OPTION VALUATION FORMULA FOR GENERAL GARCH-IN-MEAN MODELS

ZHONGMIN QIAN AND XINGCHENG XU

Abstract. We derive option pricing formulas based on general GARCH-M models by using risk-neutral arguments. These formulas are beautiful in nature and realistic for applications. We propose a parameter estimation procedure and employ Monte Carlo method to evaluate the price. Demonstrations of these formulas applying to S&P 500 index options are shown. Empirical evidence suggests that both in U.S. stock market and Chinese financial market the performances of these theoretical pricing formulas are better than the results via Black-Scholes' pricing formula with constant volatility.

Keywords. Option pricing, GARCH-in-mean, AR-GARCH, Risk Premia.

1. Introduction

In this paper, we give a solution to the question proposed by Engle (2002) for option pricing based on GARCH models, by deriving a closed formula.

The theoretical valuation formula for options was eastablished by Black and Scholes (1973) and Merton (1973) based on the geometric Brownian motion model for the return of the underlying financial instrument. A simple approach via a discrete model was given by Cox, Ross and Rubinstein (1979). Black-Scholes' formula expresses the price of an option for a financial derivative instrument in terms of the price of the underlying financial asset, the maturity, the striking price, the return of a riskless asset, and the volatility of the underlying financial instrument. The only unobservable quantity in the option formula is the volatility of the financial asset, which is assumed to be constant in Black-Scholes-Merton's model. Therefore, the theoretical valuation formula demonstrates the option price in terms of the volatility, and conversely the 'implied' volatility, in practice, provides a forecast of the option price.

Financial economists and practitioners are therefore concerned with modeling volatility in asset returns. The assumption of constant volatility in Black-Scholes-Merton's model is violated in actual financial market, and the volatility in a real market proves to be random too. Engle's model of the time-varying volatility by way of autoregressive conditional heteroskedasticity (ARCH) signified a genuine breakthrough in forecasting

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volatility by statistical models and thus evaluating options. Engle's revolutionary notion makes it possible to explain systematic features in movements of volatility over time. Since the seminal work Engle (1982), many extensions of ARCH models have been developed. The first enormously important generalization of ARCH models is the family of generalized autoregressive conditional heteroskedasticity (GARCH) models introduced by Bollerslev (1986), which allows a flexible lag length. Another very important development was a family of exponential GARCH (EGARCH) models proposed by Nelson (1991). Their models recognize asymmetrical respondencies to the past forecast errors. There are generalizations of GARCH models proposed by many researchers, and there is now a large class of ARCH models available for modelling volatilities. The importance of nonlinearity, asymmetry and long memory properties which are key features in modelling and forecasting volatility are embedded in these GARCH models. The reader may refer to the papers Engle (2002), Francq and Zakoian (2010) and etc. for further reference. For simplicity, in what follows, we call these ARCH models as 'GARCH'.

GARCH models are capable of describing and forecasting the variance of an underlying financial asset in terms of observables, so have been widely used to measure the time varying volatility at any time and to forecast its near and distant future movement. A very important application of GARCH models to finance appears in the investigation of the trade-off between risk and return for financial markets. Among them an important model is the following family of the GARCH-in-mean (GARCH-M) models

$$R_t = \mu + c\sigma_t^2 + \sigma_t z_t, \tag{1.1}$$

where R_t is the logarithm return of an underlying financial asset, σ_t^2 is a GARCH type process, $\{z_t\}$ is the Gaussian white noise in discrete time t. This family was first considered by Engle, Lilien and Robins (1987). The general form of a GARCH-M model is often written as

$$R_t = \mu_t - \frac{1}{2}\sigma_t^2 + \sigma_t z_t, \tag{1.2}$$

where μ_t may depend, linearly or non-linearly, on the past returns $\{R_{t-1}, R_{t-2}, \cdots\}$ and volatility σ_t^2. For more details, see section §2 below.

A natural question is how to value the option instruments based on these models. Many researchers use the 'plug-in' method to value options. That is, options are evaluated by using Black-Scholes pricing formula with the forecasting values of the volatility, such as the average time-varying volatility of underlying security over the life of the option in place of constant volatility σ^2. In practice, one applies Monte Carlo simulations of the future price paths and estimates the expected average per period volatility $\widehat{\sigma}_{t,T}^2$ between current time t and expired time T, then plug it into Black-Scholes pricing formula. Recently, the option pricing formula has been used by combining with stochastic volatility models, e.g. those developed in Hull and White (1987). The Hull-White price is equal to the expected Black-Scholes price integrated over the average instantaneous variance during the life of the option, which calculates the price for GARCH-M models:

$$C_t = \frac{1}{N}\sum_{n=1}^{N} C^{BS}(\widehat{\sigma}_{t,T}^2(n), S_t, K, r), \tag{1.3}$$

where $C^{BS}(\sigma^2, S_t, K, r)$ is the Black-Scholes price, and $\widehat{\sigma}^2_{t,T}(n)$ is the average volatility per period for the nth Monte Carlo simulation of the future price paths. For this approach, see e.g. Bollerslev and Mikkelsen (1996) and etc.

These approaches for option pricing formulas, even though the performance may be good, are obviously not satisfactory, and we would like to ask what is the theoretical option pricing formula for general GARCH-M models. This is one of the questons raised by Engle (2002).

Duan (1995) was the first researcher who implemented a local risk neutralization within the framework of GARCH models by using a quadratic utility function of a representative agent's risk preference. Zumbach and Fernández (2013, 2014) also adopted the risk neutralization approach for GARCH models, which we believe still deserves to be investigated further.

