Math 111A

Fangke Jiang

February 2024

1 Introduction

The rapid evolution of artificial intelligence (AI) technologies, particularly generative models like ChatGPT, has significant implications for numerous sectors, including education, customer service, and content creation. Understanding the dynamics of ChatGPT's adoption can provide valuable insights for stakeholders, including developers, policy makers, and investors, regarding its potential impact, areas of growth, and market penetration strategies. This project proposes to construct a mathematical model to analyze and forecast the adoption and expansion of ChatGPT technology within the United States.

2 Previous work

As described by Mahajan et al. [1], the classification of adopter categories can be effectively determined by applying diffusion models. The work of Bass (1969) on the diffusion of new products provides a foundational methodology for forecasting technology adoption. **The Bass diffusion Model** is especially useful for forecasting the adoption of recent products, estimating marketplace functionality, and knowing the effect of marketing efforts on the adoption manner. It is primarily based on the concept that there are forms of adapters:

Innovators (p): These are people who are the primary to adopt a brand-new product or innovation. They tend to be chance-takers and early adopters.

Imitators (q): These are people who undertake a brand-new product after seeing their pals or innovators adopt it. They are inspired by the resources of the opinions and hints of others.

The Bass Model is expressed as a differential equation and has parameters: p (the coefficient of innovation) and q (the coefficient of imitation). Previous studies have applied diffusion models to various technologies, from mobile phones to social media platforms. However, the unique characteristics of conversational AI, such as ChatGPT, necessitate a tailored approach to modeling its adoption.

Here is the basic Bass diffusion model:

$$\frac{f(t)}{1-F(t)} = p + qF(t)$$

where $F(t)$ is the cumulative fraction of adopters at time t and $f(t) = \frac{dF(t)}{dt}$ is the rate of diffusion at time t. Expressed as an ordinary differential equation,

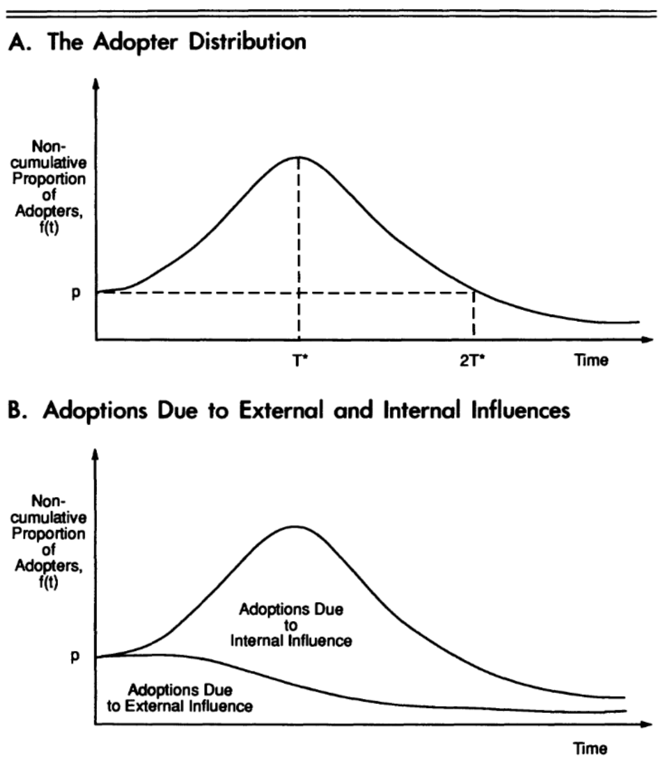
$$\frac{dF}{dt} = p(1 - F) + q(1 - F)F = (1 - F)(p + qF).$$

The new adopters $s(t)$ at time t is the rate of change of installed base, $f(t)$ multiplied by the market potential m. Under the condition $F(0) = 0$, we have that:

$$s(t) = mf(t)$$

When m is the total population of potential adopters, the cumulative number of adopters at time t is $mF(t)$. Integration of equation above yields the S-shaped cumulative adopter distribution, $F(t)$, captured by the Bass model. Here is the distribution of adopter from the Bass model.

Figure 1
DISTRIBUTION OF ADOPTERS—THE BASS MODEL



3 Research question

How will the adoption of ChatGPT evolve over time in the United States, and what are the key factors influencing its spread? For the public and academic researchers, analyzing the adoption trends and cycle can shed light on the social impacts of widespread Gen-AI integration.

