Productivity and Efficiency Analysis

2) Data envelopment analysis

c) Production theory

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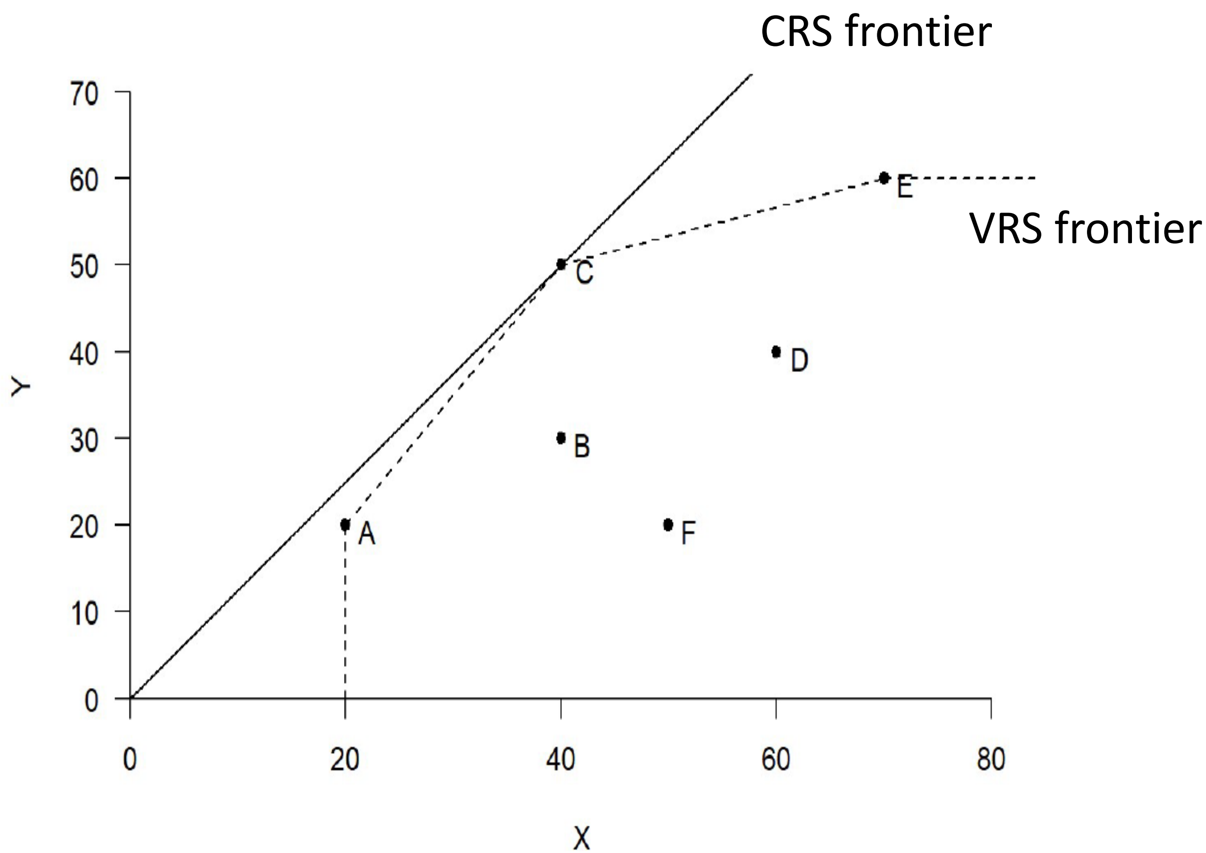
Data envelopment analysis (DEA)

Specification of the DEA formulation implies certain assumptions regarding :

1) Production technology (returns to scale)

