Advanced Digital Design

Particle Systems and ODE Solvers II, Mass-Spring Modeling

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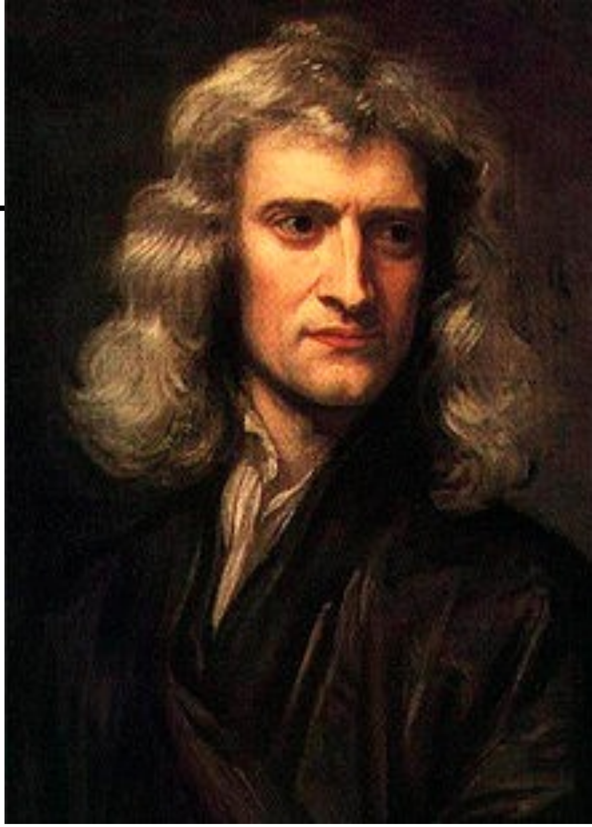
Picture: A. Selle et al.

ODEs and Numerical Integration

$$\frac{d\,\mathbf{X}(t)}{dt} = f(\mathbf{X}(t), t)$$

- Given a function f(X,t) compute X(t)
- Typically, initial value problems:
  - Given values $X(t_0)=X_0$
  - Find values X(t) for $t > t_0$
- We can use lots of standard tools

Reduction to 1st Order



- Point mass: 2nd order ODE

$$\vec{F} = m\vec{a} \quad \text{or} \quad \vec{\boldsymbol{F}} = m\frac{d^2\vec{\boldsymbol{x}}}{dt^2}$$

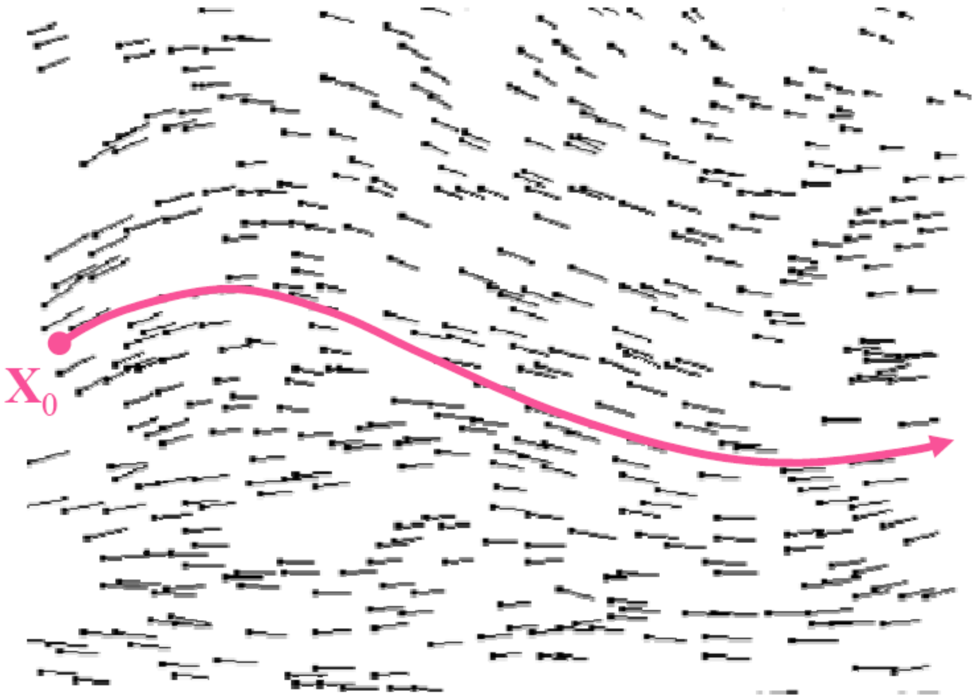
- Corresponds to system of first order ODEs

$$\begin{cases} \frac{d}{dt}\vec{\boldsymbol{x}} = \vec{\boldsymbol{v}} \\ \frac{d}{dt}\vec{\boldsymbol{v}} = \vec{\boldsymbol{F}}/m \end{cases}$$

2 unknowns (x, v) instead of just x

ODE: Path Through a Vector Field

- ***X(t)***: path in multidimensional <u>phase space</u>



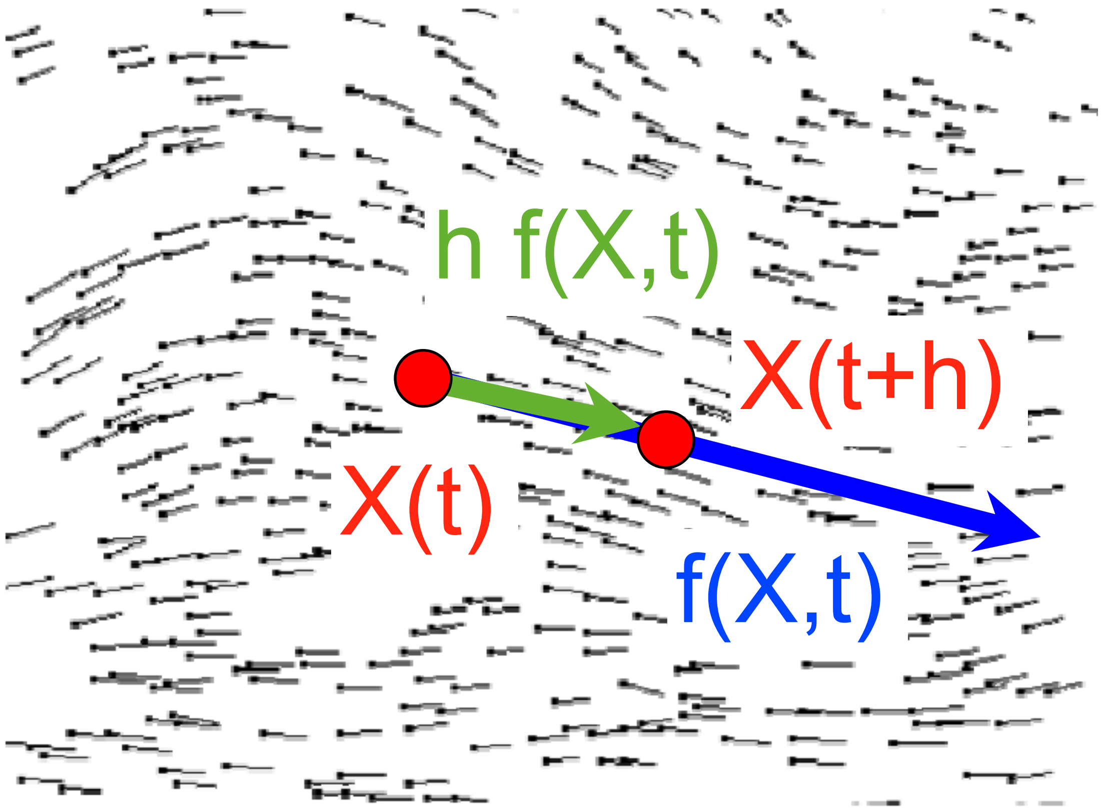
$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = f(\boldsymbol{X}, t)$$

"When we are at state X at time t, where will X be after an infinitely small time interval dt ?"

- ***f(X,t)*** is a vector that sits at each point in phase space, pointing the direction.

Euler, Visually

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X} = f(\boldsymbol{X}, t)$$



What is the inherent assumption on the velocity?

Euler's Method: Not Always Stable

- "Test equation" $f(x,t) = -kx$

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- Exact solution is a decaying exponential:

$$x(t) = x_0 e^{-kt}$$

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- Exact solution is a decaying exponential:

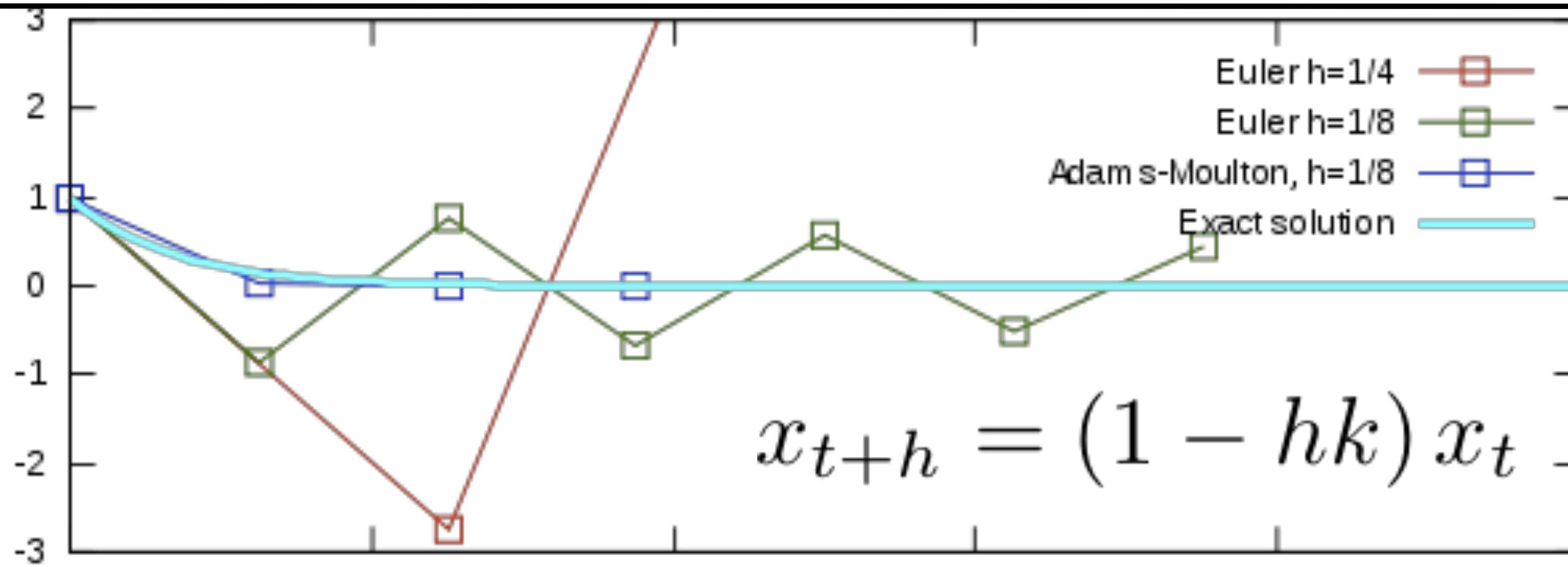
$$x(t) = x_0 e^{-kt}$$

- Let's apply Euler's method:

$$\begin{aligned} x_{t+h} &= x_t + h\,f(x_t, t) \\ &= x_t - hkx_t \\ &= \boxed{(1 - hk)\,x_t} \end{aligned}$$

Look familiar?

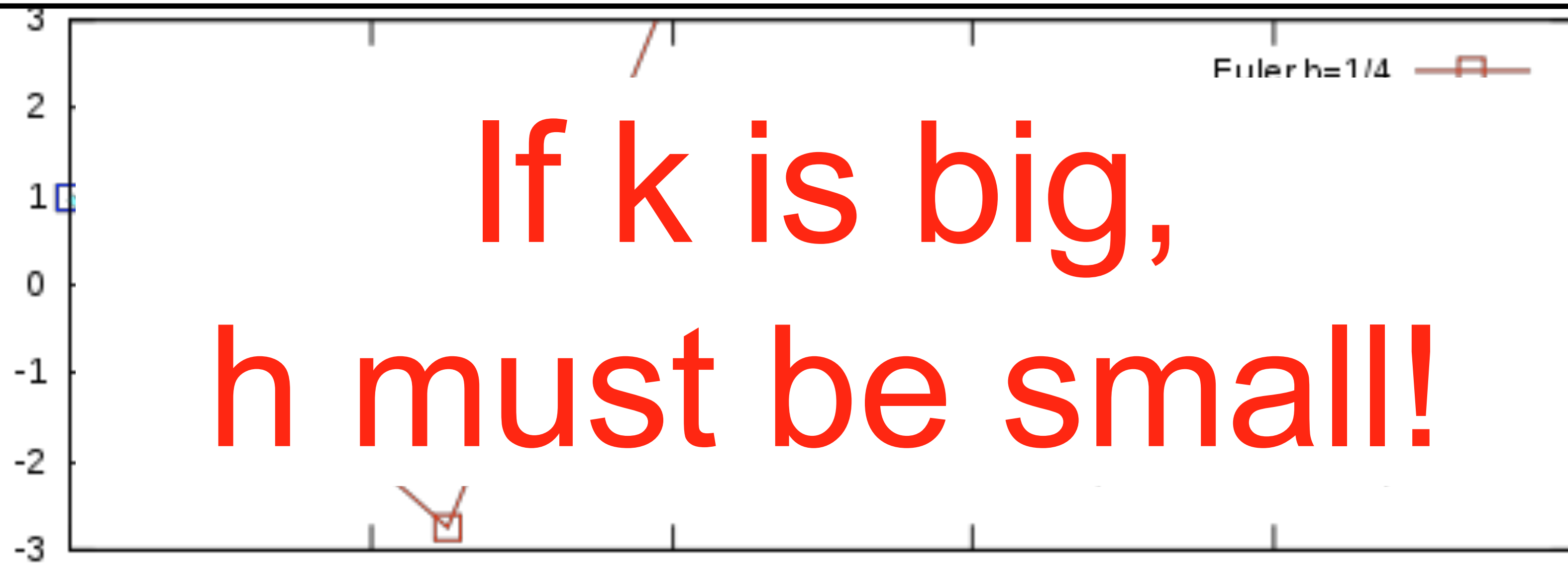
Euler’s Method: Not Always Stable



Wikipedia user Berland

- Limited step size!
  - When $0 \leq (1 - hk) < 1 \Leftrightarrow h < 1/k$ things are fine, the solution decays
  - When $-1 \leq (1 - hk) \leq 0 \Leftrightarrow 1/k \leq h \leq 2/k$ we get oscillation
  - When $(1 - hk) < -1 \Leftrightarrow h > 2/k$ things explode

Euler's Method: Not Always Stable



Wikipedia user Berland

If k is big,
h must be small!

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Analysis: Taylor Series

- Expand exact solution X(t)

$$\mathbf{X}(t_0+h)=\mathbf{X}(t_0)+h\left(\tfrac{d}{dt}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^2}{2!}\left(\tfrac{d^2}{dt^2}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^3}{3!}\left(\cdots\right)+\cdots$$

- Euler's method approximates:

$$\mathbf{X}(t_0+h)=\mathbf{X}_0+h\,f(\mathbf{X}_0,t_0)\qquad\ldots+O(h^2)\,\text{error}$$

$$h\to h/2\ \Rightarrow\ \ error\to error/4\,\text{per step}\times\text{twice as many steps}$$
$$\to error/2$$

- First-order method: Accuracy varies with h
- To get 100x better accuracy need 100x more steps

Analysis: Taylor Series Questions?