4 Approach

https://github.com/FangkeJiang/math-111a Here is the Github link with all the codes and further data visualization. The basic Bass diffusion model in 1969 has some disadvantage. Some of the factors and transition bewteen status might not be captured and it is somehow very general to all the new technology. I will try to use **differential equations** to model the adoption of ChatGPT among population specifically, which is a classic approach similar to Bass diffusion model. In the fields of epidemiology and sociology, where such models have been used to describe the spread of diseases or ideas through populations. This method focuses on capturing the dynamics of adoption through time-based change rates, providing a deterministic framework for understanding how the adoption process evolves.

4.1 ChatGPT Adoption Model

The differential equations governing the adoption model are defined as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= -\alpha S - \beta AS + \theta A - \gamma S, \\
\frac{dT}{dt} &= \alpha S + \delta O - \epsilon T - \delta T, \\
\frac{dO}{dt} &= \beta AS - \delta O, \\
\frac{dA}{dt} &= \text{retention\_factor} \times (\epsilon T + \delta O + \delta T) - \theta A + 0.4A\sin(0.59t), \\
\frac{dR}{dt} &= \gamma A,
\end{aligned}
$$

where the state variables represent the following: S represents the number of Susceptible individuals who have not yet tried ChatGPT. T represents the number of individuals who have Tried ChatGPT. O represents the number of Observers who are aware of ChatGPT but are not regular users. A represents the number of Adopters who regularly use ChatGPT. R represents the number of Resistant individuals who are unlikely to use ChatGPT again.

The parameters are interpreted as follows *Alpha*:Transition rate from susceptible to tried. *Beta*:Transition rate from susceptible to observers due to influence from adopters. *Epsilon*:Conversion rate from tried to adopters. *Delta*:Transition rate affecting movements between tried, observers, and adopters. $Gamma$:Rate at which adopters become resistant. $Theta$:Rate at which adopters revert to susceptible.

The term $0.4A\sin(0.59t)$ in the equation for $\frac{dA}{dt}$ adds a cyclical component to represent periodic fluctuations in adoption rates. Additionally, a term is subtracted from $\frac{dA}{dt}$ for $t \geq 11$ months to model a decrease in adopters, due to a loss of interest or external factors like having competitors:

$$\text{If } t \geq 11, \text{ then } \frac{dA}{dt} \text{ includes } -0.1A\cos(0.1t).$$

The model starts with the following initial conditions:

$$S(0) = 20000, \quad \text{(the potential market size)},$$
$$T(0) = 0, \quad \text{(no one has tried ChatGPT)},$$
$$O(0) = 0, \quad \text{(no observers at the start)},$$
$$A(0) = \frac{\text{actual\_data[0]}}{\text{visits\_to\_adopters}}, \quad \text{(initial adopters estimated from visit data)},$$
$$R(0) = 0, \quad \text{(no resistant individuals initially)}.$$

Here we have the first month visit data from dataset, which is 152.7 million. The visit to adopters conversion rate in my assumption is 5, which means if a people visit ChatGPT 5 times per month, he/she will be defined as adopter. The model aims to fit the actual visit data by adjusting the parameters and cyclical components to minimize the loss function, which is a combination of the sum of absolute differences and the Mean Squared Error (MSE) between the predicted and actual data. These equations are solved over a time span of one year (365 days, 12 months) using numerical methods. The solution of differential equations are solved by solver in python then plotted to show the dynamics of ChatGPT adoption number over time, illustrating how each compartment's population changes.

About derivation of the model and equations,the model equations are derived based on the logical flow of user interaction with ChatGPT, from awareness to adoption or resistance. Each equation represents the net rate of change in each state, accounting for all the ways individuals can enter or leave that state: the susceptible group decreases due to individuals trying ChatGPT or being influenced by adopters, and increases if adopters revert back. The observer group dynamics are modeled by the influence of adopters on susceptible individuals and the transitions from trying ChatGPT. The tried group is influenced by new trials from the susceptible and observer groups, transitions to adoption, or reverts back to observation. Adopters increase through conversion from the tried group, adjusted by the retention factor, which is 30 percent of those who try ChatGPT continue using it regularly and decrease as they become resistant or revert to susceptible. Resistant individuals increase as adopters stop using ChatGPT forever. This comprehensive approach allows the model to capture a wide range of dynamics in ChatGPT adoption, including external influences and user feedback loops.