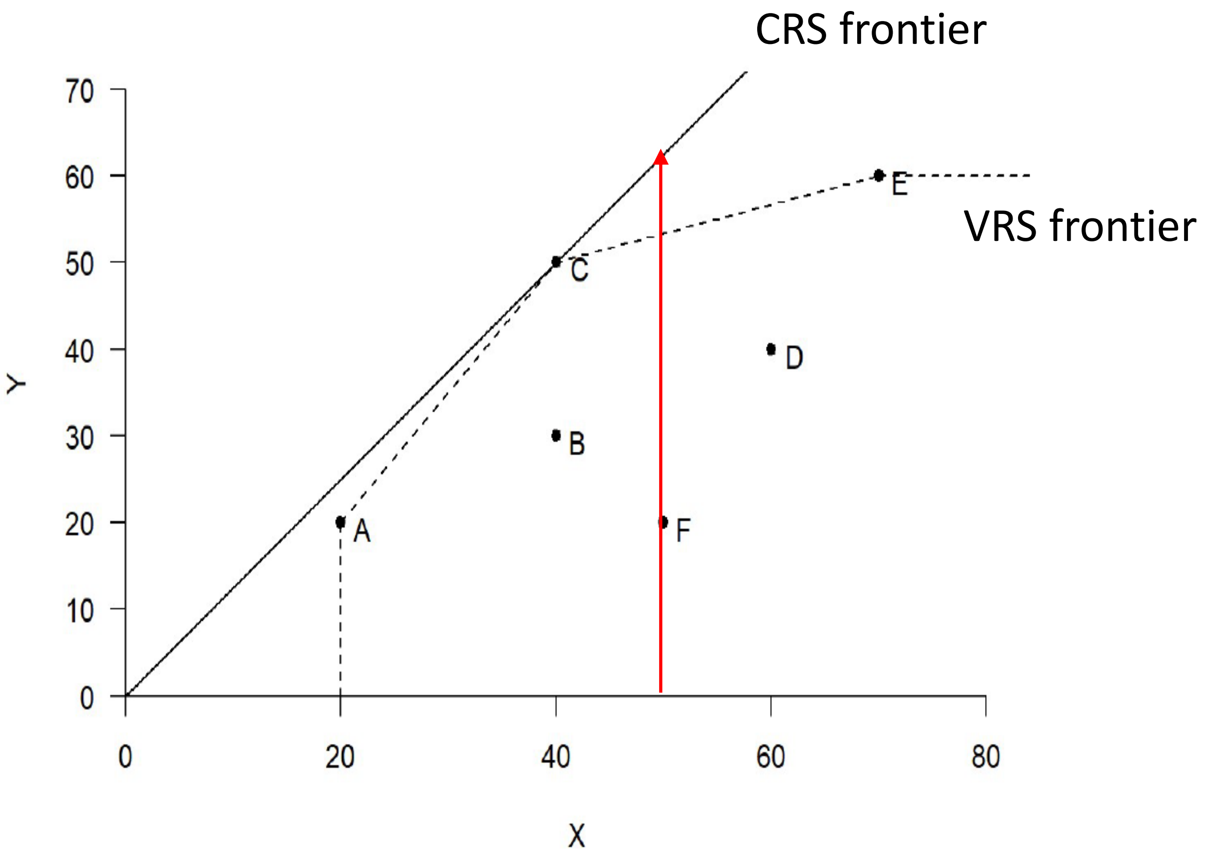
2) Efficiency metric (input, output orientation)