In this paper, we derive closed option pricing formulas (see formulas (2.10) and (2.15)) based on risk-neutral arguments. These formulas are beautiful in nature and realistic for applications, and lead to new implications such as the relationship among forecast of the future volatility, the return rate and the risk premia and so on. Empirically, the theoretical option prices based on GARCH models can also be easily employed in the real financial market. We will discuss the estimation problems for these models, too. For further details on statistical inference however one may refer to Fiorentini, Sentana and Shephard (2004), Christensen, Dahl and Iglesias (2012), Conrad and Mammen (2016) and etc. We assess the performance of our option pricing formula in the context of numerical simulations and empirical applications to Standard and Poor's (S&P) 500 composite stock price index and Chinese SSE 50 ETF, and compare them to the results obtained via Black-Scholes' pricing formula. There are many important questions about option pricing based on GARCH-M models, which should be explored and developed further. In a future work, we will investigate option valuation for the high-frequency econometric data following a GARCH model. One may refer to Shephard and Sheppard (2010), Noureldin, Shephard and Sheppard (2012) and Aït-Sahalia and Jacod (2014) for some hints on this aspect.

The plan for the rest of the paper is as follows. In section §2, we present the pricing formula for general GARCH-M models. The parameter estimation procedures are discussed in section §3, which are applied to S&P 500 composite stock price index. Section §4 gives an analysis of the option pricing behaviour based on empirical analysis of the daily S&P 500 stock index for three GARCH-M type data-generating mechanism. In section §5, we show the performance of our theoretical formula with real market prices of S&P 500 stock index options and Chinese SSE 50 ETF options using the recent market data. We also compare these prices with values given by the celebrated Black-Scholes model.

2. Option Pricing for General GARCH-M Models

In this section, we will present option pricing formulas for GARCH-M models. The first model we consider is described by the following equations

$$R_t = \mu - \frac{1}{2}\sigma_t^2 + \sigma_t z_t,\ t \in \mathbb{Z}_+, \tag{2.1}$$

$$\sigma_t^2 \sim \text{GARCH type process}, \tag{2.2}$$

where $R_t = \log \frac{S_t}{S_{t-1}}$ is the logarithm return of assets, μ is the constant return rate, and $\{z_t,\ t \in \mathbb{Z}_+\}$ is Gaussian white noise. In this paper, we also call the conditional variance process σ_t^2 as 'GARCH type' process. The model above is the simplest one in GARCH-M family, which is however a good representative of this family for option pricing. Based on risk-neutral arguments, we give the pricing formula in section §2.1.

The so-called generalized AR(k)-GARCH-in-mean (AR-GARCH-M) models are determined by the equations:

$$R_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + V(\sigma_t^2) + \sigma_t z_t, \tag{2.3}$$

$$\sigma_t^2 \sim \text{GARCH type process}, \tag{2.4}$$

where $V(\sigma_t^2)$ is the risk-return relation/risk premia describing the trade-off between risk and return, for example, $V(\sigma_t^2) = c\sigma_t^2$, $c\log(\sigma_t^2)$, $c_1\sigma_t^2 + c_2\sin(\omega + c_3\sigma_t^2)$ etc. (see e.g. Christensen, Dahl and Iglesias (2012) for a discussion). Let

$$\mu_t \triangleq \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + V(\sigma_t^2) + \frac{1}{2}\sigma_t^2. \tag{2.5}$$

Then the model can be rewritten as equations (2.1) and (2.2). An option pricing formula can be derived too by a similar method used for the first model. For the general GARCH-M model, where

$$\mu_t = \mu(\{R_s\}_{s<t}, \sigma_t^2),$$

that is, the return rate maybe depend (linearly or non-linearly) on the past returns and the volatility, the option pricing formula (2.15) is still valid.

2.1. Option Pricing for GARCH-M Models.

Let us derive an option pricing formula based on GARCH-M models, described by equations (2.1) and (2.2), where the conditional variance process σ_t^2 belongs to ARCH family. σ_t^2 may be an ARCH(p), GARCH(p,q), or EGARCH type volatility process. That is,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \sigma_{t-j}^2 z_{t-j}^2, \tag{2.6}$$

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{p} \alpha_j \sigma_{t-j}^2 z_{t-j}^2, \tag{2.7}$$

or

$$\ln(\sigma_t^2) = \alpha_t + \sum_{j=1}^{\infty} \beta_j g(z_{t-j}), \tag{2.8}$$

respectively. In general, what is important is the fact that σ_t^2 is determined by the conditional variance of $\sigma_t z_t$, that is,

$$\sigma_t^2 = \mathrm{Var}\left(\sigma_t z_t | \mathcal{F}_{t-1}\right), \tag{2.9}$$

where the sigma field (information up to time t) $\mathcal{F}_t = \sigma(\{z_t, z_{t-1}, \cdots\})$.

The option pricing formula for this discrete-time GARCH-M model is obtained by using the idea of arbitrage arguments. The derivation and proofs are given in the Appendix A.

