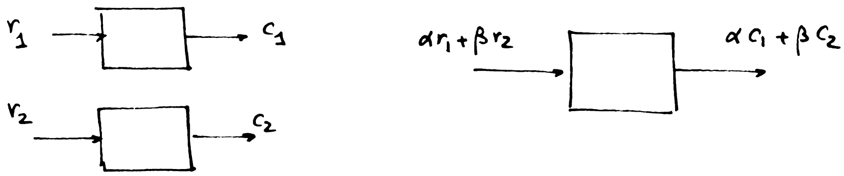
Discrete Time Systems and z- Transform

Reference: Chapter 2, textbook
Chapter 10, A. V. Oppenheim & A. S. Willsky, *Signals and Systems.*

Linear System: Satisfies the principle of superposition



Linear Time Invariant (or Linear Shift Invariant)

Input/Output relationship is independent of time

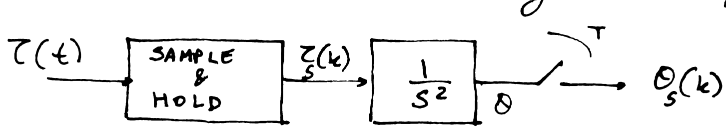
For our purposes: LTI discrete time system is described by a set of <u>difference</u> equations with constant coefficients.

$$x(k) + a_1 x(k-1) + \cdots a_{n-1} x(k-n+1) + a_n x(k-n) = b_0 e(k) + b_1 e(k-1) + \cdots b_m e(k-m)$$

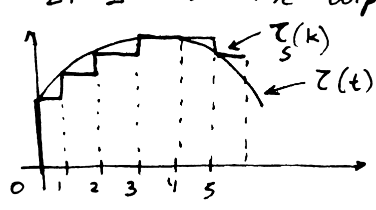
Assumption: (standing from now on) All the numbers arrive to the digital controller with the same fixed period T (the sampling period), and at the same time (single rate, synchronous sampling)

Later we will see how to remove this assumption $\Rightarrow$ nonsynchronous sampling, multirate sampling

Example: the satellite model (Assume the torque is held constant during the sampling period)



Assume $\Delta T = 1$ $\Rightarrow$ The output of the sample & hold looks like:



The differential equation relating $\theta(t)$ to $\tau_s(t)$ is

$$\ddot{\theta}(t) = \tau_s(t)$$

Integrating this equation twice in the period $kT \leq t \leq (k+1)T$ (remember that $\tau_s(t)$ is constant in between sampling instants) yields:

$$\dot{\theta}(t) = \dot{\theta}(kT) + \int_{kT}^{t} \tau_s(\lambda)\,d\lambda = \dot{\theta}(kT) + \tau(kT)(t - kT)$$

$$\theta(t) = \theta(kT) + \int_{kT}^{t} \dot{\theta}(\lambda)\,d\lambda = \theta(kT) + \dot{\theta}(kT)(t - kT) + \tau(kT)\frac{(t-kT)^2}{2}$$

Assume for simplicity that $\Delta T = 1$ and drop the T from the notation.

$$\Rightarrow \quad \theta_s(k+1) = \theta_s(k) + \dot{\theta}(k) + \frac{\tau(k)}{2} \qquad (1)$$

$$\dot{\theta}(k+1) = \theta(k) + \tau(k) \qquad (2)$$

If we want to find the transfer function from τ to θ we need to eliminate $\dot{\theta}$ from these two equations

Consider for instance (1) and its time-advanced version:

$$\theta_s(k+2) = \theta_s(k+1) + \dot{\theta}(k+1) + \frac{1}{2}\tau(k+1) \qquad (3)$$

$$-(1)+(3): \quad \theta_s(k+2) - \theta_s(k+1) = \theta_s(k+1) - \theta_s(k) + \underbrace{\dot{\theta}(k+1) - \dot{\theta}(k)}_{\tau(k) \ (\text{from } (2))} + \frac{1}{2}\left[\tau(k+1) - \tau(k)\right]$$

$$\Rightarrow \quad \boxed{\theta_s(k+2) - 2\theta_s(k+1) + \theta_s(k) = \frac{1}{2}\left[\tau(k+1) + \tau(k)\right]} \qquad (4)$$

We get a <u>second order difference equation</u> (not surprising since we have a double integrator in the loop)

Issues:

(1) We'd like to have a <u>more direct</u> way of deriving equation (4) starting from the block diagram

(2) We need to <u>solve</u> difference equations of the form (4). How? trial and error? generic solutions with adjustable parameters?

