LECTURE 4: SMALL-WORLD PHENOMENA

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CSIT 6000K: Social Networks and Social Computing: A Data Science Perspective

Thursdays 07:30 PM - 10:20 PM

Small world: a simplistic argument

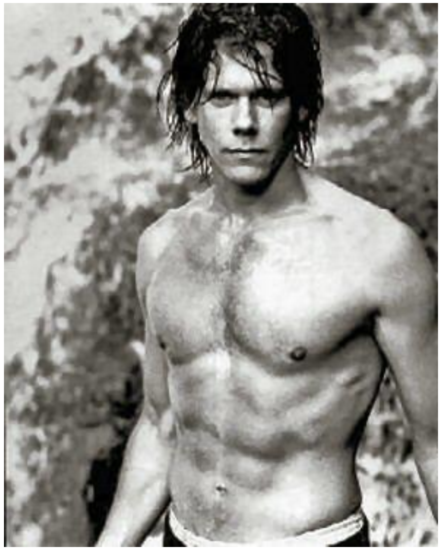
- How many people would you recognize by name?
  - ‘67 M. Gurevitch (MIT): about 500
- Roughly, how many are socially related to you?

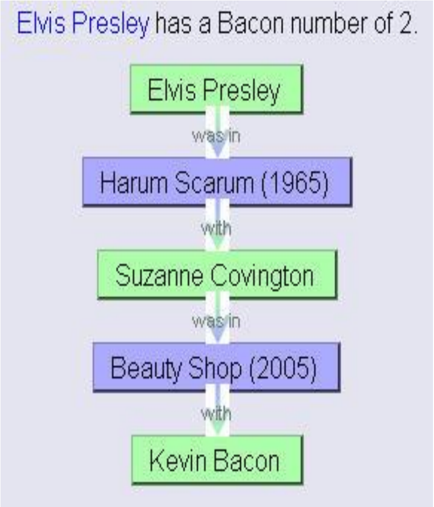
	how close to you?	Compares to	%. HK pop.
500	direct acquaintance	C.S. dept	0.0067%
250,000	share an acquaintance with you	TKO district (398,479)	3.382%
125m	share an acquaintance with a friend of yours	Guangdong Province (113m)	1691%

Six Degrees of Kevin Bacon

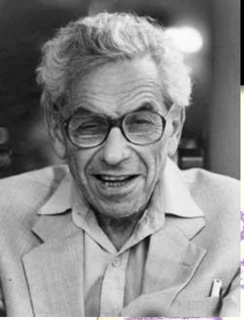
Origins of a small-world idea:

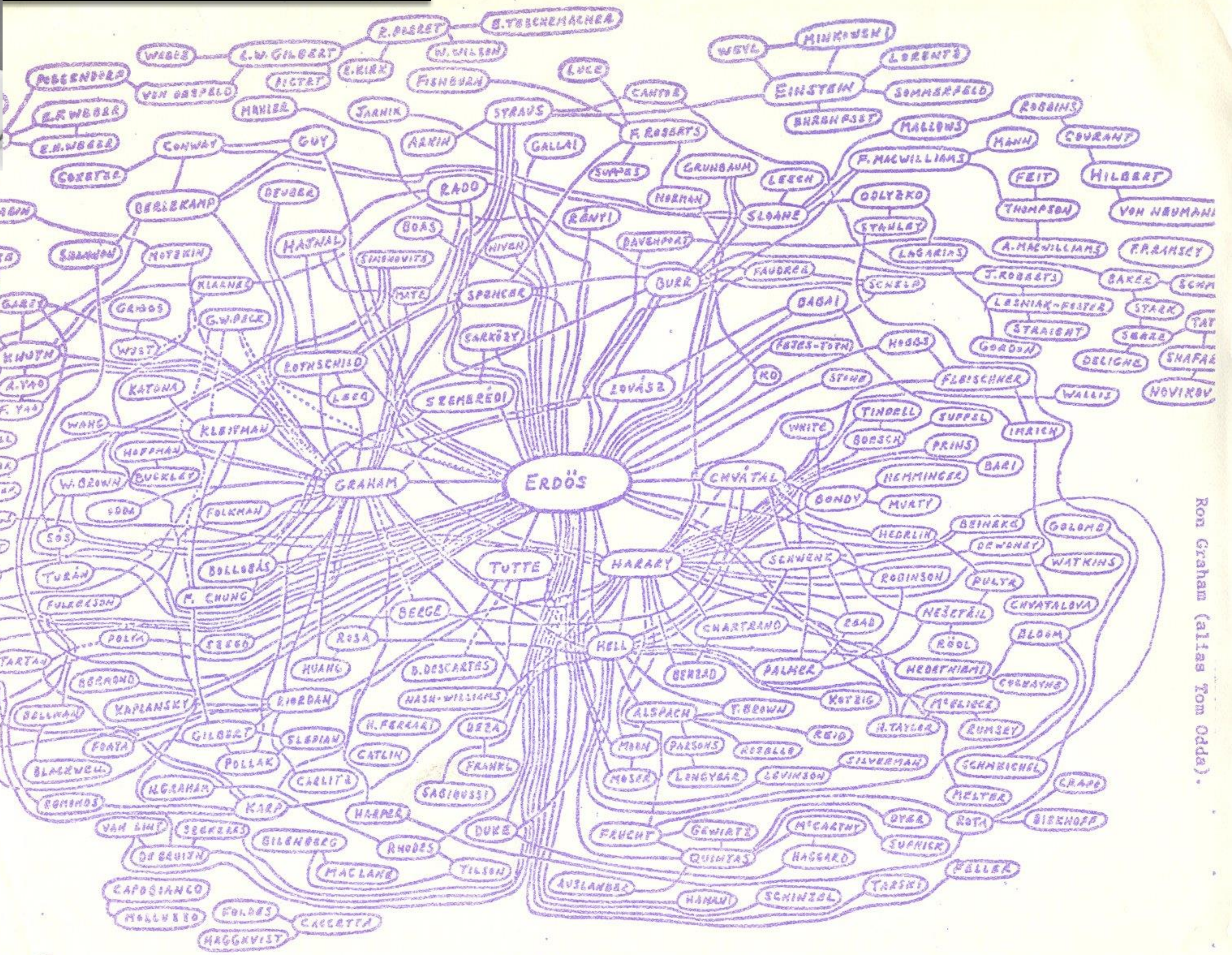
- **The Bacon number:**
  - Create a network of Hollywood actors
  - Connect two actors if they co-appeared in the movie
  - Bacon number: number of steps to Kevin Bacon
- As of Dec 2007, the highest (finite) Bacon number reported is 8
- Only approx. 12% of all actors cannot be linked to Bacon





Erdös numbers are small!



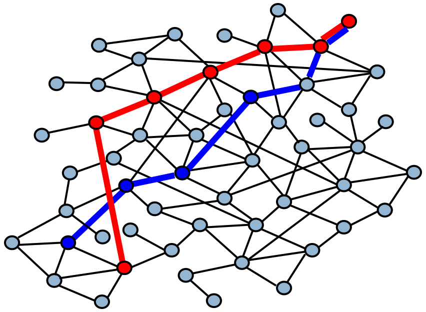


Ron Graham (alias Tom Odda).

Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

The Small-World Experiment

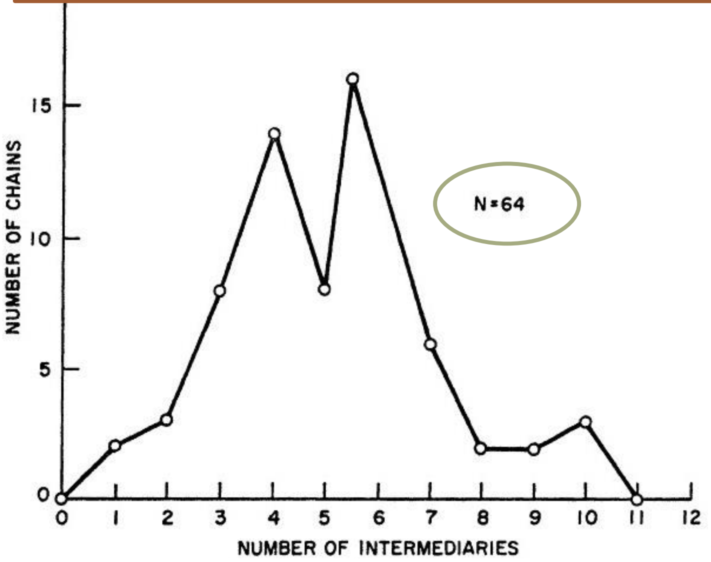
- What is the typical shortest path length between any two people?
  - Experiment on the global friendship network
    - Can’t measure, need to probe explicitly
- Small-world experiment [Milgram ’67]
  - Picked 300 people in Omaha, Nebraska and Wichita, Kansas
  - Ask them to get a letter to a stock-broker in Boston by passing it through friends
- How many steps did it take?



