Machine Learning

Lecture 01-1: Basics of Probability Theory

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Outline

Random Experiments

- Probability associated with a **random experiment** — a process with uncertain outcomes
- Often kept implicit

Tail Head

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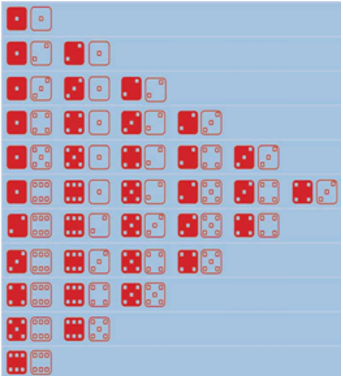
In machine learning, we often assume that data are generated by a hypothetical process (or a model), and task is to determine the structure and parameters of the model from data.

Sample Space

- **Sample space (aka population)** Ω: Set of possible outcomes and a random experiment.
- Example: Rolling two dices.

Ω



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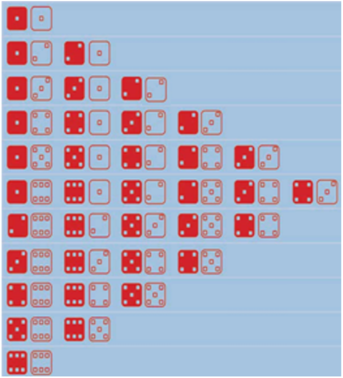
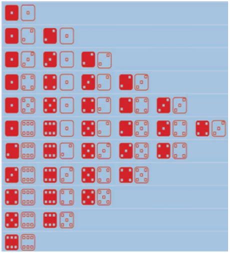


- Elements in a sample space are outcomes.

Events

- **Event**: A subset of the sample space.

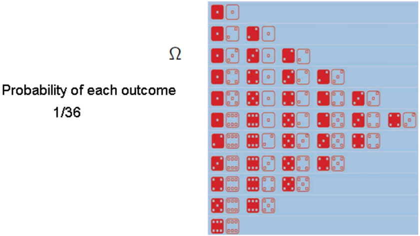
Ω



- Example: The two results add to 4.

Probability Weight Function

- A **probability weight** $P(\omega)$ is assigned to each outcome.



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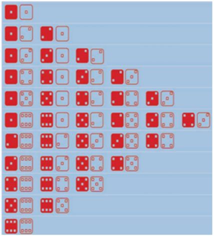
1/36



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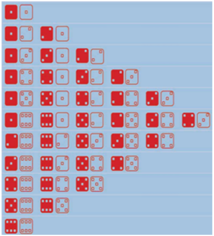

In Machine Learning, we often need to determine the probability weights, or related parameters, from data. This task is called **parameter learning.**

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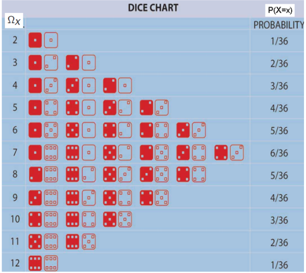
In a more advanced treatment of Probability Theory, we would start with the concept of probability measure, instead of probability weights.

Random Variables

- A **random variable** is a function over the sample space.
  - Example: X = sum of the two results. $X((2,5)) = 7; X((3,1)) = 4)$



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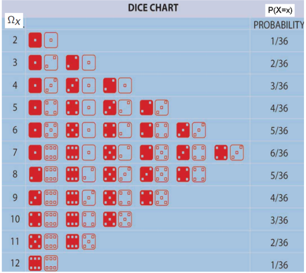

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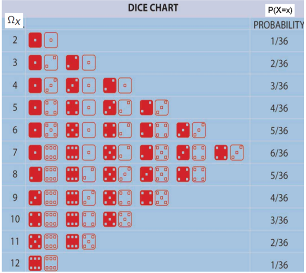
- **Domain** of a random variable: Set of all its possible values.

$$\Omega_X = \{2, 3, \ldots, 12\}$$

Random Variables and Event

- A random variable X taking a specific value x is an event:

$$\Omega_{X=x} = \{\omega \in \Omega | X(\omega) = x\}$$



- $\Omega_{X=4} = \{(1,3),(2,2,)(3,1)\}$.

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- $P(X = 4) = P(\{(1, 3), (2, 2,)(3, 1)\}) = \frac{3}{36}$.
- If X is continuous, we have a **density function** $p(X)$.