Turns out that we can address both issues using the <u>z-transform</u> pretty much the same way we used Laplace transforms in 5580

Z transform: (Reference: chapter 2 text, Chapter 10, Oppenheim & Willsky)

Given a sequence: $e_{-k} \cdots e_{-1}\ e_0\ e_1 \cdots e_k$

we define its bilateral z transform by:

$$Z(e_k) = \sum_{-\infty}^{+\infty} e_k z^{-k}$$

where z is a complex variable. In 5610 we are going to use only single-sided (or unilateral) z-transforms:

$$Z(e_k) = \sum_{0}^{\infty} e_k z^{-k} = e_0 + \frac{e_1}{z} + \frac{e_2}{z^2} + \cdots +$$

(this amounts to setting $e_k = 0$ for $k < 0$)

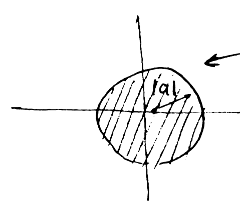
- Notation: $e_k \xleftrightarrow{Z} E(z)$

- Note that we have an infinite series. Thus $E(z)$ is likely to be defined (i.e. finite) only for some regions of the complex plane.

Example: $e_k = a^k$

$$E(z) = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots + = \begin{cases} \frac{1}{1-\left(\frac{a}{z}\right)} & \text{if } \left|\frac{a}{z}\right| < 1 \\ \infty & \text{otherwise} \end{cases}$$

In this example the z-transform is well defined only in the region $\left|\frac{a}{z}\right| < 1 \Leftrightarrow |z| > |a|$, i.e. the exterior of a disk with radius $|a|$

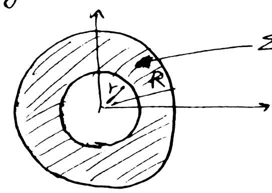


z transform defined here

- **Definition:** Region of Convergence: (ROC) The region in the z-plane where the ∞ sum $\sum_{-\infty}^{+\infty} e_k z^{-k}$ converges

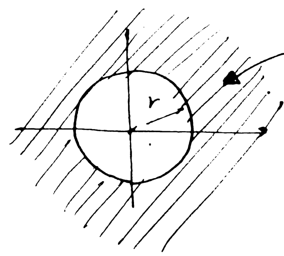
Facts

1) The ROC for general double-sided z-transforms is a ring centered about the origin: $|r| < |z| < |R|$



$\Sigma e_k z^{-k}$ well defined here

2) The ROC for unilateral z-transforms (the only ones we'll use in 429) is the exterior of a disk (i.e. $|z| > r$)



Z-transform well defined here

Q: Why do we care about ROC?

A: (a) we will need it in order to compute inverse z transforms

(b) It is related to the location of the poles of the T.F and to the concepts of stability and causality.

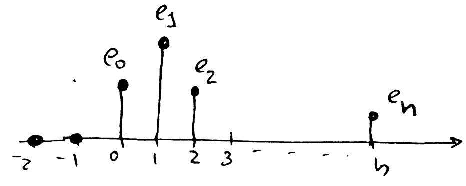
- **Properties of the Z-transform:**

1) Linearity: $Z\left[\alpha e_1(k) + \beta e_2(k)\right] = \alpha Z(e_1) + \beta Z(e_2)$