Theorem 2.1. *Let r be the one plus riskless interest rate on one period, $T-t$ be time to maturity, and K be the striking pricing. Then, for the discrete-time GARCH-M models (2.1) and (2.2), the pricing formula for European call option is given by*

$$C_t = S_t \Phi_{T-t}(D_1) - K r^{-(T-t)} \Phi_{T-t}(D_2), \tag{2.10}$$

where S_t is the spot price of the underlying asset, $\Phi_d(\cdot)$ is d-dimensional standard normal distribution function with independent components, i.e.

$$\Phi_d(D) = \int \cdots \int_D (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}\sum_{j=1}^{d} y_j^2} dy_1 \cdots dy_d,$$

the integral domains

$$D_1 = \left\{ u = (u_1, \cdots, u_{T-t}) : \sum_{j=1}^{T-t} \sigma_{t+j} u_j > -\left(\log \frac{S_t}{K r^{-(T-t)}} + \frac{1}{2} \sum_{j=1}^{T-t} \sigma_{t+j}^2 \right) \right\}$$

and

$$D_2 = \left\{ v = (v_1, \cdots, v_{T-t}) : \sum_{j=1}^{T-t} \sigma_{t+j} v_j > -\left(\log \frac{S_t}{K r^{-(T-t)}} - \frac{1}{2} \sum_{j=1}^{T-t} \sigma_{t+j}^2 \right) \right\},$$

with

$$u_j = x_j - \frac{\varrho}{\sigma_{t+j}(x)} - \sigma_{t+j}(x), \;\; v_j = x_j - \frac{\varrho}{\sigma_{t+j}(x)}, \;\; \varrho = -(\mu - \log r).$$

$\sigma_{t+j}(x)$, $j \geq 2$, are obtained by substituting $(z_{t+j-1}, \cdots, z_{t+1})$ in the conditional heteroskedasticity $\sigma_{t+j} = \sigma_{t+j}(z_{t+j-1}, \cdots, z_{t+1}, z_t, \cdots)$ with $(x_{j-1}, \cdots, x_1)$, that is,

$$\sigma_{t+j}(x) = \sigma_{t+j}(x_{j-1}, \cdots, x_1; z_t, z_{t-1}, \cdots), \;\; j = 2, \cdots, T-t,$$

and

$$\sigma_{t+1}(x) \equiv \sigma_{t+1}.$$

$\sigma_{t+j}(x)$, $j = 1, 2, \cdots$, is the conditional volatility function evaluated at time t.

By specifying the GARCH volatility process σ_t^2, we may obtain explicit option pricing formula. Begin with the case in which the volatility is a constant, i.e. $\sigma_t^2 \equiv \sigma^2$. In this case, our formula coincides with Black-Scholes' formula:

$$C_t = S_t N(d_1) - K r^{-(T-t)} N(d_2), \tag{2.11}$$

where

$$d_1 = \frac{\log \frac{S_t}{Kr^{-(T-t)}} + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log \frac{S_t}{Kr^{-(T-t)}} - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}},$$

and $N(\cdot)$ is standard normal distribution function, so that D_1 and D_2 are half planes.

If the volatility process/conditional heteroskedasticity σ_t^2 is modelled by ARCH(1) process:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2, \ \alpha_0 > 0, \ \alpha_1 \geq 0,$$

then the volatility process σ_t^2 has the following representation:

$$\sigma_{t+j}^2 = \alpha_0 + \sum_{i=1}^{j-1} \alpha_0 \alpha_1^i \prod_{\ell=1}^{i} z_{t+j-\ell}^2 + \alpha_1^j \prod_{\ell=1}^{j-1} z_{t+j-\ell}^2 \sigma_t^2 z_t^2, \ \forall \ j \geq 1,$$

where the conventions that $\sum_{i=1}^{0} := 0$, $\prod_{i=1}^{0} := 1$ have been used. The integral domains D_1 and D_2 in Theorem 2.1 are determined for this case by

$$\begin{aligned}
\sigma_{t+1}(x) &\equiv \sigma_{t+1}(z_t, z_{t-1}, \cdots) = (\alpha_0 + \alpha_1 \sigma_t^2 z_t^2)^{\frac{1}{2}}, \\
\sigma_{t+j}(x) &= \sigma_{t+j}(x_{j-1}, \cdots, x_1; z_t, z_{t-1}, \cdots) \\
&= \left(\alpha_0 + \sum_{i=1}^{j-1} \alpha_0 \alpha_1^i \prod_{\ell=1}^{i} x_{j-\ell}^2 + \alpha_1^j \prod_{\ell=1}^{j-1} x_{j-\ell}^2 \sigma_t^2 z_t^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

An exact option pricing formula for ARCH(1) model may be obtained accordingly.

Another example is the GARCH(1,1) model, where

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \ \alpha_0 > 0, \ \alpha_1, \beta_1 \geq 0.$$

The volatility process σ_t^2 has the following representation:

$$\sigma_{t+j}^2 = \alpha_0 + \alpha_0 \sum_{i=1}^{j-1} \prod_{\ell=1}^{i} (\alpha_1 z_{t+j-\ell}^2 + \beta_1) + \prod_{\ell=1}^{j-1} (\alpha_1 z_{t+j-\ell}^2 + \beta_1)(\alpha_1 \sigma_t^2 z_t^2 + \beta_1 \sigma_t^2),$$

and the integral domains D_1 and D_2 in Theorem 2.1 are determined by

$$\sigma_{t+j}(x) = \left(\alpha_0 + \alpha_0 \sum_{i=1}^{j-1} \prod_{\ell=1}^{i} (\alpha_1 x_{j-\ell}^2 + \beta_1) + \prod_{\ell=1}^{j-1} (\alpha_1 x_{j-\ell}^2 + \beta_1)(\alpha_1 \sigma_t^2 z_t^2 + \beta_1 \sigma_t^2) \right)^{\frac{1}{2}}.$$

For the GARCH(p,p) models, that is,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \sigma_{t-j}^2 z_{t-j}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2,$$

