Student ID: 06996263 Student name: Fang Zhu

Q1:

1.1

$$\Sigma = \frac{1}{4}\begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$$

$$(\Sigma - \lambda \mathrm{I})\mathrm{u} = 0$$

$$\begin{vmatrix} \frac{5}{4} - \lambda & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{7}{4} - \lambda \end{vmatrix} = 0$$

$$(\frac{5}{4} - \lambda)(\frac{7}{4} - \lambda) = \frac{3}{16}$$

So:

$\lambda_1 = 2$ and $\lambda_2 = 1$

1.2

From the first question, we knew that $\lambda_1 = 2$ and $\lambda_2 = 1$, so we can get eigenvectors of Σ.

Suppose $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$.

$$\Sigma v_i = \lambda_i v_i$$

so $\Sigma v_1 + \Sigma v_2 = \lambda_1 v_1 + \lambda_2 v_2$

$$\begin{bmatrix} 5a + \sqrt{3}b \\ \sqrt{3}a + 7b \end{bmatrix} + \begin{bmatrix} 5c + \sqrt{3}d \\ \sqrt{3}c + 7d \end{bmatrix} = \begin{bmatrix} 8a + 4c \\ 8b + 4d \end{bmatrix}$$

$$5a + \sqrt{3}b + 5c + \sqrt{3}d = 8a + 4c$$

$$7b + \sqrt{3}a + 7d + \sqrt{3}c = 8b + 4d$$

So

$$3a + 3c = \sqrt{3}b - 3\sqrt{3}d \qquad 3a - c = \sqrt{3}b + \sqrt{3}d$$

$$c = -\sqrt{3}d$$

$$b = \sqrt{3}a$$

If a = 1 and d = -1

$v_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $v_2 = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$

finally both the vectors are eigenvectors

1.3

According to the spectral theorem:

D is diagonal matrix: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

And U is combined with v_1 and v_2: $[v_1, v_2]$, v are unit eigenvectors.

From the second question, we get the answer that $v_1 = \begin{bmatrix} a \\ \sqrt{3}a \end{bmatrix}$ and $v_2 = \begin{bmatrix} -\sqrt{3}d \\ d \end{bmatrix}$.

According to the phenomenon of orthogonal matrix to find the unit eigenvectors.

$[v_1, v_2][v_1, v_2]^T = I$

$$\begin{bmatrix} a & -\sqrt{3}d \\ \sqrt{3}a & d \end{bmatrix} \begin{bmatrix} a & \sqrt{3}a \\ -\sqrt{3}d & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So: a = -d and a = + 0.5 or – 0.5

If a = 0.5

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$$

If a = -0.5

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$$

So a = + 0.5 or – 0.5 , d = - 0.5 or + 0.5

So U = $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

1.4

According to the principal of component transform

$\Sigma = \mathrm{UD}U^T$ (D = $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, U = [v_1 , v_2])

Suppose : y = $u^T x^T$; "u" is a projection vector.

Because u is the eigenvectors of covariance matrix of the sample data.

Than:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1.5

$yy^T = U^T x^T x U = U^T \Sigma U = [v1 , v2]^T (\frac{1}{4}\begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix})[v1 , v2] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = D$

1.6

According to the Mahalanobis distance, we can achieve:

$$\sqrt{X^T(\Sigma)^{-1}X} = \sqrt{X^T(X^TX)^{-1}X} = 1$$

$$X^T(X^TX)^{-1}X = 1$$

$X^T = [x1 , x2]$ $\qquad$ $X=[x1 , x2]^T$

We can get the inverse of Σ as follow:

$$\begin{bmatrix} \frac{5}{4} & \frac{\sqrt{3}}{4} & 1 & 0 \\ \frac{\sqrt{3}}{4} & \frac{7}{4} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{7}{8} & -\frac{\sqrt{3}}{4} \\ 0 & 1 & -\frac{\sqrt{3}}{4} & \frac{5}{8} \end{bmatrix}$$

Finally we get the inverse of X^TX:

$$[x1 \quad x2]\begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{5}{8} \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = 1$$

