

Note: Quantum Physics in One Dimension

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1 Basic Concepts for interacting quantum systems

1.1 Weak Interacting particles

1.1.1 Fermi liquid and free fermions

Anti-symmetric wave function: $c_k^\dagger(c_k)$ the (de)creation operator be anti commute with each other. Hamiltonian

$$H = \sum_k \epsilon_k c_k^\dagger c_k$$

and the Fermi energy in condensed matter system is about $\epsilon_F \sim 1\text{eV} \sim 12000\text{K}$. For cold atom systems, $\epsilon_F \sim 100\text{nK}$. Density of states:

$$D(E) = \sum_k \delta(E - \epsilon_k)$$

Average:

$$\langle \dots \rangle = Z^{-1} \text{Tr} [e^{-\beta H} \dots]$$

Spechual Function. The correlation function:

$$\langle \text{GS} | e^{iHt} c_{x,t} e^{-iHt} c_{0,0}^\dagger | \text{GS} \rangle$$

Green Function:

$$G(x, t) = -i\theta(t) \langle [c_{x,t}, c_{0,0}^\dagger] \rangle$$

Probability to find an excitation:

$$A(k, \omega) = -\frac{1}{\pi} \text{Im} G(k, \omega)$$

(Kallen-Lehmann) and the Green Function is

$$G(k, \omega) = \frac{1}{\omega - \xi_k + i\delta} \quad \delta = 0^+$$

\uparrow is the retarded Green Function.

Effect of interaction in metals $U \sim E_F$ so the fermi surface no longer exist. Landau Quasi particles—dressed electrons, redefine parameters, effective mass, free quasi particles. $n(\epsilon)$ of quasi particles has a jump at E_F .

In 2d and 3d, FLT can get rid of interaction.

1.1.2 Interacting Boson system

BEC: $\langle b_k^\dagger b_k \rangle = N\delta_{k,0}$ and $n_k = N_0\delta_{k,0} + n^{\text{regular}}(k)$.

Bogoliubov approximation:

$$H = \sum_k \xi_k b_k^\dagger b_k + U \sum_{kk'q} b_{k+q}^\dagger b_{k'-q}^\dagger b_{k'} b_k$$

assume that there is macroscopic occupation in $k = 0$ and the Hamiltonian will be quadratic.

1.2 Strong Correlation System

A theory on lattice, tight binding model.

$$H = -t \sum_{\langle ij \rangle} c_i^\dagger c_j$$

transform into momentum space and the result will be

$$H = \sum_k (-2t \cos k) c_k^\dagger c_k$$

in which the lattice constant $a = 1$. If we add an onsite interaction term U (Mott-Hubbard Model), it will be come a insulator.

1.3 Hubbard Model

This model is crucial to understand the strong correlated Bose system. The Hamiltonian for Bosons has hopping terms between nearest neighbours and onsite interaction:

$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}) + U \sum_i \rho_i (\rho_i - 1) - \mu \sum_i \rho_i$$

for Fermions we have:

$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}) + U \sum_i \rho_{i\uparrow} \rho_{i\downarrow} - \mu \sum_i \rho_i$$

The two Hamiltonians is called Hubbard Model(1963).

Bose-Hubbard Model. In the case Bosons without spin. The Hamiltonian on the lattice is

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i \rho_i (\rho_i - 1)$$

Mott-Insulator phase transition is neither first nor second order phase transition. For $U \gg t$, the system is in the Mott insulator phase, and for $U \ll t$, the system is in the metal phase(in high dimensions, it's Fermi liquid; in one dimension, it's Tomonaga-Luttinger liquid). Mott insulator has one electron in each site, while Band insulator has two electrons on each site

Superexchange. For Fermions with Hubbard on site interaction, an effective Hamiltonian of spin degree of freedom will be:

$$H^{\text{spin}} = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

and the parameter J is of the order $J \sim \frac{4t^2}{U}$. **Superexchange for spin one-half Fermions is the antiferromagnetic interaction.** Because the virtual process with parallel spin on neighbour sites is banned by Pauli principle.

What about the bosons with spin degree of freedom? **Superexchange interaction with spin one-half Bosons will be ferromagnetic.**

Spinless Fermions. Since we cannot have the onsite interaction, we can add interaction on neighbour sites:

$$H = -t \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}) + U \sum_{\langle ij \rangle} n_i n_j$$

2 One dimensional systems; Tomonaga-Luttinger liquid

2.1 Introduction

Physics systems with reduced dimensionality: physics at edge, interface between LaO and SrO is superconductor, but the two themselves are insulators. In one dimension, only collective excitations can exist. In one dimension the quantum fluctuation is too strong that the long range order will be zero (Mermin-Wagner theorem). Some examples of one dimension systems: Organic conductors, carbon nanotubes, quantum wires, spin chains and ladders(PRL 86 5168, PRL 101 137207, PRB 79, 020408)

In Cold Atoms system, the kinetic and interaction is easy to manipulate.