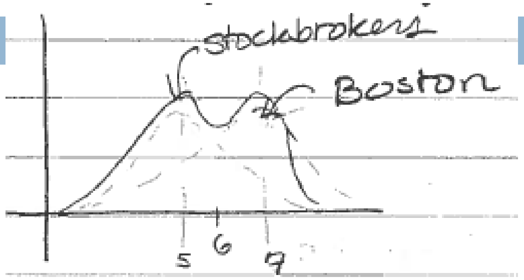
The Small-World Experiment

- **64 chains completed:**
  (i.e., 64 letters reached the target)
  - It took 6.2 steps on the average, thus
    “6 degrees of separation”
- **Further observations:**
  - People what owned stock
    had shortest paths to the stockbroker
    than random people: 5.4 vs. 5.7
  - People from the Boston area have even
    closer paths: 4.4

Milgram’s small world experiment



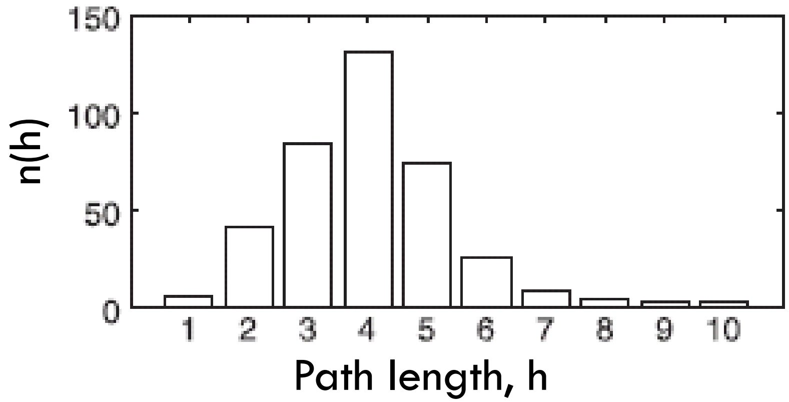
Milgram: Further Observations



- Boston vs. occupation networks:
- Criticism:
  - **Funneling:**
    - 31 of 64 chains passed through 1 of 3 people as their final step → Not all links/nodes are equal
  - Starting points and the target were non-random
  - People refused to participate (25% for Milgram)
  - **Some sort of social search:** People in the experiment follow some strategy (e.g., geographic routing) instead of forwarding the letter to everyone. **They are not finding the shortest path!**
  - There are not many samples (only 64)
  - People might have used extra information resources

Columbia Small-World Study

- In 2003 Dodds, Muhamad and Watts performed the experiment using e-mail:
  - 18 targets of various backgrounds
  - 24,000 first steps (~1,500 per target)
  - 65% dropout per step
  - 384 chains completed (1.5%)



Avg. chain length = 4.01

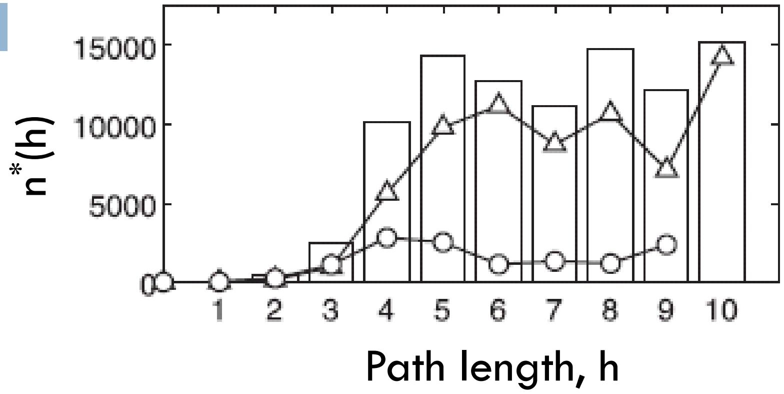
Problem: People stop participating

Correction factor: $$n^*(h) = \frac{n(h)}{\prod_{i=0}^{h-1}(1-r_i)}$$

r_i …. drop-out rate at hop *i*

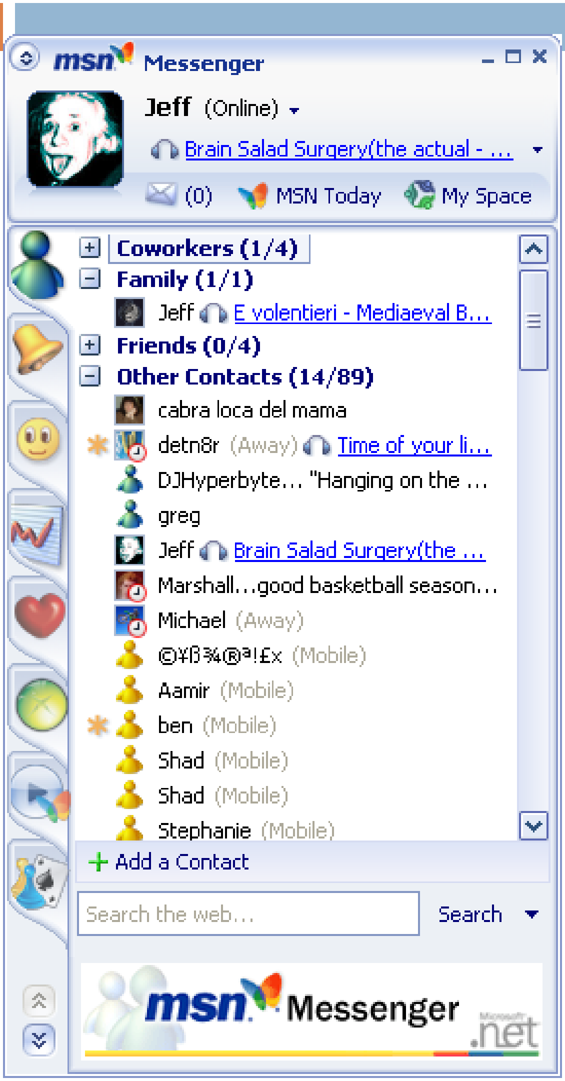
Small-World in Email Study

- After the correction:
  - **Typical path length h = 7**



- Some not well understood phenomena in social networks:
  - Funneling effect: Some target's friends are more likely to be the final step.
    - Conjecture: High reputation/authority
  - Effects of target's characteristics: Structurally why are high-status target easier to find
    - Conjecture: Core-periphery net structure

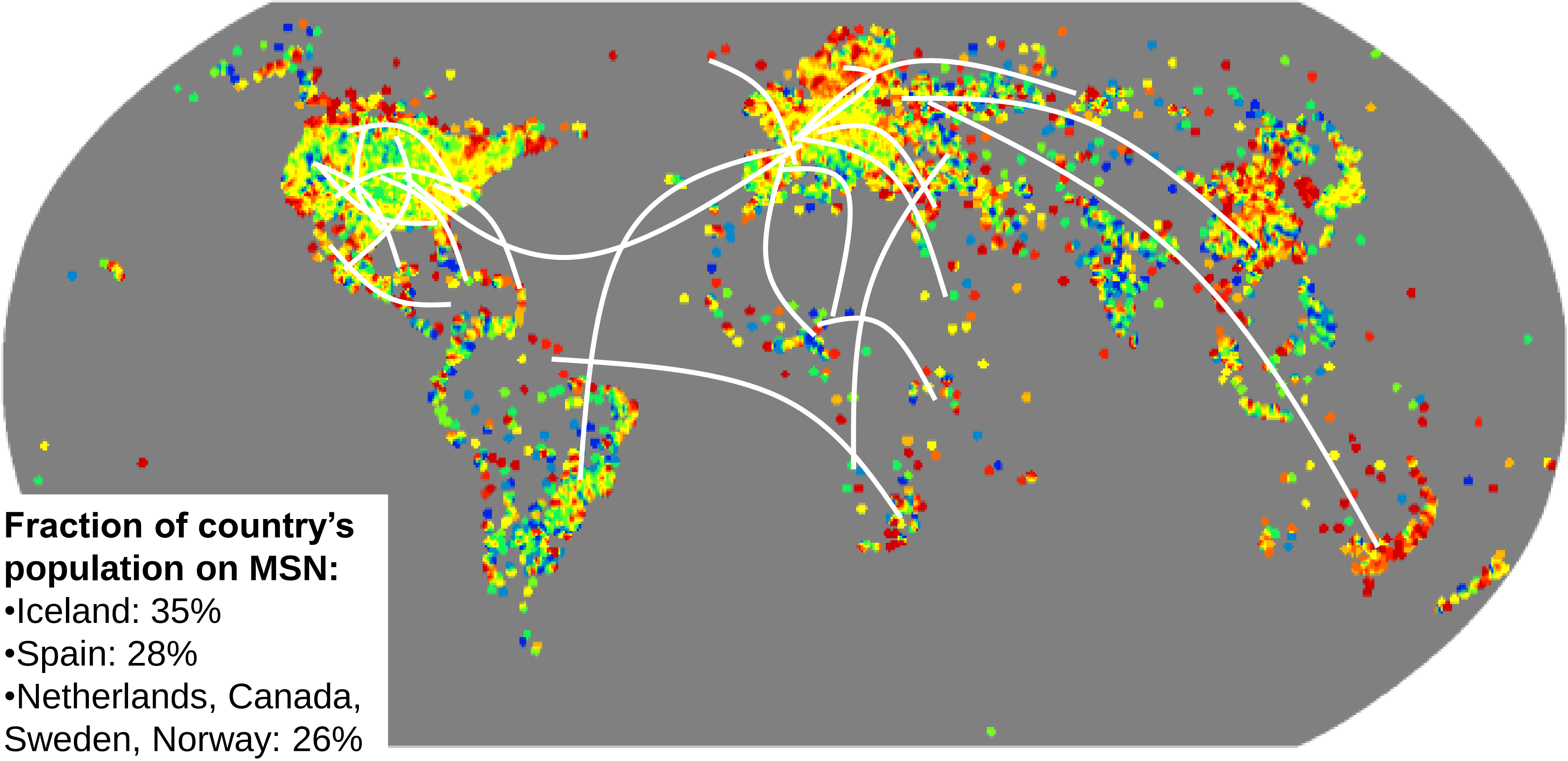
The MSN Messenger



Coworkers (1/4)
Family (1/1)
Jeff E volentieri - Mediaeval B...
Friends (0/4)
Other Contacts (14/89)
cabra loca del mama
detn8r (Away) Time of your li...
DJHyperbyte... "Hanging on the ...
greg
Jeff Brain Salad Surgery(the ...
Marshall...good basketball season...
Michael (Away)
©¥ß¾®ª!£x (Mobile)
Aamir (Mobile)
ben (Mobile)
Shad (Mobile)
Shad (Mobile)
Stephanie (Mobile)

- **MSN Messenger activity in June 2006:**
  - 245 million users logged in
  - 180 million users engaged in conversations
  - More than 30 billion conversations
  - More than 255 billion exchanged messages

