Productivity and Efficiency Analysis

6) Multiple outputs and bad outputs

6b) Distance functions and DEA

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Production theory: output sets $P(\mathbf{x})$

$$P(\mathbf{x}) = \{\mathbf{q}: \mathbf{x} \text{ can produce } \mathbf{q}\}$$

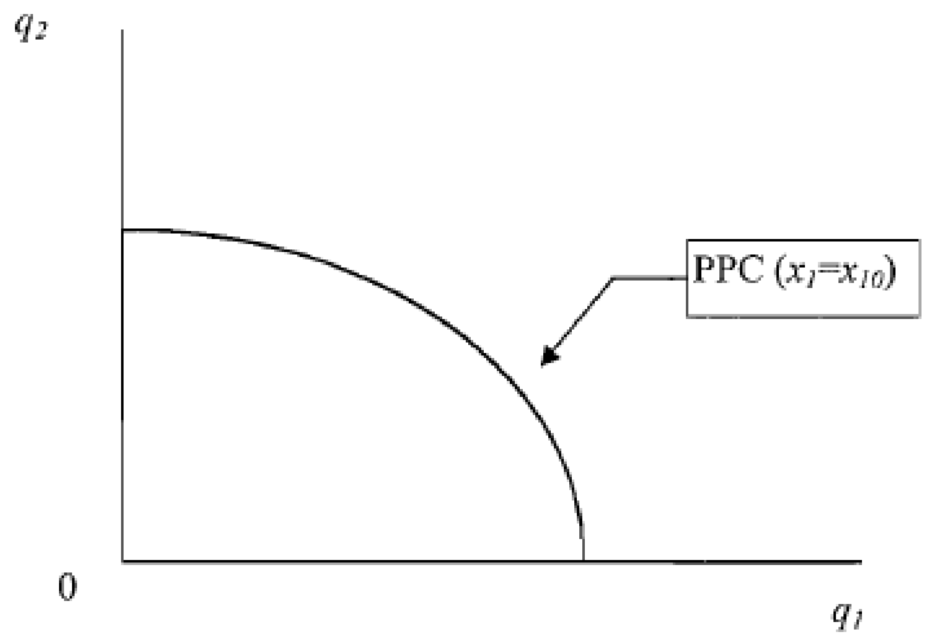
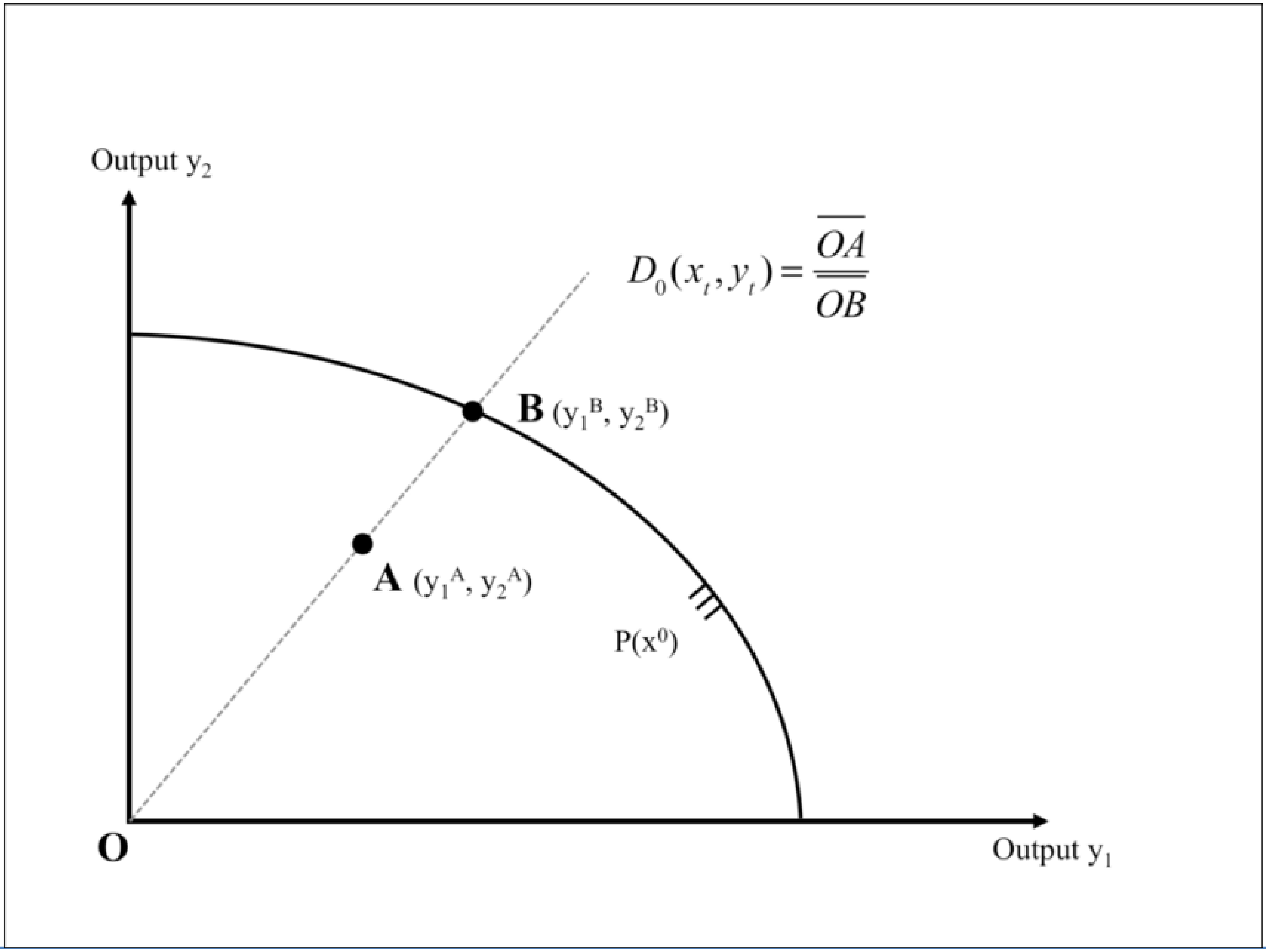


