1. The Heston Nandi Model: GARCH-HN-Gaussian

We consider a Heston model

$$\begin{cases} X_t = r + \lambda_0 h_t + \sqrt{h_t} z_t \\ h_t = F(z_{t-1}, h_{t-1}) = a_0 + a_1 \left(z_{t-1} - \gamma \sqrt{h_{t-1}}\right)^2 + b_1 h_{t-1}. \end{cases} \tag{1.1}$$

where z_t are i.i.d $\mathcal{N}(0,1)$ random. In this case, a uniaue second stationary solution exists if and only if $a_1\gamma^2 + b_1 < 1$. The average level of volatility is a comination of all the volatility parameters, as :

$$h_0 = \mathbb{E}[h_t] = \frac{a_0 + a_1}{1 - b_1 - a_1(\gamma)^2}.$$

1.1. GARCH-HN-Gaussian-Ess: $M_t^{ess} = e^{\theta_t X_t + \varepsilon_t}$

The dynamic still the same under the risk-neutral measure with the same parameter . The difference between the empirical and risk neutral dynamic that is the innovation z_{t+1} is Gaussian with scale parameter :

$$\lambda_0^* = \lambda_0 + \theta \qquad \text{and} \qquad h_{t+1}^* = h_{t+1}$$

The parameters of the Linear kernel density can be obtain from the pricing relation. We can obtain the expression of θ:

$$(\lambda_0 + \theta)h_{t+1} + \frac{h_{t+1}}{2} = 0 \qquad \Rightarrow \qquad \theta = -\frac{1}{2} - \lambda_0 \qquad \Rightarrow \qquad \lambda_0^* = -\frac{1}{2}$$

The associated risk neutral dynamics is described as follows :

$$\begin{cases} X_t = r - \frac{1}{2}h_t + \sqrt{h_t} z_t^* \\ h_t = a_0 + a_1 \left(z_{t-1}^* - \left(\gamma + \lambda_0 + \frac{1}{2}\right) \sqrt{h_{t-1}}\right)^2 + b_1 h_{t-1}. \end{cases} \tag{1.2}$$

where the first value for the variance is set to be equal to its long term value :

$$h_0^* = \frac{a_0 + a_1}{1 - b_1 - a_1(\gamma^*)^2} = \frac{a_0 + a_1}{1 - b_1 - a_1(\gamma + \lambda_0 + \frac{1}{2})^2} \tag{1.3}$$

1.2. VIX for GARCH-HN-Gaussian :

$$\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_{t+j-2}\right] = h_{t+j-1}\psi^* + h_0^*\left[1 - \psi^*\right] \tag{1.4}$$

with $\psi^* = b_1 + a_1(\gamma + \lambda_0 + \frac{1}{2})^2$.

1.3. GARCH-HN-Gaussian-Qua: $M_t^{qua} = e^{\theta_{2,t} X_t^2 + \theta_{1,t} X_t + \varepsilon_t}$

On a dans la page 98, assuming a constant proportional wedge between h_t et h_t^* i.e $\left(\dfrac{h_t^*}{h_t} = \pi > 0\right)$ we have :

$$1 + 2\theta_{2,t}^q h_t^* = \pi \qquad \text{and} \qquad 1 - 2\theta_{2,t}^q h_t = \frac{1}{\pi}$$

Thus, we obtain under $\mathbb{Q}^{Qua}$,

$$\begin{cases} X_t = r - \frac{1}{2}h_t^* + \sqrt{h_t^*} z_t^* \\ h_t = \pi a_0 + \pi^2 a_1 \left(z_{t-1}^* - \left(\dfrac{\gamma}{\pi} + \dfrac{\lambda_0}{\pi} + \dfrac{1}{2}\right) \sqrt{h_{t-1}^*}\right)^2 + b_1 h_{t-1}^*. \end{cases} \tag{1.5}$$

where z_t^* are i.i.d $\mathcal{N}(0,1)$ under $\mathbb{Q}^{Qua}$.

1.4. VIX for GARCH-HN-Gaussian :

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_{t+j-2}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{h^*_{t+j}}{\pi} \mid \mathcal{F}_{t+j-2}\right] \\
&= \frac{1}{\pi}\mathbb{E}_{\mathbb{Q}}\left[\pi a_0 + \pi^2 a_1 \left(z^*_{t+j-1} - \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2}\right)\sqrt{h^*_{t+j-1}}\right)^2 + b_1 h^*_{t+j-1}\right] \\
&= h_{t+j-1}\psi^* + h^*_0\left[1-\psi^*\right]
\end{aligned}$$

with $\psi^* = b_1 - \pi^2 a_1 \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2}\right)^2$ and $h^*_0 = \dfrac{a_0 + \pi a_1}{1-\psi^*}$.