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$$\rightarrow error/2$$

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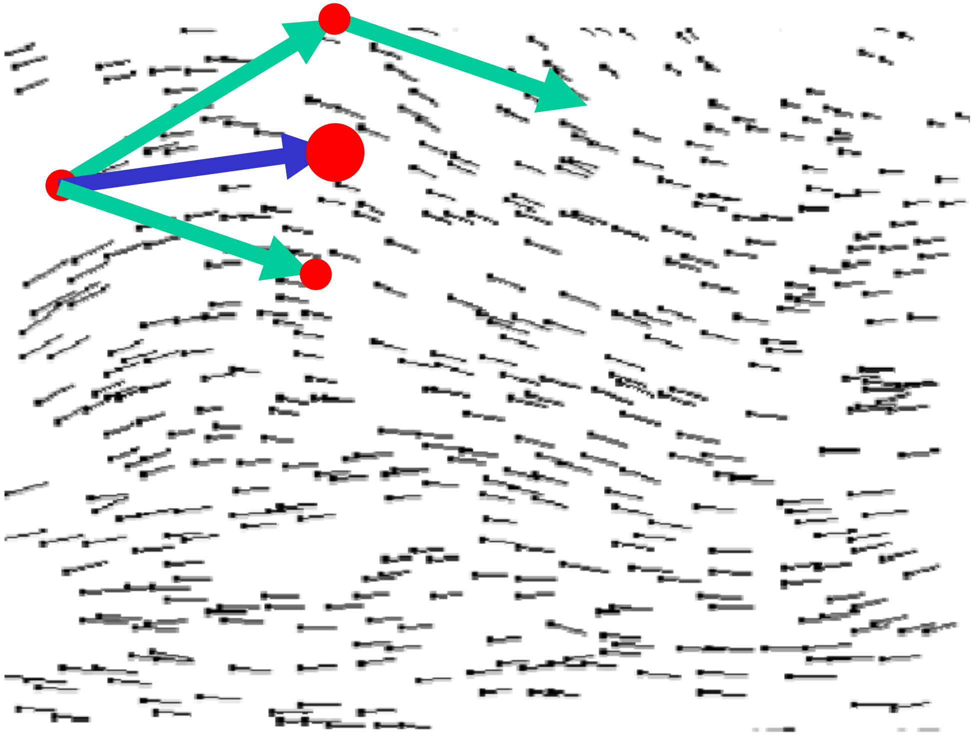
Can We Do Better?

- Problem: f varies along our Euler step
- Idea 1: look at f at the arrival of the step and compensate for variation



2nd Order Methods

- This translates to...

$$\boxed{\begin{aligned} f_0 &= f(\mathbf{X}_0, t_0) \\ f_1 &= f(\mathbf{X}_0 + h f_0, t_0 + h) \end{aligned}}$$

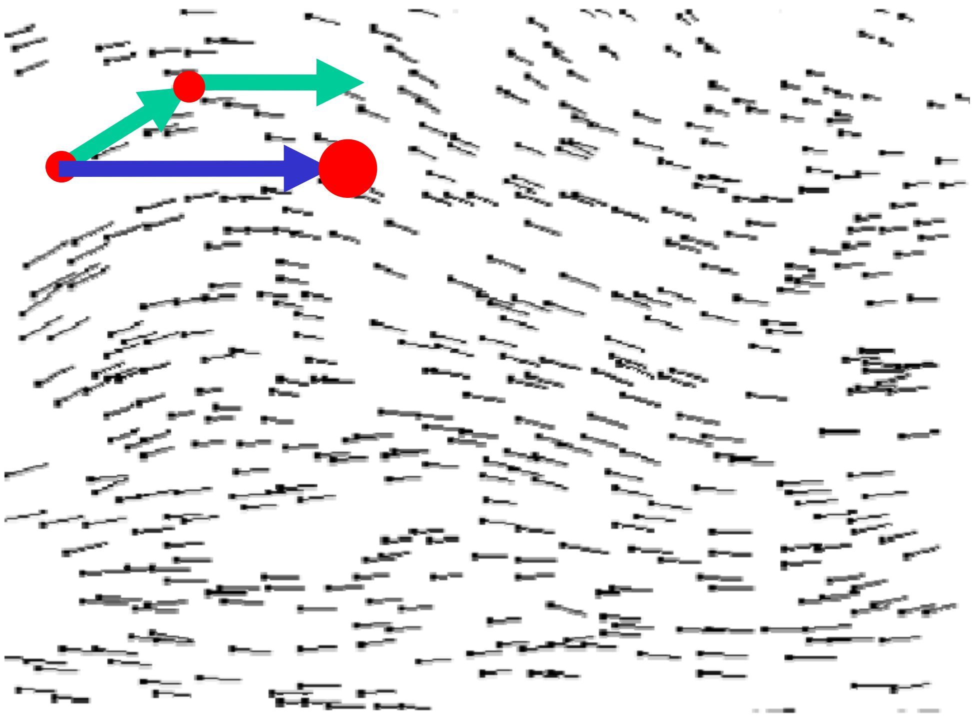
- and we get

$$\boxed{\mathbf{X}(t_0 + h) = \mathbf{X}_0 + \tfrac{h}{2}(f_0 + f_1)} + O(h^3)$$

- This is the trapezoid method
  - Analysis omitted
- Note: What we mean by "2nd order" is that the error goes down with h^2 , not h – the equation is still 1st order!

Can We Do Better?

- Problem: f has varied along our Euler step
- Idea 2: look at f after a smaller step, use that value for a full step from initial position



2nd Order Methods Cont'd

- This translates to...

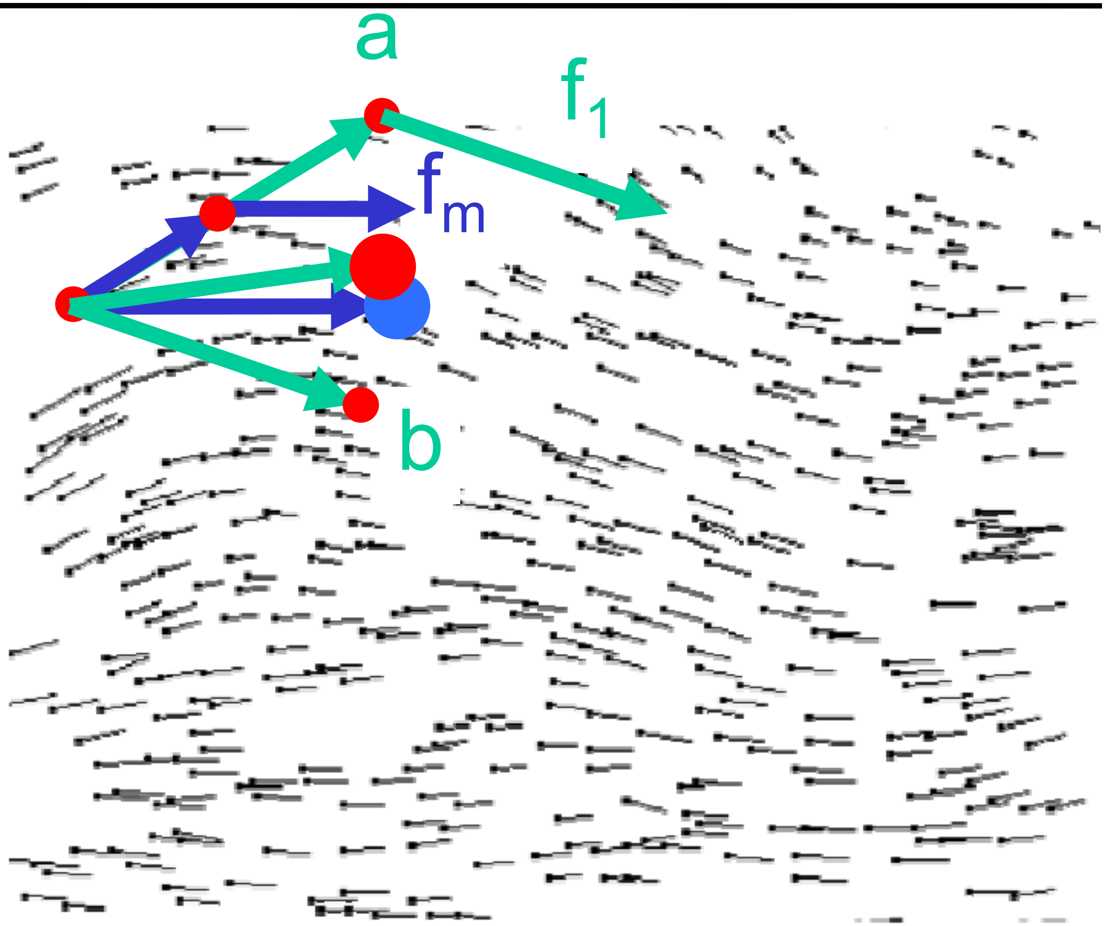
$$\boxed{\begin{aligned} f_0 &= f(\mathbf{X}_0, t_0) \\ f_m &= f(\mathbf{X}_0 + \tfrac{h}{2} f_0, t_0 + \tfrac{h}{2}) \end{aligned}}$$

- and we get $\boxed{\mathbf{X}(t_0 + h) = \mathbf{X}_0 + h\, f_m} + O(h^3)$

- This is the midpoint method
  - Analysis omitted again,
    but it's not very complicated, see
    http://en.wikipedia.org/wiki/Midpoint_method.

Comparison

- Midpoint:
  - ½ Euler step
  - evaluate f_m
  - full step using f_m
- Trapezoid:
  - Euler step (a)
  - evaluate f_1
  - full step using f_1 (b)
  - average (a) and (b)
- Not exactly same result, but same order of accuracy



Can We Do Even Better?

- You bet!
- Runge-Kutta method
- Again, see Witkin, Baraff, Kass: Physically-based Modeling Course Notes, SIGGRAPH 2001
- See eg http://www.youtube.com/watch?v=HbE3L5CIdQg

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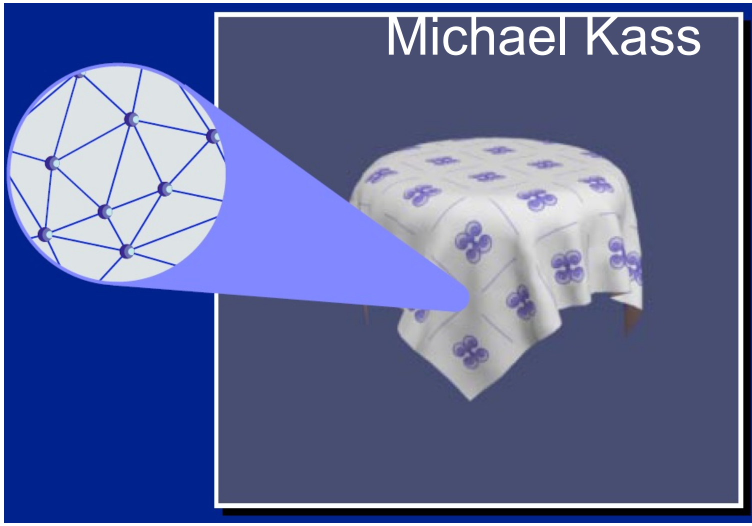
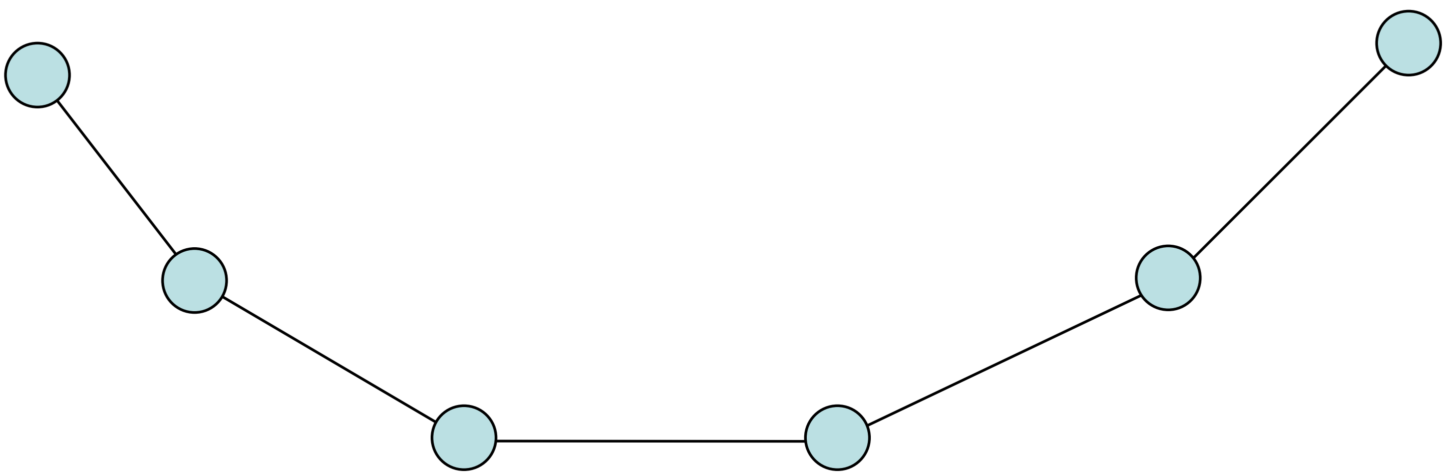
Mass-Spring Modeling

- Beyond point-like objects: strings, cloth, hair, etc.
- Interaction between particles
  - Create a network of spring forces that link pairs of particles

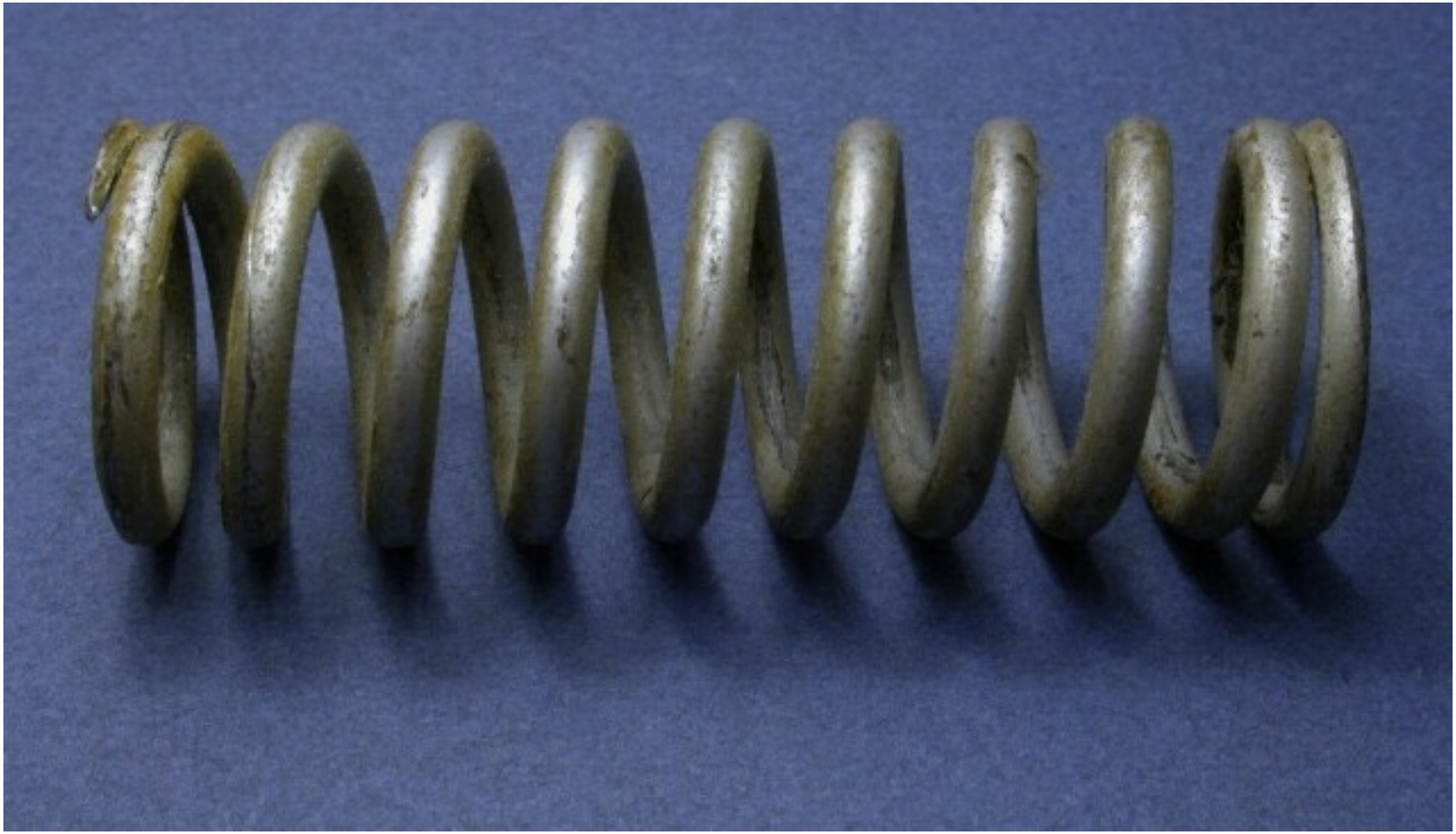


How Would You Simulate a String?