Proof: follows immediately from the definition

2) Time shift

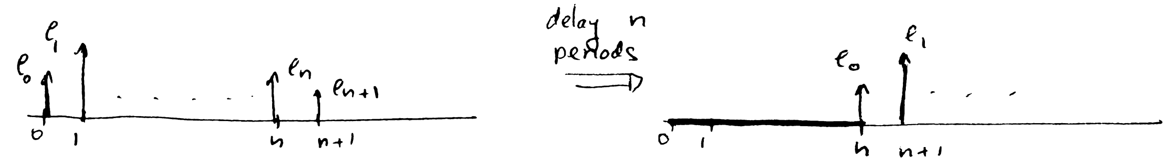
Assume that $e_k = 0$ for $k < 0$



We will consider 2 cases: (a) Time delay

(b) Time advance

(a) <u>Time delay:</u> (similar to integration in continuous time)



$\Downarrow Z$ $\qquad\qquad\qquad\qquad$ $\Downarrow Z$

$E(z)$

$$Z\left[e(k-n)\right] = \sum_0^{\infty} e(k-n) z^{-k} = \underbrace{e_{-n} + e_{-n+1} z^{-1} + \cdots e_{-1} z^{-(n+1)}}_{0} + e_0 z^{-n} + e_1 z^{-(n+1)} + \cdots +$$

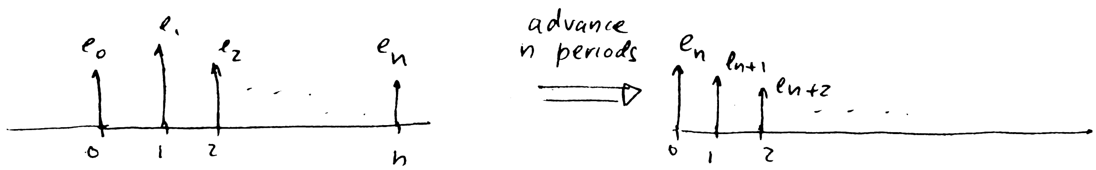
(all $e_i = 0,\ i < 0$)

$$= z^{-n}\Big[\underbrace{e_0 + e_1 z^{-1} + \cdots}_{E(z)}\Big] = z^{-n} E(z)$$

$$\Rightarrow \boxed{Z\left[e(k-n)\right] = \frac{1}{z^n} Z\left[e_k\right)}$$

Time delay of n periods $\Rightarrow$ multiply $E(z)$ times $\frac{1}{z^n}$

(b) <u>Time advance:</u> (similar to differentiation in continuos time)



Advance 1 period: $Z\left[e(k+1)\right] = e_1 + e_2 z^{-1} + \cdots = z\left(e_1 z^{-1} + e_2 z^{-2} + \cdots\right)$

$$= z\left[-e_0 + e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots\right] = \boxed{z\left[E(z) - e_0\right]}$$

Similarly:
$$\boxed{Z\left[e(k+n)\right] = z^n\left[E(z) - \sum_0^{n-1} e_k z^{-k}\right]}$$

· Note that (contrary to the delay case) here $Z\left[e(k+n)\right]$ cannot be expressed solely in terms of $E(z)$. Why the difference? Because now some of the values of $\{e_k\}$ have been last and we need to account for these.

<u>Example:</u> $Z(a^k) = \frac{z}{z-a}$. $\qquad Z(a^{k+1}) = Z(a \cdot a^k) \overset{\text{linearity}}{=} z\frac{a}{z-a} = z\left[\frac{z}{z-a} - 1\right]$ #

$$Z\left[a^{(k-1)} u(k-1)\right] = 0 + \frac{1}{z} + \frac{a}{z^2} + \cdots = \frac{1}{z-a}$$ #

- **Scaling in z-plane:**

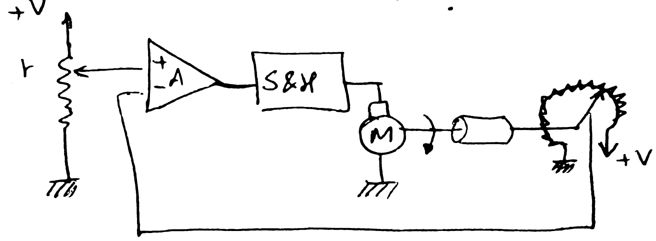
$$Z\{(r^{-k}e_k)\} = E(rz)$$

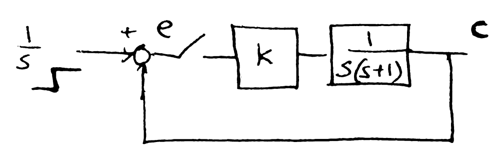
Proof: $Z\{(r^{-k}e_k)\} = \sum_0^\infty (e_r r^{-k}) z^{-k} = \sum_0^\infty e_k (rz)^{-k} = E(rz)$ #