2. The GJR Model: GARCH-GJR-Gaussian

We consider a GJR Model[1]

$$\begin{cases}
X_t = r + \lambda_0\sqrt{h_t} - \dfrac{h_t}{2} + \sqrt{h_t}z_t \\
h_t = a_0 + b_1 h_{t-1} + a_1\left(X_{t-1} - r - \lambda_0\sqrt{h_t} + \dfrac{h_t}{2}\right)^2 + a_2 \max\left(0, -\left(X_{t-1} - r - \lambda_0\sqrt{h_t} + \dfrac{h_t}{2}\right)^2\right)
\end{cases}$$

where z_t are i.i.d $\mathcal{N}(0,1)$ random variables under $\mathbb{P}$, with $a_0 > 0$ and, $a_1, a_2, b_1 \geqslant 0$ for the positive conditional variance and $\lambda_0 > 0$ for the positive equity risk-premium. The variance is weak stationary under the physical $\Psi = b_1 + a_1 + \frac{a_2}{2} < 1$. The unconditional variance under the physical measure can be expressed as $h_0 = \frac{a_0}{1-\Psi}$.

2.1. GARCH-GJR-Gaussian-Ess: $M^{ess}_t = e^{\theta_t X_t + \varepsilon_t}$

According to Duan's (1995) under the Gaussian framework, total return dynamics can be expressed under the risk-neutral measure as :

$$\begin{cases}
X_t = r - \dfrac{h_t}{2} + \sqrt{h_t}\tilde{z}_t \\
h_t = a_0 + b_1 h_{t-1} + a_1\left(X_{t-1} - r - \lambda_0\sqrt{h_t} + \dfrac{h_t}{2}\right)^2 + a_2 \max\left(0, -\left(X_{t-1} - r - \lambda_0\sqrt{h_t} + \dfrac{h_t}{2}\right)^2\right)
\end{cases}$$

where $\tilde{z}_t \sim \mathcal{N}(0,1)$. The variance is weak stationary under the risk-neutral measure if

$$\tilde{\Psi} = b_1 + \left[a_1 + a_2 N(\lambda_0)\right]\left(1 + \lambda_0^2\right) + a_2\lambda_0 n(\lambda_0) < 1$$

[1] I It is possible to used other equivalet definition as explain in the book page 35 definition 2.3.1

$$h_t = a_0 + a_+\left(\sqrt{h_t}z_t\right)^2 \mathbb{1}_{\{\sqrt{h_t}z_t \geqslant 0\}} + a_-\left(\sqrt{h_t}z_t\right)^2 \mathbb{1}_{\{\sqrt{h_t}z_t < 0\}} + b_1 h_{t-1}.$$

where $X_{t-1} - r - \lambda_0\sqrt{h_{t-1}} + \frac{h_t}{2} = \sqrt{h_t}z_t$, $a_+ = a_1$ and $a_- = a_1 + a_2$.

where $N(.)$ and $n(.)$ denote the standard normal cumulative and density distribution functions. The unconditional variance $\tilde{h}_0$ associate to the risk neutral measure :

$$\tilde{h}_0^* = \frac{a_0}{1-\tilde{\Psi}}$$

Here Ψ and $\tilde{\Psi}$ denote the volatility persistence under the physical and risk-neutral measures, respectively.

2.2. VIX for GARCH-HN-Gaussian :

$$\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_{t+j-2}\right] = h_{t+j-1}\tilde{\Psi} + h_0^*\left[1-\tilde{\Psi}\right]. \tag{2.6}$$

2.3. GARCH-GJR-Gaussian-Qua: $M_t^{qua} = e^{\theta_{2,t}X_t^2+\theta_{1,t}X_t+\varepsilon_t}$

On a dans la page 97 proposition 3.5.1 (Monfort and Pegoraro 2012), if $\forall t \in \{1,\cdots,T\}$, $\theta_{2,t}^q < \dfrac{1}{2h_t}$,

- the functional relation between $\theta_{1,t}^q$ and $\theta_{2,t}^q$ is global and explicit:

$$\frac{h_t}{2\left(1-2\theta_{2,t}^q\right)} + \frac{h_t\theta_{1,t}^q + r + \lambda_0\sqrt{h_t} - \frac{h_t}{2}}{1-2\theta_{2,t}^q} = r$$

