Productivity and Efficiency Analysis

2) Data envelopment analysis

e) Pre-history of DEA in economics

Timo Kuosmanen

Aalto University School of Business

https://people.aalto.fi/timo.kuosmanen

Data envelopment analysis (DEA)

Official history of DEA in OR/MS literature:

Example: "*DEA in its current form was first described in Charnes et al. (1978), who propose a novel method that combines and transforms multiple inputs and outputs into a single efficiency index*." Liu et al. (2013)

Reference:

Liu, Lu, Luc, Lin (2013) Data envelopment analysis 1978–2010: A citation-based literature survey, *Omega*

Activity analysis in economics

- Koopmans (1951):
  - “Analysis of Production as an Efficient Combination of Activities” http://cowles.econ.yale.edu/P/cm/m13/index.htm

- Farrell (1957, *JRSS*):
  - “The Measurement of Productive Efficiency”

- Shephard (1970):
  - “Theory of Cost and Production Functions”

- Afriat (1972, *Int. Econ. Rev*.)
  - “Efficiency Estimation of Production Functions ”

Von Neumann's model

In the actual economy, these processes P_i, $i = 1, \ldots, m$, will be used with certain *intensities* x_i, $i = 1, \ldots, m$. That means that for the total production the quantities of equations (1) must be multiplied by x_i. We write symbolically :

$$E = \sum_{i=1}^{m} x_i \, P_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$x_i = 0$ means that process P_i is not used.

3. The numerical unknowns of our problem are : (*i*) the *intensities* $x_1, \ldots, x_m$ of the processes $P_1, \ldots, P_m$; (*ii*) the *coefficient of expansion* of the whole economy α ; (*iii*) the *prices* $y_1, \ldots, y_n$ of goods $G_1, \ldots, G_n$; (*iv*) the interest factor

Von Neumann (1938; 1946 *R.E.Stud.*)

Von Neumann's efficincy index

$$\alpha = \underset{j=1,\ldots,n}{\text{Min.}} \left[\frac{\sum_{i=1}^{m} b_{ij} \; x_i}{\sum_{i=1}^{m} a_{ij} \; x_i} \right] \quad \ldots\ldots\ldots\ldots \quad (10),$$

$$\beta = \underset{i=1,\ldots,m}{\text{Max.}} \; \frac{\sum_{j=1}^{n} b_{ij} \, y_j}{\sum_{j=1}^{n} a_{ij} \, y_j} \quad \ldots\ldots\ldots\ldots \quad (11).$$

The greatest (purely technically possible) factor of expansion α' *of the whole economy is* $\alpha' = \alpha = \beta$, *neglecting prices.*

The lowest interest factor β' *at which a profitless system of prices is possible is* $\beta' = \alpha = \beta$, *neglecting intensities of production.*

Koopmans (1951)

resulting net outputs, y_n, can be written as

$$(1.5) \qquad y_n = \sum_{k=1}^{K} a_{nk} x_k, \qquad x_k \geqq 0 \quad (n = 1, \cdots, N;\, k = 1, \cdots, K).$$

The activity vectors (1.4) can be adjoined to form the *technology matrix*, or briefly the *technology*

$$(1.6) \qquad A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ \cdot & \cdot & \cdot & \cdot \\ a_{N1} & a_{N2} & \cdots & a_{NK} \end{bmatrix}.$$

T. C. KOOPMANS

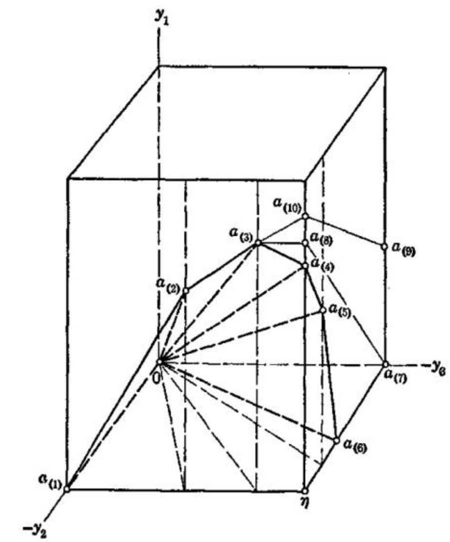

FIGURE 7

THEOREM 4.7: *A necessary and sufficient condition that the activity vector x shall lead to an efficient point $y = Ax$ in the commodity space is that there exists a vector p of positive prices such that no activity in the technology permits a positive profit and such that the profit on all activities carried out at a positive level be zero.*

Debreu (1951)

