 10$

If we try to make it a type 2: $C(s) = \frac{k}{s^2}$

Char eq: $s^4 + 5s^3 + 4s^2 + 0s + 2k$ $\implies$ always unstable

↑
missing term

If we want to make it type two we need to combine the $\frac{1}{s^2}$ term with $\underline{\text{phase lead}}$ compensation to stabilize it $\implies$

$C(s) = k\frac{(s+a)}{s^2}$ $\implies$ Char eq: $s^4 + 5s^3 + 4s^2 + 2ks + 2ka = 0$
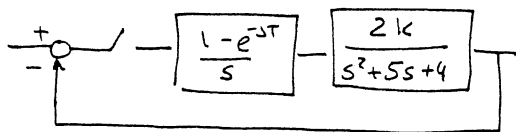
and now we need to find both k and a

Routh - Hurwitz:

s^4	1	4	2ka
s^3	5	2k	
s^2	$\frac{20-2k}{5}$	2ka	
s^1	$\frac{40k-4k^2}{5} - 10ka$		
s^0	2ka		

$a > 0$
$k > 0$
need $20 - 2k > 0 \implies k < 10$

$20 - 2k - 25a > 0$

What about the sampled data case. Assume $C(z) = k$
$T = 0.1 \, s$

Matlab yields: $\quad \mathcal{Z}\left[\dfrac{1-e^{-sT}}{s} \dfrac{2}{s^2+5s+4} \right] = \dfrac{0.0085z + 0.0072}{z^2 - 1.5752z + 0.6065}$

$\Rightarrow$ Characteristic equation:

$$z^2 + (0.0085 \cdot k - 1.5752)z + (0.6065 + 0.0072\,k) = 0$$

$\Rightarrow$ 2 roots: In this case one can solve for $z_{1,2}$ as a function of k and find out for what value we become unstable $\left(i.e. \; |z_{1,2}| = 1\right)$

$\left(\text{Turns out that } k \sim 55\right)$

However, this is fairly tedious even for a relatively simple system $\Rightarrow$
We need a <u>better way of assessing stability</u>

2 options (1) Map the z-plane to the s-plane and use continuous-time techniques (i.e. Routh Hurwitz)

(2) Derive an "equivalent" Routh Hurwitz criterion for discrete time systems
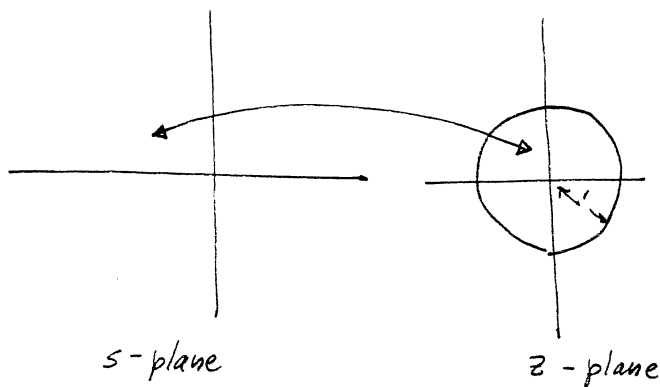
We will explore both options

- <u>Bilinear transformation</u> (section 7.3)

One way to map the z-plane to the s-plane is via the transformation

$$z = e^{sT} \iff s = \frac{1}{T} \ln z \iff Re(s) = \frac{1}{T} \ln |z|$$

$$Im(s) = \frac{1}{T} \angle \theta$$

$$|z| < 1 \iff Re(s) < 0$$

s-plane   z-plane

So in principle we could assess stability as follows

1) Use the transformation $s = \frac{1}{T} Ln(z)$ to map the discrete time char equation $D(z) = 0$ to a continuous time equivalent $D_{eq}(s)$

$$D_{eq}(s) = D(z) \Big|_{z = e^{sT}}$$

2) Assess stability of $D_{eq}(s)$ using EECE 5580 techniques

Let's try it in our laser example:

$$D(z) = z^2 + (0.0085 \, k - 1.5752) z + 0.6065 + 0.0072 \, k = 0$$

$$\Downarrow \quad z = e^{sT}$$

$$D_{eq}(s) = e^{2sT} + (0.0085 \, k - 1.5752) e^{sT} + 0.6065 + 0.0072 \, k = 0$$

But trouble!! We got a char equation that is <u>not</u> a polynomial in s (we have e^{sT} dependence) $\Rightarrow$ can't use Routh Hurwitz

$\Rightarrow$ we are stuck!

