BCS Superconductor Theory

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March 6, 2016

1 Introduction

The superconductor was discovered by Onnes in 1911, and he won the Nobel Prize in Physics in 1913.

2 Electron-Phonon Interaction Action

First we need to consider about how to quantize the oscillation of the lattice. The Hamiltonian of the phonon system can be written as:

$$H = \sum_{\mathbf{q}\lambda} \omega_{\mathbf{q}\lambda} \left(a^\dagger_{\mathbf{q}\lambda} a_{\mathbf{q}\lambda} + \frac{1}{2} \right)$$

in which λ denotes the polarization of the phonon.So the partition function of the free phonon system can be constructed by the coherent state path integral. The action is written as shown:

$$S_{\mathrm{ph}}[\bar{\phi}, \phi] = \sum_{q,\lambda} \bar{\phi}_{q\lambda} \left(-i\omega_m + \omega_{\mathbf{q}} \right) \phi_{q\lambda} \tag{1}$$

In this action, $q = (\omega_m, \mathbf{q})$ is a 4-momentum and ω_m is the Matsubara frequency for Bosons. But how does the displacement of the ion on the lattice be quantized? Just like what we do in QED, the creation-annihilation operators are the Fourier coefficients:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{q},\lambda} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{1}{\sqrt{2M\omega_{\mathbf{q}}}} \mathbf{e}_{\mathbf{q}\lambda} (a_{\mathbf{q}\lambda} + a^\dagger_{-\mathbf{q}\lambda}) \tag{2}$$

Then we have to consider how the ions interact with the electrons. Since the ions are moving, there will be electric moment in the solid and obviously $\mathbf{P} \propto \mathbf{u}$, and the charge density will be the divergence of the electric moment, say

$$\rho_{\mathrm{ion}} \propto -\nabla \cdot \mathbf{u}$$

Thus the interaction Hamiltonian can be written as

$$H_{\mathrm{el-ph}} = \gamma \int d^3x \, \rho_{\mathrm{el}}(\mathbf{x}) \nabla \cdot \mathbf{u}$$

in which γ is some positive constant with the dimension of energy. Now we change this integral into momentum space and it can be written as shown:

$$H_{\mathrm{el-ph}} = \gamma \int d^3x \, \frac{1}{L^3} \sum_{\mathbf{q}'} e^{-i\mathbf{q}'\cdot\mathbf{x}} \rho_{\mathbf{q}'} \sum_{\mathbf{q}\lambda} \frac{i\mathbf{q}\cdot\mathbf{e}_{\mathbf{q}\lambda}}{\sqrt{2M\omega_{\mathbf{q}}}} (a_{\mathbf{q}\lambda} + a^\dagger_{-\mathbf{q}\lambda}) = \gamma \sum_{\mathbf{q}\lambda} \frac{i\mathbf{q}\cdot\mathbf{e}_{\mathbf{q}\lambda}}{\sqrt{2M\omega_{\mathbf{q}}}} \rho_{\mathbf{q}} (a_{\mathbf{q}\lambda} + a^\dagger_{-\mathbf{q}\lambda}) \tag{3}$$

It is obvious that only the longitudinal component of the phonon will not vanish because $\mathbf{q}\cdot\mathbf{e}_{\mathbf{q}\lambda} = |\mathbf{q}|\delta_{\lambda,\mathrm{L}}$, we can neglect the polarization index and use the creation-annihilation operators of the longitudinal modes. Then the Hamiltonian is

$$H_{\mathrm{el-ph}} = \gamma \sum_{\mathbf{q}} \frac{i|\mathbf{q}|}{\sqrt{2M\omega_{\mathbf{q}}}} \rho_{\mathbf{q}} (a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}}) \tag{4}$$

in which the density operator is

$$\rho_{\mathbf{q}} = \sum_{\mathbf{q}\sigma} c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma}$$

Now we try to write down the Action of the interacting term. Use the Matsubara Frequencies, the Action can be written as:

$$S_{\mathrm{el-ph}}[\bar{\phi},\phi,\bar{\psi},\psi] = \gamma \sum_q \frac{i|\mathbf{q}|}{\sqrt{2M\omega_{\mathbf{q}}}} \left(\sum_k \bar{\psi}_{k+q\sigma}\psi_{k\sigma} \right) (\phi_q + \bar{\phi}_{-q}) \tag{5}$$

in which $q = (\omega_m, \mathbf{q})$ is a 4-momentum and ω_m is a Bosonic Matsubara Frequency; $k = (\omega_n, \mathbf{q})$ is also a 4-momentum and ω_n is a Fermionic Matsubara Frequency. Now we can write down the Action of the whole system in Matsubara representation:

$$S[\bar{\psi},\psi,\bar{\phi},\phi] = \sum_p \bar{\psi}_{p\sigma} \left(-i\omega_n + \frac{\mathbf{p}^2}{2m} - \mu \right) \psi_{p\sigma} + \sum_q \bar{\phi}_q(-i\omega_m + \omega_{\mathbf{q}})\phi_q + \gamma \sum_q \frac{i|\mathbf{q}|}{\sqrt{2M\omega_{\mathbf{q}}}} \left(\sum_k \bar{\psi}_{k+q\sigma}\psi_{k\sigma} \right) (\phi_q + \bar{\phi}_{-q}) \tag{6}$$

Next step is to integrate out the field of phonon and we can get the effective field theory of electrons:

$$S_{\mathrm{eff}}[\bar{\psi},\psi] = S_{\mathrm{el}}[\bar{\psi},\psi] - \ln \left(\int D\bar{\phi} D\phi \, e^{-S_{\mathrm{ph}}[\bar{\phi},\phi] - S_{\mathrm{el-ph}}[\bar{\phi},\phi,\bar{\psi},\psi]} \right) \tag{7}$$

Since the free Action of the phonon system is a quadratic form, and the interaction Action can be treated as a "source" term, we can get the result of this effective action by a Gaussian Integral. The result is:

$$\begin{aligned}
& \ln \left(\int D\bar{\phi} D\phi \, e^{-S_{\mathrm{ph}}[\bar{\phi},\phi] - S_{\mathrm{el-ph}}[\bar{\phi},\phi,\bar{\psi},\psi]} \right) \\
= \;& \ln \left(\int D\bar{\phi} D\phi \, \exp \left\{ - \sum_q \bar{\phi}_q(-i\omega_m + \omega_{\mathbf{q}})\phi_q - \gamma \sum_q \frac{i|\mathbf{q}|}{\sqrt{2M\omega_{\mathbf{q}}}} \rho_q \phi_q - (\rho_q\phi_q \leftrightarrow \rho_{-q}\bar{\phi}_q) \right\} \right) \\
= \;& -\gamma^2 \sum_q \frac{\mathbf{q}^2}{2M\omega_{\mathbf{q}}} \frac{1}{-i\omega_m + \omega_{\mathbf{q}}} \rho_q \rho_{-q} \\
= \;& -\frac{\gamma^2}{2M} \sum_q \frac{\mathbf{q}^2}{\omega_m^2 + \omega_{\mathbf{q}}^2} \rho_q \rho_{-q} \\
= \;&
\end{aligned} \tag{8}$$

3 BCS Hamiltonian

4 Cooper Pairs

5 Spontaneous Symmetry Breaking

6 Couple to the electromagnetic field

7 Anderson-Higgs Mechanism