The Typical problem in continuum Quantum systems is

$$H = \int dx \psi^\dagger \frac{-\nabla^2}{2m} \psi + \frac{1}{2} \int dx dy V(x-y) \rho(x) \rho(y) - \mu \int dx \rho(x)$$

or in the lattice systems

$$H = -J \sum_{\langle ij \rangle} b_i^\dagger b_j + U \sum_i n_i(n_i - 1) - \mu \sum_i n_i$$

So how to treat these problems?

Standard many-body theory.

- Exact solution (Bethe ansatz)
- Field theory (bosonization, CFT)
- Numerics (DMRG, MC, etc.)

2.2 Luttinger Liquid

In one dimension systems the perturbation theory failed so we need to use new methods to treat. The Hamiltonian is

$$H = \int dx \psi^\dagger \frac{-\nabla^2}{2m} \psi + \frac{1}{2} \int dx \frac{g}{2} \rho(x) \rho(x) - \mu \int dx \rho(x)$$

in which the interacting term is a δ potential. Now we can define a field ϕ_l to describe the particle distribution:

$$\rho(x) = \sum_i \delta(x - x_i) = \sum_n |\nabla \phi_l(x)| \delta(\phi_l(x) - 2\pi n)$$

and that is a unique way of labelling particles. Define

$$\phi_l(x) = 2\pi \rho_0 x - 2\phi(x)$$

and

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi} \nabla \phi(x) \right] \sum_{p \text{ integer}} e^{i2p(\pi \rho_0 x - \phi(x))} \quad \text{Poisson summation formula}$$

in which the field $\phi(x)$ varies slowly. Then how about the single particle operator?

$$\psi^\dagger = [\rho(x)]^{1/2} e^{-i\theta(x)}$$

we can find the commutation relation of these operators are

$$\left[\frac{1}{\pi} \nabla \phi(x), \theta(x') \right] = -i\delta(x - x')$$

in which $\theta(x)$ is the superfluid phase. The Hamiltonian then becomes

$$H = \frac{1}{2\pi} \int dx \left[uK(\pi \Pi(x))^2 + \frac{u}{K} (\nabla \phi(x))^2 \right]$$

in which the parametres are

$$uK = \frac{\rho_0}{2m}$$

and the explicit value of them are determined both by U, m, ρ_0 and some macroscopic values such as compressibility. Calculation result show that the correlation of Boson operators and density operators decays as power law.

For Fermions, the single particle operator is defined as(JordanWigner transformation)

$$\psi_F^\dagger = \psi_B^\dagger e^{i\frac{1}{2}\phi_l(x)}$$

so the Fermion operator can be written as

$$\psi_F^\dagger = \left[\rho_0 - \frac{1}{\pi} \nabla \phi(x) \right]^{1/2} \sum_p e^{i(2p+1)(\pi \rho_0 x - \phi(x))} e^{-i\theta(x)}$$

At finite temperature, the correlation decays as exponential function. $n_F(\xi_k)$ in one dimension does not show the behavior of Fermi liquid ($z = 0$). Spin systems can be mapped to Bosons or Fermions:

$$S^+ = (-1)^i e^{i\theta} + e^{i\theta} \cos(2\phi) \quad S_Z = \frac{-1}{\pi} \nabla \phi + (-1)^i \cos(2\phi)$$

3 Experimental realizations

Cold Atom experiments.

- science VOL 305, 20 AUG, 2004, 1125. One dimensional Tonks-Girardeau liquid
- Nature 429 277(2004). Optical lattices (dilute).
- Nat. Phys 4 489 (2008).
- PRL 108 167201.

4 Beyond Tomonaga-Luttinger liquid

Now we can add a lattice:

$$H = \int dx \psi^\dagger \frac{-\nabla^2}{2m} \psi + \frac{U}{2} \int dx \rho^2 - \mu \int dx \rho + H_{\text{lattice}}$$

and $H_{\text{lattice}} = \int dx V_0 \cos(Qx) \rho(x)$. By an approximation that we have used with field ϕ , we can find the Hamiltonian of lattice is

$$H_{\text{lattice}} = 2V_0 \rho_0 \int dx \cos(2\phi(x))$$

and interesting phase transition phenomenon will emerge.