Productivity and Efficiency Analysis

4) Unified approach: StoNED

e) Quantiles and expectiles*

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Quantile production function

Consider the multiplicative frontier model

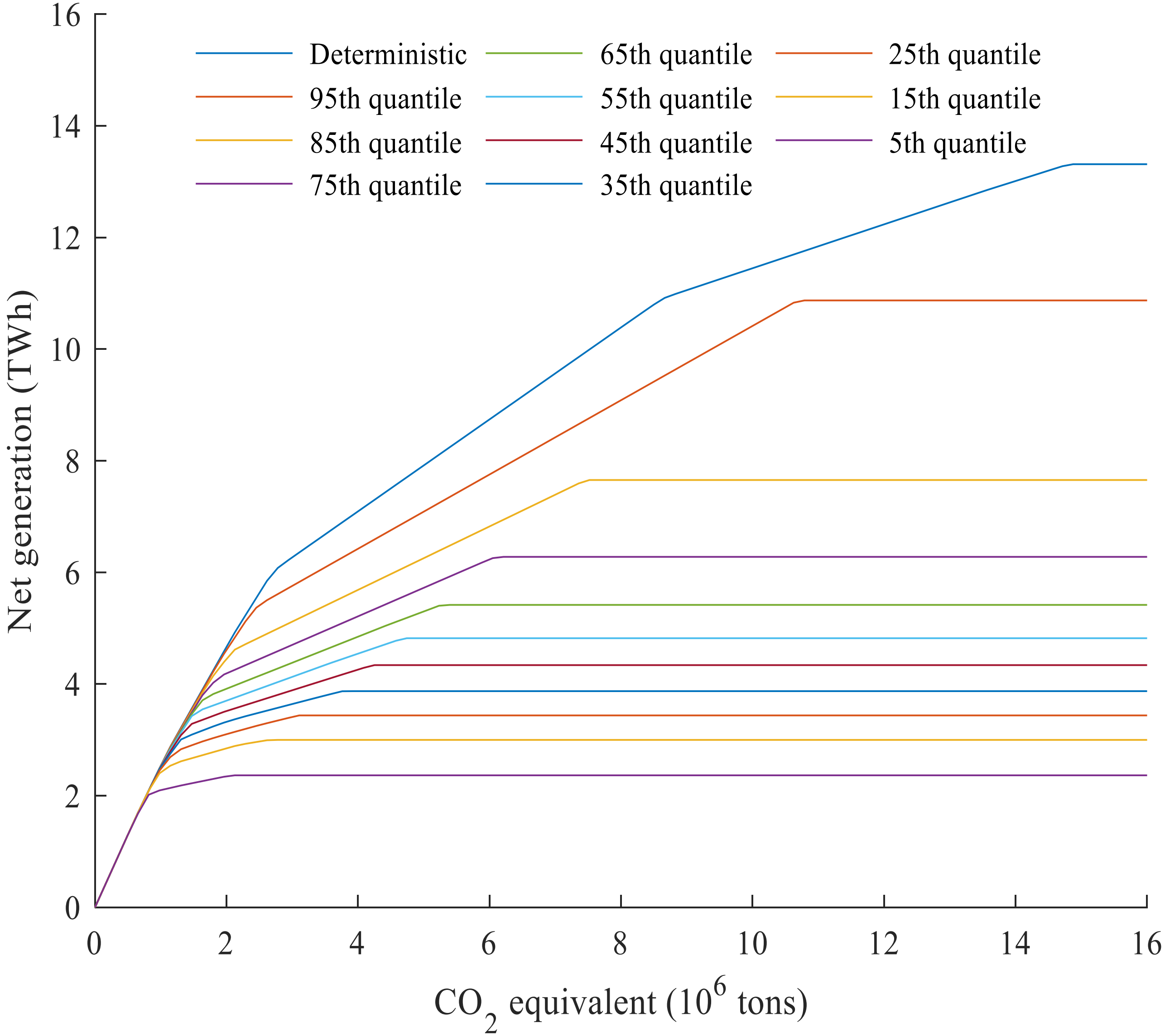
$$y_i = f(\mathbf{x}_i) \cdot \exp(v_i - u_i), \quad i = 1,\dots,n$$