Illustrative example

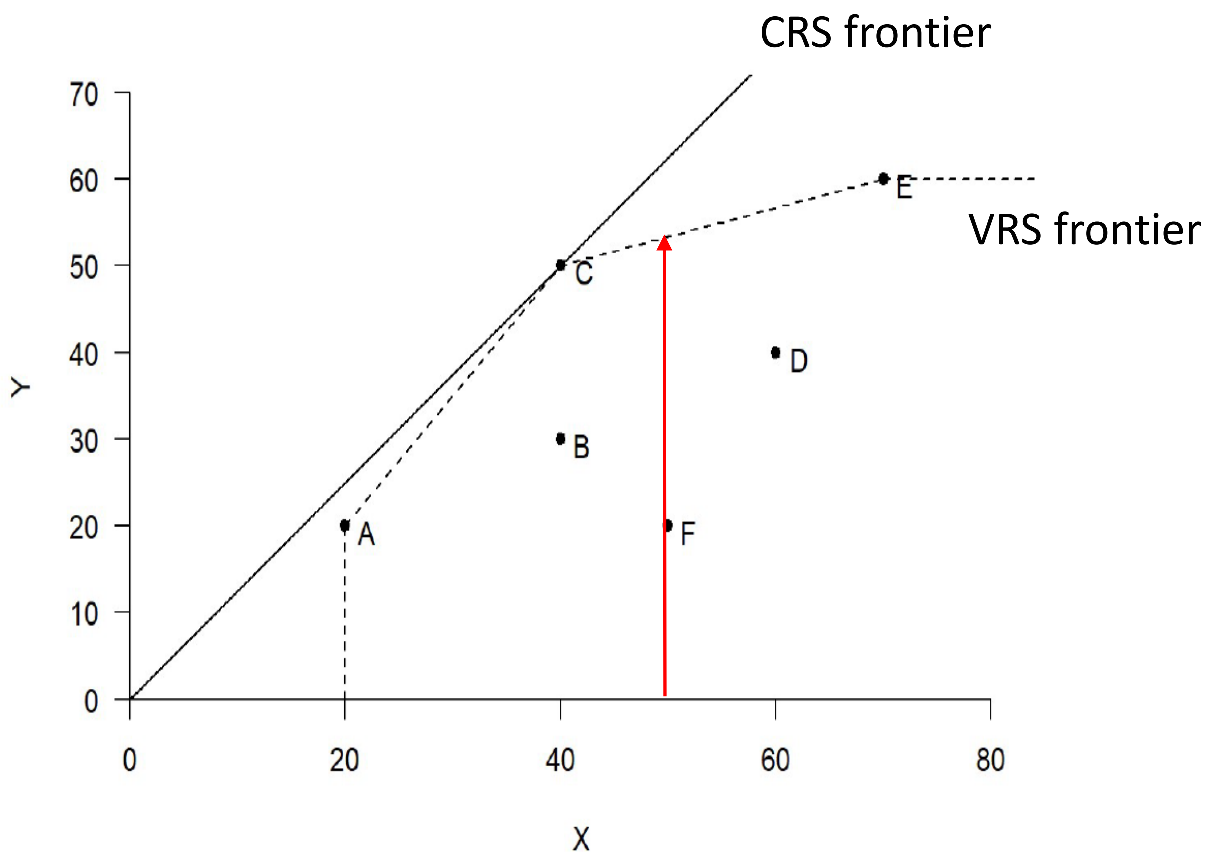


Firm	Input	Output
A	20	20
B	40	30
C	40	50
D	60	40
E	70	60
F	50	20

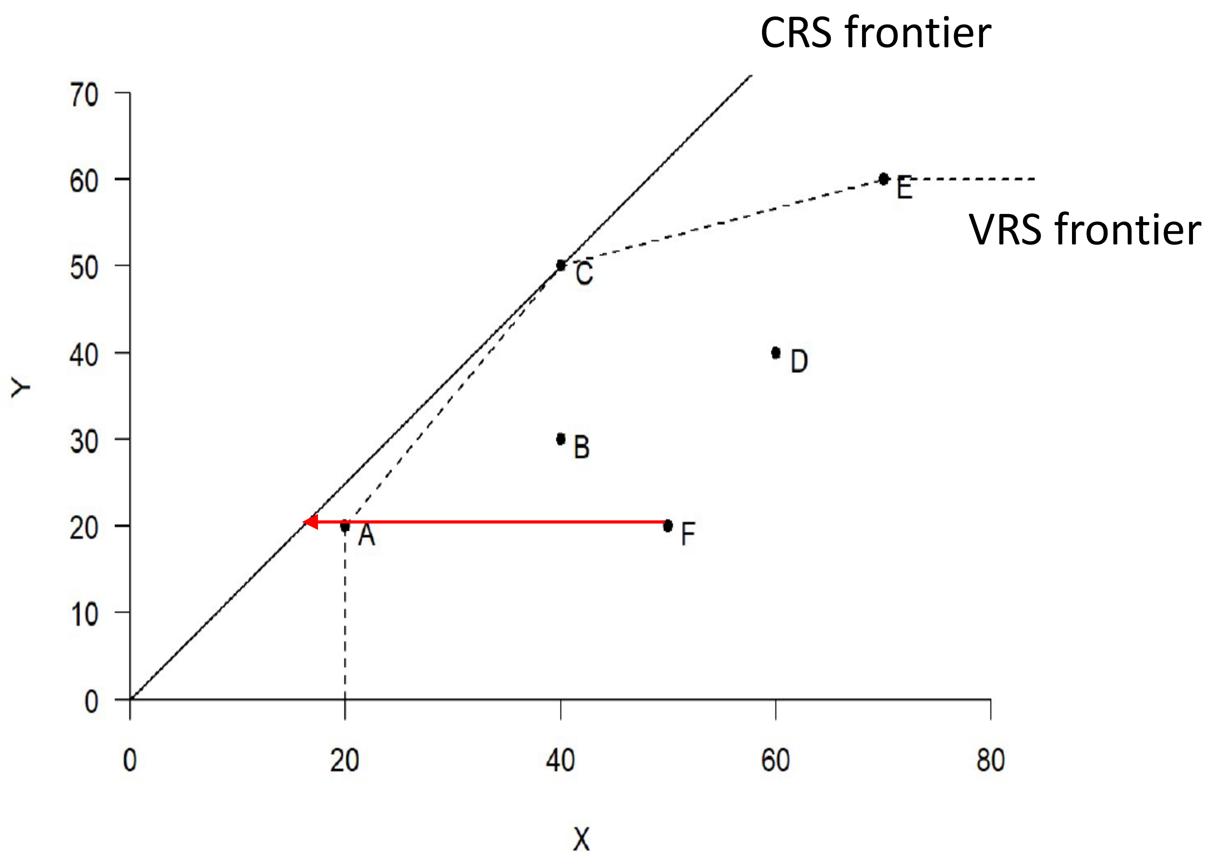
Illustrative example:
Output oriented CRS efficiency of unit F