$7x1^2 - 2\sqrt{3}x1x2 + 5x2^2 = 8$

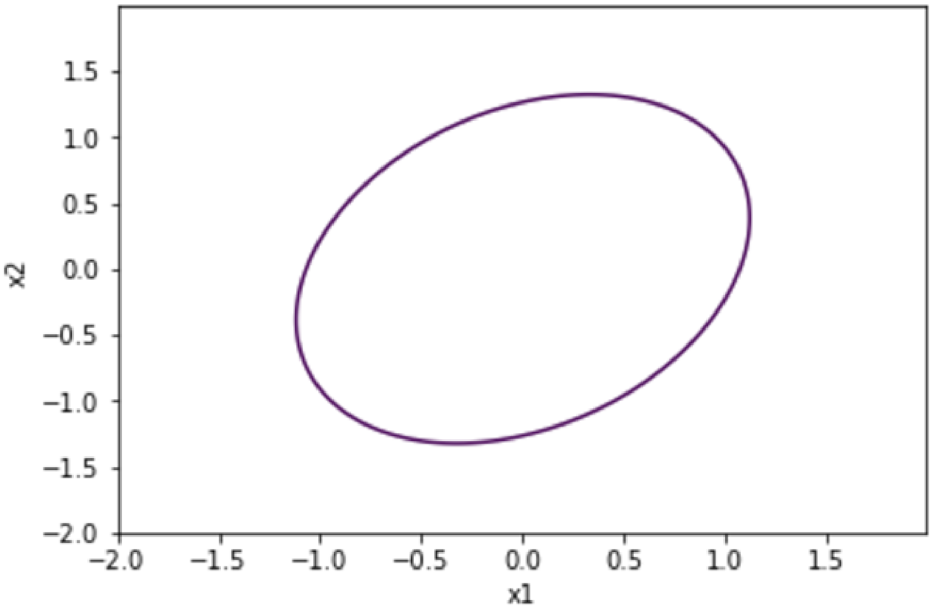
Using the python to get this plot as follow:

Code:

```
import numpy as np

import matplotlib

import matplotlib.pyplot as plt

%matplotlib inline

import math
```

```
import textwrap

import numpy as np



p3 = math.sqrt(3)

x1 = np.arange(-2,2,0.01)

x2 = np.arange(-2,2,0.01)

X,Y = np.meshgrid(x1,x2)

Z = 7*X**2+5*Y**2-2*p3*X*Y-8

plt.contour(X,Y,Z,0)

plt.ylabel('x2')

plt.xlabel('x1')

plt.show()
```

picture:



Q2

$$A^k = \left(Q \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} Q^T\right)^k$$

$= QDQ^TQDQ^T \ldots QDQ^T$

As we known that the special $Q^TQ = Q^{-1}Q = I$

So:

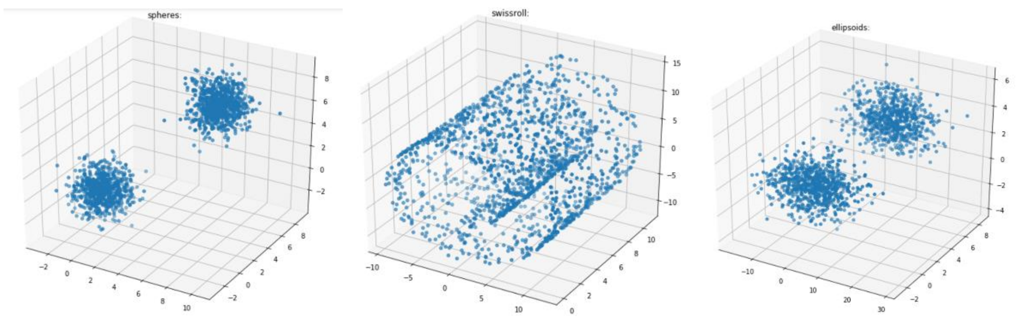
$=QD^kQ^T$

Q3.

Code:

```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
%matplotlib inline
import math
import textwrap
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
import scipy.spatial.distance as sc


spheres = np.loadtxt('spheres.txt')
swissroll = np.loadtxt('swissroll.txt')
ellipsoids = np.loadtxt('ellipsoids.txt')


print(spheres.shape)
print(swissroll.shape)
print(ellipsoids.shape)
```

```
#plot results

fig = plt.figure(figsize=(10,30))

ax = fig.add_subplot(*[3,1,1], projection='3d')

ax.scatter(spheres[:,0], spheres[:,1], spheres[:,2])

myTitle = 'spheres: ';

ax.set_title("\n".join(textwrap.wrap(myTitle, 20)))



ax = fig.add_subplot(*[3,1,2], projection='3d')

ax.scatter(swissroll[:,0], swissroll[:,1], swissroll[:,2])

myTitle = 'swissroll: ';

ax.set_title("\n".join(textwrap.wrap(myTitle, 20)))



ax = fig.add_subplot(*[3,1,3], projection='3d')

ax.scatter(ellipsoids[:,0], ellipsoids[:,1], ellipsoids[:,2])

myTitle = 'ellipsoids: ';

ax.set_title("\n".join(textwrap.wrap(myTitle, 20)))

plt.show();
```