σ_{t+j} is given by

$$\sigma_{t+j}^2 = \alpha_0 + \alpha_0 \sum_{i=1}^{j-1} \left(\sum_{k_1=1}^{p} \cdots \sum_{k_i=1}^{p} \prod_{\ell=1}^{i} \left(\alpha_{k_\ell} z^2_{t+j-\sum_{q=1}^{\ell} k_q} + \beta_{k_\ell} \right) \right)$$
$$+ \sum_{k_1=1}^{p} \cdots \sum_{k_j=1}^{p} \prod_{\ell=1}^{j} \left(\alpha_{k_\ell} z^2_{t+j-\sum_{q=1}^{\ell} k_q} + \beta_{k_\ell} \right) \sigma^2_{t+j-\sum_{q=1}^{j} k_q}.$$

The models include all ARCH(p) processes by taking $\beta_j \equiv 0$. Substituting $(z_{t+j-1}, \cdots, z_{t+1})$ in the conditional heteroskedasticity

$$\sigma_{t+j}^2 = \sigma_{t+j}^2(z_{t+j-1}, \cdots, z_{t+1}, z_t, \cdots)$$

with $(x_{j-1}, \cdots, x_1)$, we obtain the conditional volatility function $\sigma_{t+j}(x)$, which determines the integral domains $D_i,\ i = 1, 2$ in Theorem 2.1.

For the EGARCH(p,q) models,

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p} \alpha_j (|z_{t-j}| - \mathbb{E}|z_{t-j}|) + \sum_{j=1}^{p} \alpha'_j z_{t-j} + \sum_{j=1}^{q} \beta_j \ln(\sigma^2_{t-j}),$$

and

$$\ln(\sigma_{t+j}^2) = \left(\alpha_0 + \sum_{i=1}^{p} \alpha_i (|z_{t+j-i}| - \mathbb{E}|z_{t+j-i}|) + \sum_{i=1}^{p} \alpha'_i z_{t+j-i} \right)$$
$$\times \left(1 + \sum_{i=1}^{j-1} \sum_{\ell_1=1}^{q} \cdots \sum_{\ell_i=1}^{q} \beta_{\ell_1} \beta_{\ell_2} \cdots \beta_{\ell_i} \right) + \sum_{\ell_1=1}^{q} \cdots \sum_{\ell_j=1}^{q} \beta_{\ell_1} \beta_{\ell_2} \cdots \beta_{\ell_j} \ln \left(\sigma^2_{t+j-\sum_{k=1}^{j} \ell_k} \right).$$

The conditional volatility function $\sigma_{t+j}(x)$ follows. For other GARCH type processes, e.g. IGARCH, FIGARCH, IEGARCH, FIEGARCH etc. explicit computations can be done too.

Remark 2.2. *Though the conditional volatility function $\sigma_{t+j}(x)$ looks very complicated, but it is easy to be implemented iteratively by computer software, e.g. Matlab, Python and so on.*

Let us make some comments on the theoretical option valuation formula (2.10) in Theorem 2.1. The first observation is the dependence of option price on the volatility. From our formula, we can see that the option price (at time t) depends on the future volatility of the underlying security i.e. $\sigma^2_{t+1}, \sigma^2_{t+2}, \cdots, \sigma^2_T$ through the option duration. Since the volatility is varying from one period to another period, the better forecast of the future volatility is made, the more accurate the option will be valued. The GARCH type time-varying volatility is thus a method to model and forecast the dynamics of volatility. The family of GARCH models is huge, so that one has a great number of choices to select an appropriate GARCH type process for a specific underlying security to make a better forecast of the future volatility.

The second point we want to emphasize is that the return rate μ appears in the theoretical option valuation formula (2.10), which is quiet different from Black-Scholes' option pricing formula (2.11). However, by an informal analysis of sensitivity on the return rate, we find that there is a small impact on valuation of options for a large range of return rate.

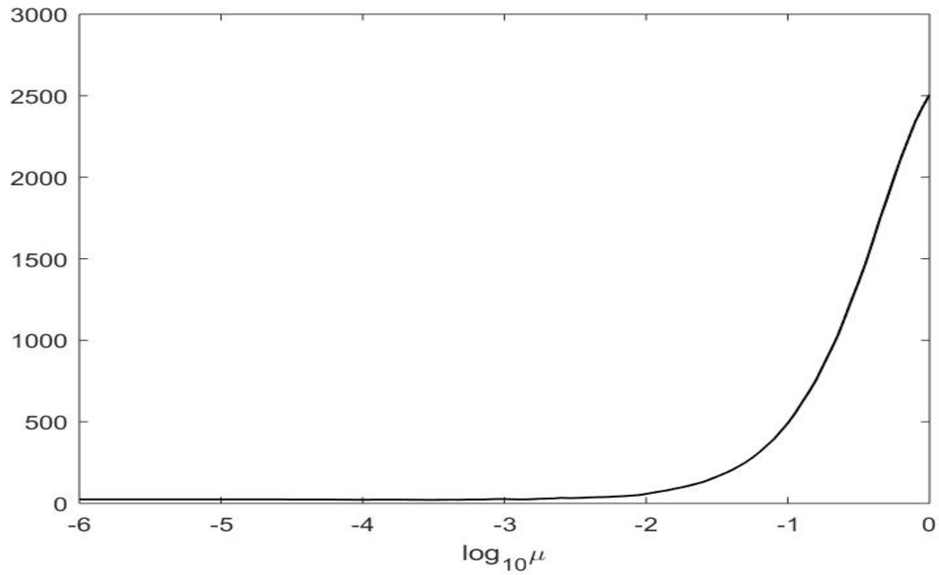

FIGURE 1. The effect of daily return rate μ on option price. Parameters are estimated from S&P 500 Index, see Model 1 in section §3.2. The annualized riskless interest rate is taken to 2.5%, string price $K = S_0$ (i.e. at-the-money option), and time-to-maturity $T = 30$ days.