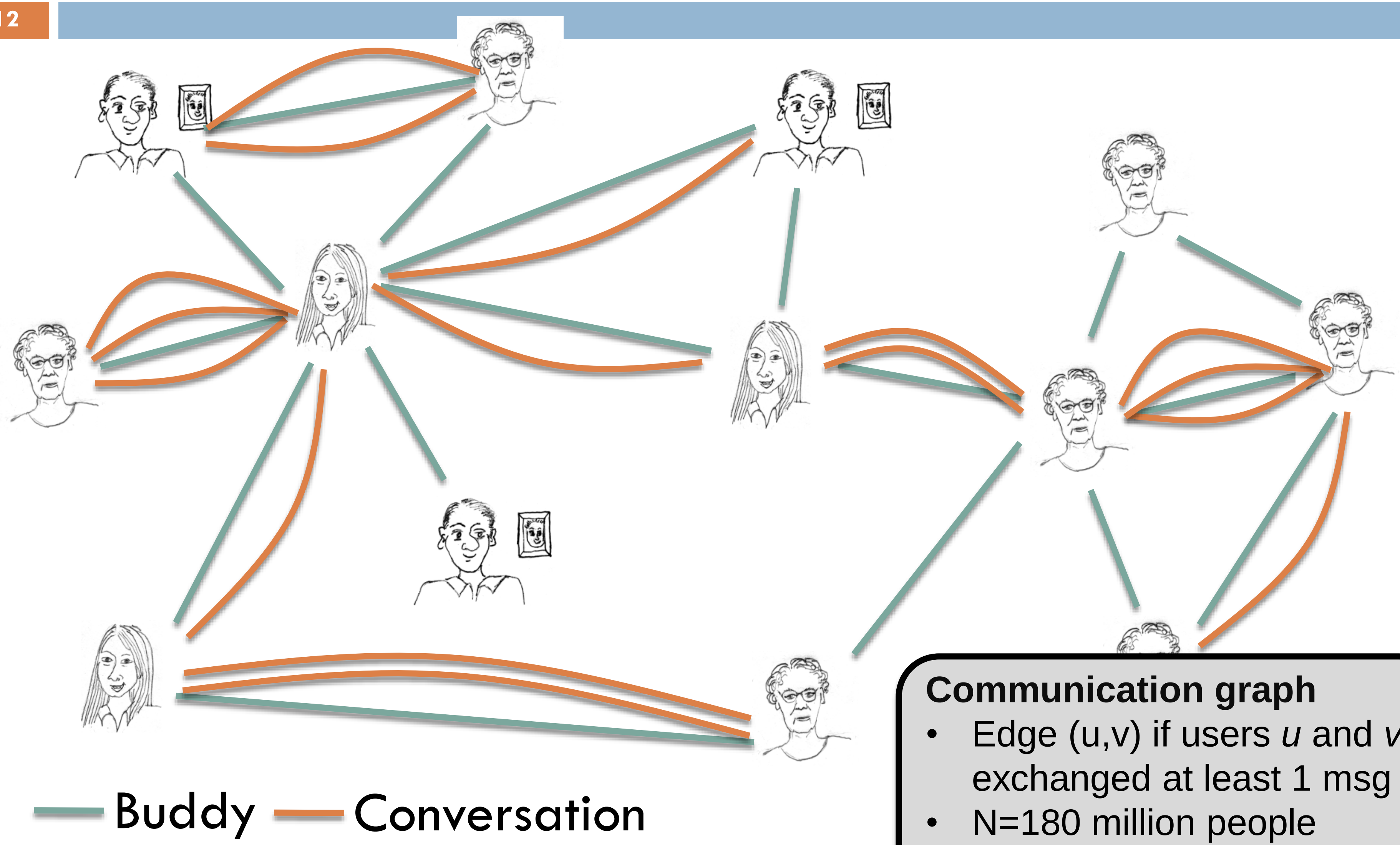
Messaging as a Network



Fraction of country's population on MSN:

- Iceland: 35%
- Spain: 28%
- Netherlands, Canada, Sweden, Norway: 26%
- France, UK: 18%
- USA, Brazil: 8%

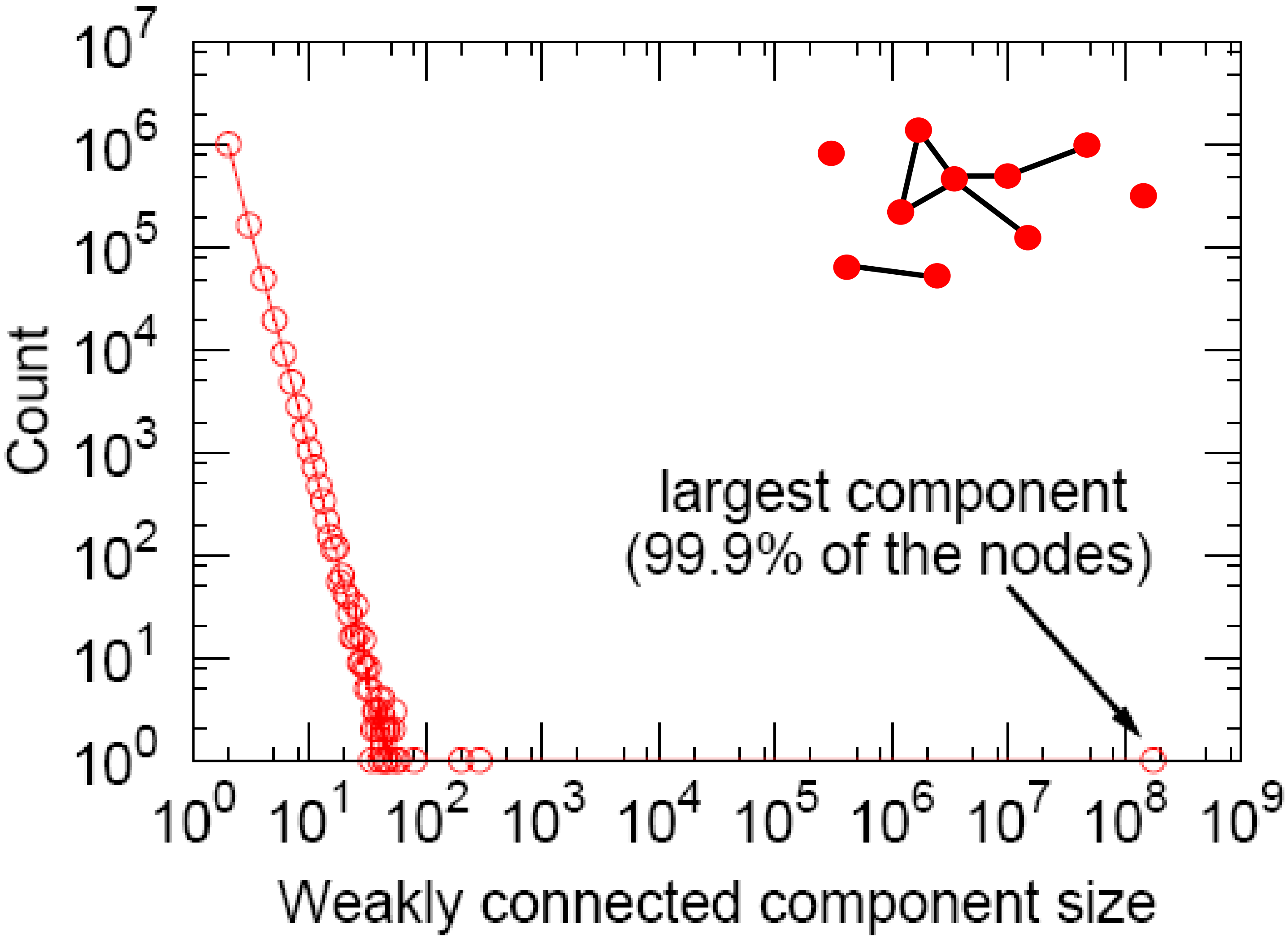
Messaging as a Network



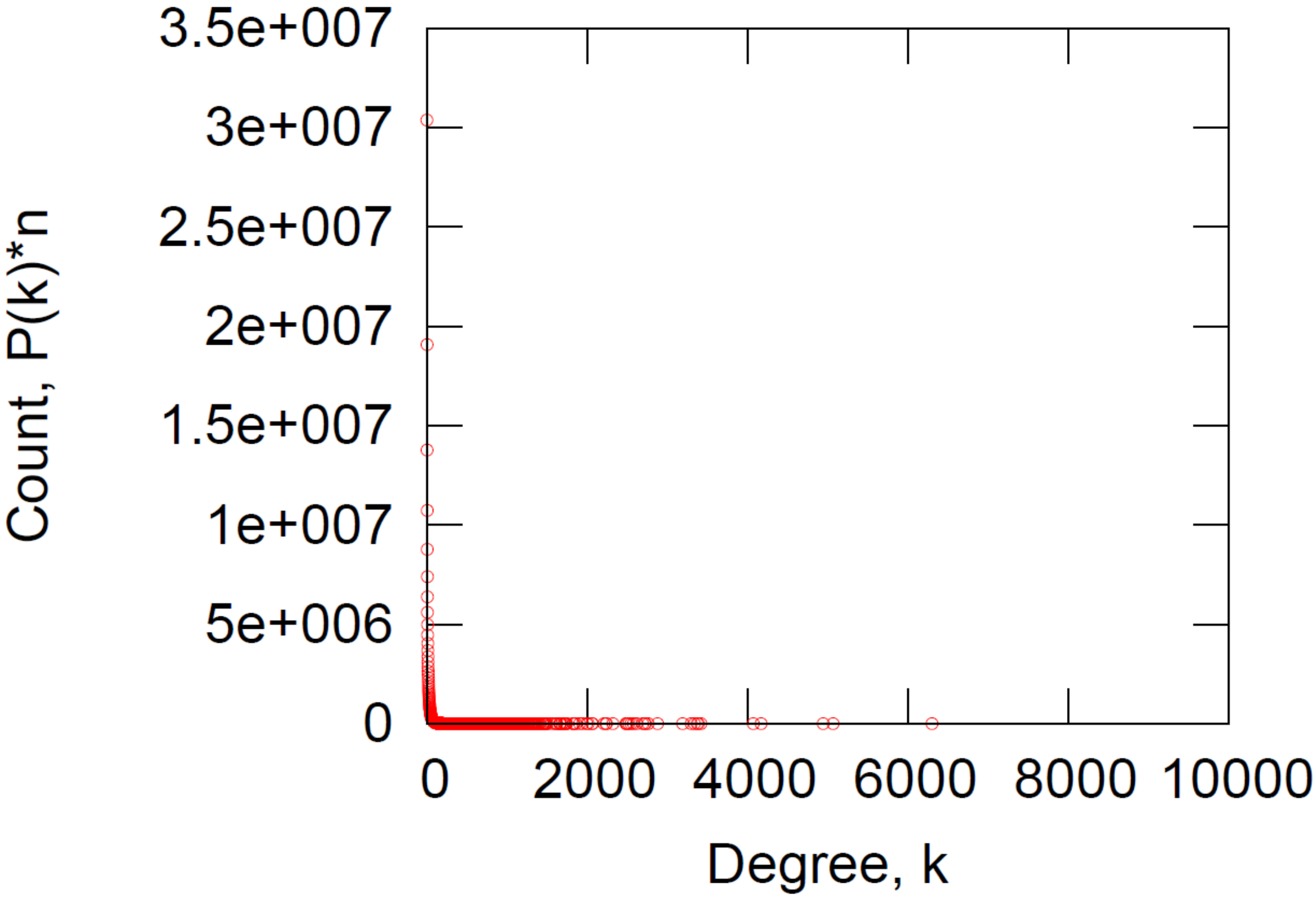
Communication graph

- Edge (u,v) if users *u* and *v* exchanged at least 1 msg
- N=180 million people
- E=1.3 billion edges