The differential equations model the rate of change of each group over time, influenced by the interactions between different groups and the parameters that govern these interactions. Individuals move from being susceptible to trying ChatGPT, influenced by their own decision (α) and the influence of current adopters (β). Once individuals have tried ChatGPT the first time, they may become adopters (ϵ), revert to being observers (δ), or influence others to observe (β). Adopters may become resistant (γ) or revert to a susceptible state (θ), possibly due to dissatisfaction or the emergence of better alternatives. The retention factor modifies the rate at which tried individuals become adopters, reflecting the percentage of users who continue using ChatGPT after trying it the first time.

5 Factors Influencing the Adoption parameter of ChatGPT

Alpha (α)

α represents the rate at which susceptible individuals decide to try ChatGPT. Factors influencing α include awareness and perception, which means how well the target audience understands ChatGPT's value proposition. As for accessibility, which is the ease of access to ChatGPT, influenced by platform availability and user interface design in internal reasons. For external reasons, there might be WIFI connection issue or if people can access to computers. This might also be influenced by age structure. According to the dataset I found in YouGov, age between 45-64 (55 percent) and 65+(59 percent) categories have the most percentage of people who never used it or seen anyone used it. In additon, the impact of advertising, marketing and promotional activities to encourage first-time use might also influence this paramter.

Beta (β)

β measures how adopters influence others to become observers. It is affected the positive feedback and recommendations from current users. Or fan of ChatGPT who recommend on social media showcasing user experiences and benefits. The public visibility of ChatGPT usage, increasing curiosity and interest among potential users by video or news. For example people age in 18-29 and 30-44 has the most amount of people who observe others usage, might because of college students and office workers have more accessibility to ChatGPT, and might easier to seen a colleague or classmate using ChatGPT and improve efficiency.

Epsilon (ϵ)

ϵ is the conversion rate from having tried ChatGPT to becoming regular adopters. Key influences include user experience, which is the satisfaction with ChatGPT's performance and utility. Also content quality which if ChatGPT provides valuable and relevant response. Or any ongoing efforts to engage users and encourage repeated use. Young generation of age between 18-29, 30-44 categories use ChatGPT to generate text the most. I also find that family income between 50-100K population use ChatGPT quite often, even double than family income below 50K, which might be a considerable factor.

Delta (δ)

δ affects the transition among tried, observed, and adopted states. Follow-Up engagement that influences users' decisions to adopt or observe further. User communities in providing help and enhancing the adoption experience.

Gamma (γ) and Theta (θ)

γ and θ represent the rates at which adopters become resistant or back to susceptible. Influencing factors might be competitive offerings, which is the emergence of alternative solutions that may lead users to abandon ChatGPT. Or there are cheaper AI tools than ChatGPT-4. Changes in user expectations and needs over time and see if Open AI can satisfy these need with training better ChatGPT generations. The ongoing service quality of ChatGPT, including updates and responsiveness to user feedback.

Time-Dependent Behavior

The cyclical component ($0.4A\sin(0.59t)$) and the decrease in adopters from month 11 ($-0.1A\cos(0.1t)$) highlight how external events and seasonal trends can impact adoption rates.

Daily Survey: ChatGPT
January 24 - 27, 2023 - 1000 US Adult Citizens

3. Ever Used ChatGPT
Which of the following best describes your familiarity with ChatGPT?
Asked of people who have heard at least a little about ChatGPT

		Gender		Age (4 category)				Race (4 category)			
	Total	Male	Female	18-29	30-44	45-64	65+	White	Black	Hispanic	Other
I've used it myself to generate text	12%	13%	11%	15%	17%	9%	5%	11%	12%	22%	8%
I've seen text it generated for someone else, but haven't used it myself	38%	41%	34%	48%	46%	27%	30%	35%	35%	60%	36%
I've never used it or seen anyone else use it	40%	39%	43%	20%	31%	55%	59%	46%	43%	10%	33%
Not sure	9%	7%	13%	17%	6%	8%	6%	7%	11%	9%	23%
Totals	99%	100%	101%	100%	100%	99%	100%	99%	101%	101%	100%
Unweighted N	(466)	(250)	(216)	(120)	(120)	(148)	(78)	(335)	(63)	(36)	(32)