(sphere, swissroll, elipsoids from left to right)

Code for 2-D and 1-D pca

****

***/

```
import numpy as np

import matplotlib

import matplotlib.pyplot as plt

%matplotlib inline

import math

import textwrap

import numpy as np
```

```
spheres = np.loadtxt('spheres.txt')

swissroll = np.loadtxt('swissroll.txt')

ellipsoids = np.loadtxt('ellipsoids.txt')


x1 = spheres[:,0]

y1 = spheres[:,1]

z1 = spheres[:,2]


x2 = swissroll[:,0]

y2 = swissroll[:,1]

z2 = swissroll[:,2]


x3 = ellipsoids[:,0]

y3 = ellipsoids[:,1]

z3 = ellipsoids[:,2]


#print(x1)

Exx1 = sum(x1)/x1.size
```

```
Exy1 = sum(y1)/y1.size

Exz1 = sum(z1)/z1.size

Mx1 = np.array([(x1[m]-Exx1) for m in range(0,1500)])

My1 = np.array([(y1[m]-Exy1) for m in range(0,1500)])

Mz1 = np.array([(z1[m]-Exz1) for m in range(0,1500)])

#print(Mx1.shape)

M1 = np.matrix([Mx1,My1,Mz1])

Cov1 = M1@M1.T / 1500

#print(Cov1)

eigen_vals_1, eigen_vecs_1 = np.linalg.eig(Cov1)

#print(eigen_vals_1.shape)

#print(eigen_vecs_1)

eigen_pairs_1 = [(np.abs(eigen_vals_1[i]),
np.array(eigen_vecs_1[:,i].T)[0]) for i in
range(len(eigen_vals_1))]

eigen_pairs_1.sort(key = lambda x : x[0],reverse=True)

#print(eigen_pairs_1)
```

```
w1 = np.hstack((eigen_pairs_1[0][1][:, np.newaxis],
eigen_pairs_1[1][1][:, np.newaxis]))
M1_pca = M1.T@w1


w12 = np.hstack((eigen_pairs_1[0][1][:, np.newaxis]))
M12_pca = M1.T@w12
fig = plt.figure(figsize = (10,20))


p11 = fig.add_subplot(*[2,1,1])
p11.scatter(np.array(M1_pca[:,0].T)[0],np.array(M1_pca[:,1].
T)[0])
p11.set_title('2 Dimension')


p12 = fig.add_subplot(*[2,1,2])
p12.scatter(np.array(M12_pca),np.zeros((1500,1)))
p12.set_title('1 Dimension')


plt.xlabel('PC 1')
```

```
plt.ylabel('PC 2')

plt.legend(loc='lower left')

plt.show()


/***********************************************************
***********************************************************/


#print(x1)

Exx2 = sum(x2)/x2.size

Exy2 = sum(y2)/y2.size

Exz2 = sum(z2)/z2.size

Mx2 = np.array([(x2[m]-Exx2) for m in range(0,1500)])

My2 = np.array([(y2[m]-Exy2) for m in range(0,1500)])

Mz2 = np.array([(z2[m]-Exz2) for m in range(0,1500)])

#print(Mx2.shape)

M2 = np.matrix([Mx2,My2,Mz2])

Cov2 = M2@M2.T/1500

#print(Cov2)
```

```
eigen_vals_2, eigen_vecs_2 = np.linalg.eig(Cov2)

#print(eigen_vals_2.shape)

#print(eigen_vecs_2)

eigen_pairs_2 = [(np.abs(eigen_vals_2[i]), np.array(eigen_vecs_2[:,i].T)[0]) for i in range(len(eigen_vals_2))]

eigen_pairs_2.sort(key = lambda x : x[0],reverse=True)

#print(eigen_pairs_2)

w2 = np.hstack((eigen_pairs_2[0][1][:, np.newaxis], eigen_pairs_2[1][1][:, np.newaxis]))

M2_pca = M2.T@w2



w22 = np.hstack((eigen_pairs_2[0][1][:, np.newaxis]))

M22_pca = M2.T@w22

fig2 = plt.figure(figsize = (10,20))



p21 = fig2.add_subplot(*[2,1,1])
```

```
p21.scatter(np.array(M2_pca[:,0].T)[0],np.array(M2_pca[:,1].
T)[0])
p21.set_title('2 Dimension')


p22 = fig2.add_subplot(*[2,1,2])
p22.scatter(np.array(M22_pca),np.zeros((1500,1)))
p22.set_title('1 Dimension')


plt.xlabel('PC 1')
plt.ylabel('PC 2')
plt.legend(loc='lower left')
plt.show()


/************************************************************
************************************************************/


#print(x1)
Exx3 = sum(x3)/x3.size
```

```
Exy3 = sum(y3)/y3.size

Exz3 = sum(z3)/z3.size

Mx3 = np.array([(x3[m]-Exx3) for m in range(0,1500)])

My3 = np.array([(y3[m]-Exy3) for m in range(0,1500)])

Mz3 = np.array([(z3[m]-Exz3) for m in range(0,1500)])

#print(Mx3.shape)

M3 = np.matrix([Mx3,My3,Mz3])

Cov3 = M3@M3.T/1500

#print(Cov3)

eigen_vals_3, eigen_vecs_3 = np.linalg.eig(Cov3)

#print(eigen_vals_3.shape)

#print(eigen_vecs_3)

eigen_pairs_3 = [(np.abs(eigen_vals_3[i]),
np.array(eigen_vecs_3[:,i].T)[0]) for i in
range(len(eigen_vals_3))]

eigen_pairs_3.sort(key = lambda x : x[0],reverse=True)

#print(eigen_pairs_3)
```

```
w3 = np.hstack((eigen_pairs_3[0][1][:, np.newaxis],
eigen_pairs_3[1][1][:, np.newaxis]))
M3_pca = M3.T@w3

w32 = np.hstack((eigen_pairs_3[0][1][:, np.newaxis]))
M32_pca = M3.T@w32
fig3 = plt.figure(figsize = (10,20))

p31 = fig3.add_subplot(*[2,1,1])
p31.scatter(np.array(M3_pca[:,0].T)[0],np.array(M3_pca[:,1].
T)[0])
p31.set_title('2 Dimension')

p32 = fig3.add_subplot(*[2,1,2])
p32.scatter(np.array(M32_pca),np.zeros((1500,1)))
p32.set_title('1 Dimension')

plt.xlabel('PC 1')
```