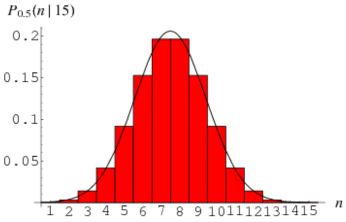
MSN Network: Connectivity



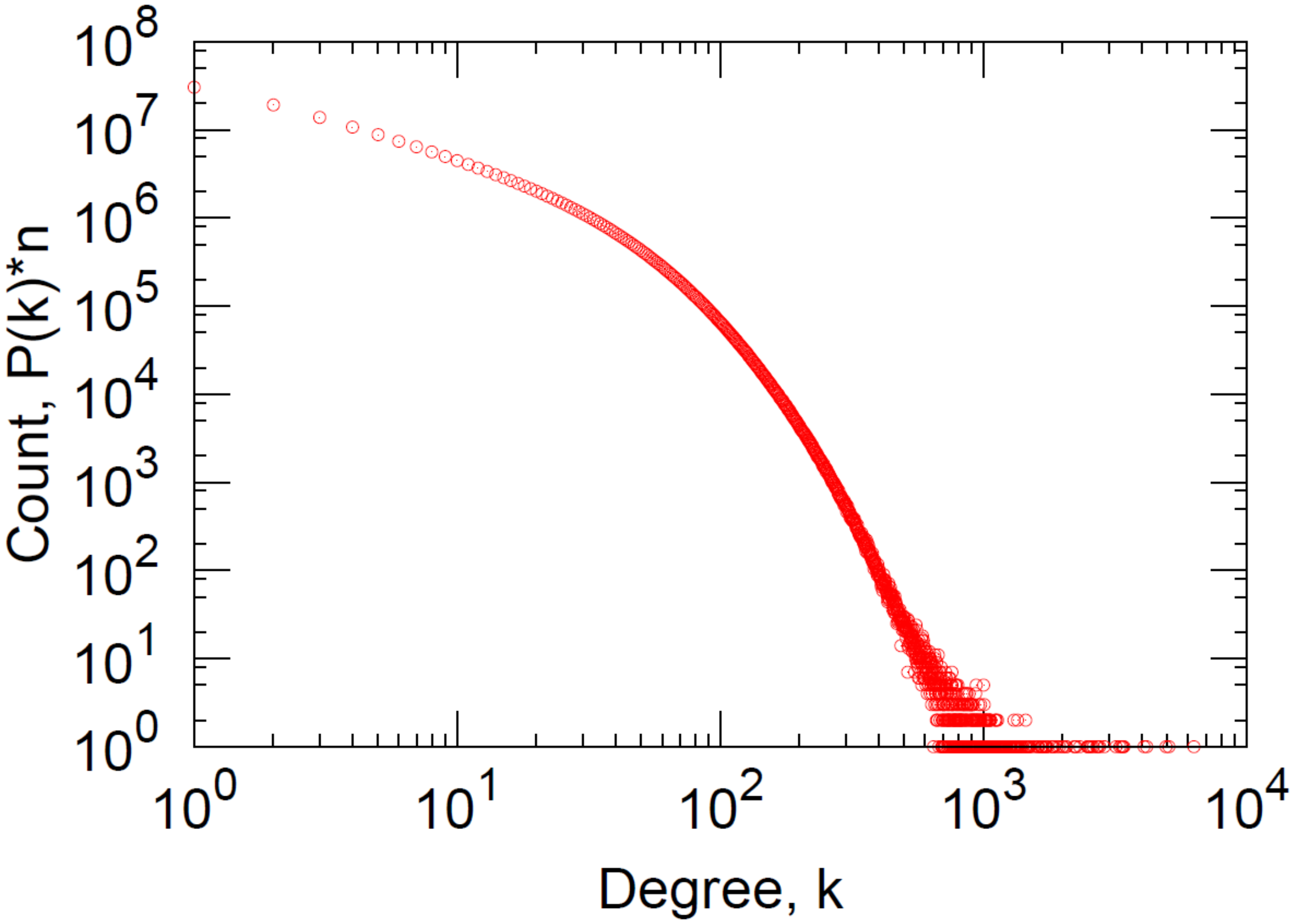
MSN: Degree Distribution



Note: Degree distribution of the MSN looks nothing like the G_{np}:

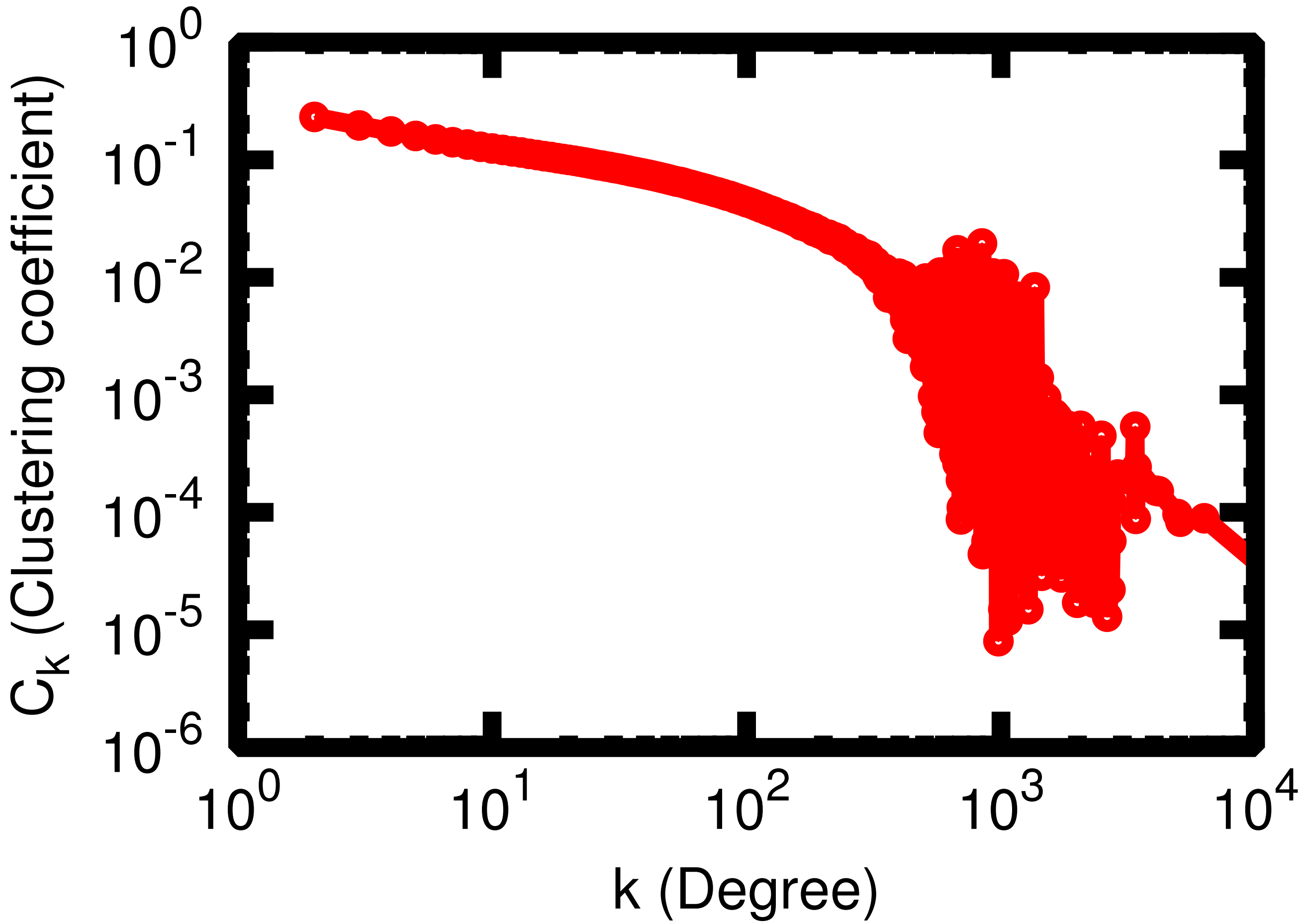


MSN: Log-Log Degree Distribution



We plot the same data as on the previous slide, just the axes are now logarithmic.