		Party ID			2020 Vote		Family Income (3 category)			Census Region			
	Total	Dem	Ind	Rep	Biden	Trump	< $50K	$50-100K	$100K+	Northeast	Midwest	South	West
I've used it myself to generate text	12%	11%	14%	12%	13%	12%	9%	13%	18%	14%	20%	8%	11%
I've seen text it generated for someone else, but haven't used it myself	38%	46%	33%	33%	39%	32%	41%	32%	44%	55%	28%	34%	41%
I've never used it or seen anyone else use it	40%	38%	40%	43%	44%	45%	37%	48%	33%	27%	43%	48%	35%
Not sure	9%	4%	13%	12%	3%	10%	13%	6%	5%	4%	9%	10%	13%
Totals	99%	99%	100%	100%	99%	99%	100%	99%	100%	100%	100%	100%	100%
Unweighted N	(466)	(207)	(150)	(109)	(212)	(140)	(152)	(141)	(124)	(88)	(95)	(191)	(92)

Figure 1: demographic factors dataset

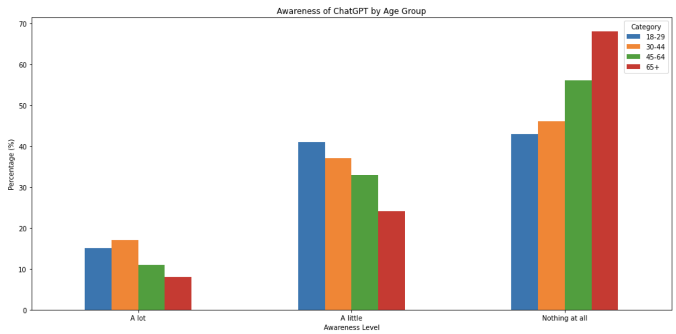


Figure 2: age group

6 Comparison with the Bass Diffusion Model

Unlike the Bass model in 1969, which only considers a cumulative adoption function and does not differentiate between active and inactive users, this model allows for a dynamic feedback loop where adopters may become inactive or resistant. This reflects a more realistic scenario where users' engagement with a technology can change over time, influenced by factors such as satisfaction with the product, competing technologies, or changing needs.

Furthermore, this new model captures the process through which non-users become users in a more granular manner, identifying specific states such as having tried the technology or having observed its use, which are not explicitly considered in the Bass model. The adoption of new technologies can often be modeled using various diffusion models that capture the dynamics of how individuals in a population begin to use a new product or service. The Bass diffusion model, which simplifies the adoption process into two main influences: innovation and imitation. The Bass model posits that the likelihood of adoption is influenced by these two factors, and it assumes a homogeneous potential market without considering the complexity of multiple stages in the adoption process or the loss of users over time.

The ChatGPT adoption model employs a system of coupled differential equations, reflecting the progression of individuals through multiple states—Susceptible, Tried, Observed, Adopters, and Resistant—providing a multifaceted view of adoption. The parameters of the ChatGPT model represent transition rates between these states, augmented by retention factors and cyclical terms, to mirror periodic variations and time-dependent aspects of adoption behavior. While the Bass Model predicts a monotonically increasing adoption curve until market saturation, the ChatGPT model incorporates dynamics that allow for the possibility of individuals ceasing to use the product, thus offering a perspective that can capture cyclical trends and resistance development. Unlike the Bass Model's static parameters, the ChatGPT model introduces time-varying parameters, enabling it to depict shifts in adoption rates corresponding to specific temporal events, such as marketing initiatives or seasonal fluctuations. The comparative analysis underscores the advanced capabilities of the ChatGPT model in capturing the complex dynamics of technology adoption, which are not considered by the traditional Bass Diffusion Model.

7 Dataset

Because of lacking direct user dataset from OpenAI, I try to find the dataset that most related to the adoption topics. (See Figure.3) This dataset comprises a record of monthly visits to the ChatGPT platform, spanning from November 2022 to December 2023. The data enumerates not only the absolute number of visits for each month but also delineates both the numerical and percentage changes in visits relative to the previous month. This progression encapsulates the fluctuating user engagement with ChatGPT over a year. Data sources such as SimilarWeb and Statista are the origins of this dataset. SimilarWeb specializes in web traffic analysis, providing insights into various metrics like the number of visits, engagement, audience geography, and other relevant data, which are derived from a mix of internet user panels, direct data integrations, and traffic data from partners. Statista aggregates statistical data and reports from multiple sources, thus offering comprehensive overviews of digital market trends, including user numbers and market forecasts. Both platforms command significant credibility in their domain, attributed to their expansive data collection network and robust analysis methodologies. However, the veracity of the data can further be substantiated by cross-referencing the presented figures with additional sources, particularly the website's direct analytics if accessible. Evaluating the reliability also entails a consideration of the sample size of the broader user base. Larger and more diverse data samples typically bolster the reliability of the conclusions drawn. Moreover, the accuracy of the data can be enhanced by transparency in the platforms' data collection methods and the presence of data validation protocols. Dataset comes from this website https://explodingtopics.com/blog/chatgpt-usersgrowth.