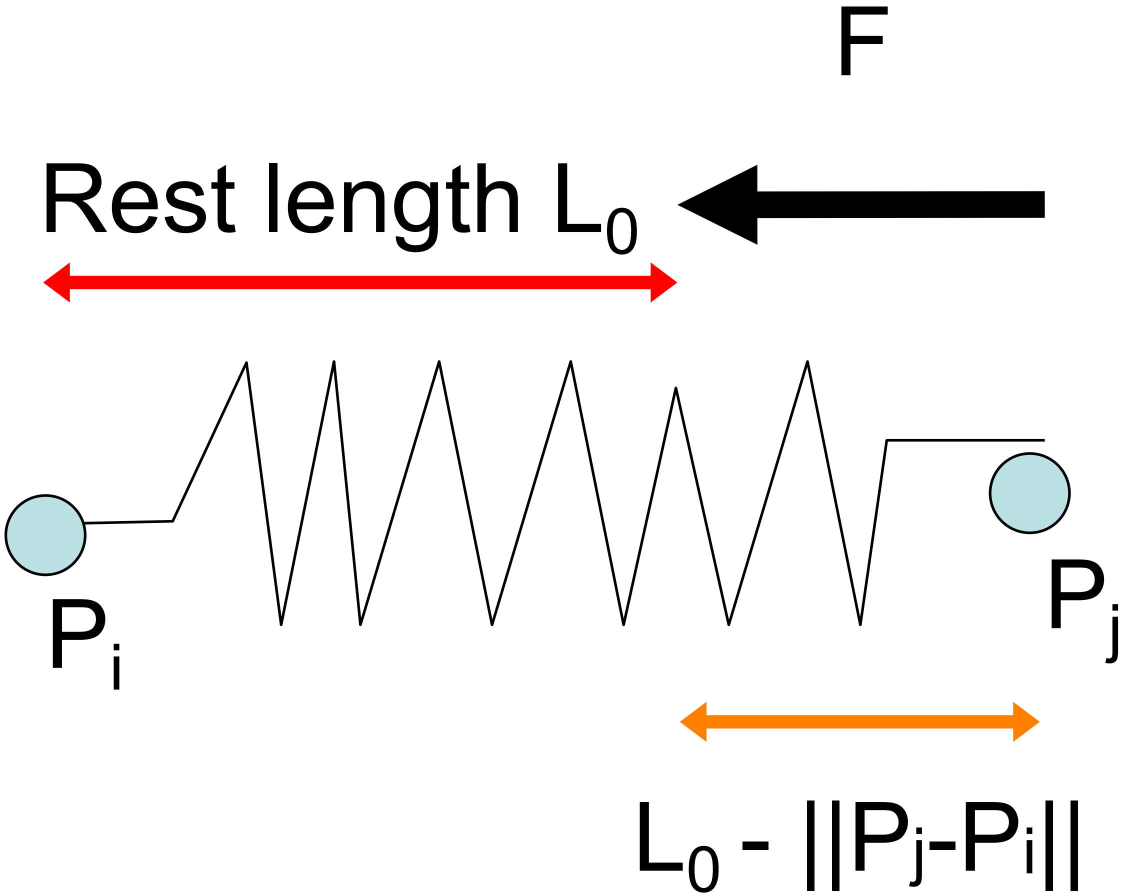
- Each particle is linked to two particles (except ends)
- Come up with forces that try to keep the distance between particles constant



Springs



Spring Force – Hooke's Law

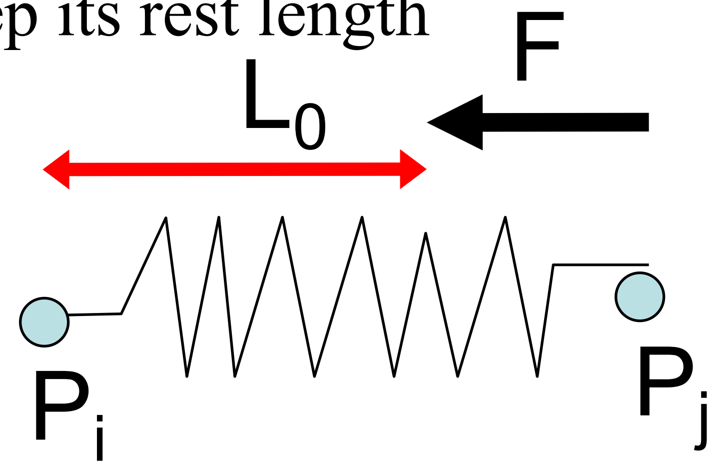


Spring Force – Hooke's Law

- Force in the direction of the spring and proportional to difference with rest length L_0.

$$F(P_i, P_j) = K(L_0 - ||\vec{P_iP_j}||)\frac{\vec{P_iP_j}}{||\vec{P_iP_j}||}$$

- K is the stiffness of the spring
  - When K gets bigger, the spring really wants to keep its rest length



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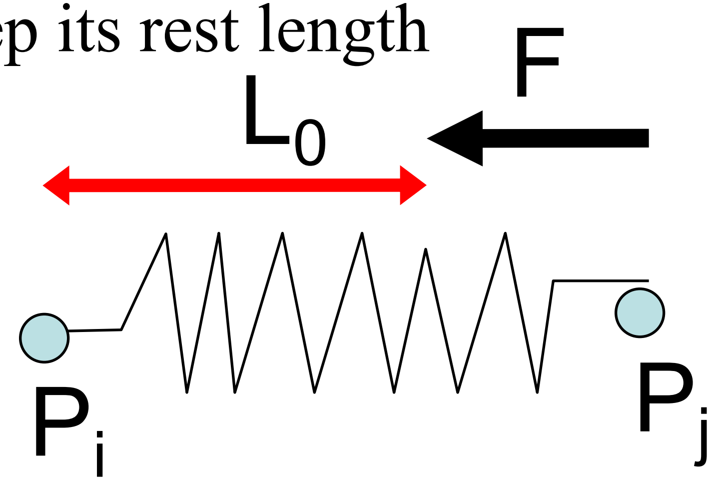
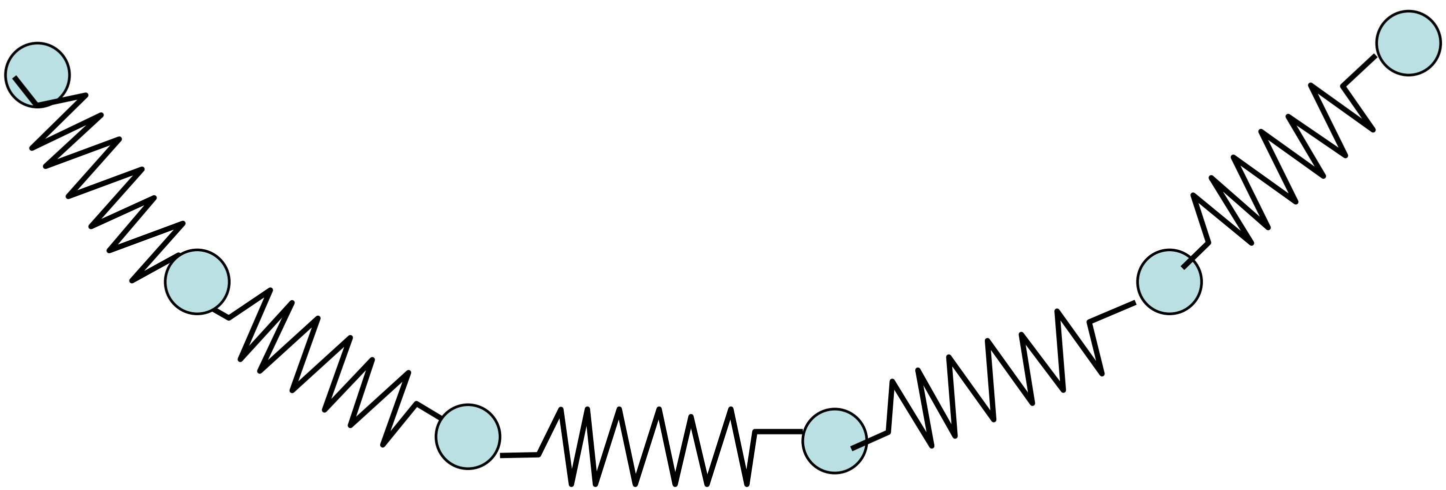


This is the force on P_j. Remember Newton: P_i experiences force of equal magnitude but opposite direction.

How Would You Simulate a String?

- Springs link the particles
- Springs try to keep their rest lengths and preserve the length of the string
- Not exactly preserved though, and we get oscillation
  - Rubber band approximation



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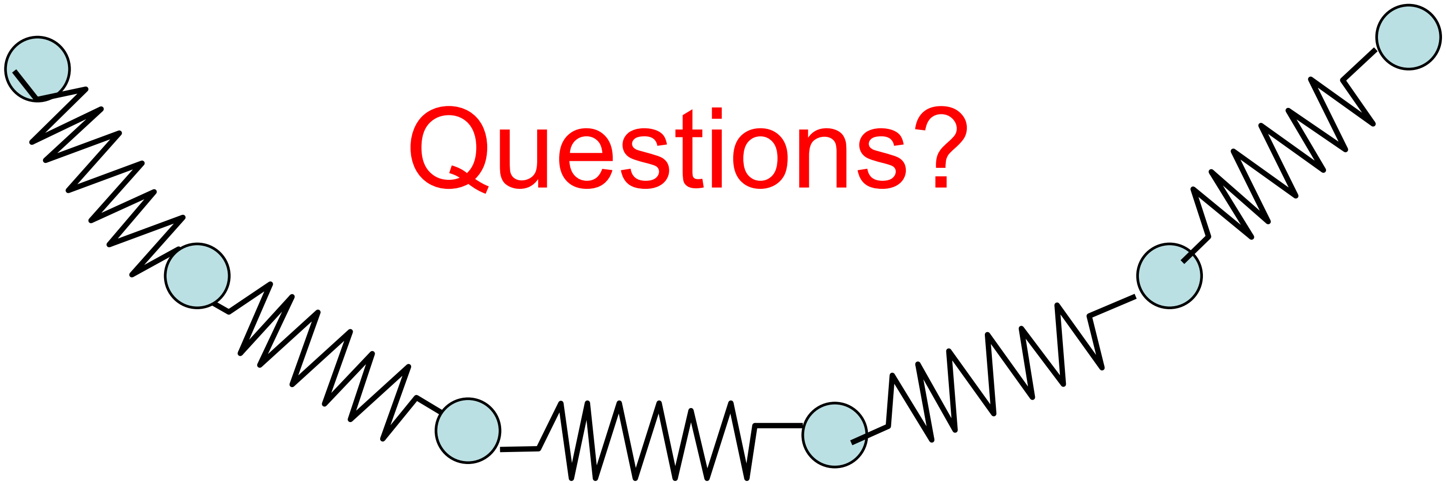
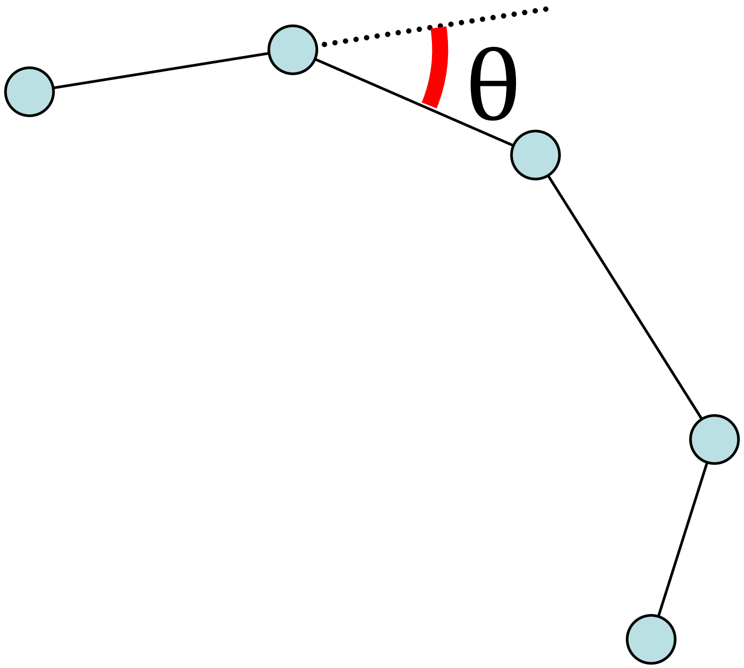


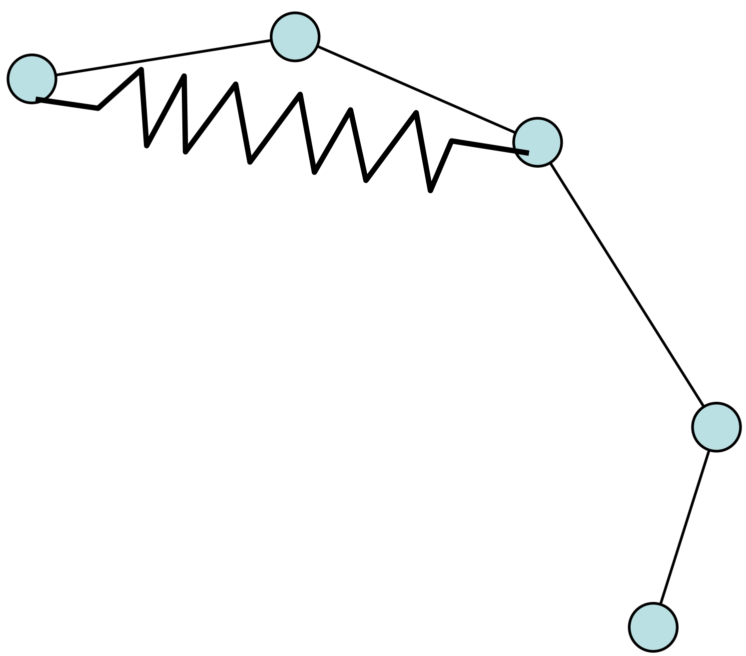
Hair

- Linear set of particles
- Length-preserving structural springs like before
- Deformation forces proportional to the angle between segments
- External forces