<u>Solution</u>: look for a different transformation.

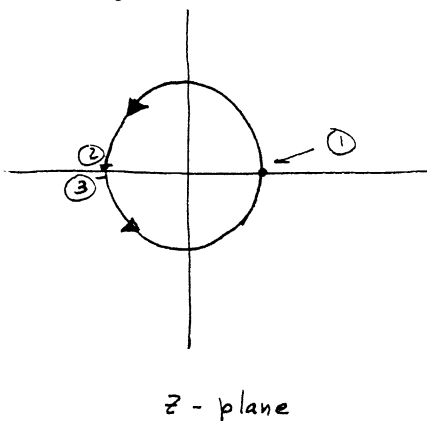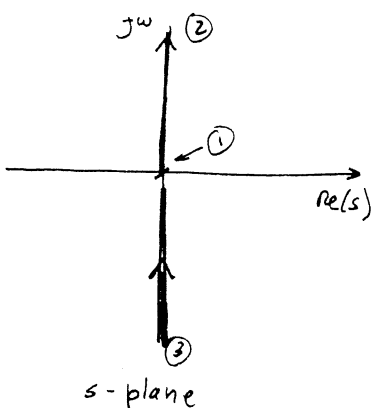Desirable properties (1) unit disk in z domain $\Longleftrightarrow$ Left half plane in s domain

(2) char eq polynomial in z $\Longleftrightarrow$ char eq polynomial in s.

One transformation that has these properties is the <u>bilinear</u> (or Tustin) transformation

$$z = \frac{1 + Ts/2}{1 - Ts/2} \quad \Longleftrightarrow \quad s = \frac{2}{T} \frac{z-1}{z+1}$$

(this is a special case of a conformal mapping: a mapping analytic in the LHP with inverse analytic in the open unit disk)

Let's look at the image of the $j\omega$ axis (in the s plane)

s-plane

z-plane

If $\quad s = j\omega \quad \Rightarrow \quad z = \dfrac{1 + j\frac{\omega T}{2}}{1 - j\frac{\omega T}{2}} \quad \Rightarrow \quad |z| = \sqrt{\dfrac{1 + \left(\frac{\omega T}{2}\right)^2}{1 + \left(\frac{\omega T}{2}\right)^2}} = 1 \quad \Rightarrow \quad z = 1 \angle 2 \tan^{-1}\left(\frac{\omega T}{2}\right)$

$\angle z = 2 \cdot \tan^{-1}\left(\frac{\omega T}{2}\right)$

$\Rightarrow$ as s moves along the $j\omega$ axis from 0 to ∞, z moves on the unit circle from 0 to π

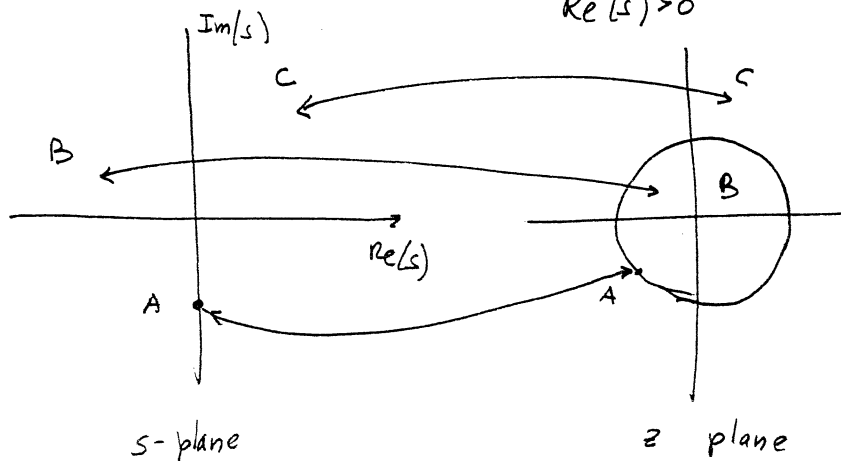(Note that the entire $j\omega$ axis gets "stuffed" into the 2π length of the unit circle)

If we have a generic point $\quad s = \sigma + j\omega \quad \Rightarrow \quad z = \dfrac{1 + \frac{\sigma T}{2} + j\frac{\omega T}{2}}{1 - \frac{\sigma T}{2} - j\frac{\omega T}{2}}$