8 Model Results

The simulation's results are in a collection of plots that juxtapose the model's prediction of adopters with the actual visit data, creating a comparison matrix for adjusting parameters. Each graph shows a distinct permutation of the parameters α and β, which have been modulated to gauge their influence on the prediction model's accuracy. The metrics of loss and Mean Squared Error (MSE) have been computed for every distinct parameter pairing, forming a quantitative evaluation of the model's precision in mirroring the actual data. An examination of the plots demonstrates that the modulation of the parameters bears significant influence on the model's capacity to capture the true pattern of the actual data. A model characterized by a lower loss and MSE is indicative

Month	Number of Visits	Change Over Previous Month	Change Over Previous Month (%)
November 2022	152.7 million	-	-
December 2022	266 million	↑ 113.3 million	↑ 74.2%
January 2023	616 million	↑ 350 million	↑ 131.58%
February 2023	1 billion	↑ 384 million	↑ 62.34%
March 2023	1.6 billion	↑ 600 million	↑ 60%
April 2023	1.8 billion	↑ 200 million	↑ 12.5%
May 2023	1.8 billion	-	-
June 2023	1.6 billion	↓ 200 million	↓ 12.5%
July 2023	1.5 billion	↓ 100 million	↓ 6.25%
August 2023	1.4 billion	↓ 100 million	↓ 6.67%
September 2023	1.5 billion	↑ 100 million	↑ 7.14%
October 2023	1.7 billion	↑ 200 million	↑ 13.33%
November 2023	1.7 billion	-	-
December 2023	1.6 billion	↓ 100 million	↓ 5.88%

Figure 3: dataset of total visit

of a good fit, which shows that the selection of these parameters is important in the accurate delineation of adoption dynamics. Disparate values of these parameters can precipitate an overestimation or underestimation of adoption, contingent on the temporal context. The results from my model can point us towards the best range of parameters that make the model's predictions match up well with the actual usage data. By looking closely at these results, we can better understand how widely ChatGPT has been adopted and how external factors may have affected its growth over time. The computations executed via the `solve_ivp` method from the `scipy.integrate` module effectively translate the intricate web of mathematical relationships into a dynamic simulation, whose output is subsequently juxtaposed against empirical data to derive actionable

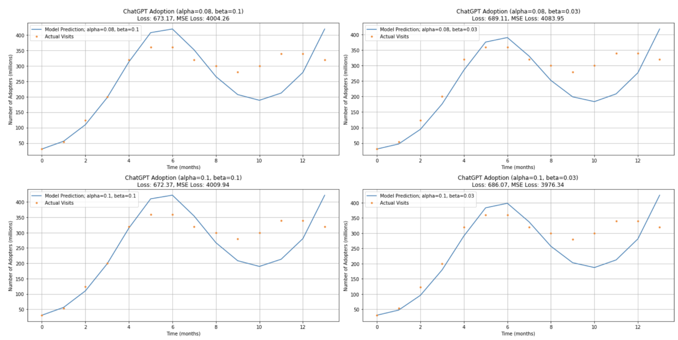


Figure 4: comparision matrix

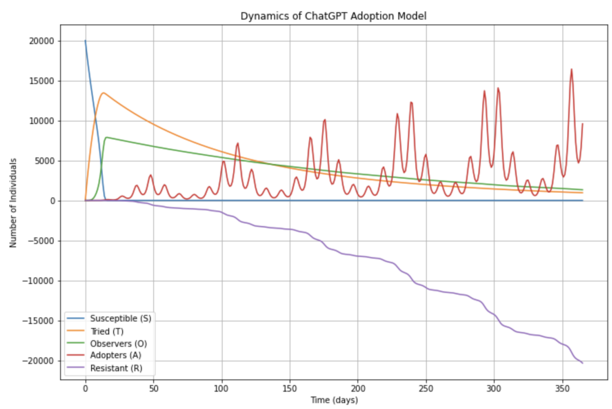


Figure 5: dynamics in 365 days

insights and optimize the model parameters. The optimized parameters are:

$$\begin{aligned}
\text{alpha} &= 0.1, \\
\text{beta} &= 0.03, \\
\text{epsilon} &= 0.008, \\
\text{delta} &= 0.0045, \\
\text{gamma} &= -0.003, \\
\text{theta} &= -0.0005.
\end{aligned}$$

The loss under this set of parameters is around 686.07, MSE loss is 3976.34.