- **Two important Theorems:** (Initial and Final value theorems)

They allow for the calculation of limits and steady state values without having to actually find the transform

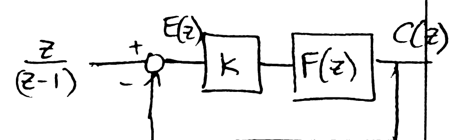
For instance, suppose that we want to analyze the steady-state error to a step input for a position control system:





We could proceed as follows

(1) find the discrete time equivalent:



(2) find $C(z)$
(3) Take the inverse z-transform $\Rightarrow \{c_k\}$
(4) find $c_{ss} = \lim_{k\to\infty} \{c_k\}$

However, this is an overkill if we are only interested in c_{ss} (and not the transient)

- **Final value theorem:**

$$\boxed{\lim_{k\to\infty} e(k) = \lim_{z\to 1} (z-1)E(z)}$$

(Physical significance: the behavior of $e(k)$ as $k\to\infty$ is related to the low frequency ($s=0 \leftrightarrow z=1$) components of $E(z)$)

Proof: Consider the sequence $\{e(k+1)-e(k)\}$. We can compute its z transform 2 different ways (a) directly
(b) using the time shift theorem

(a) yields:

$$Z\{e_{k+1}-e_k\} = \lim_{n\to\infty}\sum_{0}^{n}\{e_{k+1}-e_k\}z^{-k} = \lim_{n\to\infty}\left\{e_1-e_0+(e_2-e_1)\frac{1}{z}+\cdots(e_{n+1}-e_n)\frac{1}{z^n}\right\}$$

$$= \lim_{n\to\infty}\left\{-e_0+e_1\left(1-\frac{1}{z}\right)+\frac{e_2}{z}\left(1-\frac{1}{z}\right)+\cdots\frac{e_{n+1}}{z^n}\right\}$$

$$\Rightarrow \lim_{z\to1} Z\{e_{k+1}-e_k\} = \lim_{n\to\infty}\{-e(0)+e(n+1)\} = -e(0)+\lim_{n\to\infty}e(n) \qquad (1)$$

(b) yields:

$$Z\{e_{k+1}\} = z[E(z)-e_0] \Rightarrow Z\{e_{k+1}-e_k\} = zE(z)-ze_0-E(z)$$

$$= (z-1)E(z)-ze_0$$

$$\Rightarrow \lim_{z\to1} Z\{E(z)-e_0\} = -e(0) \qquad (2)$$

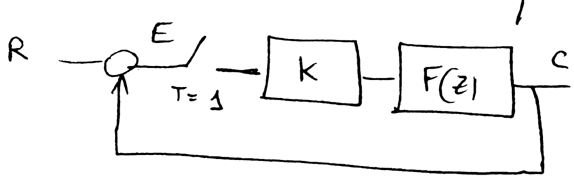
Comparing (1) and (2) we have

$$\lim_{n\to\infty}e(n)-\cancel{e(0)} = \lim_{z\to1}(z-1)E(z)-\cancel{z e_0} \Rightarrow \boxed{\lim_{n\to\infty}e(n)=\lim_{z\to1}(z-1)E(z)}$$

Examples (1) unit step: $e_k=1$ for all k

$$E(z)=\frac{1}{1-1/z}=\frac{z}{z-1} \qquad \lim_{z\to1}(z-1)E(z)=\lim_{z\to1}\frac{z\cancel{(z-1)}}{\cancel{(z-1)}}=1=e_\infty \ \#$$