Hair - Alternative Structural Forces

- Springs between mass n & n+2 with rest length $2L_0$
  - Wants to keep particles aligned



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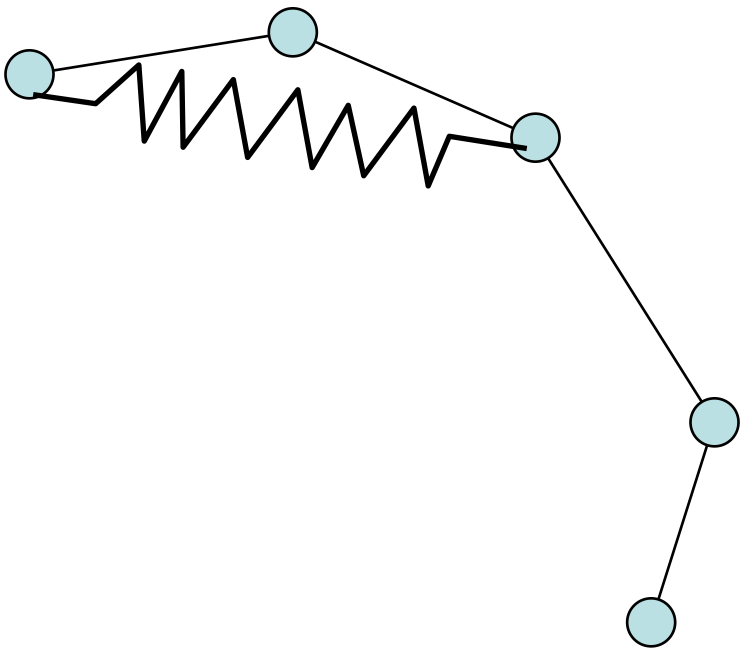
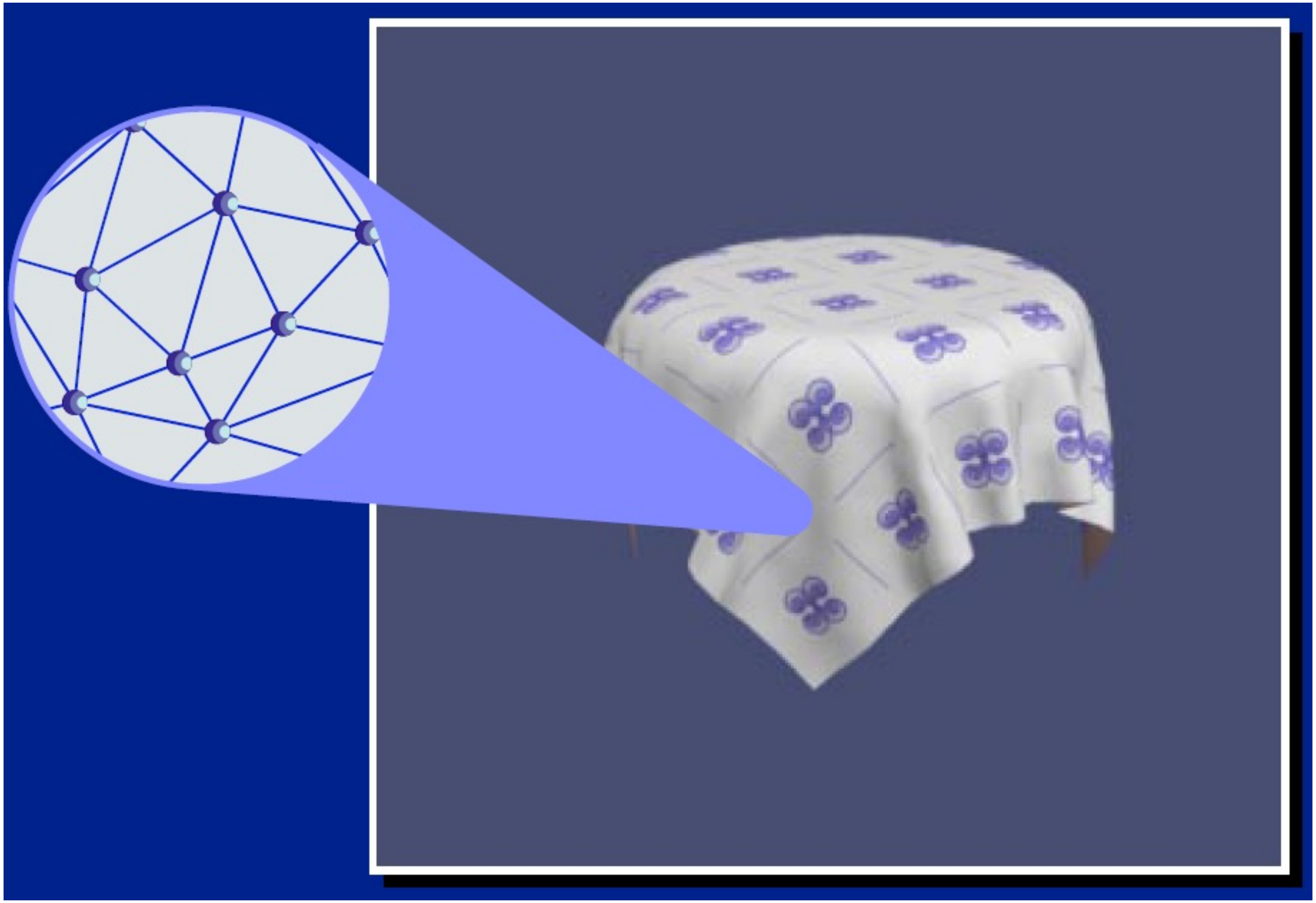


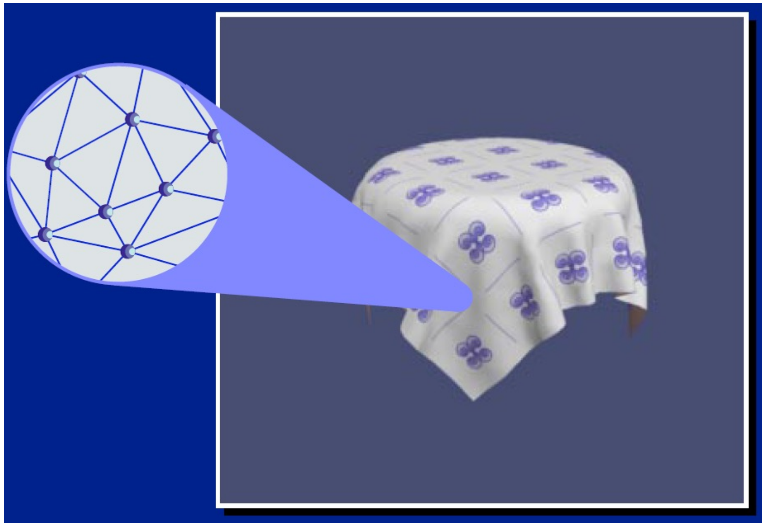
Questions?

Mass-Spring Cloth



Michael Kass

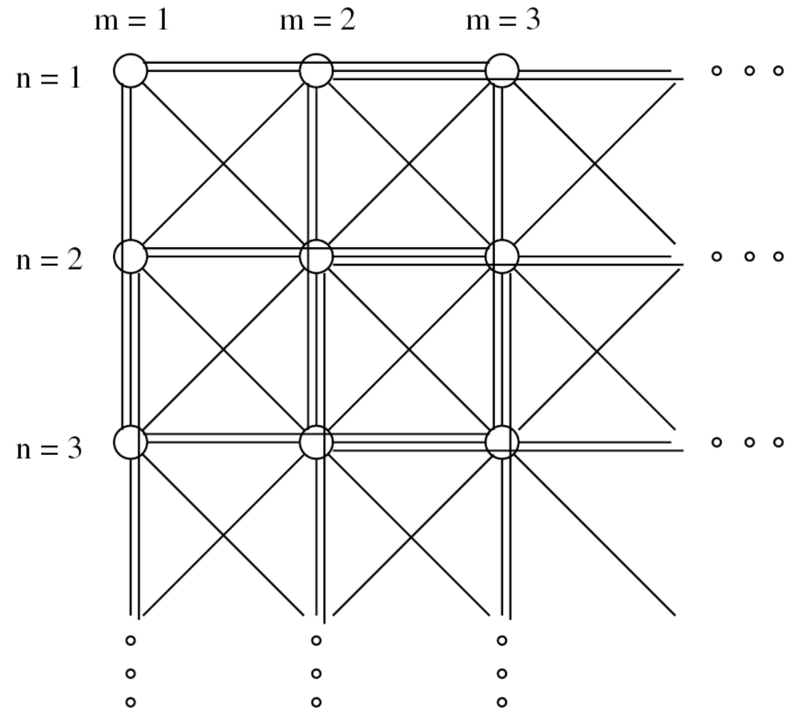
Cloth – Three Types of Forces



- Structural forces
  - Try to enforce invariant properties of the system
    - E.g. force the distance between two particles to be constant
  - Ideally, these should be constraints, not forces
- Internal deformation forces
  - E.g. a string deforms, a spring board tries to remain flat
- External forces
  - Gravity, etc.

Springs for Cloth

- Network of masses and springs
- Structural springs:
  - link (i j) and (i+1, j); and (i, j) and (i, j +1)
- Deformation:
  - Shear springs
    - (i j) and (i+1, j+1)
  - Flexion springs (prevent bending)
    - (i,j) and (i+2,j); (i,j) and (i,j+2)
- See Provot's Graphics Interface '95 paper for details

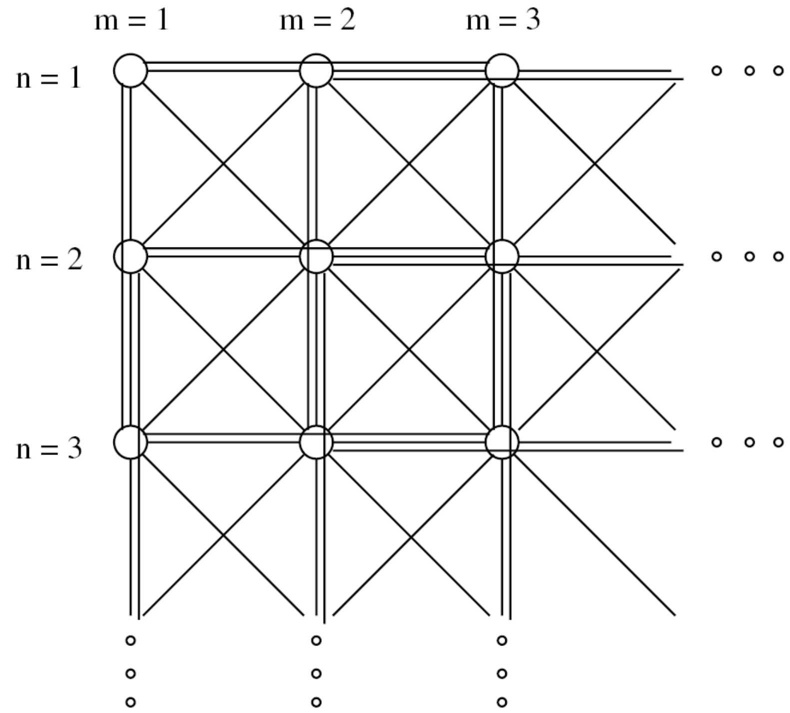


Provot 95

http://graphics.stanford.edu/courses/cs468-02-winter/Papers/Rigidcloth.pdf

External Forces

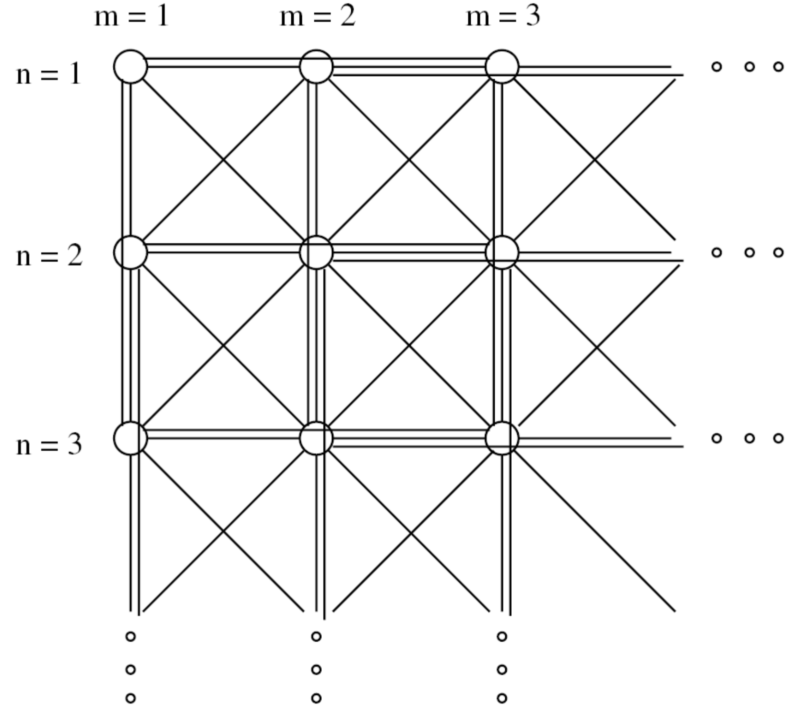
- Gravity G
- Friction
- Wind, etc.



Provot 95

Cloth Simulation

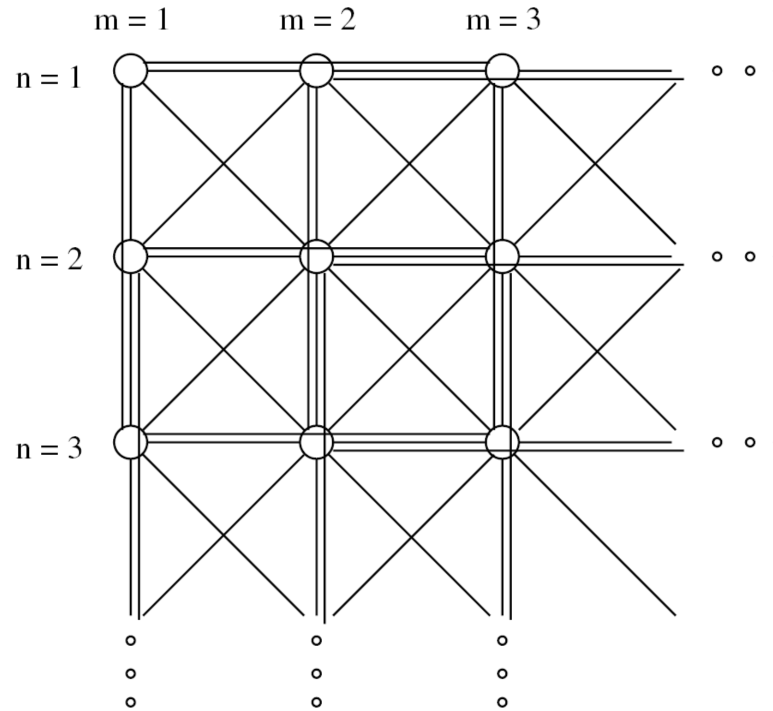
- Then, the all trick is to set the stiffness of all springs to get realistic motion!
- Remember that forces depend on other particles (coupled system)
- But it is sparse (only near neighbors)
  - This is in contrast to e.g. the N-body problem.