The quantile production function is defined as

$$Q(\tau|\mathbf{x}) = f(\mathbf{x}) \cdot F^{-1}_{\exp(v-u)}(\tau)$$

where τ: $0 < \tau < 1$ is the order of quantile and F is the cumulative distribution function of the composite error term.

Quantile production function: illustration



Quantile convex regression

Following Wang et al. (2014) *EJOR*, we can estimate the τ quantile production function as

$$\min_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\varepsilon}^{-},\boldsymbol{\varepsilon}^{+}} (1-\tau)\sum_{i=1}^{n}\varepsilon_i^{-} + \tau\sum_{i=1}^{n}\varepsilon_i^{+}$$

$$s.t.$$

$$y_i = \alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_i - \varepsilon_i^{-} + \varepsilon_i^{+} \ \forall i$$

$$\alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_i \le \alpha_h + \boldsymbol{\beta}_h'\mathbf{x}_i \ \forall i,h$$

$$\boldsymbol{\beta}_i \ge \mathbf{0} \ \forall i$$

$$\varepsilon_i^{-} \ge 0, \varepsilon_i^{+} \ge 0 \ \forall i$$

Quantile convex regression: properties

Denote the number of observations with strictly positive residual ($e_i^+ > 0$) by n_τ^+ and the number of observations with strictly negative residual ($e_i^- > 0$) by n_τ^-.

Property 1: *For any* $\tau \in [0,1]$, *the optimal solution to the convex quantile regression problem satisfies the inequalities* $\frac{n_\tau^+}{n} \le 1-\tau$ and $\frac{n_\tau^-}{n} \le \tau$.

Property 2: *Suppose the observed data* $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{b}_i)$, $i = 1,\ldots,n$, *are i.i.d. random variables. As* $n \to \infty$, *the optimal solution to the convex quantile regression problem converges to the following theoretical bounds with probability* 1*:*

$$\frac{n_\tau^+}{n} \to 1-\tau \text{ and } \frac{n_\tau^-}{n} \to \tau .$$

Quantile expectile regression

Kuosmanen et al. (2015) propose the following quadratic programming formulation

$$\min_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\varepsilon}^-,\boldsymbol{\varepsilon}^+} (1-\tilde{\tau})\sum_{i=1}^{n}(\varepsilon_i^-)^2 + \tilde{\tau}\sum_{i=1}^{n}(\varepsilon_i^+)^2$$

$$s.t.$$

$$y_i = \alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_i - \varepsilon_i^- + \varepsilon_i^+ \;\; \forall i$$

$$\alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_i \le \alpha_h + \boldsymbol{\beta}_h'\mathbf{x}_i \;\; \forall i,h$$

$$\boldsymbol{\beta}_i \ge \mathbf{0} \;\; \forall i$$

$$\varepsilon_i^- \ge 0, \varepsilon_i^+ \ge 0 \;\; \forall i$$

Quantile expectile regression: properties

For any $\tilde{\tau} \in (0,1)$, *the optimal solution to the convex expectile regression problem yields unique residuals* e^{+} e^{-} *that satisfy the following equations*

$$e_i^{+} e_i^{-} = 0,$$

$$\tilde{\tau} = \frac{\sum_{i=1}^{n} e_i^{-}}{\sum_{i=1}^{n} e_i^{-} + \sum_{i=1}^{n} e_i^{+}}.$$