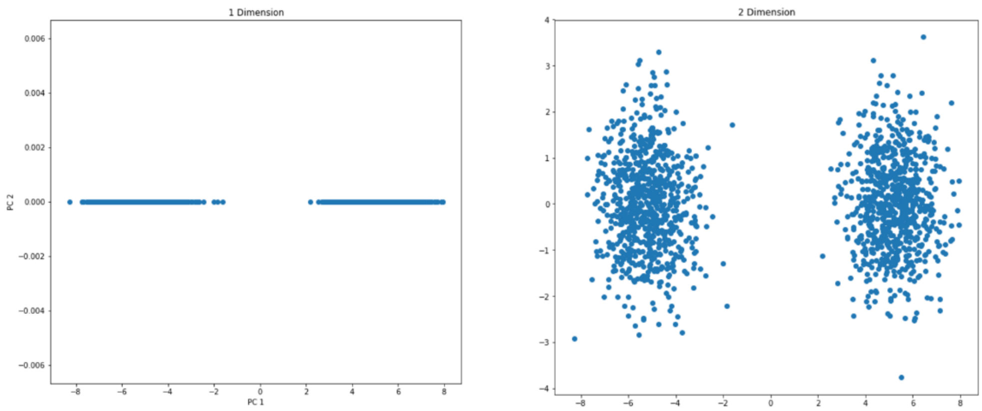
plt.ylabel('PC 2')

plt.legend(loc='lower left')

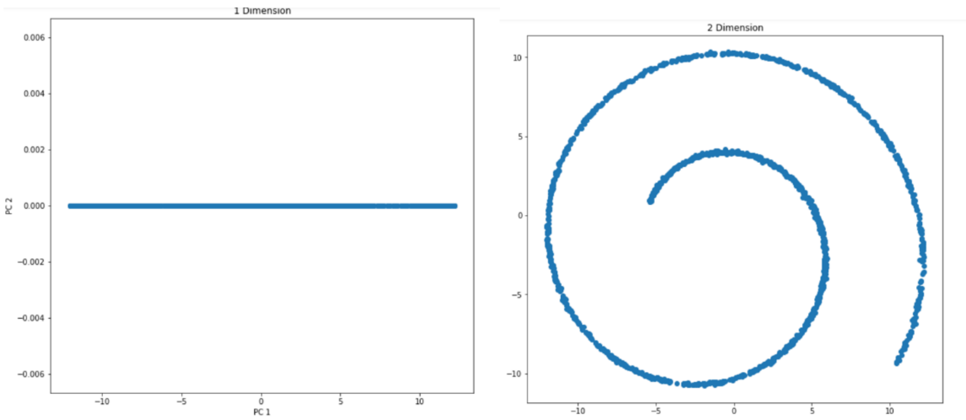
plt.show()

