Reading Note of Majorana Fermion Positivity

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May 27, 2016

1 Introduction

In this note I will talk about the proof of **reflection positivity of Majoranas**, and the conditions of the theorem. Finally I will talk about some examples, including Ising model and Heisenberg Model.

The theorem we want to prove is:

$$\mathrm{Tr}\big[A\vartheta(A)e^{-H}\big] \geq 0$$

if the Hamiltonian H satisfy some conditions. Operator ϑ is anti-linear. In the proof we can find out the condition. We will talk about these in detail in the following sections.

2 Proof

The proof of the theorem can be divided into the following parts:

(1) Define the (Hilbert) space and algebra we are consider about.

(2) Study the monomial basis on our algebra.

(3) Study the Hamiltonian of the problem.

(4) Prove the reflection positivity of an inner-product.

(5) Use the result of (4) to get the main result.

Now the previous steps will be shown in detail in the following sections.

2.1 Definition of the space and algebra

First we have to deal with the concept "Majorana". As an example, we can consider about a lattice, and the creation-annihilation operators can be decomposed into to parts, say two Majoranas:

$$c_{2j-1} = a_j + a_j^\dagger, \quad c_{2j} = i(a_j - a_j^\dagger)$$

Thus it is easy to find the Majoranas obey the Clifford Algebra:

$$\{c_i, c_j\} = 2\delta_{ij} \tag{1}$$

Now we consider the geometrical significants of these operators as a lattice Λ, and define a reflection operator ϑ, by which the lattice is divided into two sectors:

$$\Lambda = \Lambda_+ \cup \Lambda_- \,, \Lambda_+ \cap \Lambda_- = \varnothing \,, \vartheta(\Lambda_+) = \vartheta(\Lambda_-)$$

Now we need the operator ϑ is anti-linear which means that

$$\theta(f+\lambda g)=\vartheta(f)+\bar{\lambda}\vartheta(g)\,,\lambda\in\mathbb{C}$$

in which f and g are vectors in the Hilbert space. In other words, the operator will take the complex conjugate of the target vector.

Now we will introduce some of the space and algebra. $\mathfrak{A}(\mathcal{B})$ denotes the algebra defined on the set $\mathcal{B}\subset\Lambda$; $\mathfrak{A}_\pm=\mathfrak{A}(\Lambda_\pm)$ is the algebra on $\Lambda_\pm$; $\mathfrak{A}_\pm^{\text{even}}$ is the algebra with even power on Λ. In conclusion, the definition is

$$\forall A\in\mathfrak{A}_-, A=\sum a_\beta M_\beta, M_\beta \text{ is the product of an arbitrary number of Majoranas } c_j\in\Lambda_-$$
$$\forall A\in\mathfrak{A}_-^{\text{even}}, A=\sum a_\beta M_\beta, M_\beta \text{ is the product of an even number of Majoranas } c_j\in\Lambda_-$$

That is all we need to define before we talk about the problem.

2.2 Hamiltonian

We hope the Hamiltonian can be written as the following form:

$$H=H_-+H_0+H_+ \tag{2}$$

and we hope $H_-=\vartheta(H_+)$. The H_0 term will have the following form:

$$H_0=\sum_{\mathfrak{J}}J_{\mathfrak{J}}i^{\sigma(\mathfrak{J})}C_{\mathfrak{J}}\vartheta(C_{\mathfrak{J}}) \tag{3}$$

each $\mathfrak{J}$ denotes a distinct product of many Majoranas $C_{\mathfrak{J}}$, and $\sigma(\mathfrak{J})$ is defined as:

$$\sigma(\mathfrak{J})=n(\mathfrak{J}) \bmod 2$$

in which $n(\mathfrak{J})$ is the number of Majoranas in the product $C_{\mathfrak{J}}$. At the end of this section we will see the coefficients $J_{\mathfrak{J}}$ satisfy the following condition if we want reflection positivity is correct:

$$\begin{cases}\text{all}J_{\mathfrak{J}}\geq 0 & \sigma(\mathfrak{J})=1\\ \text{or all}J_{\mathfrak{J}}\leq 0 & \\ \text{all}J_{\mathfrak{J}}\leq 0 & \sigma(\mathfrak{J})=0\end{cases}$$

The reason why these inequalities should be satisfied will be shown in section 2.4.

2.3 Reflection positivity of the inner-product

Consider about an operator $A\in\mathfrak{A}(\Lambda)$, it can be written as a linear combination of monomial basis:

$$A=\sum_\beta a_\beta M_\beta \tag{4}$$

in which M_β is a monomial. So easily we can find there are 2^N different monomials in $\mathfrak{A}(\Lambda)$.

2.4 The reflection positivity of Majoranas

2.5 Reflection Bounds (Generalized Cauchy-Schwartz inequility)

3 Ising Model and Heisenberg Model as examples

References

[1] Jaffe, A., & Pedrocchi, F. L. (2014). Reflection Positivity for Majoranas. Annales Henri Poincar, 16(1), 189203. http://doi.org/10.1007/s00023-014-0311-y