Coefficient of resource utilization

We have proved that, $\mathfrak{Z}$ being convex,

$$\rho = \operatorname*{Min}_{\mathbf{z}\in\mathfrak{Z}} \operatorname*{Max}_{\bar{\mathbf{p}}\in\bar{\mathfrak{P}}} \bar{\mathbf{p}}\cdot\mathbf{z} = \operatorname*{Max}_{\bar{\mathbf{p}}\in\bar{\mathfrak{P}}} \operatorname*{Min}_{\mathbf{z}\in\mathfrak{Z}} \bar{\mathbf{p}}\cdot\mathbf{z}.$$

The saddle points are $\mathbf{z}^*$, all of whose components are equal to ρ, associated with any normal $\bar{\mathbf{p}}^*$ to $\mathfrak{Z}^{\min}$ at $\mathbf{z}^*$. They appear to be the result of the antagonistic activities of a central agency which chooses $\bar{\mathbf{p}}$ in $\bar{\mathfrak{P}}$ so as to *maximize* $\bar{\mathbf{p}}\cdot\mathbf{z}$ and of production units (resp. consumption units) which choose $\mathbf{y}_j$ (resp. $\mathbf{x}_i$) in $\mathfrak{Y}_j$ (resp. $\mathfrak{X}_i$) so as to *minimize* $\bar{\mathbf{p}}\cdot\mathbf{z}$.[22]

[21] If we were interested only in the fact that the operations Min and Max can be inverted, we could give the very short following proof: choose $\mathbf{z}^*$ and one of the $\mathbf{p}^*$ and show that this is a saddle point [**17**, Section 13]. This is indeed immediate but hardly enlightening.

[22] The structure of the set $\mathfrak{Z}$ makes these antagonistic activities formally different from a zero-sum two-person game in the von Neumann-Morgenstern [**17**, Section 17] sense.

Farrell (1957, *J. Royal Stat. Soc.*)

- Decomposition (left diagram):

  Cost efficiency (OR/OP)

  = Technical efficiency (OQ/OP) x Allocative efficiency (OR/OQ)
- Piece-wise linear isoquant (right diagram)

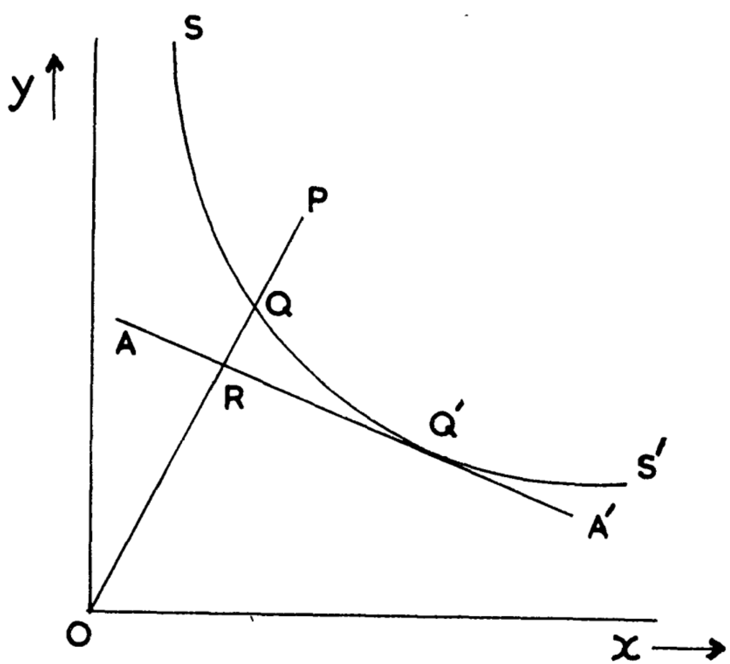

DIAGRAM 1.