- Under $\mathbb{Q}^{Qua}$:

$$\begin{cases} X_t &= r - \dfrac{h_t^*}{2} + \sqrt{h_t^*}\tilde{z}_t \\ \dfrac{h_t^*}{1+2\theta_{2,t}^q h_t^*} &= F\left(\sqrt{1+2\theta_{2,t}^q h_t^*}\left[-\dfrac{m_{t-1}}{\sqrt{h_{t-1}^*}} - \dfrac{\sqrt{h_{t-1}^*}}{2} + \tilde{z}_t\right], \dfrac{h_{t-1}^*}{1+2\theta_{2,t-1}^q h_{t-1}^*}\right) \end{cases}$$

where $h_t^* = \dfrac{h_t}{1-2\theta_{2,t}^q h_t}$, and $\tilde{z}_t$ are i.i.d $\mathcal{N}(0,1)$ random variables, with

$$F\left(X_{t-1},h_t\right) = a_0+b_1h_{t-1}+a_1\left(X_{t-1}-r-\lambda_0\sqrt{h_t}+\frac{h_t}{2}\right)^2+a_2\max\left(0,-\left(X_{t-1}-r-\lambda_0\sqrt{h_t}+\frac{h_t}{2}\right)^2\right).$$

3. The Inverse-Gaussian-GARCH Model: IG-GARCH

We consider

$$\begin{cases} X_t &= r + \nu h_t + \eta y_t \\ h_t &= w + bh_{t-1} + cy_{t-1} + a\frac{h_{t-1}^2}{y_t} \end{cases} \tag{3.7}$$

where the $(y_t)_{t\in\{1,\ldots,T\}}$ are random variables generating an information filtration denoted by $(\mathcal{F}_t)_{t\in\{0,\ldots,T\}}$ where $\mathcal{F}_0 = \{\emptyset,\Omega\}$ and $(\mathcal{F}_t = \sigma(y_u; 1\leq u\leq t))_{t\in\{1,\ldots,T\}}$. Moreover, we suppose that, given $\mathcal{F}_{t-1}$, y_t follows an Inverse Gaussian distribution with degree of freedom $\delta_t = \frac{h_t}{\eta^2}$.

3.1. IG-GARCH-Esscher : $M_t^{ess} = e^{\theta_t X_t + \varepsilon_t}$

Assuming that the process $(X_t)_t$ is defined by 3.7, then, Under $\mathbb{Q}^{ess}$, the process $(X_t)_t$ is again an IG-GARCH model with changed parameters :

$$\begin{cases} X_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right) & = & r + \nu^* h_{t+1}^* + \eta^* y_{t+1}^* \\ h_{t+1}^* & = & w^* + b^* h_t^* + c^* y_t^* + a^* \frac{(h_t^*)^2}{y_t^*} \end{cases} \tag{3.8}$$

$$\text{where} \qquad \begin{aligned} \nu^* &= \nu\left(\frac{\eta^*}{\eta}\right)^{-\frac{3}{2}}, \quad y_{t+1}^* = y_{t+1}\left(\frac{\eta^*}{\eta}\right)^{-1}, \\ w^* &= w\left(\frac{\eta^*}{\eta}\right)^{\frac{3}{2}}, \quad c^* = c\left(\frac{\eta^*}{\eta}\right)^{\frac{5}{2}}, \quad a^* = a\left(\frac{\eta^*}{\eta}\right)^{-\frac{5}{2}}, \end{aligned}$$

with $\eta^* = \dfrac{\eta}{1 - 2\theta^* \eta}$ and where, given $\mathcal{F}_{t-1}$, y_t^* follows an Inverse Gaussian distribution with degree of freedom $\delta_t^* = \frac{h_t^*}{(\eta^*)^2}$ and $(\theta_t^*, \varepsilon_t^*)$ by :

$$\begin{aligned} \theta_t^* &= \theta^* = \tfrac{1}{2}\left[\eta^{-1} - \tfrac{1}{\nu^2\eta^3}\left[1 + \tfrac{\nu^2\eta^3}{2}\right]^2\right] \\ \varepsilon_t^* &= -r(\theta^* + 1) - \theta^* \nu h_t - \left[\delta_t\left(1 - \sqrt{(1 - 2\theta^*\eta)}\right)\right]. \end{aligned}$$