Provot 95

Forces: Structural vs. Deformation

- Structural forces are here just to enforce a constraint
- Ideally, the constraint would be enforced strictly
  - at least a lot more than we can afford
- We'll see that this is the root of a lot of problems
- In contrast, deformation forces actually correspond to physical forces



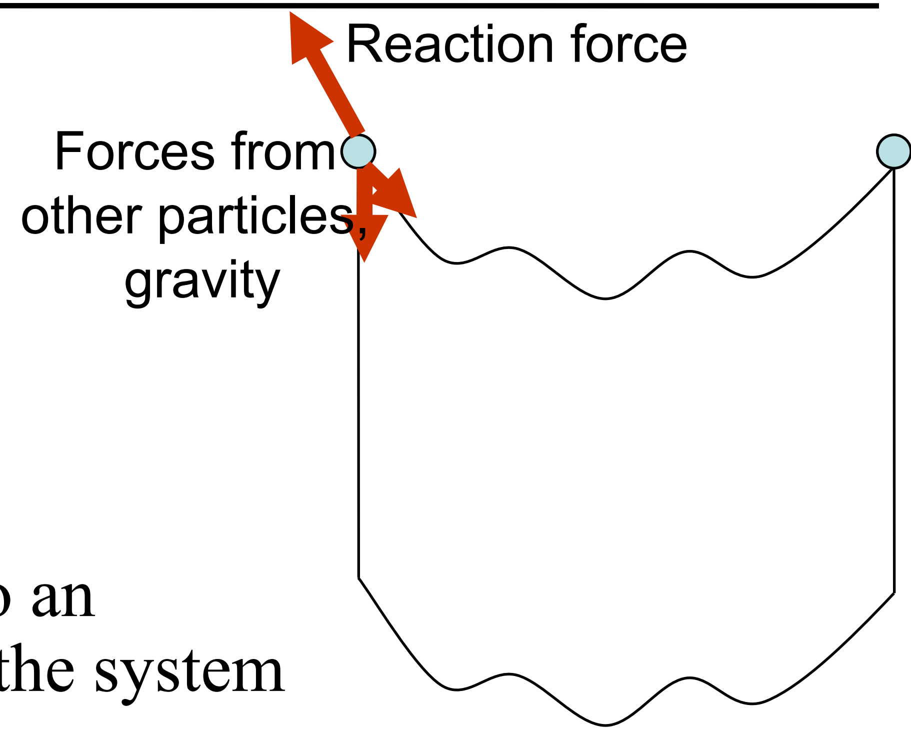
Provot 95

Practical Issues

- Contact forces
- Discretization problem
- Stiffness

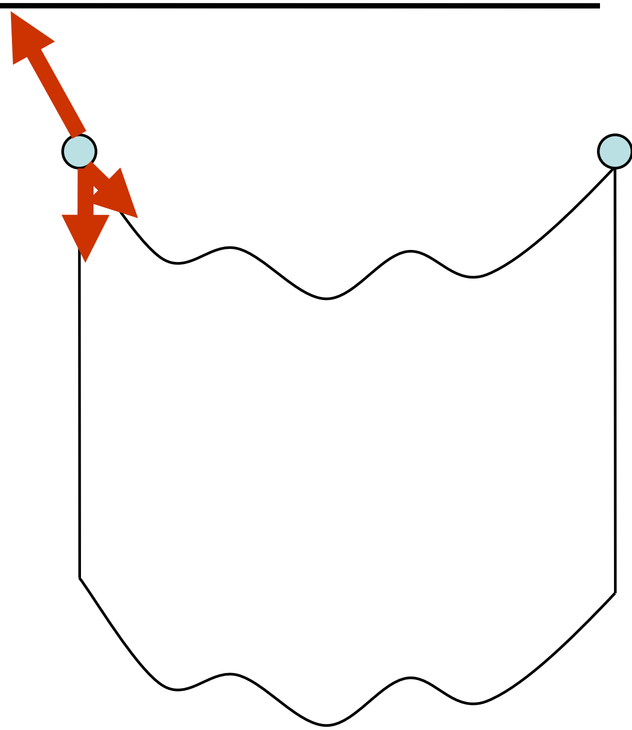
Contact Forces

- Hanging curtain:
  - 2 contact points stay fixed
- What does it mean?
  - Sum of the forces is zero
- How so?
  - Because those point undergo an external force that balances the system
- What is the force at the contact?
  - Depends on all other forces in the system
  - Gravity, wind, etc.



Contact Forces

- How can we compute the external contact force?
  - Inverse dynamics!
  - Sum all other forces applied to point
  - Take negative
- Do we really need to compute this force?
  - Not really, just ignore the other forces applied to this point!



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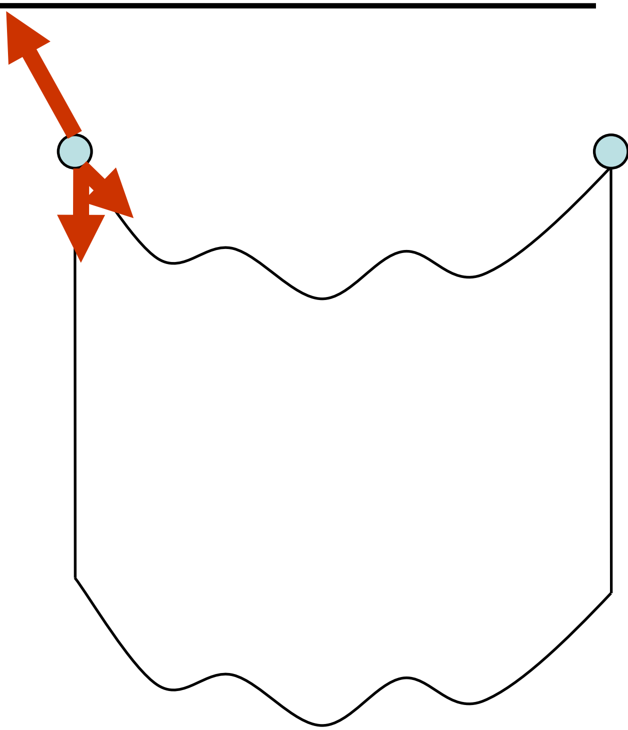
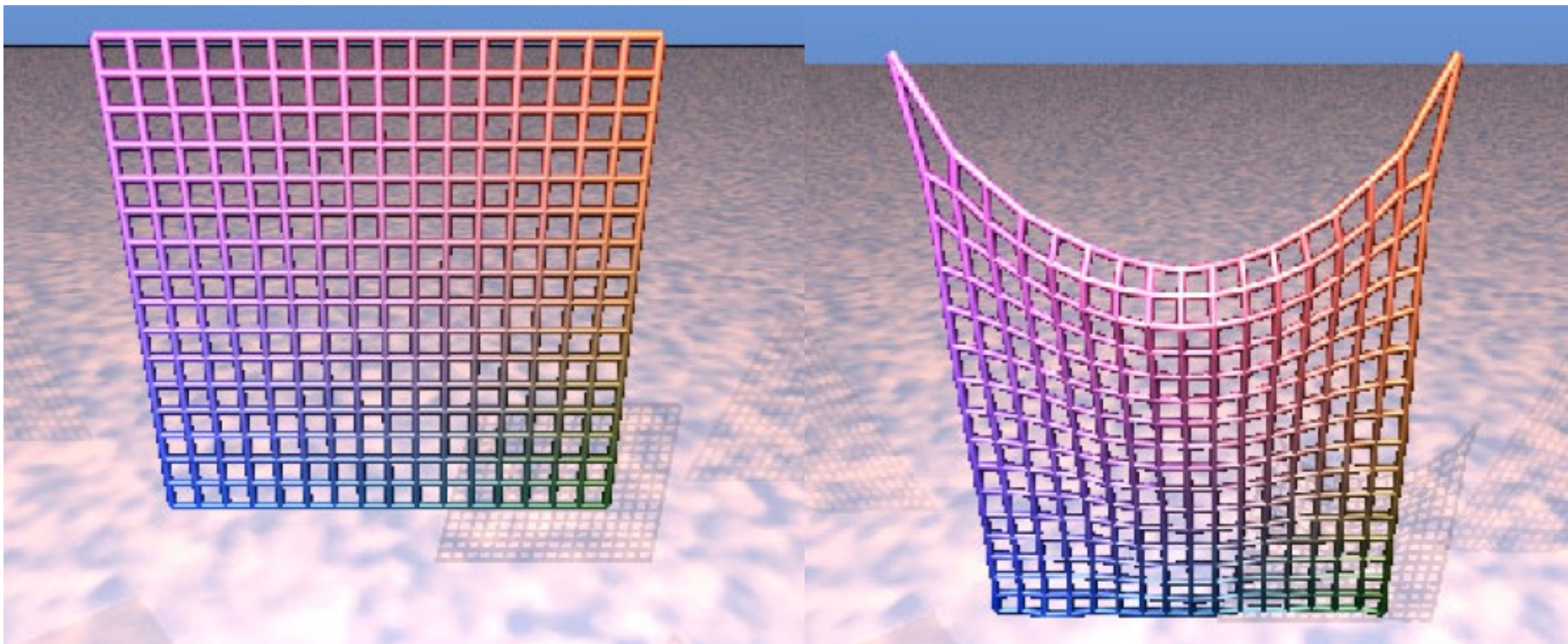


Questions?

Stiffness: Example

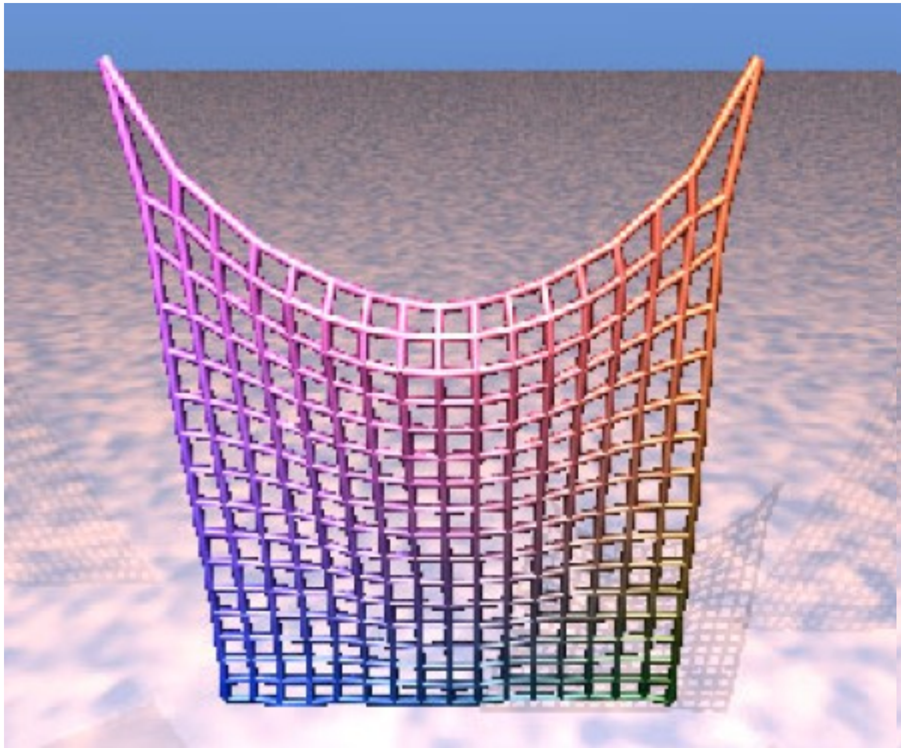
- Excessive rubbery deformation:
  the strings are not stiff enough



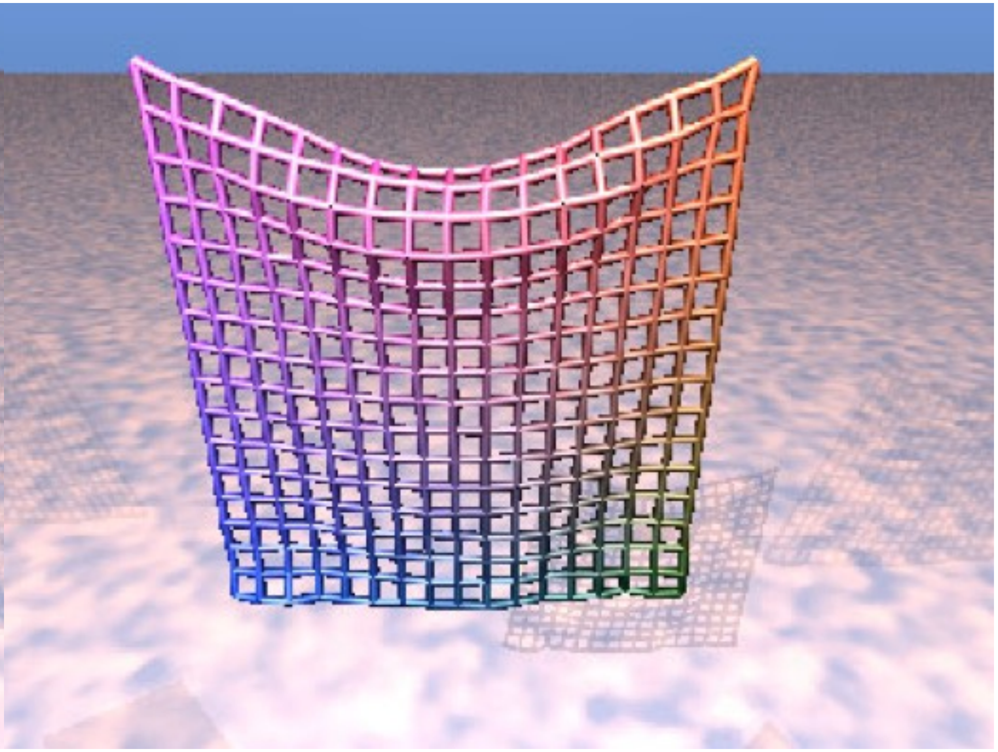
Initial position After 200 iterations

One Solution

- Constrain length to increase by less than 10%
  - A little hacky



Simple mass-spring system

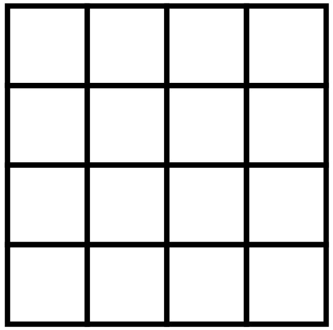


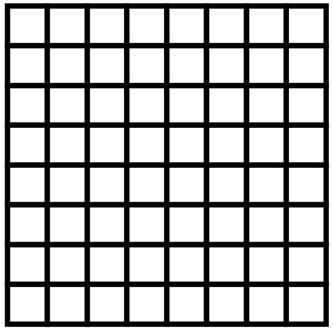
Improved solution
(see Provot Graphics Interface 1995)

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.84.1732&rep=rep1&type=pdf

The Discretization Problem

- What happens if we discretize our cloth more finely?
- Do we get the same behavior?
- Usually not! It takes a lot of effort to design a scheme that is mostly oblivious to the discretization.





