# Appendix A: Mathematical DQQ model

This section introduces the proposed Double Quadratic Queue (DQQ) model. By adopting the fluid queue derivation process, the model considers the two distinct stages and differentiates itself from a typical single-stage fluid queue model. The fluid queue modeling approach is a method used to analyze and model the behavior of queues in a dynamic setting, taking into account varying arrival and discharge rates. Newell's approach combines Taylor expansion, calculus, and geometric representation to derive an algebraic expression for the queue length at any given time. The DQQ model incorporates both geometric representation and algebraic expression. This fluid queue process helps visualize the relationships between arrival and discharge rates, queue length, and the passage of time. The ultimate goal of the subsequent derivation process is to obtain an algebraic expression for the queue length or cumulative change, which can be employed to study the queue's or system's behavior analytically.

**Table 1** summarizes the notations used in the DQQ model.

**Table 1** Notations and definitions used in this paper

|  |  |
| --- | --- |
| **Notations** | **Definitions** |
|  | net flow rate at time |
|  | ridership rate at time , |
|  | start time of disruption, |
|  | time instance with maximum negative net flow rate |
|  | time instance with maximum cumulative ridership change |
|  | time instance with maximum positive net flow rate |
|  | end time of recovery process, where reaching a new equilibrium condition |
|  | disruption duration, |
|  | recovery duration, |
|  | entire duration of two normal/equilibriums, |
|  | Relative disruption duration ratio, |
|  | recovery to disruption (ROD) ratio, , which is the proportion of recovery in relation to the disruption, indicating how well the system recovers from the disruption |
|  | maximum ridership changes at time |
|  | permanent loss at end time |
|  | parameter in the single quadratic-form net flow rate function |
|  | parameter in the cubic-form net flow rate function |
|  | parameters in the double quadratic-form net flow rate function |

To maintain generality, we normalize both time and ridership together such that their cumulative changes from the baseline condition at the beginning of the disruption are set to 0, i.e., . By doing so, the entire duration of the disruption and recovery can be consistently represented as:

|  |  |
| --- | --- |
|  | (1) |
|  | (2) |
|  | (3) |

Typically, parameters , , , , , are observable directly from time series datasets. In the DQQ model, we assume the net flow rate, , can be represented as double quadratic forms, as shown in **Figure 1**, which has three intersection points, , , and , with the horizontal axis.

Diagram

Description automatically generated

**Figure 1** Graphical illustration of DQQ model for a single wave in transit cumulative ridership changes

The first-stage net flow rate, , could be approximated by Taylor’s Theorem with at :

|  |  |
| --- | --- |
|  | (4) |

Considering the parabola opens downwards, we have:

|  |  |
| --- | --- |
|  | (5) |

At the first stage of DQQ model, .

|  |  |
| --- | --- |
|  | (6) |
|  | (7) |

We have

|  |  |
| --- | --- |
|  | (8) |
|  | (9) |

By substituting **Eq. (8)** and **Eq. (9)** to **Eq. (4)**, the factored form of can be represented by:

|  |  |
| --- | --- |
|  | (10) |

Similarly, considering Taylor’s Theorem with at , we let , then have .

Thus, the DQQ model can be represented as **Eq. (4)**. This paper aims to calibrate the unknown second-order gradient parameters, and .

|  |  |
| --- | --- |
|  | (4) |

As depicted in **Figure 1(a)**, the disruption occurs, and ridership starts to decline at time . At time , the flow loss rate reaches its peak. Consequently, the transit system experiences the greatest passenger loss, , at time , as illustrated in **Figure 2(b)**. Following , transit ridership begins to recover. At , the net flow recovery rate attains its maximum value. Eventually, the transit system reaches a new normal/equilibrium at , where the represents the gap between this new equilibrium and the initial equilibrium status. The duration of disruption and recovery processes determines the horizontal temporal interval and the vertical steepness of the adaptivity evolution curve, i.e. the relative location of roots in the DQQ model. Generally, the durations and the permanent loss can be used to assess the system’s recovery capability.

As shown in **Figure 1**, the boundary conditions can be represented as Eq. (7)-(14).

|  |  |
| --- | --- |
|  | (5) |
|  | (6) |
|  | (7) |
|  | (8) |
|  | (9) |

**(1) When** , the disruption process is represented by .

The first order derivative of net flow rate in disruption process, , is obtained:

|  |  |
| --- | --- |
|  | (10) |

Substituting Eq. (4) and leads to the following symmetrical relationships between and :

|  |  |
| --- | --- |
|  | (11) |

Virtual queue length or cumulative flow change at time t, can be obtained:

|  |  |
| --- | --- |
|  | (12) |

By substituting function in Eq. (4), can be represented in terms of and :

|  |  |
| --- | --- |
|  | (13) |

As stated above, the maximum queue length is achieved at time *t*2:

|  |  |
| --- | --- |
|  | (14) |

The Eq. (14) dedicates that can be represented by the time instance and the curvature parameter . Since and can be observed from real-life data, the curvature rate can be calibrated.

**(2) When** , the recovery process is represented by .

The first order derivative of is derived as:

|  |  |
| --- | --- |
|  | (15) |

Bysubstituting Eq. (6) and , and are symmetric about :

|  |  |
| --- | --- |
|  | (16) |

The queue length during the recovery process is expressed as:

|  |  |
| --- | --- |
|  | (17) |

Substituting Eq. (14) and function in Eq. (4) leads to expression:

|  |  |
| --- | --- |
|  | (18) |

The final permanent loss is determined at time , i.e. :

|  |  |
| --- | --- |
|  | (19) |

Since can be observed from real-life data, and the can be obtained using Eq. (14), the can be calibrated.

Especially, when the ridership is fully recovered at the end of the adaptative evolution, i.e. . Substituting Eq. (19) leads to the internal relationship between the and in terms of parameter or .

|  |  |
| --- | --- |
|  | (20) |

**3.2 Comparison among** **single-stage quadratic model, cubic model, and DQQ model**

It is intriguing to compare the proposed DQQ model with the single-stage quadratic queue model and the cubic queue model, as all three models fall within the overarching modeling framework of fluid queues. As demonstrated in Figure 2, these three models share similar notations for various critical time instances during the recovery-to-disruption process. Table 2 presents a comparison of the boundary conditions and derivation results across these three distinct formulations. The single-stage quadratic queue model is incapable of describing the new equilibrium point with the net flow rates at , and the single-stage cubic queue model can only depict the fast recovery process with , accompanied by possible permanent loss. In contrast, the DQQ model can characterize a severe, prolonged disruption with a feasible range of between 0 and 1 by permitting a first-order gradient jump at . Meanwhile, the single-stage models are continuous and smooth throughout their entire domain.

A picture containing diagram

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(1) (2)

**Figure 2** Graphical illustration of net flow rate functions in (1) singe quadratic form and (2) cubic form

Diagram

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(1) (2)

**Figure 2** Graphical illustration of net flow rate functions in (1) singe quadratic form and (2) cubic form with different values of

**Table 2** Comparison in different forms of net flow rate

|  |  |  |  |
| --- | --- | --- | --- |
| **Net flow rate functions** | **Boundary condition** | **Derivation results** | **Remark** |
| Single quadratic forms | ,  ,  , | , | (1) , full recovery, i.e. ;  (2) , disruption process only. |
| Cubic forms | ,  ,  ,  , | ,  , | (1) , full recovery, i.e. ;  (2) , disruption process only. |
| Double quadratic forms | , | (1) , similar to Newell’s PAQ model;  (2) , if , full recovery, i.e. ;  (3) , disruption process only. |