TABLE 1. Daily and annualized return rate μ

1 day	10^{-5}	10^{-4}	5×10^{-4}	10^{-3}	5×10^{-3}	10^{-2}	10^{-1}
1 year	0.36%	3.65%	18.25%	36.48%	1.82	3.63	34.79

Figure 1 plots the relationship between daily return rate μ and option price C computed from formula (2.10). We find that for reasonable return rate μ, it has little impact on the option price. Table 1 shows daily return rates and their annualized ones.

The mathematical reason for appearance of return rate in our formula is that the martingale $\widehat{M}$ (M with drift) in equation (A.8) under risk-neutral measure $\mathbf{Q}$ is not same in law with the martingale M in equation (A.5) under physical measure $\mathbb{P}$, which is different from the 'Brownian motion' case, i.e. Brownian motion with drift under measure $\mathbf{Q}$ is still Brownian motion. In econometric words, for the GARCH-M case we are considering, the return innovation distribution is changed to another distribution by changing physical measure $\mathbb{P}$ to risk-neutral measure $\mathbf{Q}$. This is different from the constant volatility case, in which the return innovation distribution keeps unchanged under the transformation of measures.

Another point we want to explain is about the integral domains D_1 and D_2. The dimension of the integral domains changes according to time-to-maturity. When the

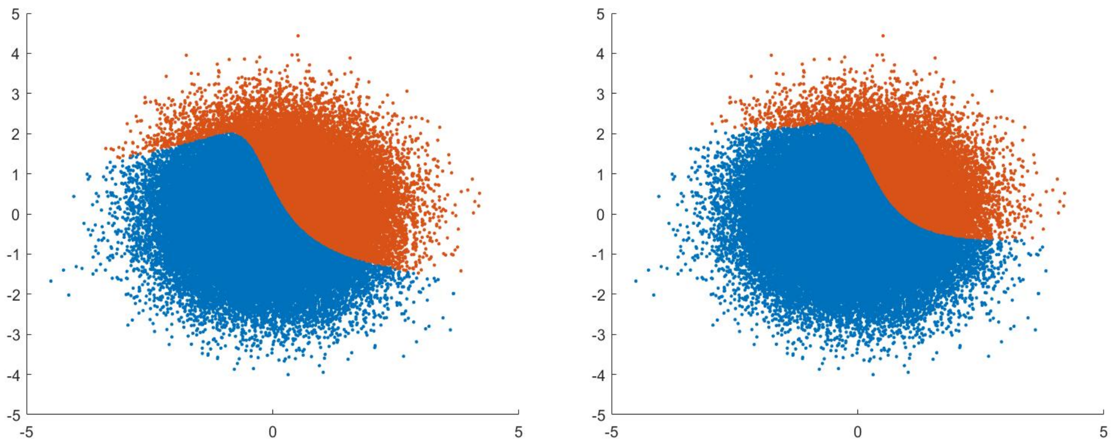

FIGURE 2. Integral domains D_1 (left) and D_2 (right) in 2 dimension

volatility is constant, $\Phi_{T-t}(D_i)$ become $N(d_i)$, $i = 1, 2$, due to the Gaussianity of linear combination of Gaussian/normal random variables. The integral domains of constant volatility case are always half planes with linear partition. However, for the GARCH type volatility, the borderline is no longer linear/flat but a curve/surface. A 2-dimensional example (with GARCH(1,1) volatiltiy) may provide a heuristic knowledge of the shape. In Figure 2, we simulated 100,000 points (all the colored points) with 2-dimensional standard normal distribution. According to the three-sigma rule of thumb, there is 99.73% of the values which lie within three standard deviations of the mean. So all the colored points cluster as shown in the Figure 2. The red parts are the integral domains D_1 (left) and D_2 (right). One can see very clearly the borderline and the shape of domains from this figure. By the way, one can actually compute out values of $\Phi_{T-t}(D_i)$, which are the ratio of numbers of red points to all colored points by Manto Carlo method.

2.2. **Option Pricing for AR(k)-GARCH-M Models.** Now we consider the more general AR(k)-GARCH-M models:

$$R_t = \mu_t - \frac{1}{2}\sigma_t^2 + \sigma_t z_t,\ t \in \mathbb{Z}_+, \tag{2.12}$$

$$\mu_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j}, \tag{2.13}$$

$$\sigma_t^2 \sim \text{GARCH type process}, \tag{2.14}$$

where $R_t = \log \frac{S_t}{S_{t-1}}$ is the logarithm return of assets, $\{z_t,\ t \in \mathbb{Z}_+\}$ is the Gaussian white noise, and σ_t^2 belongs to ARCH family. The return rate μ_t can be more general as long as μ_t is measurable with respect to $\mathcal{F}_{t-1}$. The riskless interest rate may depend on the time but should not be random.

Then we have the following European call option pricing formula, the proof of which is given in the Appendix B. (For simplicity, we still assume that the riskless interest rate is a constant.)

