Calculations of Greeks in the Black and Scholes Formula

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1 Non-dividend paying stock

In the Black and Scholes model the price of an European call option on a non-dividend paying stock is

$$C = S\,\mathrm{N}(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) \ , \tag{1}$$

where S is the stock's price at valuation date, K is the strike price, r is the (constant) spot rate, $\tau = T - t$ is the time to maturity, T the expiry, t the valuation date and

$$d_1 = \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \ , \tag{2}$$

$$d_2 = \frac{\log \frac{S}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} \ , \tag{3}$$

where σ is the stock's volatility.

Theorem 1. *The greeks for the call option are:*

delta: $\Delta_C = \dfrac{\partial C}{\partial S} = \mathrm{N}(d_1) \ ,$

gamma: $\Gamma_C = \dfrac{\partial^2 C}{\partial S^2} = \dfrac{\mathrm{N}'(d_1)}{S\sigma\sqrt{\tau}} = \dfrac{K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{S^2\sigma\sqrt{\tau}} \ ,$

theta: $\Theta_C = \dfrac{\partial C}{\partial t} = -rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) - \dfrac{\sigma S N'(d_1)}{2\sqrt{\tau}} = -K\,\mathrm{e}^{-r\tau}\left[r\,\mathrm{N}(d_2) + \dfrac{\sigma\,\mathrm{N}'(d_2)}{2\sqrt{\tau}}\right] \ ,$

rho: $\rho_C = \dfrac{\partial C}{\partial r} = \tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) \ ,$

vega: $\mathcal{V}_C = \dfrac{\partial C}{\partial \sigma} = \sqrt{\tau}S\,\mathrm{N}'(d_1) = \sqrt{\tau}K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) \ .$

In order to prove the theorem we collect some common calculations in the following

Lemma 1. *It holds*

$$S\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) = 0 \ , \tag{4}$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{\tau}} \ , \tag{5}$$

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r} = \frac{\sqrt{\tau}}{\sigma} \ , \tag{6}$$

$$\frac{\partial d_2}{\partial t} - \frac{\partial d_1}{\partial t} = \frac{\sigma}{2\sqrt{\tau}} \ , \tag{7}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{\tau} \ . \tag{8}$$

Proof. First of all, we remember that

$$\mathrm{N}'(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2} \quad .$$

Statement (4) holds if and only if

$$S\,\mathrm{N}'(d_1) = K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) \quad\Longleftrightarrow\quad \frac{S}{K}\,\mathrm{e}^{r\tau} = \frac{\mathrm{N}'(d_2)}{\mathrm{N}'(d_1)} \quad\Longleftrightarrow\quad \log\frac{S}{K} + r\tau = \frac{d_1^2 - d_2^2}{2} \quad .$$

Notice that the right hand side of the last condition is

$$\begin{aligned}\frac{d_1^2 - d_2}{2} &= \frac{1}{2}(d_1 + d_2)(d_1 - d_2) = \frac{1}{2}(2d_1 - \sigma\sqrt{\tau})\sigma\sqrt{\tau} = \log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau - \frac{1}{2}\sigma^2 \\ &= \log\frac{S}{K} + r\tau\end{aligned}$$

and this completes the proof of (4).

The proofs of the other statements are straightforward calculations. □

Proof of theorem 1. For the *delta*, we have that

$$\begin{aligned}\Delta_C = \frac{\partial C}{\partial S} &= \mathrm{N}(d_1) + S\,\mathrm{N}'(d_1)\frac{\partial d_1}{\partial S} - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\frac{\partial d_2}{\partial S} \\ &= \mathrm{N}(d_1) + \frac{\partial d_1}{\partial S}\left[S\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\right] && \text{by (5)} \\ &= \mathrm{N}(d_1) && \text{by (4).}\end{aligned} \tag{9}$$

Using (9) and (5) the gamma is

$$\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_C}{\partial S} = \mathrm{N}'(d_1)\frac{\partial d_1}{\partial S} = \frac{\mathrm{N}'(d_1)}{S\sigma\sqrt{\tau}} \quad .$$

By (4) it can be also written in the form

$$\Gamma_C = \frac{\frac{K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{S}}{S\sigma\sqrt{\tau}} = \frac{K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{S^2\sigma\sqrt{\tau}} \quad .$$

The *theta* is

$$\begin{aligned}\Theta_C = \frac{\partial C}{\partial t} &= S\,\mathrm{N}'(d_1)\frac{\partial d_1}{\partial t} - rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\frac{\partial d_2}{\partial t} \\ &= -rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) + \frac{\partial d_1}{\partial t}\left[S\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\right] - \frac{\sigma K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{2\sqrt{\tau}} && \text{by (7)} \\ &= -rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) - \frac{\sigma S N'(d_1)}{2\sqrt{\tau}} && \text{by (4)} \\ &= -rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) - \frac{\sigma K\,\mathrm{e}^{-r\tau} N'(d_2)}{2\sqrt{\tau}} && \text{by (4)} \\ &= -K\,\mathrm{e}^{-r\tau}\left[r\,\mathrm{N}(d_2) + \frac{\sigma\,\mathrm{N}'(d_2)}{2\sqrt{\tau}}\right] \quad .\end{aligned}$$

For the *rho* we have

$$\begin{aligned}\rho_C = \frac{\partial C}{\partial r} &= S\,\mathrm{N}'(d_1)\frac{\partial d_1}{\partial r} + \tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\frac{\partial d_2}{\partial r} \\ &= \tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) + \frac{\partial d_1}{\partial r}\left[S\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\right] && \text{by (6)} \\ &= \tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) && \text{by (4).}\end{aligned}$$