/**

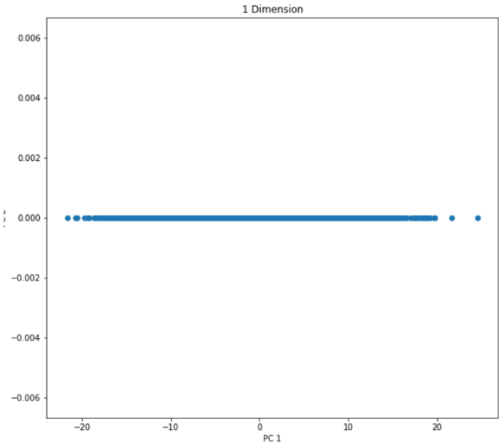
***/

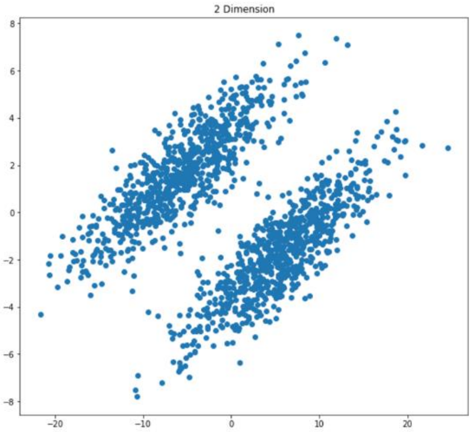


Sphere



Swissroll





Ellipsoids

1. Matrix

Sphere:

```
[[ 15045.16477116  13452.5115079   13631.98710877]
 [ 13452.5115079   14876.92584157  13592.65574337]
 [ 13631.98710877  13592.65574337  15383.70237092]]
```

Swissroll:

```
[[ 64889.0265442     230.10763148   6678.83066677]
 [   230.10763148  15859.39242309    231.42171507]
 [  6678.83066677    231.42171507  70685.01799753]]
```

Ellipsoids:

[[86741.32255707   22025.42123755   11042.18121689]
 [22025.42123755   14784.6022647   6785.12588853]
 [11042.18121689   6785.12588853   5032.15322094]]

2. Eigenvalues and eigenvectors

Sphere:

Values:

[42222.06446919   1584.5399621   1499.1885523
6]

Vectors:

[[-0.57604721 -0.62074794   0.53182855]
 [-0.57310984 -0.15721136 -0.80425723]
 [-0.58285051   0.76808632   0.26519558]]

Swissroll:

Values:

[75069.2129182   60506.65397977   15857.5700668
6]

Vectors:

[[0.54862373   0.83605889   0.00418753]
 [0.00539983   0.00146519 -0.99998435]

[0.83605194 -0.54863776   0.00371074]]

Ellipsoids:

Values:

[94684.71190217   10321.58611625   1551.7800242
9]

Vectors:

[[0.95175893   0.30681239   0.00459253]
[0.27407997 -0.8433001   -0.46230412]
[0.13796775 -0.4412608   0.88670954]]

3.

Ratio(sphere 2 dimension) = $(\lambda_1 + \lambda_2)/ (\sum\lambda_i)$ = (42222.06446919+1584.5399621)/(42222.06446919+1584.5399621+1499.18855236) = 0.97>0.9

Ratio(sphere 1 dimension) = $(\lambda_1)/ (\sum\lambda_i)$ = (42222.06446919)/(42222.06446919+1584.5399621+1499.18855236) = 0.93>0.9

Ratio(Swissroll 2 dimension) = $(\lambda_1 + \lambda_2) / (\sum\lambda_i)$ =

(75069.2129182+60506.65397977)/(75069.2129182+ 60506.65397977+15857.57006686) = 0.89<0.9

Ratio(Swissroll 1 dimension) = $(\lambda_1) / (\sum\lambda_i)$ =

(75069.2129182)/(75069.2129182+60506.65397977+ 15857.57006686) = 0.50<0.9

Ratio(Ellipsoids 2 dimension) = $(\lambda_1 + \lambda_2) / (\sum\lambda_i)$ =

(94684.71190217+10321.58611625)/(94684.71190217 +10321.58611625+1551.78002429) = 0.98>0.9

Ratio(Ellipsoids 1 dimension) = $(\lambda_1) / (\sum\lambda_i)$ =

(94684.71190217)/(94684.71190217+10321.58611625 +1551.78002429) = 0.89<0.9

If we define the ratio bigger than 0.9 is good, we can saw that both 1 and 2 dimension for sphere, 2 dimension for ellipsoids are good PCA, whereas the rest of them are not good.