Figure 3.1 Production Possibility Curve

Source: Coelli, Prasada Rao, O'Donnell & Battese (2005)

Output distance function $D_O(\mathbf{x,y})$



Source: Färe and Primont (1995)

Output distance function $D_O(\mathbf{x},\mathbf{y})$

- Duality theory:

$$D_O(\mathbf{x},\mathbf{y}) \leq 1 \quad \leftrightarrow (\mathbf{x},\mathbf{y}) \in T$$

- Properties of T carry over to D
- *Important*: Distance function is not only a measure of efficiency, it is a valid functional representation of the technology
  - "generalized production function"

Output distance function in DEA

$$T^{DEA\text{-}VRS} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{y} \le \mathbf{Y}\boldsymbol{\lambda};\ \mathbf{x} \ge \mathbf{X}\boldsymbol{\lambda};\ \boldsymbol{\lambda} \ge \mathbf{0};\ \mathbf{1}'\boldsymbol{\lambda} = 1\}$$

$$D_O^{DEA\text{-}VRS}(\mathbf{x},\mathbf{y}) = \max \theta$$

s.t.

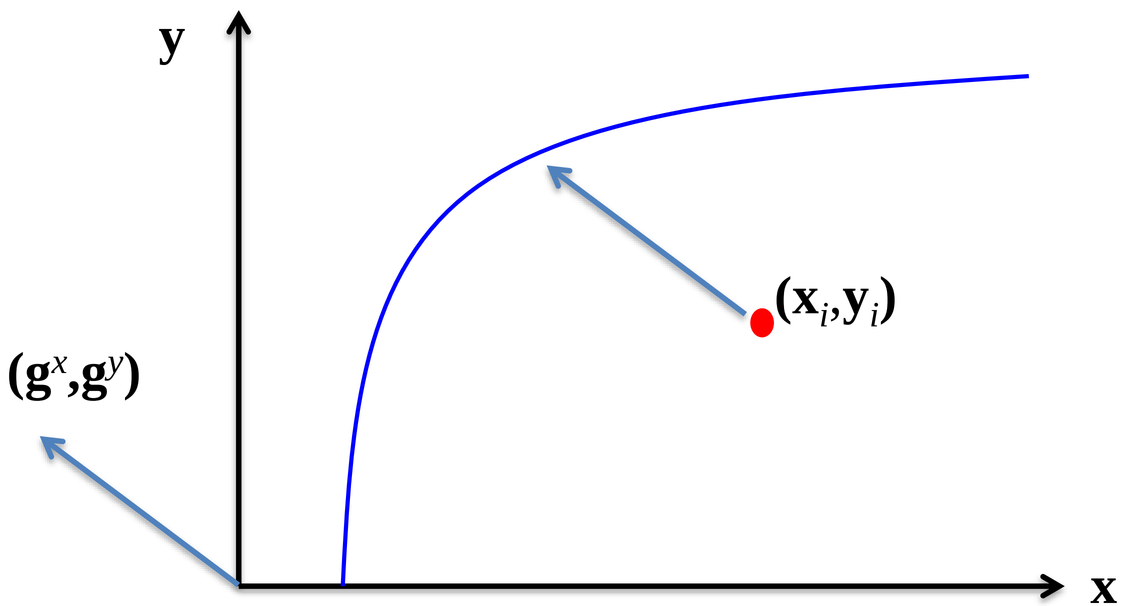
$$\theta\mathbf{y} \le \mathbf{Y}\boldsymbol{\lambda}$$