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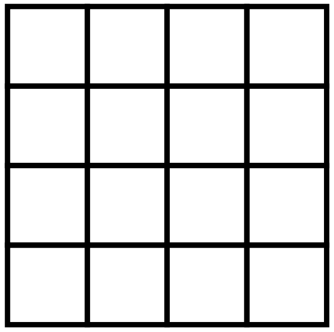
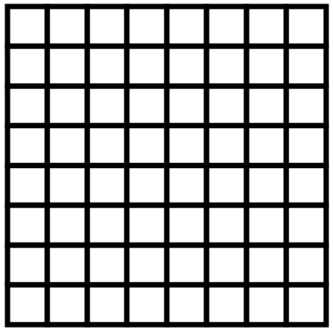




Questions?

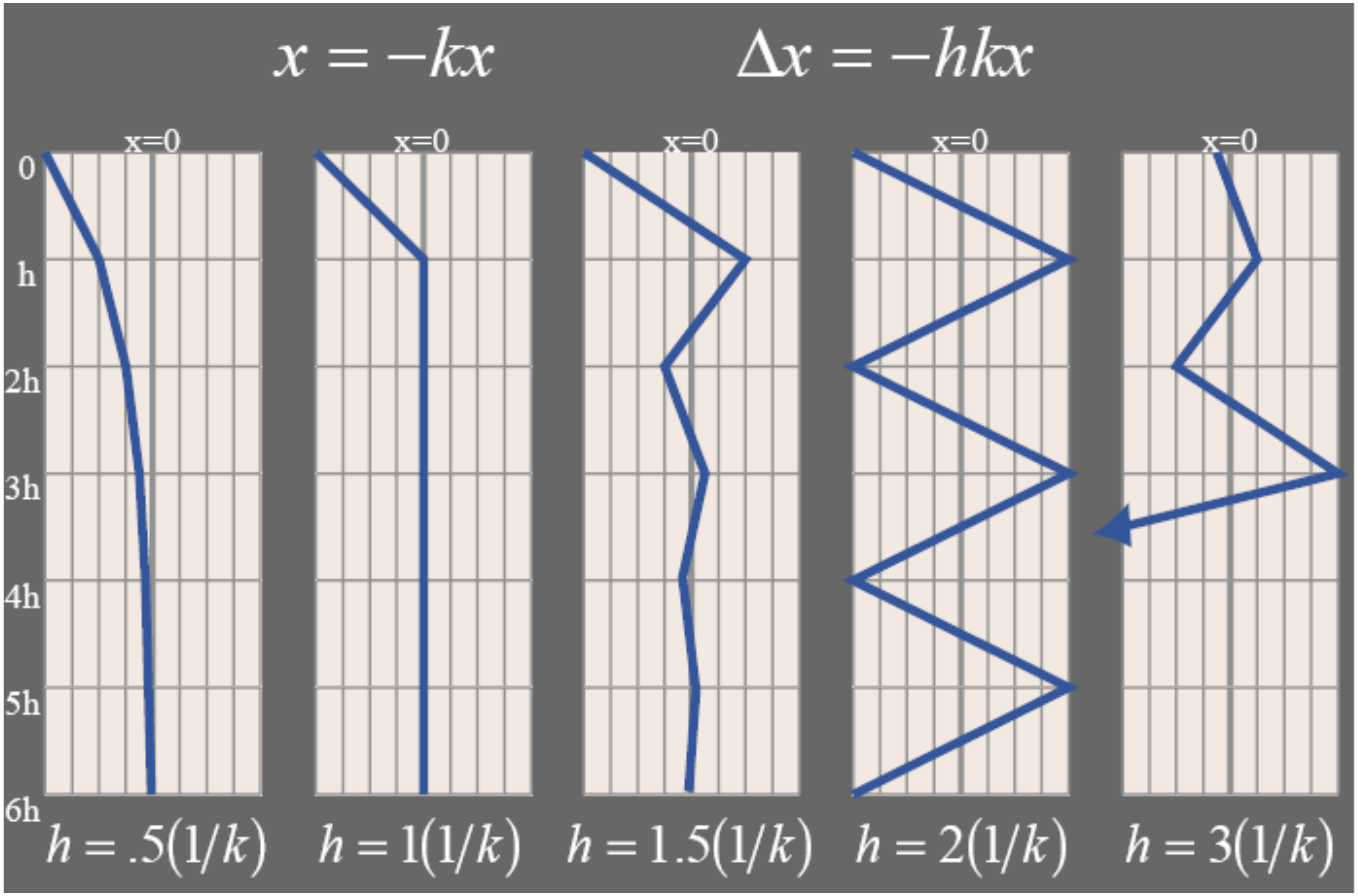
The Stiffness Issue

- We use springs while we really mean constraint
  - Spring should be super stiff, which requires tiny Δt
  - Remember $x' = -kx$ system and Euler speed limit!
    - The story extends to N particles and springs (unfortunately)

- Many numerical solutions
  - Reduce Δt (well, not a great solution)
  - Actually use constraints
  - Implicit integration scheme

Euler Has a Speed Limit!

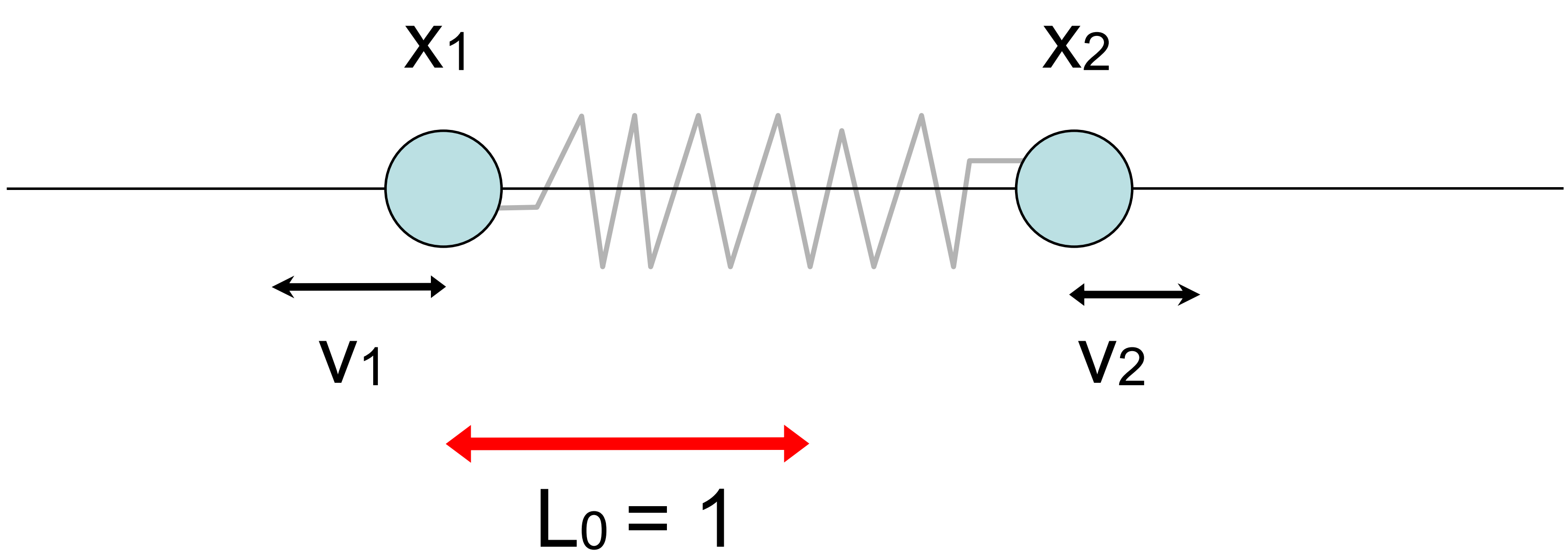
- h > 1/k: oscillate. h > 2/k: explode!



From the SIGGRAPH PBM notes

Why Stiff Springs Are Difficult

- 1D example, with two particles constrained to move along the x axis only, rest length $L_0 = 1$
- Phase space is 4D: (x_1, v_1, x_2, v_2)
  - Although spring force only depends on x_1, x_2 and L_0.



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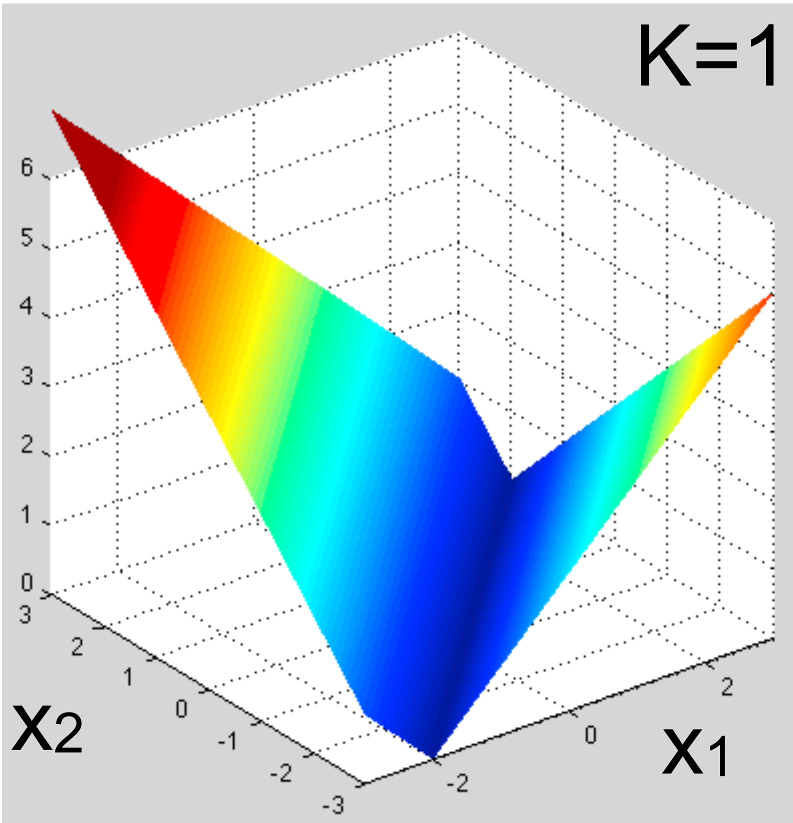
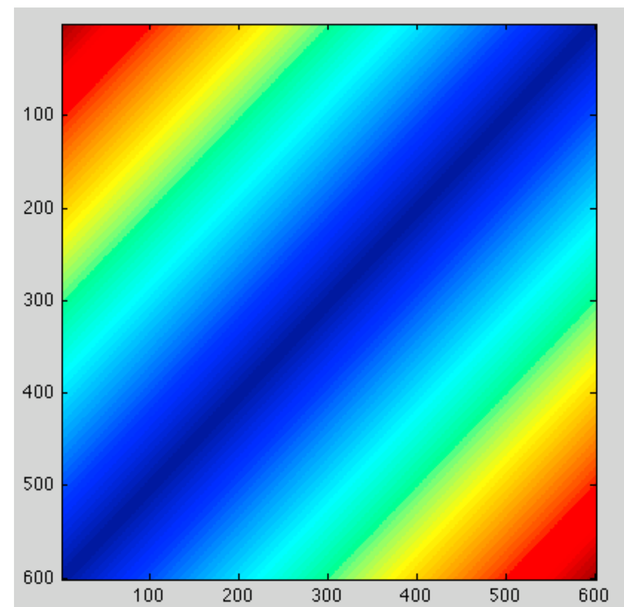


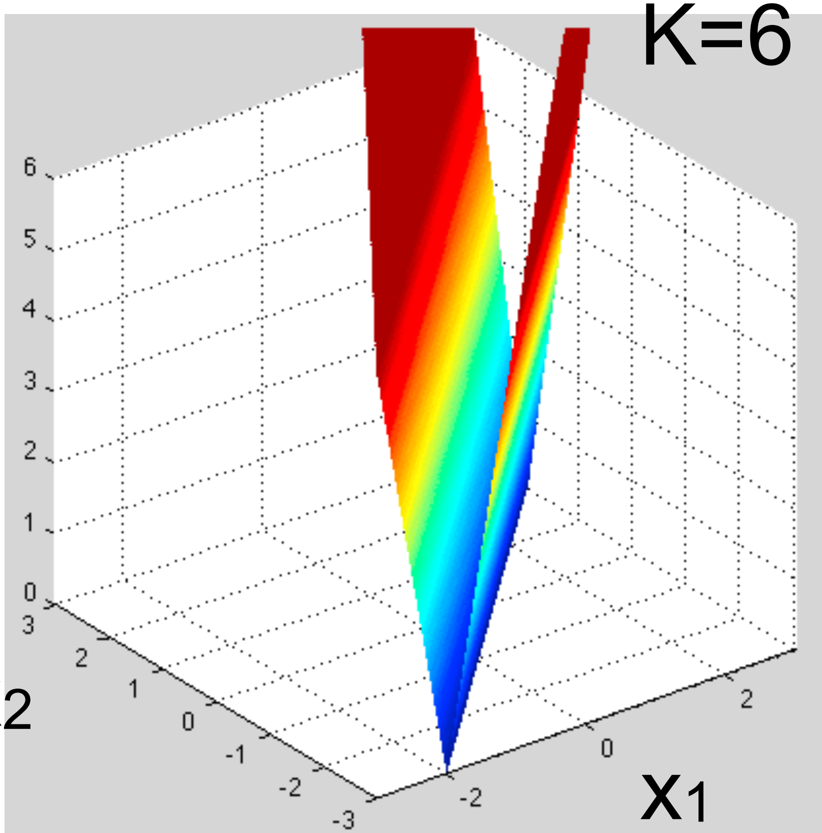
height=magnitude
of spring force

x2



x1

Why Stiff Springs Are Difficult



x_2

x_1

height=magnitude
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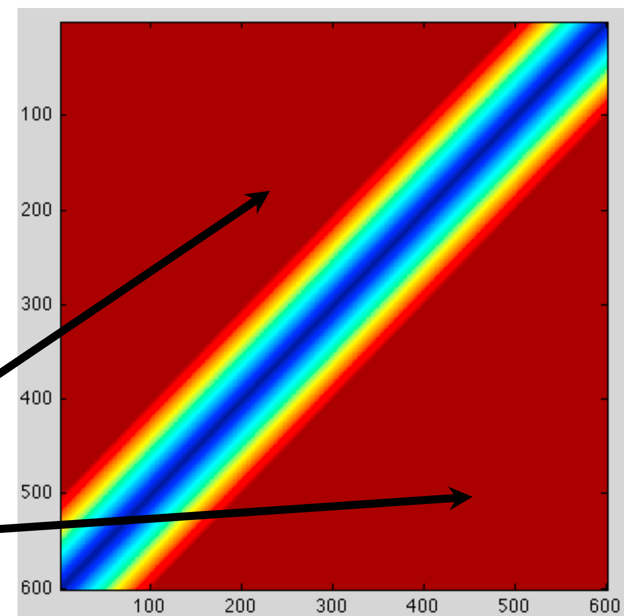
x_2