9 Limitation and Next step

There are limitations which makes the loss stay in 686.07. The prediction after 8th month start to underestimate the adoption and didn't capture the decrease

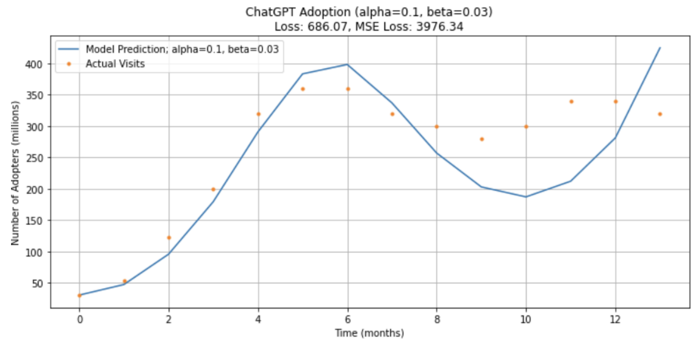


Figure 6: simulation of loss function

after 12th month. My model excludes certain complexities such as demographic variations and social network effects to maintain analytical tractability. While the ChatGPT adoption model provides valuable insights into user engagement trends, it is not without its limitations. One significant constraint is the model's reliance on a fixed set of parameters that do not account for the evolving nature of technology and user behavior. Real-world phenomena such as market competition, technological advancements, and changing user preferences can substantially influence adoption patterns, yet these factors are not dynamically integrated into the model. Furthermore, the model's parameters, are based on estimations that may not capture the granular nuances of ChatGPT's actual adoption landscape. For instance, the conversion rates from trial to regular use, represented by the retention factor, are assumed to be constant over time. However, in reality, these rates could fluctuate due to a multitude of factors, including improvements in the technology or changes in user experience. The model also simplifies the adoption process by considering a homogeneous user base, which overlooks the diversity of user segments that may exhibit distinct adoption behaviors. Additionally, external factors such as socio-economic conditions, policy changes, and cultural trends are not explicitly modeled, despite their potential impact on technology adoption. Another limitation arises from potential inaccuracies in the underlying visit data. If the visit metrics do not accurately reflect active and consistent use, the model's predictions may not truly represent the adoption curve. The data's reliability depends on the accuracy and representativeness of the sources from which it was obtained, and any errors or biases in the data collection process could propagate through the model's forecasts.

10 Conclusion

The simulation of the ChatGPT adoption model provides an illustrative forecast of the technology's uptake within the United States, delineating a trajectory marked by robust initial growth followed by phases of stabilization and fluctuations. This trend, emergent from the simulation results, can be attributed to a complex interplay of factors, notably the parameters that encapsulate social influence and individual propensity to engage with novel technology.

The model postulates that the inception of ChatGPT's market penetration is primarily governed by the rate at which potential users trial the platform, as well as the persuasive power exerted by existing adopters. However, as the novelty wanes and the user base expands, the influence of these factors naturally attenuates, giving way to a more gradual growth curve that may plateau or even recede. This indicates that the initial surge in adoption is tempered over time by variables such as market saturation and the diminishing returns of user acquisition efforts.

For future iterations of the model, a multi-faceted approach could be employed to refine its predictive accuracy. Enriching the model with data that captures demographic diversity and user segmentation could yield insights into adoption behaviors that the current model may not fully illuminate. Furthermore, allowing for time-varying dynamics in the parameters representing social influence and individual decision-making processes could align the model more closely with real-world phenomena. Inclusion of external datasets reflecting economic conditions, technological advancements, and competitive actions could provide a more holistic understanding of the factors steering ChatGPT's adoption.

References

[1] Mahajan, Vijay, et al. "Determination of Adopter Categories by Using Innovation Diffusion Models." *Journal of Marketing Research*, vol. 27, no. 1, 1990, pp. 37–50. JSTOR, https://doi.org/10.2307/3172549. Accessed 16 Mar. 2024.

Github appendix for codes and visualization: https://github.com/FangkeJiang/math-111a