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Frequentist interpretation

- Probabilities are **long term relative frequencies**.
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- The frequentist interpretation is meaningful only when experiment can be repeated under the same condition.

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  - Subjectivist: degree of belief based on state of knowledge
    - Primary school student: 0.5
    - Me: 0.8
    - Geographer: 1 or 0
- Arguments such as **Dutch book** are used to explain why one's probability beliefs must satisfy Kolmogorov's axioms.

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- The subjectivist interpretation makes concepts such as conditional independence easy to understand.

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$$Bin(X = k|n, \theta) = \begin{cases} \binom{n}{k}\theta^k(1-\theta)^{n-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k < 0 \text{ or } k > n \end{cases}$$

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- If $n = 1$, then X follows the **Bernoulli distribution**, written as $X \sim Ber(\theta)$

$$Ber(X = x|\theta) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \end{cases}$$

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- Let $\mathbf{x} = (x_1, ..., x_K)$ be a random vector, where x_j is the number of times side j of the die occurs. Then $\mathbf{x}$ follows the **multinomial distribution**, written as $\mathbf{x} \sim Multi(n, \boldsymbol{\theta})$

$$Multi(\mathbf{x}|n, \boldsymbol{\theta}) = \binom{n}{x_1, \ldots, x_K} \prod_{j=1}^{K} \theta_k^{x_j},$$

where $\binom{n}{x_1, \ldots, x_K} = \frac{n!}{x_1! \ldots x_K!}$ is the multinomial coefficient

Categorical Distribution

- In the previous slide, if $n = 1$, $\mathbf{x} = (x_1, ..., x_K)$ has one component being 1 and the others are 0. In other words, it is a **one-hot** vector.
- In this case, $\mathbf{x}$ follows the **categorical distribution**, written as $\mathbf{x} \sim Cat(\boldsymbol{\theta})$

$$Cat(\mathbf{x}|\boldsymbol{\theta}) = \prod_{j=1}^{K} \theta_j^{\mathbf{1}(x_j=1)},$$

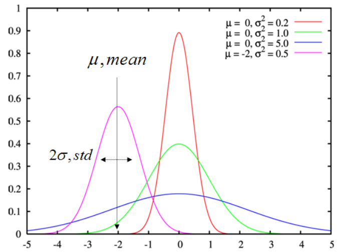
where $\mathbf{1}(x_j = 1)$ is the indicator function, whose value is 1 when $x_j = 1$ and 0 otherwise.

Gaussian (Normal) Distribution

- The most widely used distribution in statistics and machine learning is the Gaussian or normal distribution.
- Its probability density is given by

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Here $\mu = E[X]$ is the mean (and mode), and $\sigma^2 = var[X]$ is the variance



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- A **joint probability mass function**, $P(X_1, X_2, \ldots, X_n)$, over those random variables is:
  - a function defined on the Cartesian product of their state spaces:

$$\prod_{i=1}^{n} \Omega_{X_i} \rightarrow [0,1]$$

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(\Omega_{X_1=x_1} \cap \Omega_{X_2=x_2} \cap \ldots \cap \Omega_{X_n=x_n}).$$

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	public	private	others
low	.17	.01	.02
medium	.44	.03	.01
upper medium	.09	.07	.01
high	0	0.14	0.1

Multivariate Gaussian Distributions

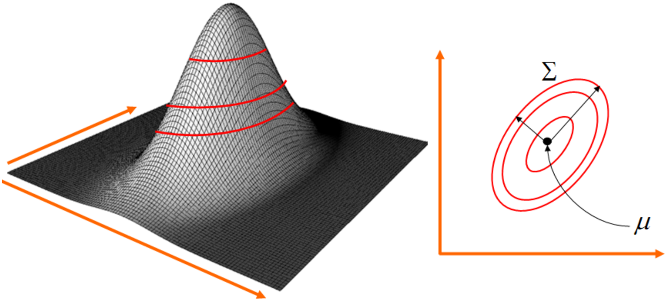
- For continuous variables, the most commonly used joint distribution is the multivariate Gaussian distribution: $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} exp\left[-\frac{(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})}{2}\right]$$

  - D: dimensionality.
  - $\mathbf{x}$: vector of D random variables, representing data
  - $\boldsymbol{\mu}$: vector of means
  - Σ: covariance matrix. $|\Sigma|$ denotes the determinant of Σ.

