Productivity and Efficiency Analysis

2) Data envelopment analysis

b) DEA formulations

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Data envelopment analysis (DEA)
Charnes, Cooper & Rhodes (1978), EJOR

Application: juvenile delinquency programs

Problem: how to aggregate multiple inputs and outputs

Fractional programming problem:

$$Eff_i = \max_{\beta,\gamma} \frac{\mathbf{y}_i'\boldsymbol{\gamma}}{\mathbf{x}_i'\boldsymbol{\beta}}$$

$$s.t. \qquad \frac{\mathbf{y}_h'\boldsymbol{\gamma}}{\mathbf{x}_h'\boldsymbol{\beta}} \leq 1 \quad \forall h$$

$$\boldsymbol{\gamma} \geq \mathbf{0}, \boldsymbol{\beta} \geq \mathbf{0}$$

β is the vector of input weights

γ is the vector of output weights

Data envelopment analysis (DEA)

$\mathbf{y}_i$ is the output vector of the evaluated unit *i* (data)

$\mathbf{x}_i$ is the input vector of the evaluated unit *i* (data)

$\boldsymbol{\beta}$ is the vector of input weights (variables)

$\boldsymbol{\gamma}$ is the vector of output weights (variables)

Our objective is to maximize the ratio of weighted output to weighted inputs (= productivity ratio)

$$Eff_i = \max \frac{\mathbf{y}'_i \boldsymbol{\gamma}}{\mathbf{x}'_i \boldsymbol{\beta}} = \frac{\sum_{s=1}^{S} y_{si} \gamma_s}{\sum_{m=1}^{M} x_{mi} \beta_m}$$

Data envelopment analysis (DEA)

Our objective is to maximize the ratio of weighted output to weighted inputs (= productivity ratio)

$$Eff_i = \max \frac{\mathbf{y}_i'\boldsymbol{\gamma}}{\mathbf{x}_i'\boldsymbol{\beta}}$$

Subject to the constraints that the weights are non-negative

$$\boldsymbol{\gamma} \geq \mathbf{0}, \boldsymbol{\beta} \geq \mathbf{0}$$

and the maximum efficiency is 1:

$$\frac{\mathbf{y}_h'\boldsymbol{\gamma}}{\mathbf{x}_h'\boldsymbol{\beta}} \leq 1 \quad \forall h$$

Data envelopment analysis (DEA)

Charnes, Cooper & Rhodes (1978), EJOR

Charnes-Cooper transformation (Charnes&Cooper, 1962) to a linear programming (LP) problem:

$$
\begin{aligned}
& Eff_i = \max \mathbf{y}_i' \boldsymbol{\gamma} \\
& s.t. \qquad \mathbf{x}_i' \boldsymbol{\beta} = 1 \\
& \qquad\qquad \mathbf{y}_h' \boldsymbol{\gamma} - \mathbf{x}_h' \boldsymbol{\beta} \le 0 \quad \forall h \\
& \qquad\qquad \boldsymbol{\gamma} \ge \mathbf{0}, \boldsymbol{\beta} \ge \mathbf{0}
\end{aligned}
$$

This LP problem is equivalent to the previous fractional programmig problem, but can be solved by standard LP solvers.

This LP problem is usually solved separately for each firm *i*

Data envelopment analysis (DEA)

Charnes, Cooper & Rhodes (1978), EJOR

Equivalent dual formulation using intensity weights:

$$
\begin{aligned}
Eff_i \quad &= \min \phi \\
s.t. \qquad &\phi \mathbf{x}_i \geq \mathbf{X}\boldsymbol{\lambda} \\
&\mathbf{y}_i \leq \mathbf{Y}\boldsymbol{\lambda} \\
&\boldsymbol{\lambda} \geq \mathbf{0}
\end{aligned}
$$

X is *nxm* matrix of inputs (units *h* = 1,…,*n*)

Y is *nxs* matrix of outputs (units *h* = 1,…,*n*)

Data envelopment analysis (DEA)

Dual formulation using the sum notation:

$$Eff_i = \min \phi$$

$$s.t. \qquad \phi x_{mi} \geq \sum_{h=1}^{n} x_{mh} \lambda_h \ \forall m$$

$$y_{si} \leq \sum_{h=1}^{n} y_{sh} \lambda_h \ \forall s$$

$$\lambda_h \geq 0 \ \forall h$$

Basic DEA formulations

Input orientation

$$\min \phi$$

$$s.t.$$

$$\phi \mathbf{x}_i \geq \mathbf{X\lambda}$$

$$\mathbf{y}_i \leq \mathbf{Y\lambda}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

Output orientation

$$\max \theta$$

$$s.t.$$

$$\mathbf{x}_i \geq \mathbf{X\lambda}$$

$$\theta \mathbf{y}_i \leq \mathbf{Y\lambda}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

VRS:

Add constraint $\mathbf{1}'\boldsymbol{\lambda} = 1$

Data envelopment analysis (DEA)

Specification of the DEA formulation implies certain assumptions regarding :

1) Production technology (returns to scale)

2) Efficiency metric (input or output orientation)

Next lesson

2c) Production theory