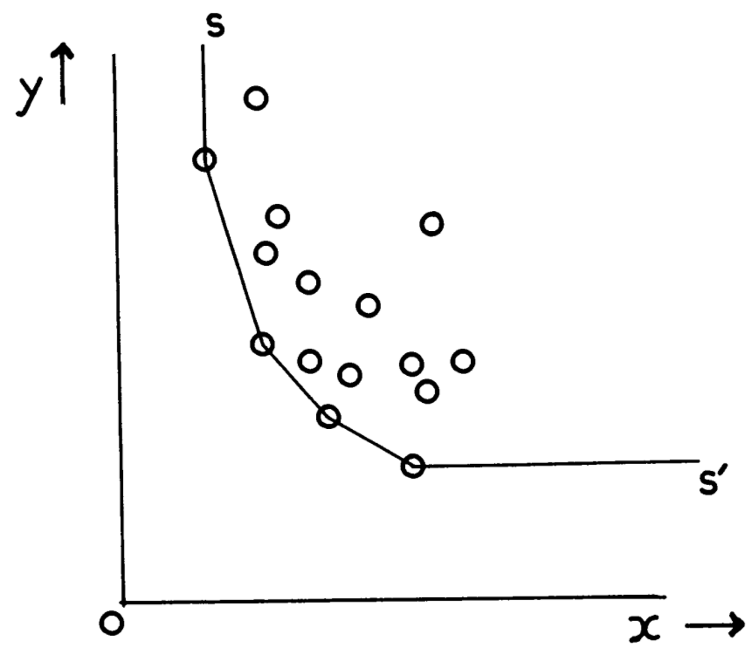

DIAGRAM 2.

Afriat (1972, *Int. Econ. Rev.*)

"Efficiency Estimation of Production Functions "

- **Axioms:**

(P1) f non-decreasing (free disposal of input)
(P2) f non-decreasing concave (classical)
(P3) f non-decreasing concave conical (classical constant returns to scale)

Afriat proves minimal functions that satisfy axioms (P1, P2, P3) and envelop all data points

(Minimum extrapolation theorems)

(P1) => Free disposable hull (FDH: Deprins, Simar, Tulkens, 1984)

(P2) => DEA VRS (Banker, Charnes, Cooper, 1984; *Man. Sci.*)

(P3) => DEA CRS (Charnes, Cooper, Rhodes, 1978, *EJOR*)

Afriat (1972, *Int. Econ. Rev.*)

"Efficiency Estimation of Production Functions"

- **Minimum extrapolation theorems (convex, VRS case):**

THEOREM 1.2. *For any given* $(x_t, y_t)(t = 1, \ldots, k)$, *the function*

$$F(x) = \max[\Sigma y_t\lambda_t \colon \Sigma x_t\lambda_t \leqq x,\ \Sigma\lambda_t = 1,\ \lambda_t \geqq 0]$$

is non-decreasing concave and such that $y_t \leqq F(x_t)$. *There exists a non-decreasing concave function* $f(x)$ *such that* $y_t = f(x_t)$ *for all* t *if and only if*

$$\Sigma x_t\lambda_t \leqq x_s \Longrightarrow \Sigma y_t\lambda_t \leqq y_s\ (\lambda_t \geqq 0,\ \Sigma\lambda_t = 1)\ ,$$

and this is if and only if $y_t = F(x_t)$. *In this case* $F(x)$ *is itself one such function, and it is everywhere not greater than any other. In any case* $F(x)$ *is everywhere not greater than any other non-decreasing concave function* $f(x)$ *such that* $y_t \leqq f(x_t)$.

Afriat (1972, *Int. Econ. Rev.*)

"Efficiency Estimation of Production Functions"

A random sample of k individuals from a population run as fast as they can over 100 yards, and the average speed of each is recorded, say $v_1, \ldots, v_k$. It could be asked on such data what is the maximum possible speed v_m with which 100 yards can be covered. This speed must be greater than any in the sample. If it could be assigned then $e_t = v_t/v_m$ is a 100-yards running efficiency.

Let $\rho(x)$ be a probability density on $<0, 1>$ representing a fixed hypothesis about the distribution of efficiency. Then v_m can be determined to make the likelihood $\Pi_t\ \rho(v_t/v_m)$ of the efficiencies in the sample a maximum. More generally $\mu_\theta(x)$ can be a probability density carrying parameters θ, representing a hypothesis about the form of the efficiency distribution. Then both v_m and θ can be determined to make $\Pi_t\ \rho_\theta(v_t/v_m)$ a maximum.

Afriat (1972, *Int. Econ. Rev.*)

"Efficiency Estimation of Production Functions "

- **Cost function and duality**

the cost function is

$$C_F(p, y) = \min\,[px\colon \Sigma y_t\lambda_t \geqq y,\ \Sigma x_t\lambda_t \leqq x,\ \Sigma\lambda_t = 1,\ \lambda_t \geqq 0]\,.$$

A dual expression is

$$C_F(p, y) = \max\,[sy - \theta\colon \theta \geqq sy_t - p_t x_t\,,\ s \geqq 0]$$

- **Profit efficiency**
- **Constant returns to scale**
- **Parametric programming (Cobb-Douglas) subject to axioms P1, P2, P3**

Next topic

3) Stochastic frontier analysis (SFA)