Theorem 2.3. *Let r be one plus riskless interest rate on one period, $T-t$ be time to maturity, and K be the striking pricing. Then, for the discrete-time AR(k)-GARCH-M models, the pricing formula for European call option is*

$$C_t = S_t \Phi_{T-t}(D_1) - K r^{-(T-t)} \Phi_{T-t}(D_2), \tag{2.15}$$

where

$$D_1 = \left\{ u = (u_1, \cdots, u_{T-t}) : \sum_{j=1}^{T-t} \sigma_{t+j} u_j > -\left(\log \frac{S_t}{K r^{-(T-t)}} + \frac{1}{2} \sum_{j=1}^{T-t} \sigma_{t+j}^2 \right) \right\},$$

$$D_2 = \left\{ v = (v_1, \cdots, v_{T-t}) : \sum_{j=1}^{T-t} \sigma_{t+j} v_j > -\left(\log \frac{S_t}{K r^{-(T-t)}} - \frac{1}{2} \sum_{j=1}^{T-t} \sigma_{t+j}^2 \right) \right\},$$

and

$$u_j = x_j - \frac{\varrho_{t+j}(x)}{\sigma_{t+j}(x)} - \sigma_{t+j}(x), \;\; v_j = x_j - \frac{\varrho_{t+j}(x)}{\sigma_{t+j}(x)}, \;\; \varrho_{t+j}(x) = -(\mu_{t+j}(x) - \log r).$$

$\sigma_{t+j}(x)$, $j = 1, 2, \cdots, T-t$, are conditional volatility functions evaluated at time t, and $\mu_{t+j}(x)$, $j = 1, 2, \cdots, T-t$, are conditional return rate functions evaluated at time t which are defined in the same way as conditional volatility functions.

Consider as an example, the AR(1)-GARCH(1,1)-M model:

$$\begin{aligned} R_t &= \gamma_0 + \gamma_1 R_{t-1} - \frac{1}{2}\sigma_t^2 + \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{aligned}$$

Theorem 2.3 gives the theoretical option pricing formula C_t for this model, where the conditional return rate functions $\mu_{t+j}(x)$, $j = 1, 2, \cdots, T-t$, can be computed explicitly.

In fact, since $\mu_t = \gamma_0 + \gamma_1 R_{t-1}$,

$$\mu_{t+j} = \sum_{i=0}^{j-1} \gamma_0 \gamma_1^i + \gamma_1^j R_t + \sum_{i=1}^{j-1} \gamma_1^{j-i} \left(-\frac{1}{2}\sigma_{t+i}^2 + \sigma_{t+i} z_{t+i} \right). \tag{2.16}$$

Substituting $(z_{t+j-1}, \cdots, z_{t+1})$ in the return rate μ_{t+j} with $(x_{j-1}, \cdots, x_1)$, we may obtain the conditional return rate function

$$\mu_{t+j}(x) = \frac{\gamma_0(1-\gamma_1^j)}{1-\gamma_1} + \gamma_1^j R_t + \sum_{i=1}^{j-1} \gamma_1^{j-i} \left(-\frac{1}{2}\sigma_{t+i}^2(x) + \sigma_{t+i}(x) x_i \right), \tag{2.17}$$

where $\{\sigma_{t+i}(x)\}$ are the conditional volatility functions defined in section §2.1.

For a general AR(k)-GARCH-M model given by

$$R_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} - \frac{1}{2}\sigma_t^2 + \sigma_t z_t,$$
$$\sigma_t^2 \sim \text{GARCH type processes},$$

the return rate
(2.18)

$$\begin{aligned}\mu_{t+j} &= \gamma_0 + \gamma_0 \sum_{i=1}^{j-1} \left(\sum_{\ell_1=1}^{k} \cdots \sum_{\ell_i=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_i} \right) + \sum_{\ell_1=1}^{k} \cdots \sum_{\ell_j=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j} R_{t+j-\sum_{q=1}^{j}\ell_q} \\ &\quad + \sum_{i=1}^{j-1} \left(\sum_{\ell_1=1}^{k} \cdots \sum_{\ell_i=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_i} \left(-\frac{1}{2}\sigma^2_{t+j-\sum_{q=1}^{i}\ell_q} + \sigma_{t+j-\sum_{q=1}^{i}\ell_q} z_{t+i-\sum_{q=1}^{j}\ell_q} \right) \right).\end{aligned}$$

By replacing $(z_{t+j-1}, \cdots, z_{t+1})$ in the return rate μ_{t+j} with $(x_{j-1}, \cdots, x_1)$, and replacing conditional volatility σ_{t+j} with associated conditional volatility functions $\sigma_{t+j}(x)$, we may calculate the conditional return rate function $\mu_{t+j}(x)$ at time t.

The risk-return relation can be linear or nonlinear in σ_t^2. For example, $V(\sigma_t^2) = c\sigma_t^2$, $c\log(\sigma_t^2)$, $c_1\sigma_t^2 + c_2\sin(\omega + c_3\sigma_t^2)$ etc. (see e.g. Christensen, Dahl and Iglesias (2012) and references therein) have been discussed. The AR(k)-GARCH-M models can be therefore generalized, such as

$$R_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + V(\sigma_t^2) + \sigma_t z_t, \tag{2.19}$$

$$\sigma_t^2 \sim \text{GARCH type process}, \tag{2.20}$$

where $V(\sigma_t^2)$ is the risk premia describing the tradeoff between risk and return. We may define

$$\mu_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + V(\sigma_t^2) + \frac{1}{2}\sigma_t^2, \tag{2.21}$$

so that the model is still described by equations (2.12) and (2.14).