MSN: Clustering

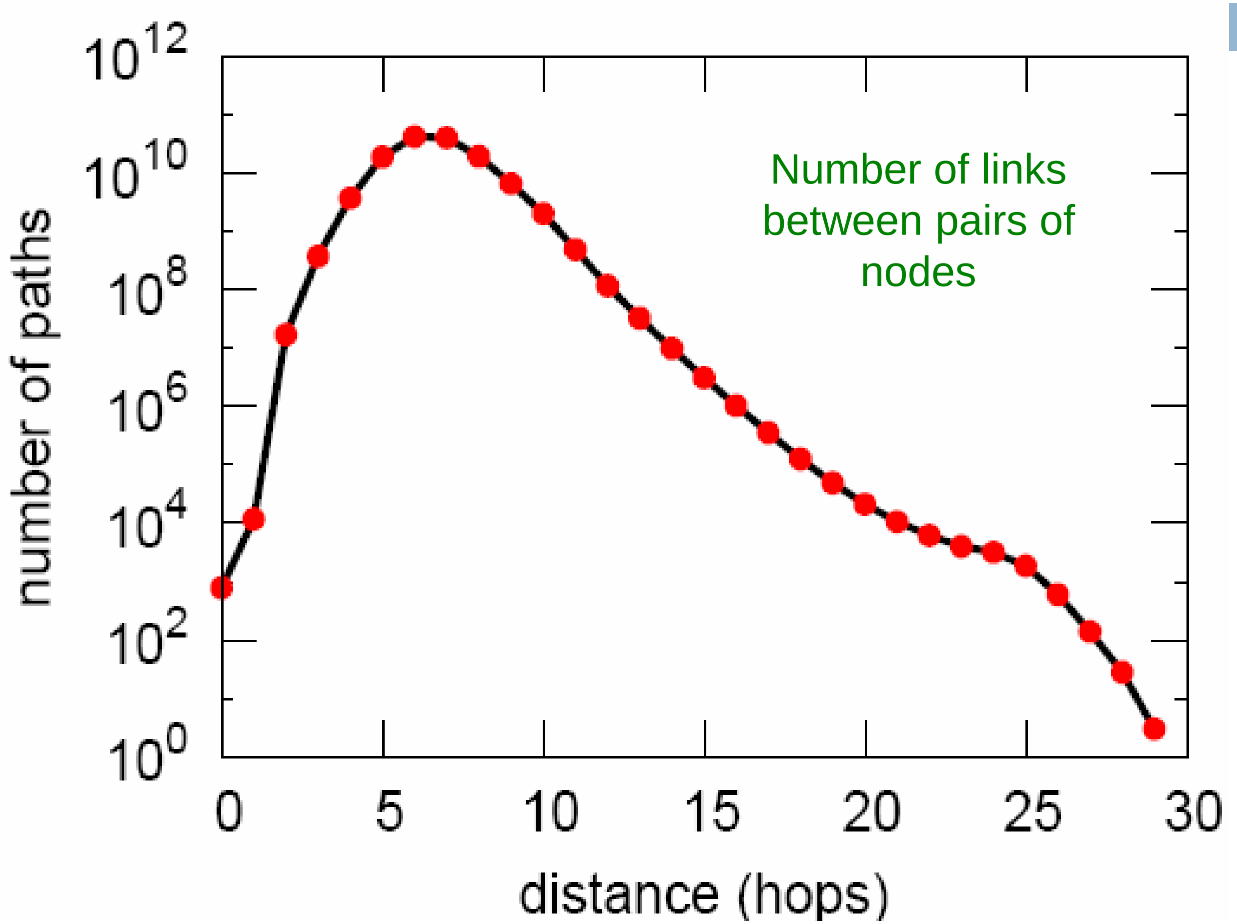


Avg. clustering of the MSN:
C = 0.1140

Avg. clustering of corresponding G_{np}:
$C = \overline{k}/n \approx 8 \cdot 10^{-8}$

C_k: average C_i of nodes i of degree k: $$C_k = \frac{1}{N_k} \sum_{i:k_i=k} C_i$$

MSN: Diameter



Number of links between pairs of nodes

Avg. path length **6.6**

90% of the people can be reached in < 8 hops

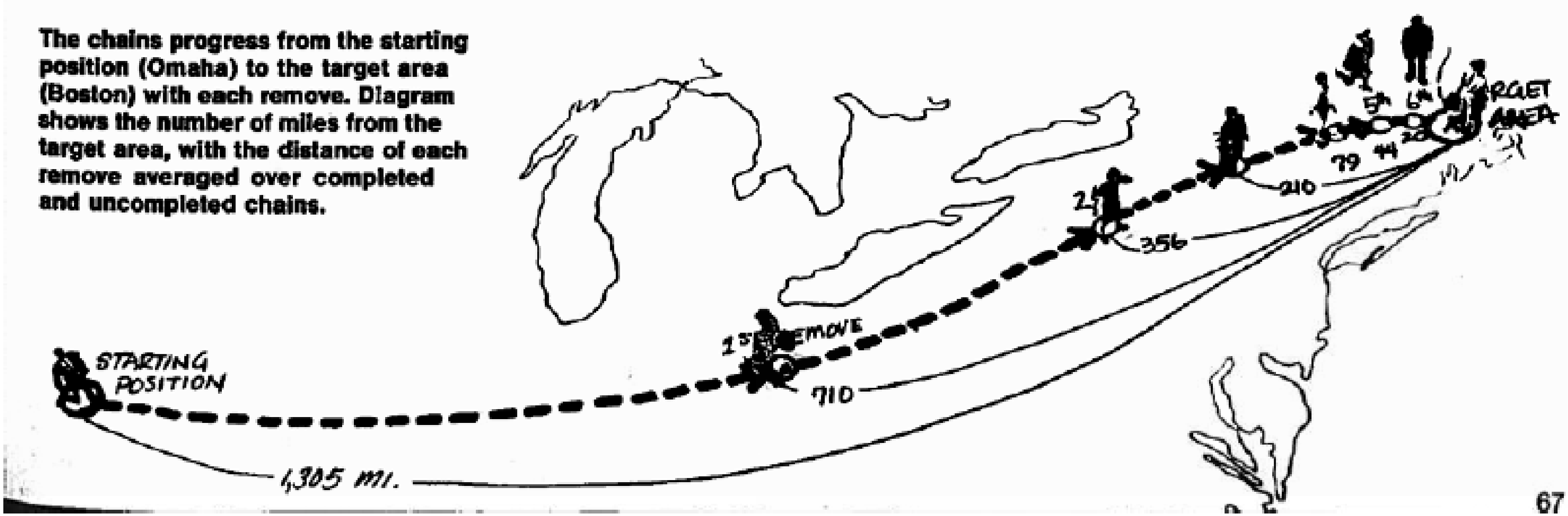
# nodes as we do BFS out of a rand

Steps	#Nodes
0	1
1	10
2	78
3	3,96
8	52,993,778
9	10,321,008
10	1,955,007
11	518,410
12	149,945
13	44,616
14	13,740
15	4,476
16	1,542
17	536
18	167
19	71
20	29
21	16
22	10
23	3
24	2
25	3

Table: use the expansion terminology: j and Sj.

Two Questions

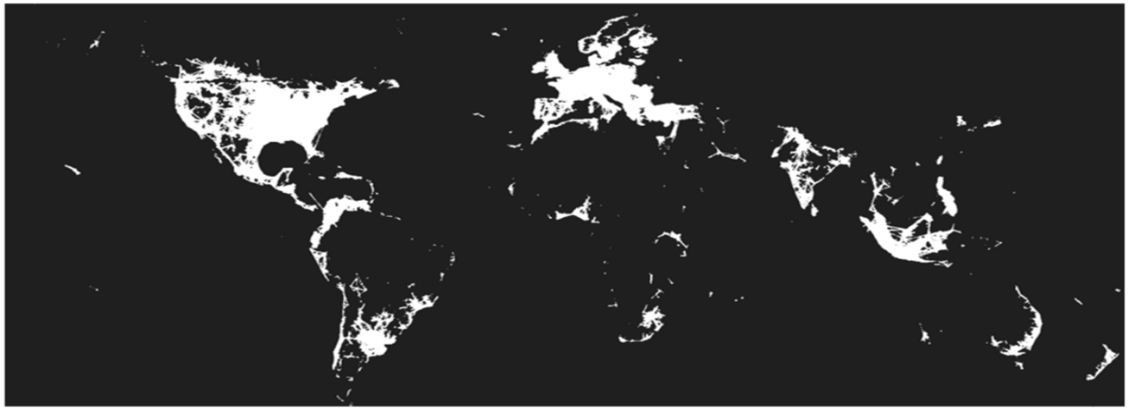
- □ (Today) What is the structure of a social network?
- □ (Later) Which mechanisms do people use to route and find the target?

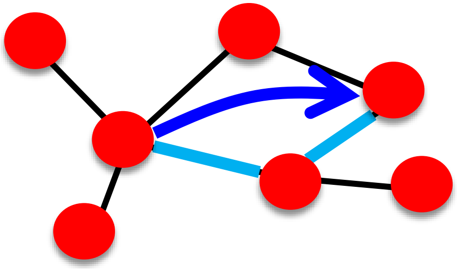


The chains progress from the starting position (Omaha) to the target area (Boston) with each remove. Diagram shows the number of miles from the target area, with the distance of each remove averaged over completed and uncompleted chains.

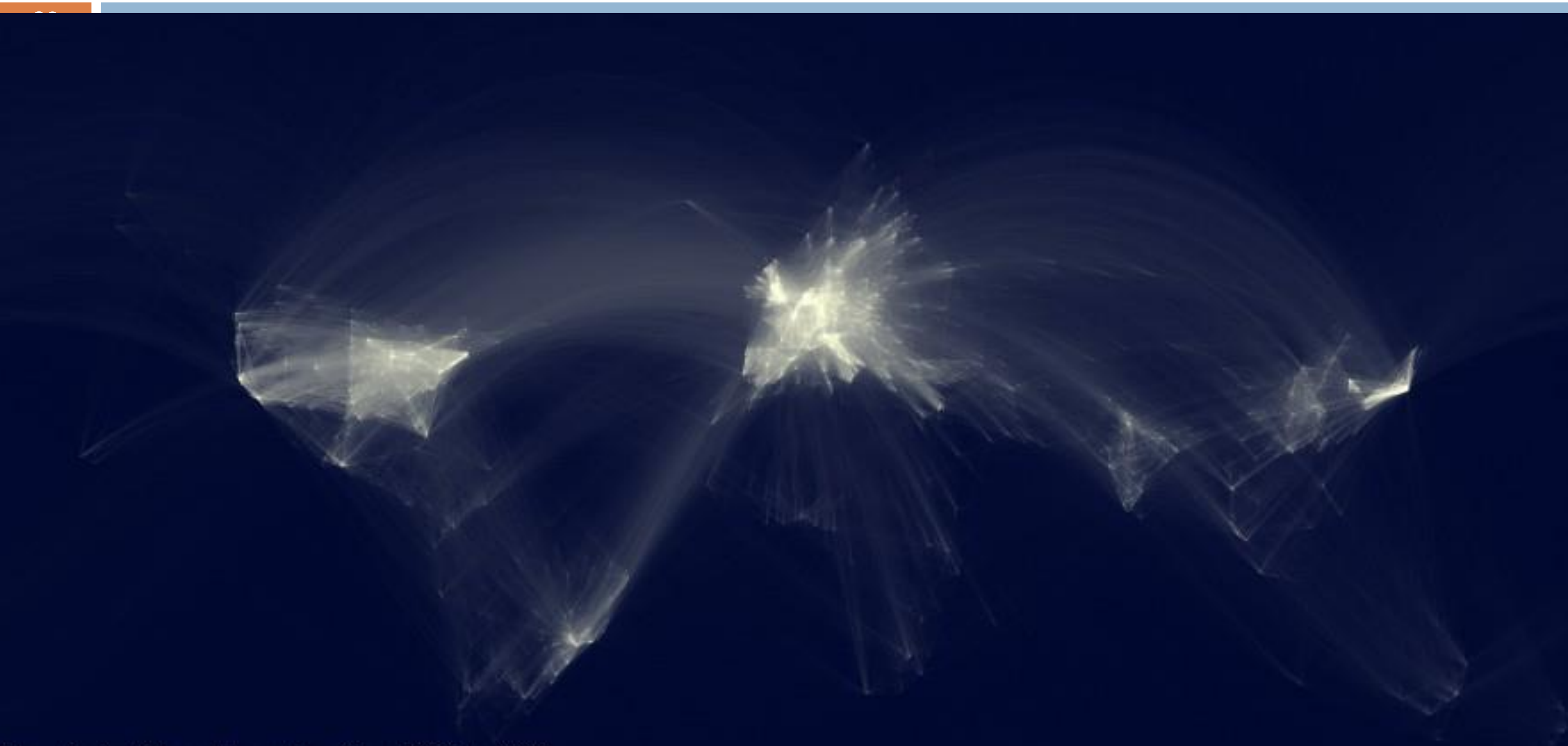
6-Degrees: Should We Be Surprised?