Finally, the *vega* is

$$\begin{aligned}
\mathcal{V}_C = \frac{\partial C}{\partial \sigma} &= S\,\mathrm{N}'(d_1)\frac{\partial d_1}{\partial \sigma} - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\frac{\partial d_2}{\partial \sigma} \\
&= \sqrt{\tau}K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) + \frac{\partial d_1}{\partial \sigma}\left[S\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)\right] && \text{by (8)} \\
&= \sqrt{\tau}K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) && \text{by (4)} \\
&= \sqrt{\tau}S\,\mathrm{N}'(d_1) && \text{by (4).}
\end{aligned}$$

□

Consider now a forward contract, with strike K and maturity T, i.e. with payoff at time T given by $F(T) = S(T) - K$. Denote by $F = F(t) = S(t) - K\,\mathrm{e}^{-r(T-t)} = S - K\,\mathrm{e}^{-r\tau}$ its price at time t.

Exercise. *The Greeks of the forward contract are*

$$\begin{aligned}
&\textit{delta:} && \Delta_F = \frac{\partial F}{\partial S} = 1 \ , \\
&\textit{gamma:} && \Gamma_F = \frac{\partial^2 F}{\partial S^2} = 0 \ , \\
&\textit{theta:} && \Theta_F = \frac{\partial F}{\partial t} = -rK\,\mathrm{e}^{-r\tau} \ , \\
&\textit{rho:} && \rho_F = \frac{\partial F}{\partial r} = \tau K\,\mathrm{e}^{-r\tau} \ , \\
&\textit{vega:} && \mathcal{V}_F = \frac{\partial F}{\partial \sigma} = 0 \ .
\end{aligned}$$

By using the put-call parity relation $C - P = F$ and the previous exercise it is straightforward to compute the Greeks for a put option.

Exercise. *The Greeks of the put option are*

$$\begin{aligned}
&\textit{delta:} && \Delta_P = \frac{\partial P}{\partial S} = -\,\mathrm{N}(-d_1) \ , \\
&\textit{gamma:} && \Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{\mathrm{N}'(d_1)}{S\sigma\sqrt{\tau}} = \frac{K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{S^2\sigma\sqrt{\tau}} \ , \\
&\textit{theta:} && \Theta_P = \frac{\partial P}{\partial t} = rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(-d_2) - \frac{\sigma S N'(d_1)}{2\sqrt{\tau}} = K\,\mathrm{e}^{-r\tau}\left[r\,\mathrm{N}(-d_2) - \frac{\sigma\,\mathrm{N}'(d_2)}{2\sqrt{\tau}}\right] \ , \\
&\textit{rho:} && \rho_P = \frac{\partial F}{\partial r} = -\tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(-d_2) \ , \\
&\textit{vega:} && \mathcal{V}_P = \frac{\partial C}{\partial \sigma} = \sqrt{\tau}S\,\mathrm{N}'(d_1) = \sqrt{\tau}K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) \ .
\end{aligned}$$

(In order to better interpret the formaulae, recall that for every x, $N'(x) = N'(-x)$).

2 Dividend paying stock

Assume now the stock pays dividends at a constant dividend yield δ. We know that the call option price Black and Scholes formula becomes

$$C = S\,\mathrm{e}^{-\delta\tau}\,\mathrm{N}(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) \ . \tag{10}$$

Exercise. *It holds*

$$S\,\mathrm{e}^{-\delta\tau}\,\mathrm{N}'(d_1) - K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) = 0$$

and formulae (5), (6), (7) *and* (8) *remain the same in the non-dividend paying case.*

Exercise. *The greeks for the call option are:*

$$\begin{aligned}
&\textit{delta:} && \Delta_C = \frac{\partial C}{\partial S} = \mathrm{N}(d_1)\,\mathrm{e}^{-\delta\tau} \ , \\
&\textit{gamma:} && \Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{\mathrm{N}'(d_1)\,\mathrm{e}^{-\delta\tau}}{S\sigma\sqrt{\tau}} = \frac{K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2)}{S^2\sigma\sqrt{\tau}} \ , \\
&\textit{theta:} && \Theta_C = \frac{\partial C}{\partial t} = S\,\mathrm{e}^{-\delta\tau}\left[\delta\,\mathrm{N}(d_1) - \frac{\sigma N'(d_1)}{2\sqrt{\tau}}\right] - rK\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) \\
& && \qquad = \delta S\,\mathrm{e}^{-\delta\tau}\,\mathrm{N}(d_1) - K\,\mathrm{e}^{-r\tau}\left[r\,\mathrm{N}(d_2) + \frac{\sigma\,\mathrm{N}'(d_2)}{2\sqrt{\tau}}\right] \ , \\
&\textit{rho:} && \rho_C = \frac{\partial C}{\partial r} = \tau K\,\mathrm{e}^{-r\tau}\,\mathrm{N}(d_2) \ , \\
&\textit{vega:} && \mathcal{V}_C = \frac{\partial C}{\partial \sigma} = \sqrt{\tau} S\,\mathrm{e}^{-\delta\tau}\,\mathrm{N}'(d_1) = \sqrt{\tau} K\,\mathrm{e}^{-r\tau}\,\mathrm{N}'(d_2) \ .
\end{aligned}$$

We know the forward price in the dividend paying case to be $F = S\,\mathrm{e}^{-\delta\tau} - K\,\mathrm{e}^{-r\tau}$.

Exercise. *Deduce the Greeks of the forward contract.*

Put call parity relation remains formally the same: $C - P - F$; of course all the quantities involved have to be computed by the formulae for the dividend paying case.

Exercise. *Using put-call parity and the previous results, obtain the Greeks of the put option.*