Since μ_t is measurable with respect to $\mathcal{F}_{t-1}$, the option pricing formula (2.15) in Theorem 2.3 still works for this generalized model, with the conditional volatility functions $\sigma_{t+j}(x)$ evaluated at time t as in the section §2.1. The conditional return rate

$$\begin{aligned}\mu_{t+j} &= \gamma_0 + V(\sigma^2_{t+j}) + \frac{1}{2}\sigma^2_{t+j} + \sum_{\ell_1=1}^{k} \cdots \sum_{\ell_j=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_j} R_{t+j-\sum_{q=1}^{j}\ell_q} \\ &\quad + \left(\gamma_0 + V(\sigma^2_{t+j}) + \frac{1}{2}\sigma^2_{t+j} \right) \sum_{i=1}^{j-1} \left(\sum_{\ell_1=1}^{k} \cdots \sum_{\ell_i=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_i} \right) \\ &\quad + \sum_{i=1}^{j-1} \left(\sum_{\ell_1=1}^{k} \cdots \sum_{\ell_i=1}^{k} \gamma_{\ell_1}\gamma_{\ell_2}\cdots\gamma_{\ell_i} \left(-\frac{1}{2}\sigma^2_{t+j-\sum_{q=1}^{i}\ell_q} + \sigma_{t+j-\sum_{q=1}^{i}\ell_q} z_{t+i-\sum_{q=1}^{j}\ell_q} \right) \right).\end{aligned}$$

Remark 2.4. *Like the conditional volatility σ_{t+j}, the conditional return rate μ_{t+j} looks very complicated, it is in fact very easy to be implemented like σ_{t+j} by e.g. Matlab, Python and so on. The informal analysis for the sensitivity of theoretical valuation formula with respect to return rate μ reveals little sensitivity in a large range. Thus, in real applications, we do not really need the calculation of these conditional return rate functions for the purpose of saving computation costs.*

Remark 2.5. *We would like to point out that our option valuation formula in Theorem 2.3 is still valid. For example, for the following model:*

$$R_t = \mu_t - \frac{1}{2}\sigma_t^2 + \sigma_t z_t, \tag{2.22}$$

$$\mu_t = \mu(\{R_s\}_{s<t}, \sigma_t^2), \tag{2.23}$$

$$\sigma_t^2 \sim \textit{GARCH type process}, \tag{2.24}$$

our formula is applicable. The return rate μ_t may depend on the past returns and the volatility linearly or both nonlinearly.

3. Parameter Estimation

In this section, we describe one estimation algorithm for the models we have discussed in the previous section.

3.1. **Estimation Illustration.** Consider the first model GARCH-M with constant return rate, for example,

$$R_t = \mu - \frac{1}{2}\sigma_t^2 + \sigma_t z_t, \tag{3.1}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \tag{3.2}$$

There are four parameters $\mu, (\alpha_0, \alpha_1, \beta_1)$ to be estimated. The iterated estimation algorithm is suggested below.

Step 1: Provide a set of initial parameters $\widehat{\theta}^{(0)} = (\widehat{\alpha}_0^{(0)}, \widehat{\alpha}_1^{(0)}, \widehat{\beta}_1^{(0)})$ and $\widehat{\mu}^{(0)}$ by fitting the data using a standard GARCH model with a nonzero mean, i.e.

$$R_t = \mu + \sigma_t z_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

Step 2: Compute the conditional volatility process $\widehat{\sigma}_t^{2,(i)}$ for $t = 1, 2, \cdots, T$ by equation (3.2) based on the parameters $\widehat{\theta}^{(i)} = (\widehat{\alpha}_0^{(i)}, \widehat{\alpha}_1^{(i)}, \widehat{\beta}_1^{(i)})$ obtained in the last step.

Step 3: Update $\widehat{\theta}^{(i)}$ and $\widehat{\mu}^{(i)}$. That is, find $\widehat{\theta}^{(i+1)}$ and $\widehat{\mu}^{(i+1)}$ by the standard GARCH model with a nonzero mean

$$\widehat{R}_t^{(i)} \triangleq R_t + \frac{1}{2}\widehat{\sigma}_t^{2,(i)} = \mu + \sigma_t z_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

Step 4: Repeat Steps 2 and 3 for a finite fixed number of iterations or until convergence.

More generally, for AR-GARCH-M models with linear risk premia, i.e.

$$R_t = \mu + c\sigma_t^2 + \sigma_t z_t, \tag{3.3}$$

and

$$R_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + c\sigma_t^2 + \sigma_t z_t, \tag{3.4}$$

with σ_t^2, for example, a GARCH(1,1) process, the iterated estimation algorithm is suggested below.
Step 1: Provide a set of initial parameters $\widehat{\theta}^{(0)} = (\widehat{\alpha}_0^{(0)}, \widehat{\alpha}_1^{(0)}, \widehat{\beta}_1^{(0)})$ by fitting the data using a standard AR(k)-GARCH model

$$R_t = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + \sigma_t z_t.$$

Step 2: Compute the conditional volatility process $\{\widehat{\sigma}_t^{2,(i)},\ t = 1, 2, \cdots, T\}$ by equation (3.2) based on the parameters $\widehat{\theta}^{(i)} = (\widehat{\alpha}_0^{(i)}, \widehat{\alpha}_1^{(i)}, \widehat{\beta}_1^{(i)})$ obtained in the last step.
Step 3: Estimate $\widehat{\gamma}^{(i)} = (\widehat{\gamma}_0^{(i)}, \widehat{\gamma}_1^{(i)}, \cdots, \widehat{\gamma}_k^{(i)})$ and $\widehat{c}^{(i)}$ by the linear regression model

$$\mathbb{E}(R_t|\mathcal{F}_{t-1}) = \gamma_0 + \sum_{j=1}^{k} \gamma_j R_{t-j} + c\widehat{\sigma}_t^{2,(i)}. \tag{3.5}$$

Step 4: Update $\widehat{\theta}^{(i)}$, and find $\widehat{\theta}^{(i+1)}$ via the standard GARCH model

$$\begin{aligned} \widehat{R}_t^{(i)} &= \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{aligned}$$

where

$$\widehat{R}_t^{(i)} \triangleq R_t - \widehat{\gamma}_0^{(i)} - \sum_{j=1}^{k} \widehat{\gamma}_j^{(i)} R_{t-j} - \widehat{c}^{(i)} \widehat{\sigma}_t^{2,(i)}.$$

Step 5: Repeat Steps 2, 3 and 4 for a finite fixed number of iterations or until convergence.