- Assume each human is connected to 100 other people. **Then:**
  - Step 1: reach 100 people
  - Step 2: reach 100*100 = 10,000 people
  - Step 3: reach 100*100*100 = 1,000,000 people
  - Step 4: reach 100*100*100*100 = 100M people
  - **In 5 steps we can reach 10 billion people**
- **What's wrong here?**
  - **92% of new FB friendships are to a friend-of-a-friend** [Backstom-Leskovec '11]





Scientific Collaborations



Map of scientific collaborations from 2005 to 2009
Computed by Olivier H. Beauchesne @ Science-Metrix, Inc.
Data from Scopus, using books, trade journals and peer-reviewed journals

Clustering Implies Edge Locality

- **MSN network has 7 orders of magnitude larger clustering than the corresponding G_{np}!**
- **Other examples:**

Actor Collaborations (IMDB): N = 225,226 nodes, avg. degree $\overline{k}$ = 61
Electrical power grid: N = 4,941 nodes, $\overline{k}$ = 2.67
Network of neurons: N = 282 nodes, $\overline{k}$ = 14

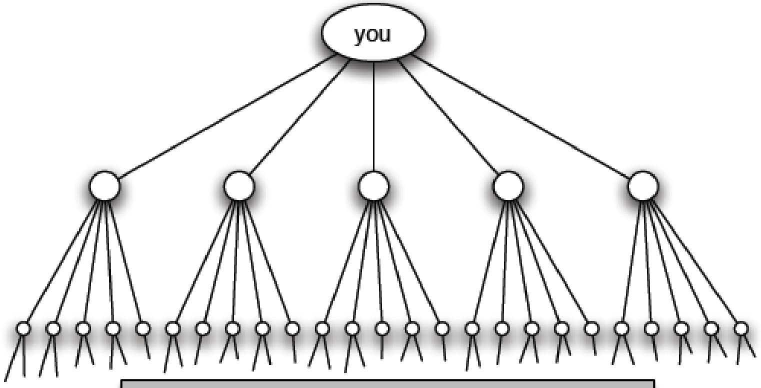
Network	h_{actual}	h_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power Grid	18.70	12.40	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

h ... Average shortest path length
C ... Average clustering coefficient

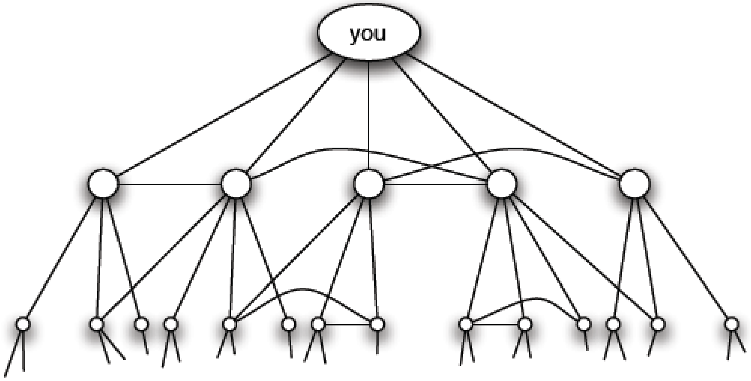
Back to the Small-World

- **Consequence of expansion:**
  - Short paths: $O(\log n)$
    - This is “best” we can do if the have a constant degree
    - and there are *n* nodes
- **But networks have local structure:**
  - **Triadic closure:**

    Friend of a friend is my friend
- **How can we have both?**



Pure exponential growth



Triadic closure reduces growth rate

Clustering vs. Randomness

Where should we place social networks?

Clustered?

Random?

Simplest Model of Graphs

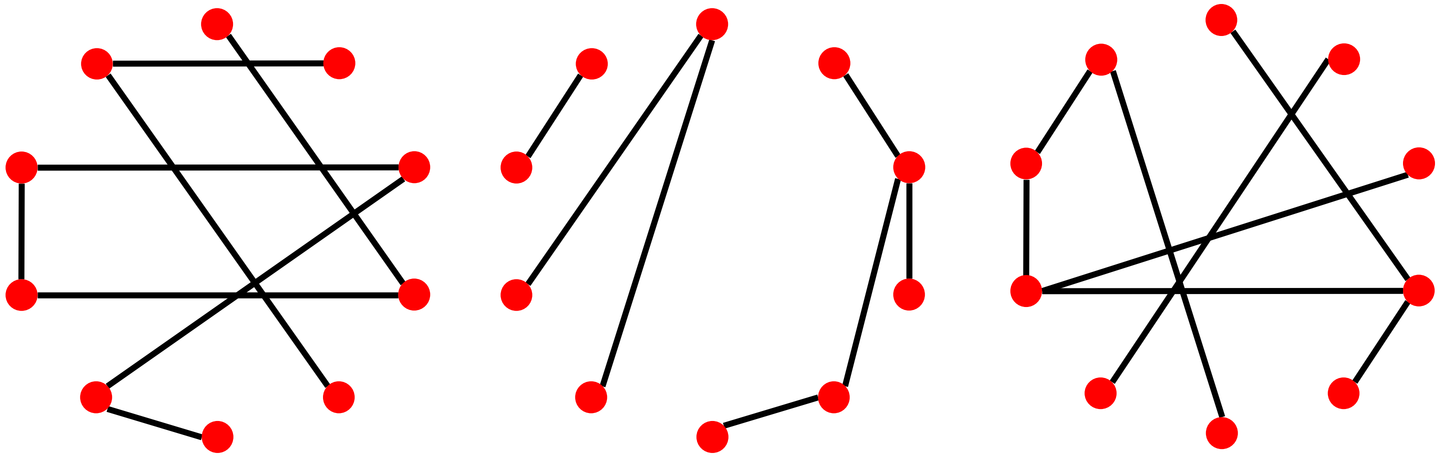
- **Erdös-Renyi Random Graphs** [Erdös-Renyi, '60]

- Two variants:
  - $G_{n,p}$: undirected graph on n nodes and each edge (u,v) appears i.i.d. with probability p
  - $G_{n,m}$: undirected graph with n nodes, and m uniformly at random picked edges

What kinds of networks does such model produce?

Random Graph Model

- ***n* and *p* do not uniquely determine the graph!**
  - The graph is a result of a random process
- We can have many different realizations



n = 10
p= 1/6

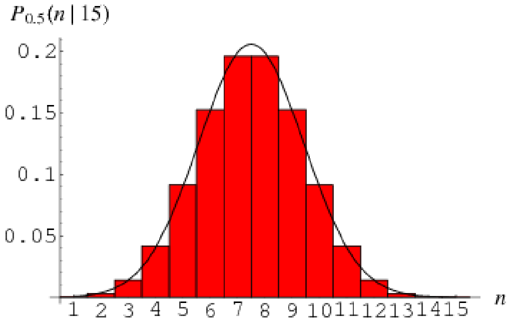
Random Graph Model: Edges

- **How likely is a graph on E edges?**
- $P(E)$: the probability that a given G_{np} generates a graph on exactly E edges:

$$P(E) = \binom{E_{\max}}{E} p^{E} (1-p)^{E_{\max}-E}$$

where $E_{max}=n(n-1)/2$ is the maximum possible number of edges in an undirected graph of n nodes

Binomial distribution >>>



Node Degrees in a Random Graph

- **What is expected degree of a node?**
  - Let X_v be a rnd. var. measuring the degree of node v
  - **We want to know:** $E[X_v] = \sum_{j=0}^{n-1} j\, P(X_v = j)$
    - **For the calculation we will need: Linearity of expectation**
      - For any random variables $Y_1, Y_2, \ldots, Y_k$
      - If $Y = Y_1 + Y_2 + \ldots Y_k$, then $E[Y] = \sum_i E[Y_i]$
- **Easier way:**
  - Decompose X_v to $X_v = X_{v,1} + X_{v,2} + \ldots + X_{v,n-1}$
    - where $X_{v,u}$ is a $\{0,1\}$-random variable which tells if edge (v,u) exists or not

$$E[X_v] = \sum_{u=1}^{n-1} E[X_{vu}] = (n-1)p$$

How to think about this?

- Prob. of node u linking to node v is p
- u can link (flips a coin) to all other $(n\text{-}1)$ nodes
- Thus, the expected degree of node u is: $p(n\text{-}1)$

Properties of G_{np}

Degree distribution:	$P(k)$
Path length:	h
Clustering coefficient:	C

What are values of these properties for G_{np}?

Degree Distribution

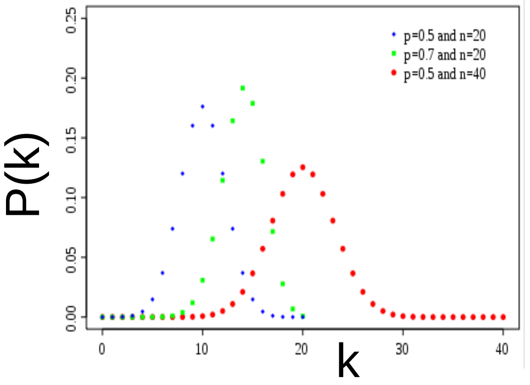
- **Fact: Degree distribution of G_{np} is Binomial.**
- Let $P(k)$ denote a fraction of nodes with degree k:

$$P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Select k nodes out of n-1

Probability of having k edges

Probability of missing the rest of the n-1-k edges



Mean, variance of a binomial distribution

$$\bar{k} = p(n-1)$$

$$s^2 = p(1- p)(n- 1)$$

$$\frac{\sigma}{\bar{k}} = \left[\frac{1-p}{p}\frac{1}{(n-1)}\right]^{1/2} \approx \frac{1}{(n-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of k.