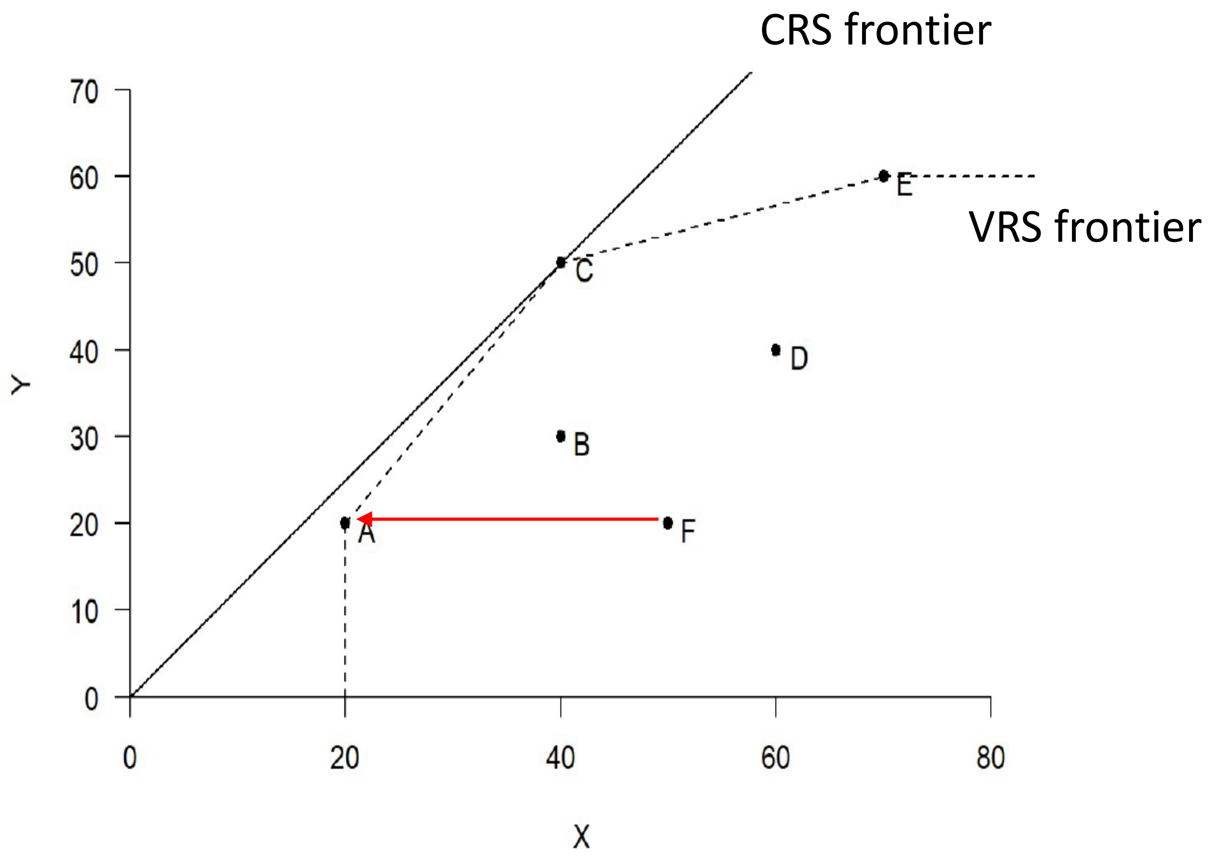
Illustrative example:
Output oriented VRS efficiency of unit F



Illustrative example: Input oriented CRS efficiency of unit F



Illustrative example:
Input oriented CRS efficiency of unit F



Production technology

General representation of technology: Production Possibility Set (PPS)

$T = \{(\mathbf{x},\mathbf{y}) \mid \text{inputs } \mathbf{x} \text{ can produce outputs } \mathbf{y}\}$

Connection to production function: in the single output case where output y is a scalar, the PPS can be stated as

$T = \{(\mathbf{x},\mathbf{y}) \mid f(\mathbf{x}) \geq y\}$

In other words, production function f characterizes the boundary of the PPS.