$$\mathbf{x} \ge \mathbf{X}\boldsymbol{\lambda}$$

$$\mathbf{1}'\boldsymbol{\lambda} = 1$$

$$\boldsymbol{\lambda} \ge \mathbf{0}$$

Directional distance function (DDF)



Directional distance function (DDF)

Production possibility set:

$$T = \{(\mathbf{x}, \mathbf{y}) \in \Re_+^{m+s} | \mathbf{x} \text{ can produce } \mathbf{y}\}.$$

Directional distance function (DDF) (Chambers et al., 1996):

$$\vec{D}_T(\mathbf{x}, \mathbf{y}, \mathbf{g}^x, \mathbf{g}^y) = \sup\{\theta | (\mathbf{x} - \theta \mathbf{g}^x, \mathbf{y} + \theta \mathbf{g}^y) \in T\},$$

Note:

$$T = \{(\mathbf{x}, \mathbf{y}) \in \Re_+^{m+s} | \vec{D}_T(\mathbf{x}, \mathbf{y}, \mathbf{g}^x, \mathbf{g}^y) \geq 0\} \text{ for any } (\mathbf{g}^x, \mathbf{g}^y) \in \Re_+^{m+s}$$

Directional distance function in DEA

$$T^{DEA\text{-}VRS} = \{(\mathbf{x},\mathbf{y}) \mid \mathbf{y} \le \mathbf{Y}\boldsymbol{\lambda};\ \mathbf{x} \ge \mathbf{X}\boldsymbol{\lambda};\ \boldsymbol{\lambda} \ge \mathbf{0};\ \mathbf{1}'\boldsymbol{\lambda} = 1\}$$

$$DDF(\mathbf{x},\mathbf{y};\mathbf{g}^{\mathbf{x}},\mathbf{g}^{\mathbf{y}}) = \max \theta$$

s.t.

$$\mathbf{y} + \theta\mathbf{g}^{\mathbf{y}} \le \mathbf{Y}\boldsymbol{\lambda}$$

$$\mathbf{x} - \theta\mathbf{g}^{\mathbf{x}} \ge \mathbf{X}\boldsymbol{\lambda}$$

$$\mathbf{1}'\boldsymbol{\lambda} = 1$$

$$\boldsymbol{\lambda} \ge \mathbf{0}$$

Cost function C

- Another common approach to model multiple outputs is to use a cost aggregate of inputs
- *Definition:* Cost function is the minimum cost of producing the given outputs **y** at given input prices **w:**

$$C(\mathbf{y},\mathbf{w}) = \min \{\mathbf{w}'\mathbf{x} \mid (\mathbf{x},\mathbf{y}) \in T\}$$

Cost function C

- Duality theory:

$$C(\mathbf{y},\mathbf{w}) \le \mathbf{w}'\mathbf{x} \quad \leftrightarrow (\mathbf{x},\mathbf{y}) \in T$$

- Properties of T carry over to C
- Cost function is a valid dual representation of the technology using input prices and output quantities.

Cost function in DEA

$$C^{DEA\text{-}VRS}(\mathbf{w},\mathbf{y}) = \min \mathbf{w}'\boldsymbol{\chi}$$

s.t.

$$\mathbf{y} \leq \mathbf{Y}\boldsymbol{\lambda}$$

$$\boldsymbol{\chi} \geq \mathbf{X}\boldsymbol{\lambda}$$

$$\mathbf{1}'\boldsymbol{\lambda} = 1$$

$$\boldsymbol{\lambda} \geq \mathbf{0}$$

Cost function in DEA

Special case: 1) all firms take the same prices w as given, 2) total cost aggregate *x* observed

$C^{DEA\text{-}VRS}(\mathbf{y}) = \min x$

s.t.

$\mathbf{y} \leq \mathbf{Y\lambda}$

$x \geq \mathbf{X\lambda}$

$\mathbf{1}'\boldsymbol{\lambda} = 1$

$\boldsymbol{\lambda} \geq \mathbf{0}$

Next lesson

6c) Bad outputs