Multivariate Gaussian Distributions

- A 2-D Gaussian distribution.
- μ: center of contours
- Σ: orientation and size of contours



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	public	private	others	P(Rent)
low	.17	.01	.02	.2
medium	.44	.03	.01	.48
upper medium	.09	.07	.01	.17
high	0	0.14	0.1	.15
P(Type)	.7	.25	.05	

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medium	.44	.03	.01	.48
upper medium	.09	.07	.01	.17
high	0	0.14	0.1	.15
P(Type)	.7	.25	.05	

- Called marginal probability because written on the margins.

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In contrast:

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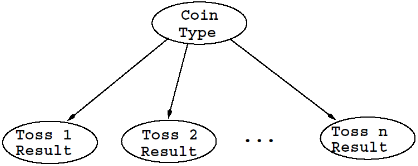

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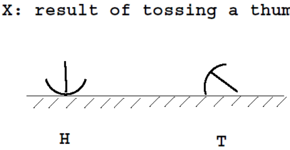
- Task: To estimate parameter $\theta = P(X{=}H)$.



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  - $\theta = 0.01$
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  It explains the data the best, and is hence the most likely.

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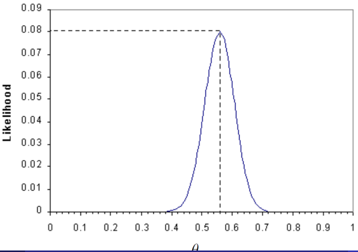
MLE best explains data or best fits data.



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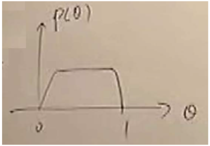
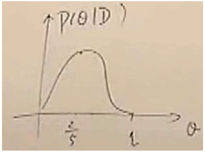
	Before Seeing Data	After Seeing Data: {2H, 3T}
MLE	?	2/5
Bayesian Estimation		

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- **Bayesian MAP**:

$$P(D_{m+1}=H|\mathcal{D}) = \theta^* = \arg\max p(\theta|\mathcal{D})$$

Calculating Bayesian Estimation

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p(\theta|\mathcal{D}) \propto \theta^{m_h+\alpha_h-1}(1-\theta)^{m_t+\alpha_t-1} \tag{2}
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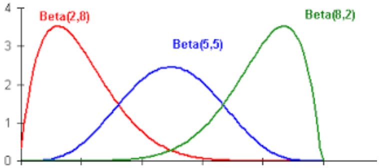
Beta Distribution

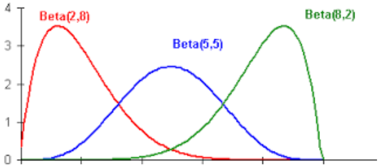


- The normalization constant for the Beta distribution $B(\alpha_h, \alpha_t)$

$$\frac{\Gamma(\alpha_t + \alpha_h)}{\Gamma(\alpha_t)\Gamma(\alpha_h)}$$

where $\Gamma(.)$ is the **Gamma function**. For any integer α, $\Gamma(\alpha) = (\alpha - 1)!$. It is also defined for non-integers.

Beta Distribution



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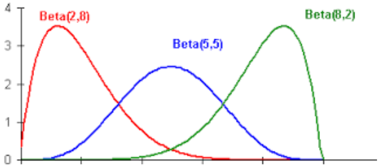

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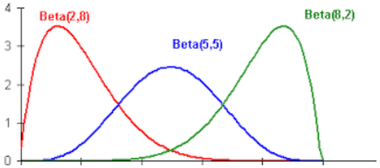

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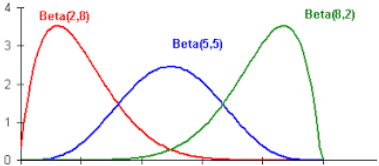

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- The larger the equivalent sample size, the more confident we are in our prior.

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- Binomial Likelihood: $\theta^{m_h}(1-\theta)^{m_t}$

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- Conjugate families allow closed-form for posterior distribution of parameters and closed-form solution for prediction.

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- Consequently,

$$P(D_{m+1} = T|\mathcal{D}) = \frac{m_t+\alpha_t}{m+\alpha}$$

- After taking data $\mathcal{D}$ into consideration, now our **updated belief** on $X = T$ is $\frac{m_t+\alpha_t}{m+\alpha}$.

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  - In case 2,

    $$P(D_{m+1} = H|\mathcal{D}) = \frac{30,000+100}{100,0000+100+100} \approx 0.3$$

    Data prevail.

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- In this course, we will focus on MLE.