Expectiles are not the same as quantiles

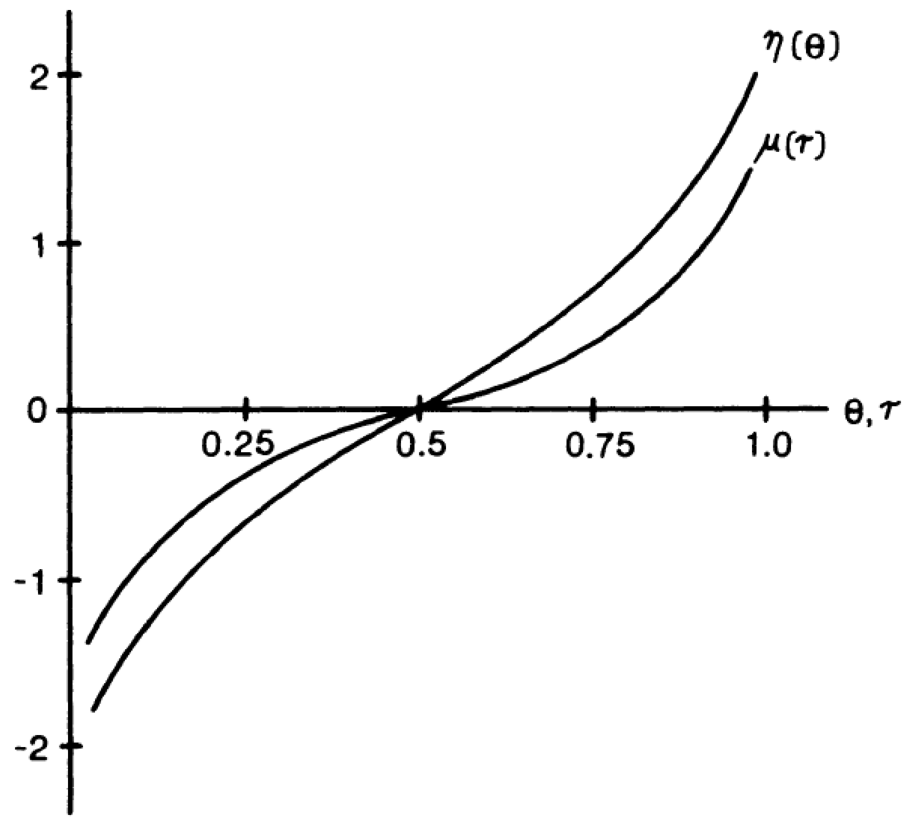


FIGURE 1.—Plot of quantile ($\eta(\theta)$) and expectile ($\mu(\tau)$) functions for the standard journal distribution.

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Source: Newey and Powell (1987) *Econometrica*

But they can be converted to quantiles

There is a unique transfer function h such that

$$h(\tau) = \tilde{\tau}$$

That is. quantiles can be transformed to expectiles. and vice versa.

It is possible to estimate the quantile of interest indirectly using the corresponding expectile.

There is some evidence to suggest that indirect estimation of quantiles using expectiles can help to decrease the mean squared error (MSE).

Monte Carlo simulation

Example

Scenario with 2 inputs. 1 output. 200 observations:

	Expectile		Quantile	
tau	**MSE**	**BIAS**	**MSE**	**BIAS**
0.1	0.095	0.069	0.121	0.052
0.3	0.051	0.024	0.072	0.022
0.5	0.051	0.002	0.066	0.013
0.7	0.070	-0.022	0.079	-0.004
0.9	0.142	-0.089	0.141	-0.041

Monte Carlo simulation

Comparison with FDH-based order-α method

Scenario with 2 inputs. 1 output. 200 observations:

	Order-alpha		Monotonic Quantile	
tau	**MSE**	**BIAS**	**MSE**	**BIAS**
0.1	2.079	-1.116	0.491	0.376
0.3	1.785	-1.123	0.246	0.103
0.5	1.541	-1.036	0.198	0.037
0.7	1.383	-1.011	0.221	-0.106
0.9	1.206	-0.948	0.389	-0.350

Next lesson

5) Contextual variables