(2) Back to the position control system. It can be shown (we will see it later in the semester) that if $T=1$ then the discrete time equivalent is



with $F(z)=\dfrac{0.368z+0.264}{z^2-1.368z+0.368}$

$R=\dfrac{z}{z-1}$

The overall transfer function: $G(z)=\dfrac{C}{R}=\dfrac{kF}{1+kF}$, $H(z)=\dfrac{E}{R}=\dfrac{1}{1+kF}$

Based on our experience from 5580 we expect $E_{ss} = \lim_{t\to\infty} e(t) = 0$ (since we have integral control)

Let's try the final value theorem:

$$E_{ss} = \lim_{z\to 1} (z-1)\,E(z) = \lim_{z\to 1} (z-1) \frac{1}{1+kF(z)} \cdot R(z) = \lim_{z\to 1} \cancel{(z-1)} \frac{1}{1+kF(z)} \frac{z}{\cancel{(z-1)}}$$

$$= \frac{1}{1+k\lim_{z\to 1} F(z)} = 0 \quad \#$$

Since $F(z) = (z-1)(z-e^{-1})$

$\Rightarrow F(1) = \infty$

According to this, we should get zero steady state error regardless of the value of k!

Let's try a few simulations:

$k=1$ fine

$k=2$ fine

$k=2.5$ oopps!!

For $k=2.5$ we get $e_k = \infty$ rather than zero! FVT fails

(indeed you can show that this is the case for all $k > 2.39$)

What happened here? The final value theorem is NOT applicable

Warning: The formula (and the proof) are correct only if $\lim_{k\to\infty} e_k$ exists. We may have situations like the one above where $\lim_{z\to 1}(z-1)\,E(z)$ exists even though $\lim_{k\to\infty} e(k)$ does not. Obviously the FVT is not applicable then. So the theorem should be restated as follows:

$$\boxed{\lim_{n\to\infty} e(n) = \lim_{z\to 1} (z-1)\,E(z) \quad \text{provided that both limits exist}}$$

equivalently, $E(z)\cdot(z-1)$ must have all its poles inside the unit circle so that the series converges in $|z| \geq 1$

- <u>Initial value theorem</u>:

$$\boxed{\lim_{k \to 0} e(k) = \lim_{z \to \infty} E(z)}$$

<u>Proof</u>: follows immediately from the definition:

$$E(z) = e_0 + \frac{e_1}{z} + \frac{e_2}{z^2} + \cdots \frac{e_n}{z^n} + \cdots$$

when $z \to \infty$ all the terms in the RHS except the first drop out

- <u>Convolution of time sequences</u>:

Given two time sequences $e_1(k)$ and $e_2(k)$ its convolution $f = e_1 * e_2$ is defined as:

$$f_k = \sum_{l=-\infty}^{l=+\infty} e_1(l)\, e_2(k-l)$$

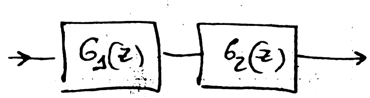
If $e_1(k)$ and $e_2(k) = 0$ for $k<0$ (as in this course) then the formula above reduces to:

$$f_k = \sum_{l=0}^{k} e_1(l) \cdot e_2(k-l)$$

$$\boxed{\mathcal{Z}\{e_1 * e_2\} = E_1(z) \cdot E_2(z)}$$

convolution in time domain $\Longleftrightarrow$ product in z-domain

Very useful result: allows for analyzing the combination of dynamic systems by just using linear algebra



<u>Proof</u>:

$$f_k z^{-k} = \sum_{l=0}^{k} e_1(l)\, e_2(k-l)\, z^{-k} = \sum_{l=0}^{k} e_1(l) z^{-l} \cdot e_2(k-l) z^{-(k-l)}$$

$$= \sum_{l=0}^{\infty} e_1(l) z^{-l} \cdot e_2(k-l) z^{-(k-l)}$$

(where we exploited the fact that $e_2(n) = 0$ for $n<0$) $\Rightarrow$

$$F(z) = \sum_{k=0}^{\infty} f_k z^{-k} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} e_1(l) z^{-l}\, e_2(k-l)\, z^{-(k-l)}$$