Production technology

Three classical axioms:

1) Free disposability: If $(\mathbf{x}',\mathbf{y}') \in T$, then

$(\mathbf{x},\mathbf{y}) = (\mathbf{x}' + \mathbf{a},\mathbf{y}' - \mathbf{b}) \in T$ for all $\mathbf{a}\geq\mathbf{0}, \mathbf{b}\geq\mathbf{0}$.

2) Convexity: If $(\mathbf{x}',\mathbf{y}') \in T$ and $(\mathbf{x}'',\mathbf{y}'') \in T$, then

$(\mathbf{x},\mathbf{y}) = c(\mathbf{x}',\mathbf{y}') + (1\text{-}c)\,(\mathbf{x}'',\mathbf{y}'') \in T$ for all $c\geq0$.

3) Constant returns to scale (CRS): If $(\mathbf{x}',\mathbf{y}') \in T$ then

$(\mathbf{x},\mathbf{y}) = d(\mathbf{x}',\mathbf{y}') \in T$ for all $d\geq0$.

Stated in terms of production function *f*, these axioms imply that 1) *f* is monotonic increasing, 2) *f* is concave, and 3) *f* is linearly homogenous.

Classic DEA technology (CRS)

Denote the observed data by input matrix **X** and output matrix **Y**. The classic DEA technology is

$T^{DEA\text{-}CRS} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x} \geq \mathbf{X}\boldsymbol{\lambda};\ \mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda};\ \boldsymbol{\lambda} \geq \mathbf{0}\}$.

Minimum extrapolation theorem 1:

$T^{DEA\text{-}CRS}$ is the smallest set that

- contains all observed data points and
- satisfies axioms 1-3

DEA technology (VRS)

We can relax the CRS axiom 3. The resulting *variable returns to scale* (VRS) technology is

$T^{DEA\text{-}VRS} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x} \geq \mathbf{X}\boldsymbol{\lambda};\ \mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda};\ \boldsymbol{\lambda} \geq \mathbf{0};\ \mathbf{1}'\boldsymbol{\lambda} = 1\}$.

Note the sum of $\boldsymbol{\lambda}$ weights must be equal to 1.

Minimum extrapolation theorem 2:

$T^{DEA\text{-}VRS}$ is the smallest set that

- contains all observed data points and
- satisfies axioms 1-2

Free disposable hull (FDH)

We can relax the convexity axiom 2, too. The resulting free disposable hull (FDH) technology is

$T^{FDH} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x} \geq \mathbf{X\lambda};\ \mathbf{y} \leq \mathbf{Y\lambda};\ \mathbf{1}'\boldsymbol{\lambda} = 1;\ \lambda \in \{0,1\}\ \}.$

Note the λ weights must be either 0 or 1.

Minimum extrapolation theorem 3:

T^{FDH} is the smallest set that

- contains all observed data points and
- satisfies axiom 1

Efficiency metrics

In DEA, efficiency is measured as a distance from the evaluated unit ($\mathbf{x}_i$, $\mathbf{y}_i$) to the boundary of T^{DEA}

The most commonly used efficiency metric is Farrell's (1957) radial input and output oriented efficiency measures

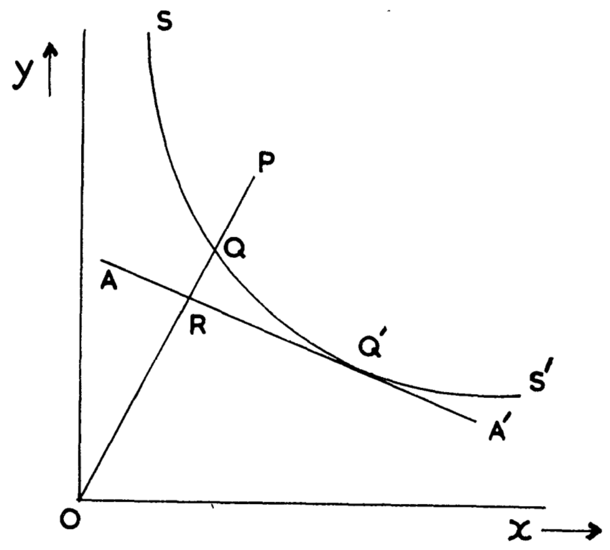


DIAGRAM 1.

Next lesson

2d) Statistical approach to DEA