Clustering Coefficient of G_{np}

□ **Remember:** $$C_i = \frac{2e_i}{k_i(k_i-1)}$$

□ Edges in G_{np} appear i.i.d with prob. p

□ **So:** $$e_i = p\,\frac{k_i(k_i-1)}{2}$$

Each pair is connected with prob. *p*

No. of distinct pairs of neighbors of node *i* of degree k_i

□ **Then:** $$C = \frac{p \cdot k_i(k_i-1)}{k_i(k_i-1)} = p = \frac{\bar{k}}{N}$$

Clustering coefficient of a random graph is small.
For a fixed avg. degree, *C* decreases with the graph size *N*.

Real Networks vs. G_{np}

- **Are real networks like random graphs?**
  - Giant connected component: ☺
  - Average path length: ☺
  - Clustering Coefficient: ☹
  - Degree Distribution: ☹
- **Problems with the random network model:**
  - Degreed distribution differs from that of real networks
  - Giant component in most real network does NOT emerge through a phase transition
  - No local structure – clustering coefficient is too low
- **Most important: Are real networks random?**
  - The answer is simply: **NO!**

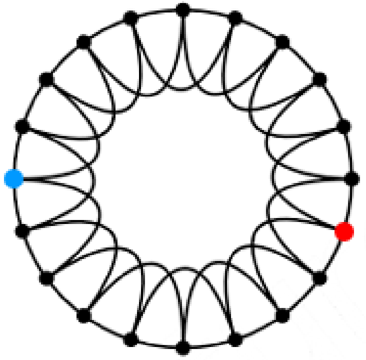
Real Networks vs. G_{np}

- **If G_{np} is wrong, why did we spend time on it?**
  - It is the reference model for the rest of the class.
  - It will help us calculate many quantities, that can then be compared to the real data
  - It will help us understand to what degree is a particular property the result of some random process

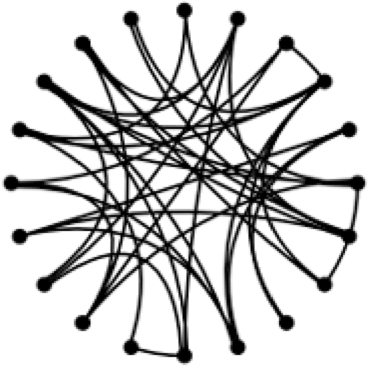
So, while G_{np} is WRONG, it will turn out to be extremly USEFUL!

Small-World: How?

- **Could a network with high clustering be at the same time a small world?**
  - How can we at the same time have **high clustering** and **small diameter?**



High clustering
High diameter



Low clustering
Low diameter

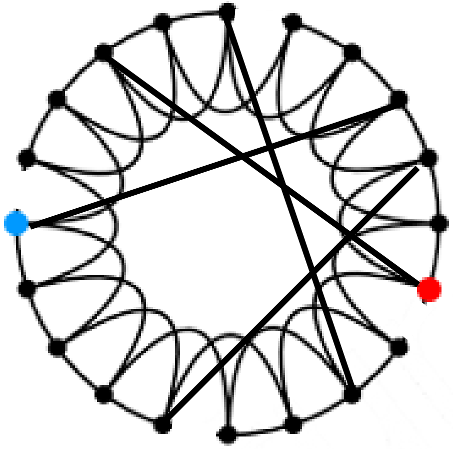
  - Clustering implies edge "locality"
  - Randomness enables "shortcuts"

Solution: The Small-World Model

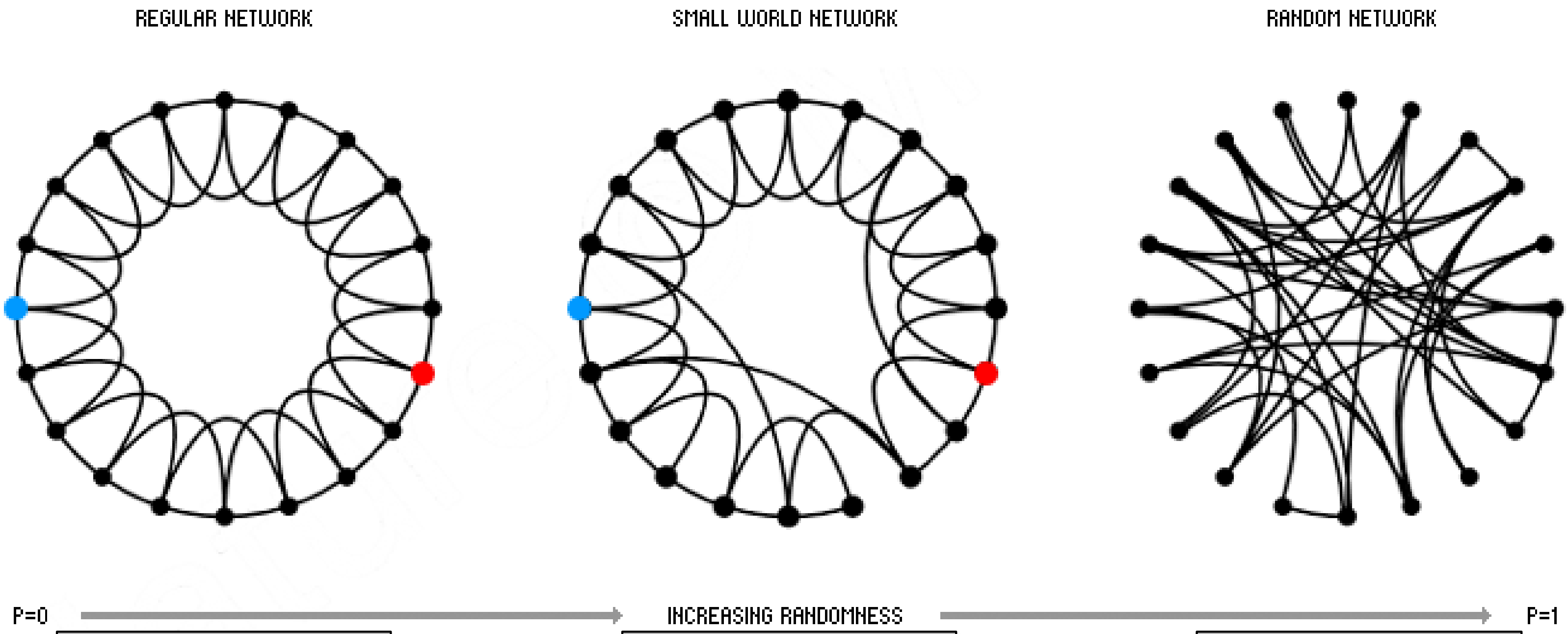
Small-world Model [Watts-Strogatz '98]:

2 components to the model:

- (1) Start with a **low-dimensional regular lattice**
  - Has high clustering coefficient
- Now introduce randomness ("shortcuts")
- (2) **Rewire:**
  - Add/remove edges to create shortcuts to join remote parts of the lattice
  - For each edge with prob. p move the other end to a random node