Now let $k-\ell=m \Rightarrow F(z)=\sum_{k=0}^{\infty} e_1(\ell)\, z^{-\ell} \sum_{m=0}^{\infty} e_2(m) z^{-m} = E_1(z)\cdot E_2(z)$ #

- <u>Solution of difference equations</u>

Recall that one of the motivations for looking into the z-transform was the expectation that it will help us solve difference equations (as the Laplace transform helped with differential equations). Next we will see how to accomplish this

<u>Example:</u> consider the difference equation:

$$m(k) = e(k) - e(k-1) - m(k-1) \qquad k \geq 0$$

Assume initial conditions $m(-1)=0$, $e(-1)=0$ and take z-transforms on both sides:

$$\begin{array}{ccccccc} m(k) & = & e(k) & - & e(k-1) & - & m(k-1) \\ z\downarrow & & z\downarrow & & \downarrow z & & \downarrow z \\ M(z) & = & E(z) & - & \frac{1}{z}E(z) & - & \frac{1}{z}M(z) \end{array}$$

$$\Rightarrow \left(1+\frac{1}{z}\right)M(z) = \left(1-\frac{1}{z}\right)E(z)$$

solving for $M(z)$ yields: $\boxed{M(z) = \left(\frac{z-1}{z+1}\right)E(z)}$

Assume that $e_k = \begin{cases} 0 & k<0 \\ 1 & k \geq 0 \end{cases} \Rightarrow E(z) = \frac{z}{z-1}$

$$\Rightarrow M(z) = \frac{z}{z+1} = \frac{1}{1+1/z} \Rightarrow \boxed{m_k = (-1)^k \quad \#}$$

Sanity check:

$$m(0) = e(0) - e(-1) - m(-1) = 1$$
$$m(1) = e(1) - e(0) - m(0) = -1$$
$$\vdots$$
$$m(k) = \cancel{e(k)} - \cancel{e(k-1)} = -m(k-1)$$

Features: (1) We solve the difference equation by solving a single algebraic equation

(2) method incorporates initial conditions

<u>Caveat:</u> we need to find an inverse z-transform

In general we have:

$$x(k+n) + a_1 x(k+n-1) + \cdots + a_n x(k) = b_0 e(k+n) + \cdots + b_m e(k+n-m)$$

Taking z transforms on both sides yields:

(Recall that $x(k+n) \xleftrightarrow{z} z^n \left[X(z) - \sum_{0}^{n-1} x(k) z^{-k} \right]$)

$$\left(z^n + a_1 z^{n-1} + \cdots + a_n\right) X(z) + (IC) = \left(b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}\right) E(z) + (IC)$$

where $IC = \begin{Bmatrix} x(0), & x(1) & \cdots & , x(n-1) \\ e(0), & e(1) & \cdots & e(n-1) \end{Bmatrix}$

If $IC = 0$ then $X(z) = \left[\dfrac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} \right] E(z)$

- <u>Concept of Transfer Function:</u>

Transfer function: Ratio of the z-transform of the output to the z-transform of the input when <u>all</u> initial conditions are set to zero

From linearity it follows that we can represent the I/O relationship for an LTI system as:

$X(z) = G(z) E(z)$ $\qquad E(z) \longrightarrow \boxed{G(z)} \longrightarrow X(z)$

<u>Fact</u>: For finite dimensional systems $G(z)$ is rational, i.e.

$G(z) = \dfrac{N(z)}{D(z)}$ where N and D are polynomials

Moreover, if the system is causal then $G(z)$ is proper, i.e. degree $(N) \leq$ Degree (D)

<u>Notation:</u> Roots of $N(z) = 0$: zeros of the system (related to performance)

Roots of $D(z) = 0$: poles of the system (related to stability)

- Suppose that we compute $X(z) = H(z) E(z)$. Now we need to go back to the time domain $\Rightarrow$ need to figure out a way of computing <u>inverse z-transforms</u>.