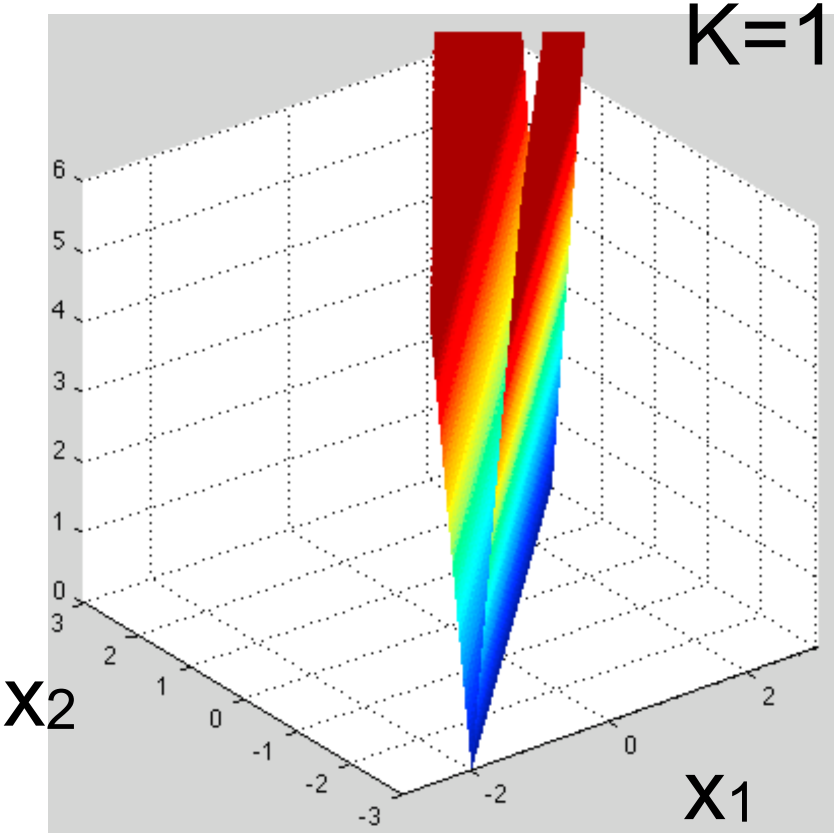


x_1

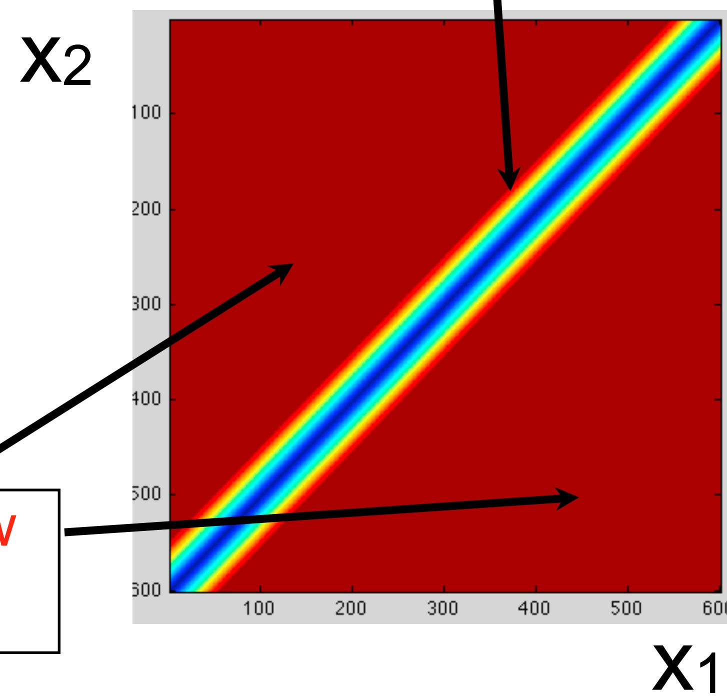
Forces grow
really big!

Why Stiff Springs Are Difficult

K=11

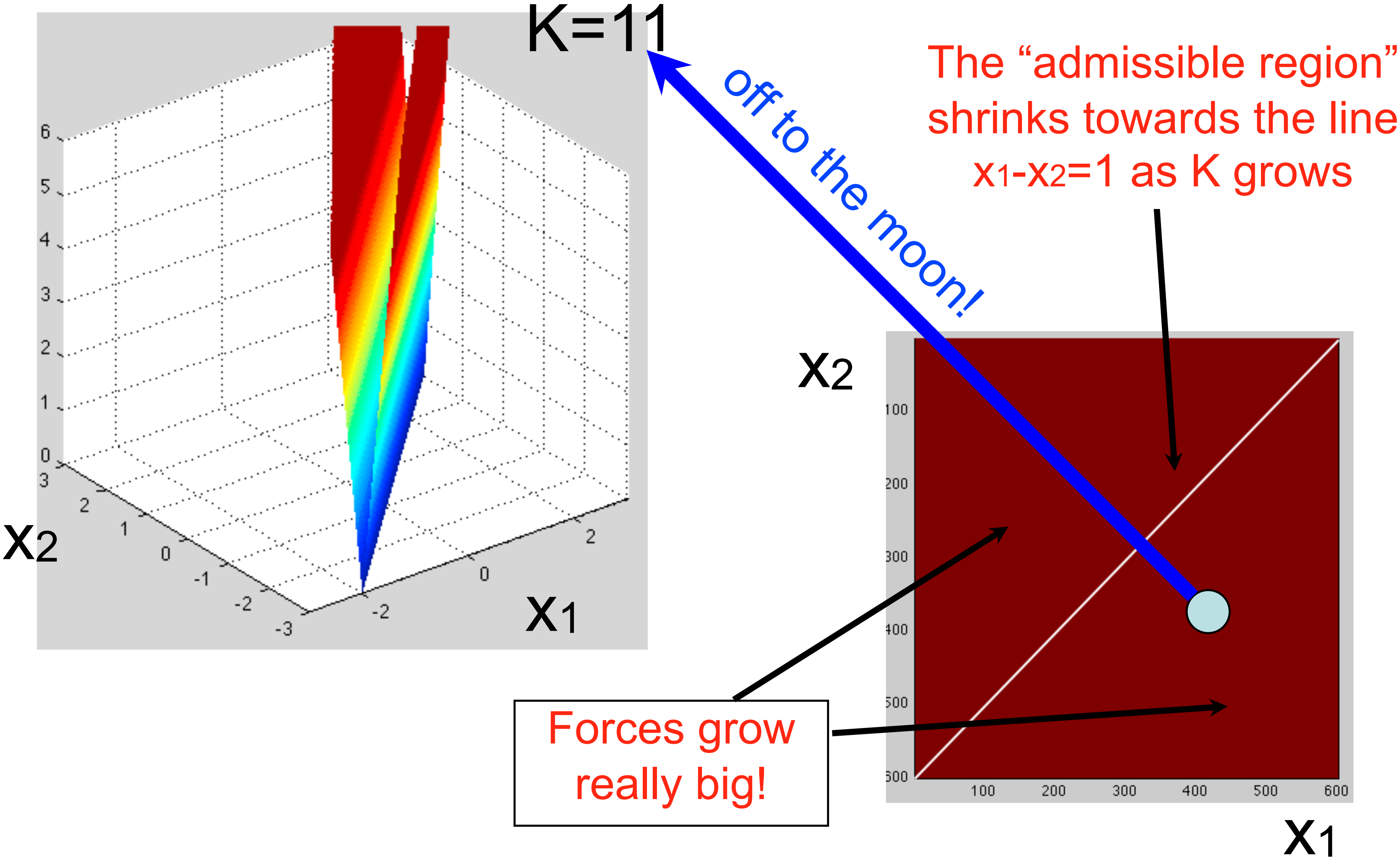


The "admissible region" shrinks towards the line x_1-x_2=1 as K grows



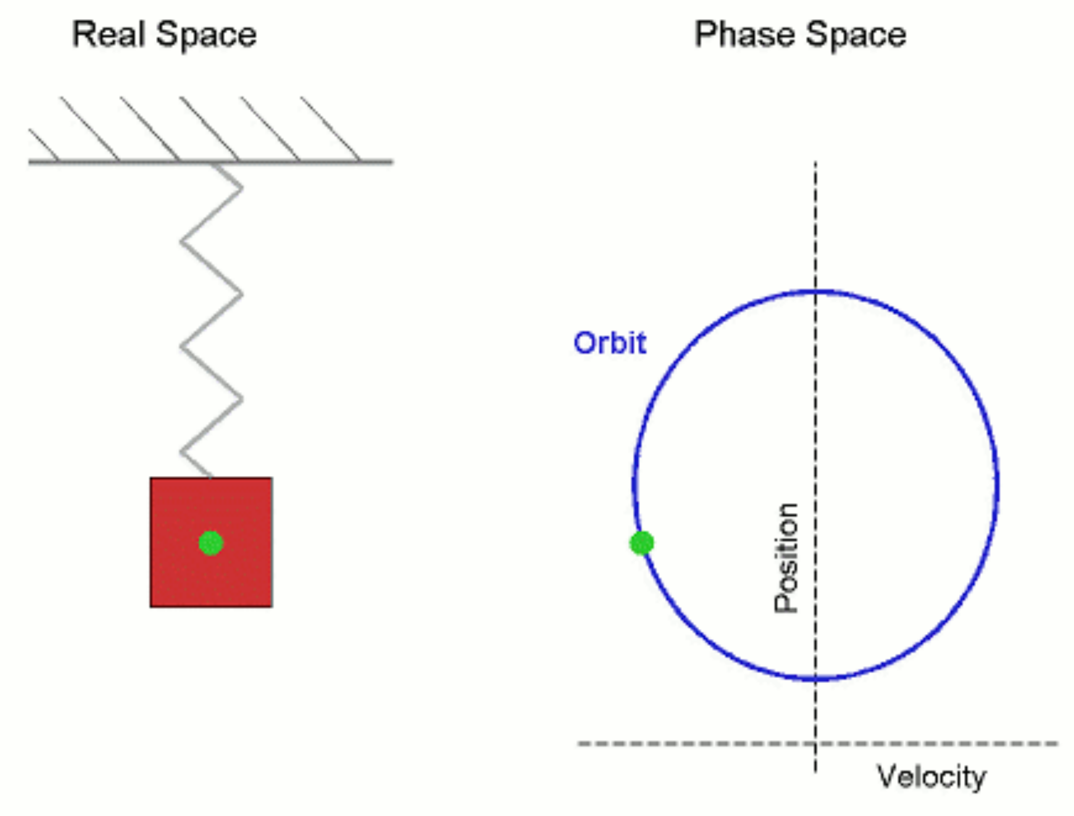
Forces grow really big!

Why Stiff Springs Are Difficult



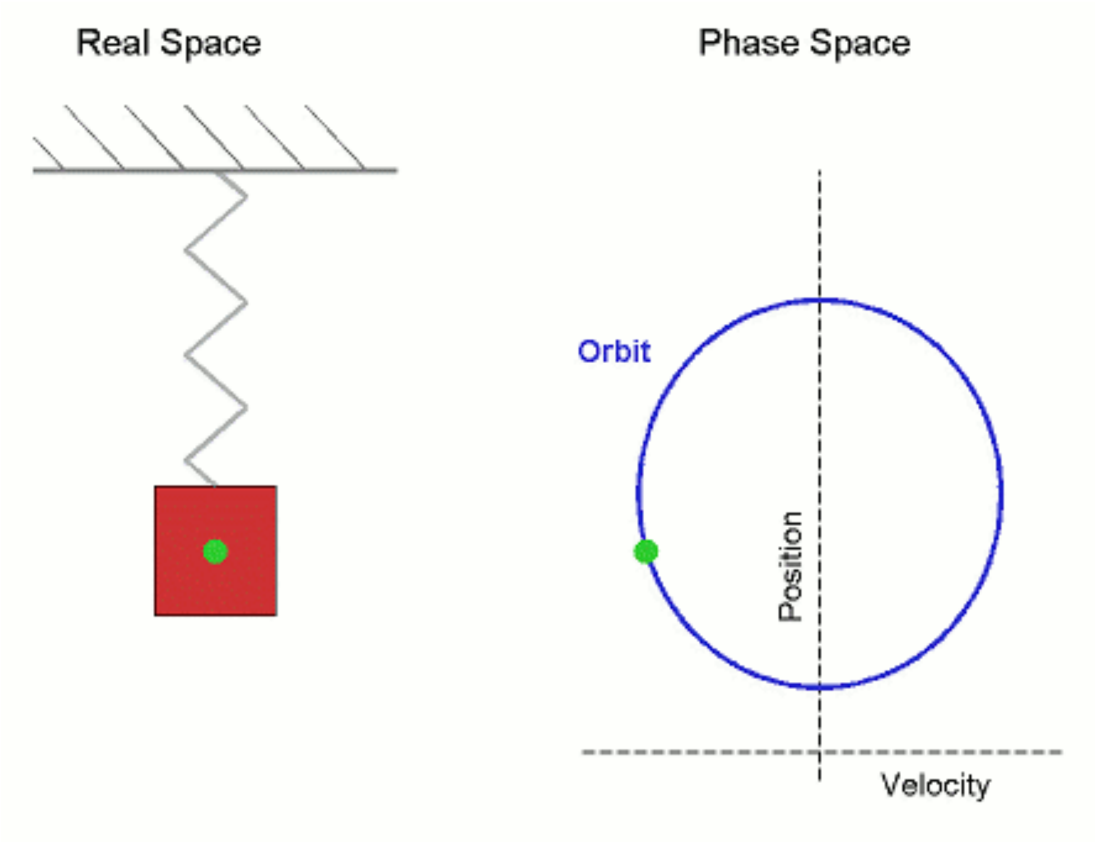
Mass on a Spring, Phase Space

- State of system (phase) : velocity & position
  - similar to our X=(x v) to get 1st order