3.2. IG-GARCH-Ushaped : $M_t^{Ushp} = e^{\theta_t X_t + \varepsilon_t + \frac{\rho_t}{y_t}}$

Under the risk-neutral probability $\mathbb{Q}^{Ushp}$ associated to $(M_t^{Ushp})_{t\in\{1,\cdots,T\}}$, the overall dynamics of the log-return is, once again similar the historical one:

$\forall t \in \{1, \cdots, T\}$, if we assume a constant proportional wedge between h_t and h_t^* (i.e $\dfrac{h_t^*}{h_t} = \pi$) the dynamics of Y_t under $\mathbb{Q}^{Ushp}$is of the form:

$$\begin{cases} X_{t+1} & = & r + \nu^* h_{t+1}^* + \eta^* y_{t+1}^* \\ h_{t+1}^* & = & w^* + b h_t^* + c^* y_t^* + a^* \dfrac{(h_t^*)^2}{y_t^*} \end{cases} \tag{3.9}$$

$$\text{where} \qquad \begin{aligned} &\nu^* = \frac{\nu}{\pi}, \quad w^* = w\pi, \quad c^* = \frac{c\pi\eta^*}{\eta}, \quad a^* = \frac{a\eta}{\pi\eta^*}, \\ &\eta^* = \sqrt[3]{\tfrac{\pi^2}{\nu^2}\left(-1 + \sqrt{1 + \tfrac{8\nu}{27\pi}}\right)} + \sqrt[3]{\tfrac{\pi^2}{\nu^2}\left(-1 - \sqrt{1 + \tfrac{8\nu}{27\pi}}\right)}, \end{aligned}$$

and where, given $\mathcal{F}_t$, y_{t+1}^* follows an IG distribution with degree of freedom $\delta_{t+1}^* = \dfrac{h_{t+1}^*}{(\eta^*)^2}$.

3.3. VIX for IG-GARCH-Ushaped :

Under both specifications of the pricing kernel, the risk-neutral dynamics of the IG-GARCH model may be written as

$$\begin{cases} X_{t+1} & = & r + \nu^* h_{t+1}^* + \eta^* y_{t+1}^* \\ h_{t+1}^* & = & w^* + b^* h_t^* + c^* y_t^* + a^* \frac{(h_t^*)^2}{y_t^*} \end{cases}$$

where, given $\mathcal{F}_t$, y_{t+1}^* follows an IG distribution with parameter $\frac{h_{t+1}^*}{\eta^*}$ under the risk-neutral probability $\mathbb{Q}$. Thus[2],

[2]Using the fact that an IG random variable Z with degree of freedom δ fulfills $E[\frac{1}{Z}] = \frac{1}{\delta} + \frac{1}{\delta^2}$.

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_{t+j-2}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{h^*_{t+j}}{\pi} \mid \mathcal{F}_{t+j-2}\right] \\
&= \frac{1}{\pi}\left[w^* + bh^*_{t+j-1} + \frac{c^*}{(\eta^*)^2}h^*_{t+j-1} + a^*\mathbb{E}_{\mathbb{Q}}\left[\frac{(h^*_{t+j-1})^2}{y^*_{t+j-1}} \mid \mathcal{F}_{t+j-2}\right]\right] \\
&= \frac{1}{\pi}\left[w^* + \left[b + \frac{c^*}{(\eta^*)^2} + a^*\,(\eta^*)^2\right]h^*_{t+j-1} + a^*\,(\eta^*)^4\right] \\
&= \frac{1}{\pi}\left[h^*_{t+j-1}\psi^* + h^*_0\,[1-\psi^*]\right] = h_{t+j-1}\psi^* + h_0\,[1-\psi^*]
\end{aligned}
$$

4. Comparing predictibility of time series VIX :

4.1. The mean of pricing errors (MPE):

$$MPE_{VIX} = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{VIX^m_j}{VIX^M_j} - 1\right)$$

where VIX^m_j is the computed VIX and VIX^M_j the market VIX for date j.

4.2. The mean of absolute pricing errors (MAE):

$$MAE_{VIX} = \frac{1}{N}\sum_{j=1}^{N}\left(\left|\frac{VIX^m_j}{VIX^M_j} - 1\right|\right)$$

where VIX^m_j is the computed VIX and VIX^M_j the market VIX for date j.

4.3. The root mean of square pricing errors (RMSE):

$$RMSE_{VIX} = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(VIX^m_j - VIX^M_j\right)^2}$$

where VIX^m_j is the computed VIX and VIX^M_j the market VIX for date j.