The Small-World Model



High clustering
High diameter

$$h = \frac{N}{2\bar{k}} \qquad C = \frac{3}{4}$$

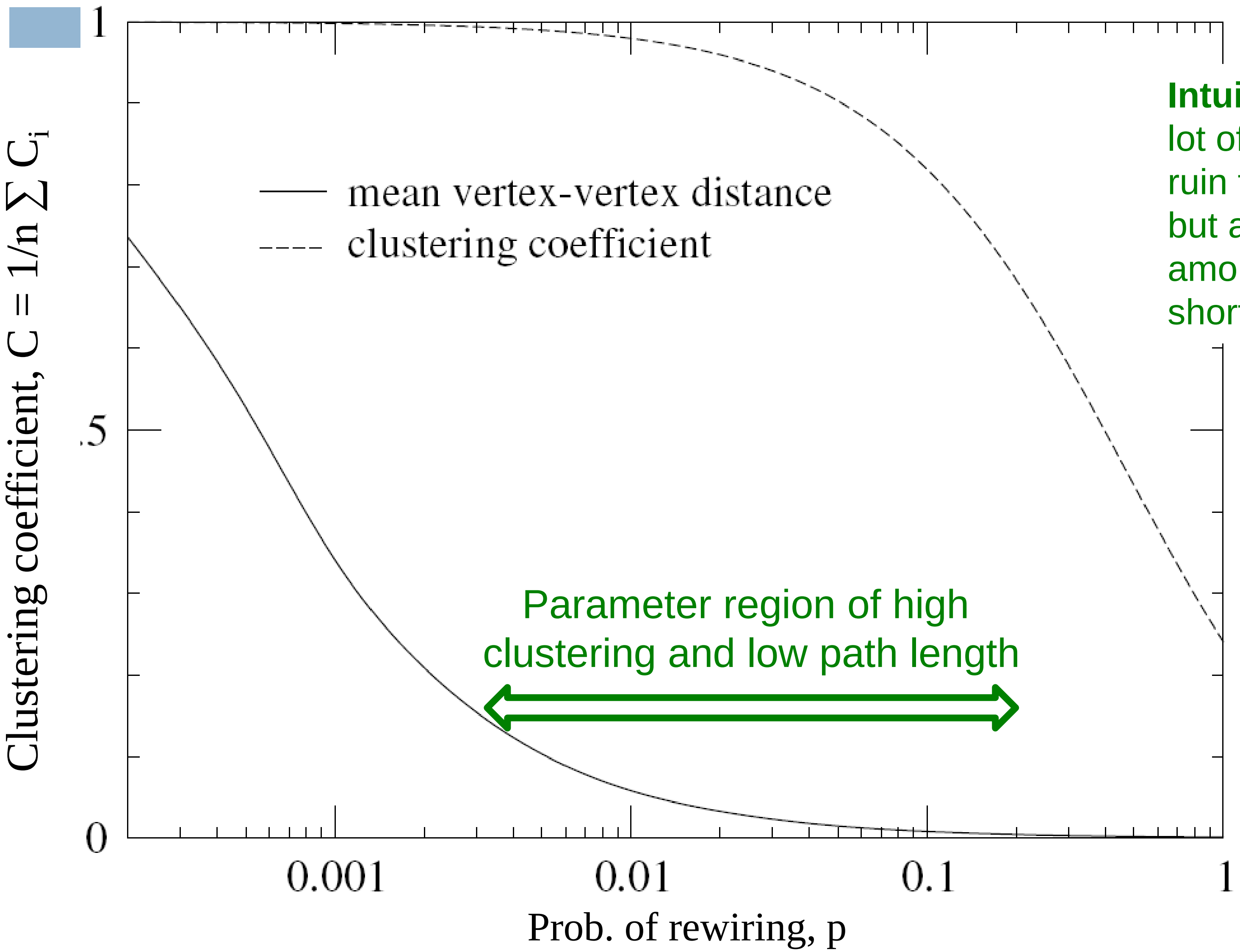
High clustering
Low diameter

Low clustering
Low diameter

$$h = \frac{\log N}{\log \alpha} \qquad C = \frac{\bar{k}}{N}$$

Rewiring allows us to “interpolate” between
a regular lattice and a random graph

The Small-World Model

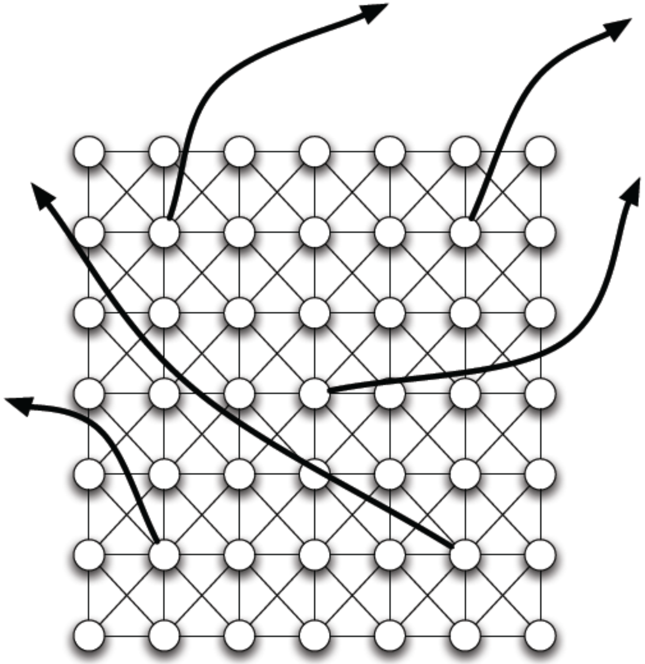


Parameter region of high clustering and low path length

Intuition: It takes a lot of randomness to ruin the clustering, but a very small amount to create shortcuts.

Diameter of the Watts-Strogatz

- **Alternative formulation of the model:**
  - Start with a square grid
  - Each node has 1 random long-range edge
    - Each node has 1 spoke. Then randomly connect them.



$$C_i = \frac{2 \cdot e_i}{k_i(k_i - 1)} = \frac{2 \cdot 12}{9 \cdot 8} \geq 0.33$$

There are already 12 triangles in the grid and the long-rage edge can only close more.

What's the diameter?

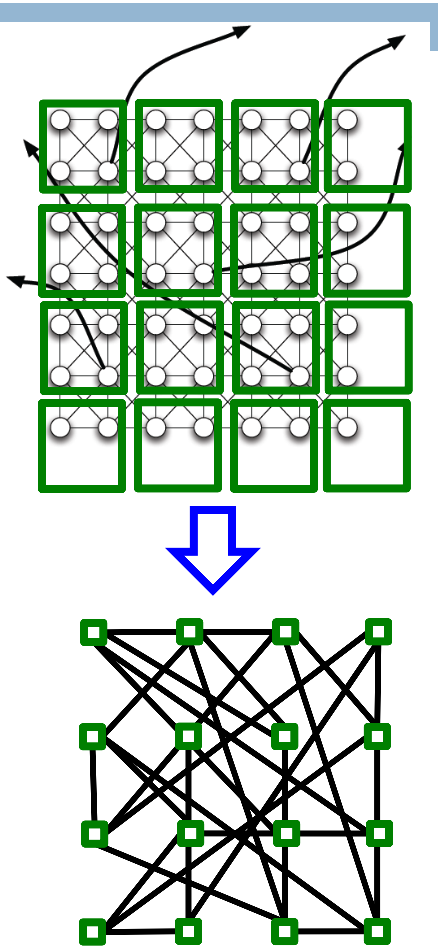
It is *log(n)*

Why?

Diameter of the Watts-Strogatz

- **Proof:**
  - Consider a graph where we contract 2x2 subgraphs into supernodes
  - Now we have 4 edges sticking out of each supernode
    - **4-regular random graph!**
  - From Thm. we have short paths between super nodes
  - We can turn this into a path in a real graph by adding at most 2 steps per hop

⇒ **Diameter of the model is** $O(2 \log n)$



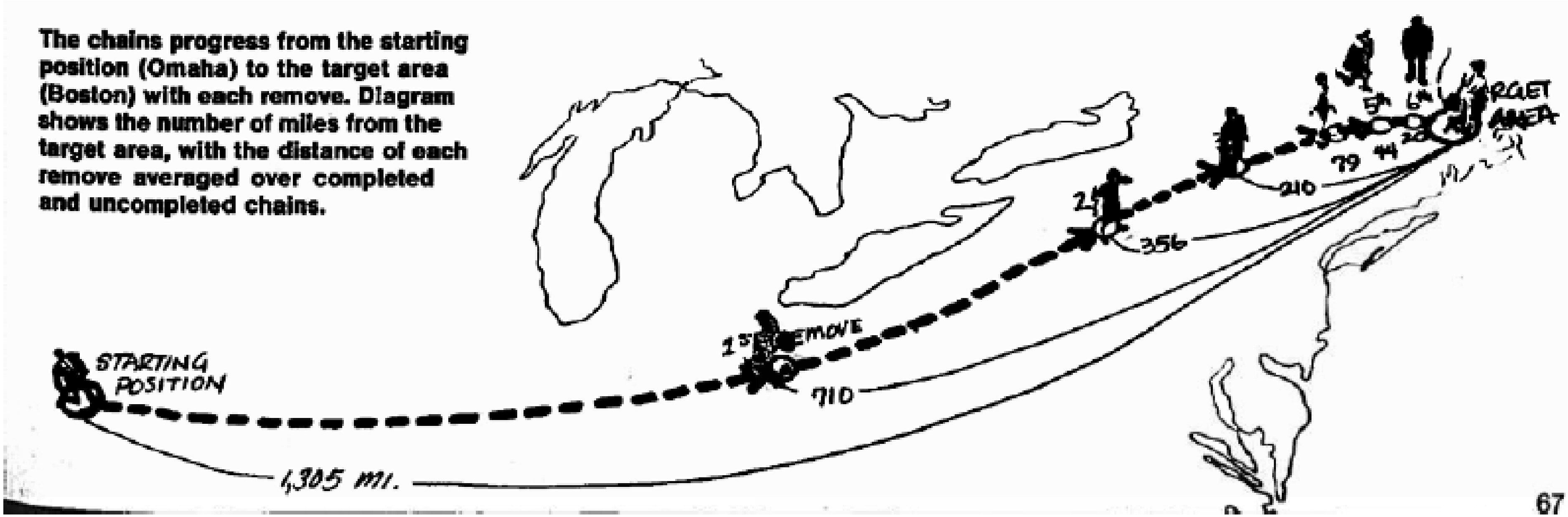
4-regular random graph

Small-World: Summary

- **Could a network with high clustering be at the same time a small world?**
  - Yes. You don't need more than a few random links.
- **The Watts Strogatz Model:**
  - Provides insight on the interplay between clustering and the small-world
  - Captures the structure of many realistic networks
  - Accounts for the high clustering of real networks
  - Does not lead to the correct degree distribution
  - Does not enable **navigation** **(next lecture)**

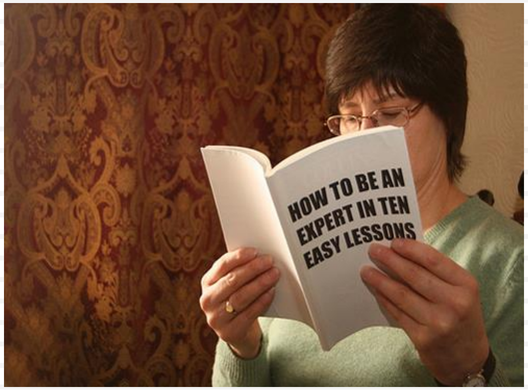
How to Navigate the Network?

- □ (1) What is the structure of a social network?
- □ (Next) Which mechanisms do people use to route and find the target?



The chains progress from the starting position (Omaha) to the target area (Boston) with each remove. Diagram shows the number of miles from the target area, with the distance of each remove averaged over completed and uncompleted chains.

The 10 papers that will make you a social expert



10 sociological must-reads

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2. M. Granovetter, “The strength of weak ties: A network theory revisited,” *Sociological theory*, vol. 1, pp. 201–233, 1983.
3. M. McPherson, L. Smith-Lovin, and J. M. Cook, “Birds of a Feather: Homophily in Social Networks,” *Annual review of sociology*, vol. 27, pp. 415–444, Jan. 2001.
4. M. O. Lorenz, “Methods of measuring the concentration of wealth,” *Publications of the American Statistical Association*, vol. 9, no. 70, pp. 209–219, 1905.
   + H. Simon, “On a Class of Skew Distribution Functions,” Biometrika, vol. 42, no. 3, pp. 425–440, 1955.
5. R. I. M. Dunbar, “Coevolution of Neocortical Size, Group-Size and Language in Humans,” *Behav Brain Sci*, vol. 16, no. 4, pp. 681–694, 1993.
6. D. Cartwright and F. Harary, “Structural balance: a generalization of Heider's theory.,” *Psychological Review*, vol. 63, no. 5, pp. 277–293, 1956.
7. M. Granovetter, “Threshold Models of Collective Behavior,” *The American Journal of Sociology*, vol. 83, no. 6, pp. 1420–1443, May 1978.
8. B. Ryan and N. C. Gross, “The diffusion of hybrid seed corn in two Iowa communities,” *Rural sociology*, vol. 8, no. 1, pp. 15–24, 1943.
   + S. Asch, “Opinions and social pressure,” *Scientific American*, 1955.
9. R. S. Burt, *Structural Holes: The Social Structure of Competition*. Harvard University Press, 1992.
10. F. Galton, “Vox Populi,” *Nature*, vol. 75, no. 1949, pp. 450–451, Mar. 1907.