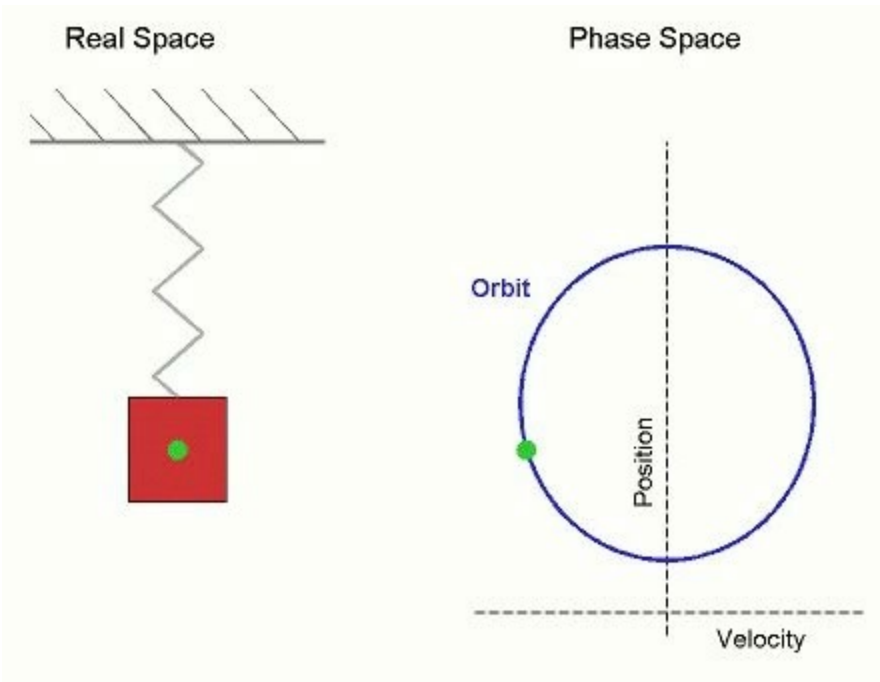
Mass on a Spring, Phase Space

- Guess how well Euler will do...
  always diverge



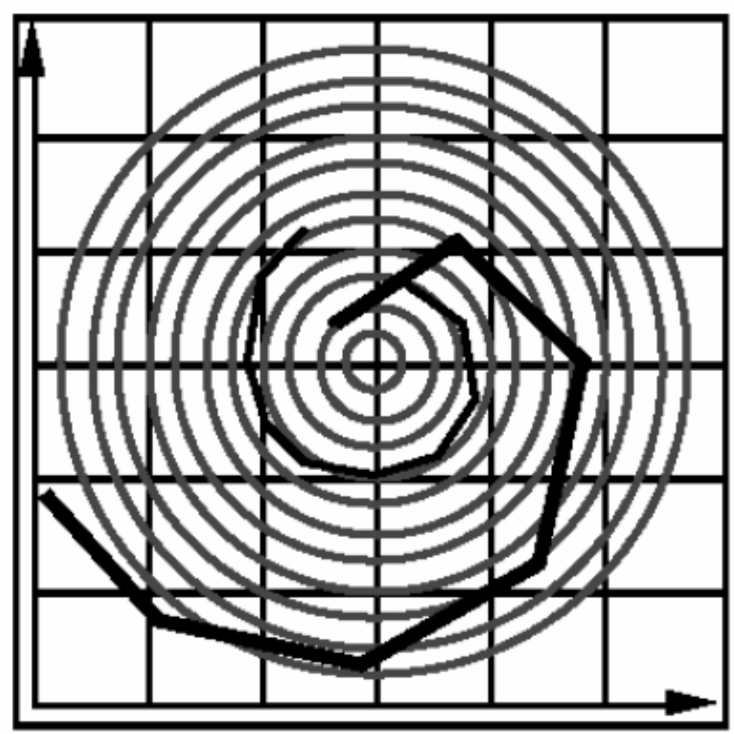
Mass on a Spring, Phase Space

- Guess how well Euler will do...
  always diverge



Phase Space

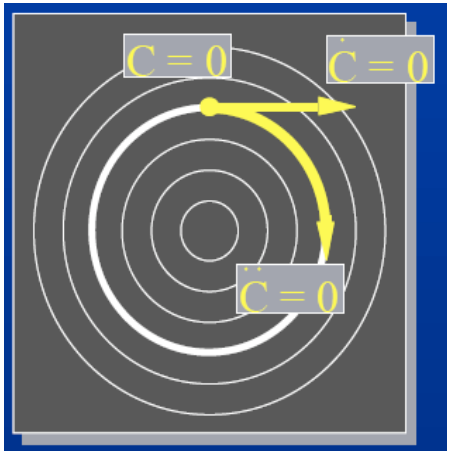
v



x

Constrained Dynamics (Advanced)

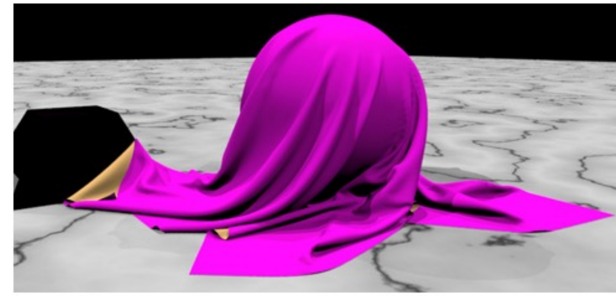
- In our mass-spring cloth, we have "encouraged" length preservation using springs that want to have a given length
- Constrained dynamic simulation: force it to be constant!
- How it works
  - Start with constraint equation
    - E.g., $(x_2-x_1)-1 = 0$ in the previous 1D example
  - Derive extra forces that will exactly enforce constraint
    - This means projecting the external forces (like gravity) onto the "subspace" of phase space where constraints are satisfied
    - Fancy name for this: "Lagrange multipliers"
  - See the SIGGRAPH 2001 Course Notes for more details



Questions?

- Further reading
  - Stiff systems: http://en.wikipedia.org/wiki/Stiff_equation
  - Explicit vs. implicit solvers: http://en.wikipedia.org/wiki/Explicit_and_implicit_methods
  - Again, consult the SIGGRAPH 2001 course notes!
    - http://www.pixar.com/companyinfo/research/pbm2001/

The Collision Problem



- More advanced topic
- A cloth has many points of contact
- Requires
  - Efficient collision detection
  - Efficient numerical

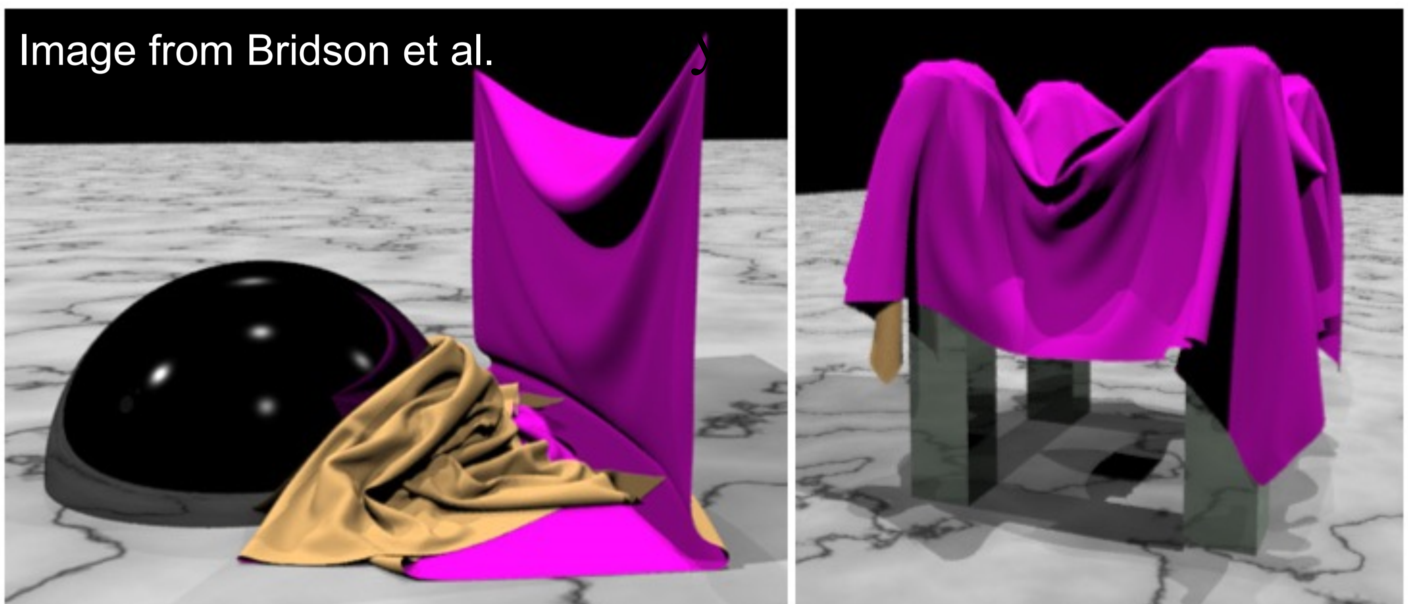


Image from Bridson et al.

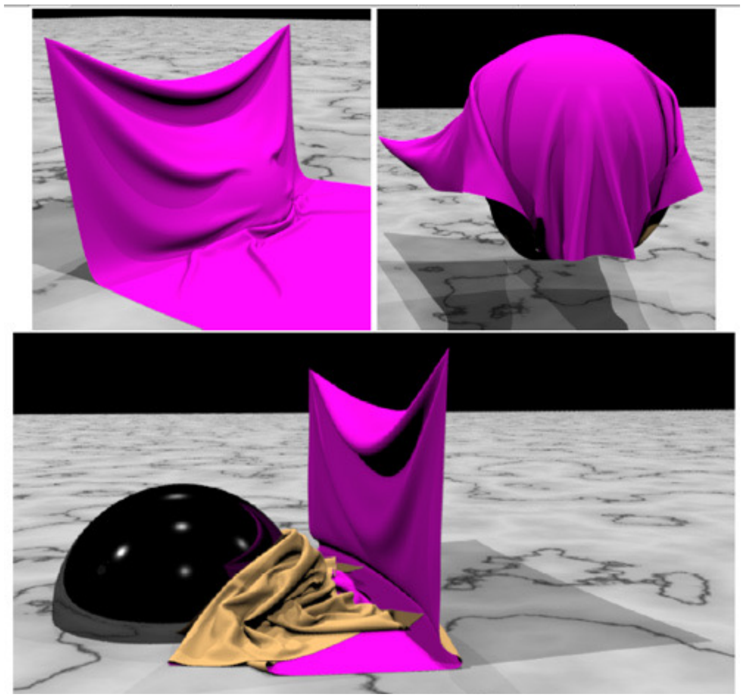
Collisions

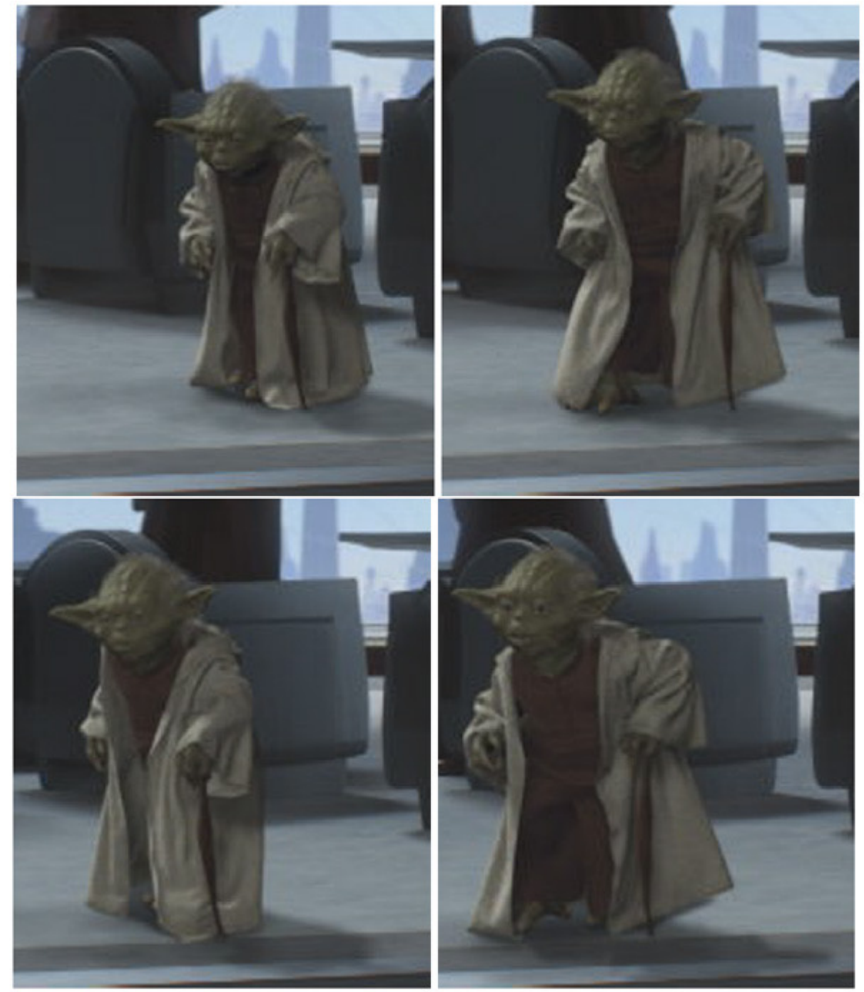
[Robert Bridson, Ronald Fedkiw & John Anderson](#)

Robust Treatment of Collisions, Contact and Friction for Cloth Animation

SIGGRAPH 2002

- Cloth has many points of contact
- Need efficient collision detection and stable treatment





Cool Cloth/Hair Demos

- Robert Bridson, Ronald Fedkiw & John Anderson: Robust Treatment of Collisions, Contact and Friction for Cloth Animation SIGGRAPH 2002
- Selle. A, Su, J., Irving, G. and Fedkiw, R., "Robust High-Resolution Cloth Using Parallelism, History-Based Collisions, and Accurate Friction," IEEE TVCG 15, 339-350 (2009).
- Selle, A., Lentine, M. and Fedkiw, R., "A Mass Spring Model for Hair Simulation", SIGGRAPH 2008, ACM TOG 27, 64.1-64.11 (2008).

Cool Cloth/Hair Demos

Selle, A., Lentine, M. and Fedkiw, R., "A Mass Spring Model for Hair Simulation", SIGGRAPH 2008, ACM TOG 27, 64.1-64.11 (2008).

Cool Cloth/Hair Demos Questions?

- Selle. A, Su, J., Irving, G. and Fedkiw, R., "Robust High-Resolution Cloth Using Parallelism, History-Based Collisions, and Accurate Friction," IEEE TVCG 15, 339-350 (2009).

Implementation Notes

- It pays off to abstract (as usual)
  - It's easy to design your "Particle System" and "Time Stepper" to be unaware of each other
- Basic idea
  - "Particle system" and "Time Stepper" communicate via floating-point vectors X and a function that computes f(X,t)
    - "Time Stepper" does not need to know anything else!

Implementation Notes

- Basic idea
  - “Particle System” tells “Time Stepper” how many dimensions (N) the phase space has
  - “Particle System” has a function to write its state to an N-vector of floating point numbers (and read state from it)
  - “Particle System” has a function that evaluates f(X,t), given a state vector X and time t
  - “Time Stepper” takes a “Particle System” as input and advances its state

Particle System Class

```
class ParticleSystem
{
      virtual int getDimension()
      virtual setDimension(int n)
      virtual float* getStatePositions()
      virtual setStatePositions(float* positions)
      virtual float* getStateVelocities()
      virtual setStateVelocities(float* velocities)
      virtual float* getForces(float* positions, float* velocities)
            virtual setMasses(float* masses)
            virtual float* getMasses()

      float* m_currentState
}
```

Time Stepper Class

```
class TimeStepper
{
      virtual takeStep(ParticleSystem* ps, float h)
}
```

Forward Euler Implementation

```
class ForwardEuler : TimeStepper
{
        void takeStep(ParticleSystem* ps, float h)
         {
         velocities = ps->getStateVelocities()
         positions = ps->getStatePositions()
         forces = ps->getForces(positions, velocities)
         masses = ps->getMasses()
         accelerations = forces / masses
         newPositions = positions + h*velocities
         newVelocities = velocities  + h*accelerations
         ps->setStatePositions(newPositions)
         ps->setStateVelocities(newVelocities)
         }
}
```

Mid-Point Implementation

```
class MidPoint : TimeStepper
{
        void takeStep(ParticleSystem* ps, float h)
         {
         velocities = ps->getStateVelocities()
         positions = ps->getStatePositions()
         forces = ps->getForces(positions, velocities)
         masses = ps->getMasses()
         accelerations = forces / masses
         midPositions = positions + 0.5*h*velocities
         midVelocities = velocities  + 0.5*h*accelerations
         midForces = ps->getForces(midPositions, midVelocities)
         midAccelerations = midForces / masses
         newPositions = positions + h*midVelocities
         newVelocities = velocities  + h*midAccelerations

         ps->setStatePositions(newPositions)
         ps->setStateVelocities(newVelocities)
         }
}
```

Particle System Simulation

```
        ps = new MassSpringSystem(particleCount, masses, springs,
externalForces)
        stepper = new ForwardEuler()
        time = 0
        while time < 1000
              stepper->takeStep(ps, 0.0001)
              time = time + 0.0001
              // render
```

Particle System Simulation

```
        ps = new MassSpringSystem(particleCount, masses, springs,
externalForces)
        stepper = new MidPoint()
        time = 0
        while time < 1000
              stepper->takeStep(ps, 0.0001)
              time = time + 0.0001
              // render
```

Questions?