$\Rightarrow \quad |z|^2 = \dfrac{\left(1 + \frac{\sigma T}{2}\right)^2 + \left(\frac{\omega T}{2}\right)^2}{\left(1 - \frac{\sigma T}{2}\right)^2 + \left(\frac{\omega T}{2}\right)^2} \quad \Rightarrow \quad \begin{array}{l} |z| < 1 \iff \sigma < 0 \\ > 1 \iff \sigma > 0 \end{array}$

In other words, the region $\mathrm{Re}(s) < 0 \quad \overset{\text{mapped to}}{\iff} \quad |z| < 1$

$\mathrm{Re}(s) > 0 \qquad\qquad\quad |z| > 1$

s-plane

z plane

s-domain		z-domain
$j\omega$ axis	$\iff$	unit circle
LHP	$\iff$	int of unit disk
RHP	$\iff$	exterior of unit disk.

- ## Physical motivation for Tustin's method
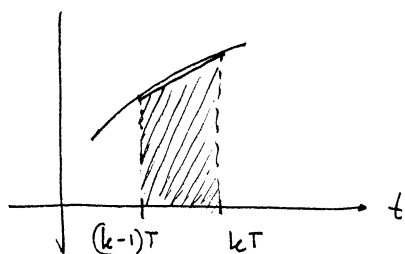
Suppose that we have a continuous time system and we decide to find a discrete time equivalent by numerical integration

$$E \longrightarrow \boxed{\dfrac{a}{s+a}} \longrightarrow C$$
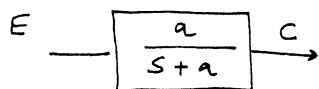
$$C(s) = \frac{a}{s+a} E(s) \quad /\!/ \quad \overset{\circ}{c} + a c(t) = a e(t)$$

$$C(kT) = \int_{(k-1)T}^{kT} \left[a\, e(\lambda) - a c(\lambda) \right] d\lambda + c\left[(k-1)T \right]$$

$$= c\left[(k-1)T \right] + \int_{0}^{T} f(\lambda)\, d\lambda$$

If we approximate the integral using the trapezoidal rule, we get:

$$\int_{(k-1)T}^{kT} \sim \left[\frac{f\left[(k-1)T\right] + f(kT)}{2} \right] T$$

$$c(kT) = c\left[(k-1)T\right] + \frac{aT}{2}\left[e\left[(k-1)T\right] + e(kT) - c\left[(k-1)T\right] - c(kT) \right]$$

$$\left(1 + \frac{aT}{2}\right) c(kT) - \left(1 - \frac{aT}{2}\right) c\left[(k-1)T\right] = \frac{aT}{2}\left[e(kT) + e\left[(k-1)T\right] \right]$$

Thus the corresponding __discrete__ transfer function is :

$$\left[\left(1 + \frac{aT}{2}\right) - \left(1 - \frac{aT}{2}\right) \frac{1}{z} \right] C(z) = \frac{aT}{2}\left[1 + \frac{1}{z} \right] E(z)$$

$$\frac{C(z)}{E(z)} = \left(\frac{z+1}{z}\right) \frac{z}{z\left(1 + \frac{aT}{2}\right) + \frac{aT}{2} - 1} = \frac{aT(z+1)}{(2 + aT)z + aT - 2}$$
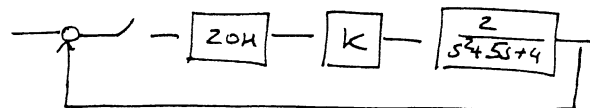
$$= \frac{a}{\frac{2}{T}\left(\frac{z-1}{z+1}\right) + a} \qquad \#$$

Compared with the original TF in the s-domain $\quad H(s) = \dfrac{a}{s+a}$

we see that the trapezoidal rule amounts to the substitution:

$$s \longleftarrow \frac{2}{T} \frac{z-1}{z+1} \qquad \#$$

- <u>Back to the laser example</u>:

Recall that for $T=0.1$ we found experimentally that for $k \approx 55$ it goes unstable. Let's see if we can show this analytically

$$G(z) = \mathfrak{Z}\left[\frac{1-e^{-\delta T}}{S}\frac{2}{s^2+5s+4}\right] = 2\left(\frac{z-1}{z}\right)\mathfrak{Z}\left[\frac{1}{s(s^2+5s+4)}\right]$$

$$= \frac{0.0085z + 0.0072}{z^2 - 1.5752z + 0.6065}$$

The <u>discrete time</u> characteristic equation is given by:

$$1 + K\,G(z) = 0$$

Using the bilinear transformation $z = \dfrac{1+\frac{Ts}{2}}{1-\frac{Ts}{2}} = \dfrac{1+0.05s}{1-0.05s}$

yields: $\quad G(s) = \dfrac{-0.0004s^2 - 0.0904s + 1.9721}{s^2 + 4.9467s + 3.9442}$

and the corresponding "continuous time equivalent" char equation is:

$$1 + K\,G(s) = 0 \quad \Longleftrightarrow \quad (1 - 4\cdot10^{-3}k)s^2 + (4.9467 - 9.04\cdot10^{-2}k)s + (3.9442 + 1.9721k) = 0$$

Routh Hurwitz array:

s^2	$1 - 4\cdot10^{-3}k$	$3.9442 + 1.9721k$
s^1	$4.9467 - 9.04\cdot10^{-2}k$	
s^0	$3.9442 + 1.9721k$	

$\Rightarrow$ stable iff:

$1 - 4\cdot10^{-3}k > 0 \quad \Rightarrow \quad k < 2.5\cdot10^{+3}$

$4.9467 - 9.04\cdot10^{-2}k > 0 \quad \Rightarrow \quad k < 54.7128$

$3.9442 + 1.9721k > 0 \quad \Rightarrow \quad k > -2$

$\boxed{-2 < k < 54.713}$

Suppose that we want to find out the point where the system becomes marginally stable and the frequency of oscillation. From the Routh Hurwitz array we have that the system is marginally stable for $k = 54.713$

The corresponding auxiliary equation is: $(1 - 4\cdot10^{-3}k)s^2 + 3.9442 + 1.9721k = 0$

$$\Rightarrow 0.9776s^2 + 111.844 = 0$$

$$\boxed{s = \pm j\,10.7}$$

We should expect an oscillation with frequency $\omega = 10.7$ rad/sec __provided__ that $\omega \ll \omega_S = \frac{2\pi}{T}$ (say $\omega \sim \frac{\omega_s}{10}$)

In our case $\omega_S = 62.83$ (so $\omega \sim \frac{\omega_s}{6}$ and the approx should be ok)

- Q: What happens if we increase the sampling interval?

  A: Intuitively the system should become __less__ stable   (due to increased time delay)

  Consider the same system as before, but let $T=1$, rather than $T=0.1$

$$G(z) = 2\left(\frac{z-1}{z}\right) 3\left[\frac{1}{s(s^2+5s+4)}\right] = \frac{0.2578z + 0.0525}{z^2 - 0.3862z + 0.0067}$$

$$\Downarrow \quad z = \frac{1 + 0.5s}{1 - 0.5s}$$

$$G(s) = \frac{-0.1474s^2 - 0.1507s + 0.8910}{s^2 + 2.8523s + 1.7820}$$

Char. eq: $(1 - 0.1474k)s^2 + (2.8523 - 0.1507k)s + 1.7820 + 0.8910k = 0$

stable iff:
$$1 - 0.1474k > 0 \implies k < 6.784$$
$$2.8523 - 0.1507k > 0 \implies k < 18.93$$
$$1.782 + 0.891k > 0 \implies k > -2$$

$$\boxed{-2 < k < 6.784}$$

Comparison :

$T = 0.1$	$T = 1$	cont. time
$-2 < k < 54.713$	$-2 < k < 6.784$	$-2 < k < \infty$

Note: recall that from the bilinear transf we have

$$s = j\omega_c \iff z = 1 \angle 2\tan^{-1}\left(\frac{\omega_c T}{2}\right)$$

A cont time frequency of oscillation ω_c corresponds to a discrete time oscillation with frequency $\omega_d = \frac{2}{T} \tan^{-1}\left(\frac{\omega_c T}{2}\right)$

$$\implies \omega_d \sim \omega_c \iff T\omega_c \ll 1$$

(in this case $\tan^{-1}\frac{\omega_c T}{2} \simeq \frac{\omega_c T}{2}$ and $\frac{2}{T}\tan^{-1}\left(\frac{\omega_c T}{2}\right) \sim \omega_c$)

$$\omega_c T \ll 1 \iff \omega_c \ll \omega_{sampling}$$