For the estimation of the general model

$$R_t = \mu(R, \sigma_t^2) + \sigma_t z_t, \tag{3.6}$$

$$\sigma_t^2 \sim \text{GARCH type process}, \tag{3.7}$$

one should first present a semiparametric estimation for $\mu(R, \sigma_t^2)$, and then use iterated estimation procedure as suggested above. As our main aim of this work is not to present the estimation problems for these models, we refer to Christensen, Dahl and Iglesias (2012), Conrad and Mammen (2016) and references therein for the details of estimation and convergence problems. For the exact likelihood inference of these models, see for example Fiorentini, Sentana and Shephard (2004), in which they suggested a Markov chain Monte Carlo algorithm to carry out the estimation probelms.

3.2. **Empirical Example.** The data set analyzed here consists of daily prices on the Standard and Poor's 500 composite stock index from January 3 through December 29, 2017, for a total of $T = 251$ observations. The daily prices are denoted $\{S_t,\ t = 1, 2, \cdots, T\}$ and logarithm return as $R_t = \log(S_t) - \log(S_{t-1})$. The time t subscript refers to trading days.

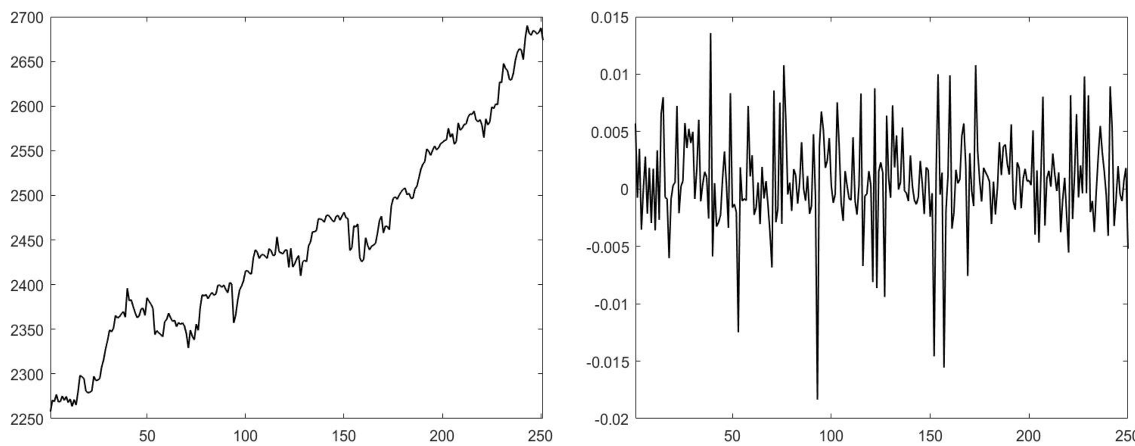

Figure 3. Prices and log-Return of S&P 500 Index during 2017

Figure 3 plots the prices S_t on the left and the logarithm return R_t on the right for the whole year. Here we use three represented GARCH type models to fit these historical data.

The first model (say, Model 1) is

$$R_t = \mu - \frac{1}{2}\sigma_t^2 + \sigma_t z_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

The iterated estimation procedure above leads to identification of the following model for $R_t = \log(S_t) - \log(S_{t-1})$:

$$\begin{aligned} R_t &= \underset{(2.6145\times 10^{-4})}{6.6488 \times 10^{-4}} - 0.5\sigma_t^2 + \sigma_t z_t, \\ \sigma_t^2 &= \underset{(8.9816\times 10^{-7})}{8.753 \times 10^{-7}} + \underset{(0.0176)}{0.05}\sigma_{t-1}^2 z_{t-1}^2 + \underset{(0.0165)}{0.9}\sigma_{t-1}^2. \end{aligned}$$

The numbers in the parenthesis are standard errors.

The second model (say, Model 2) is

$$R_t = \mu + c\sigma_t^2 + \sigma_t z_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

which is specified to

$$\begin{aligned} R_t &= 5.8333 \times 10^{-4} + 5.5693\sigma_t^2 + \sigma_t z_t, \\ &\quad (9.2205 \times 10^{-5}) \quad (5.4418) \\ \sigma_t^2 &= 8.7397 \times 10^{-7} + 0.05\sigma_{t-1}^2 z_{t-1}^2 + 0.9\sigma_{t-1}^2. \\ &\quad (8.9682 \times 10^{-7}) \quad (0.0175) \qquad (0.0162) \end{aligned}$$

The third model (say, Model 3) is

$$\begin{aligned} R_t &= \gamma_0 + \gamma_1 R_{t-1} + c\sigma_t^2 + \sigma_t z_t, \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{aligned}$$

The specification for this model is

$$\begin{aligned} R_t &= 8.0621 \times 10^{-4} - 0.1344 R_{t-1} - 2.216\sigma_t^2 + \sigma_t z_t, \\ &\quad (0.0015) \qquad (0.0632) \qquad (31.7553) \\ \sigma_t^2 &= 8.5598 \times 10^{-7} + 0.05\sigma_{t-1}^2 z_{t-1}^2 + 0.9\sigma_{t-1}^2. \\ &\quad (8.7837 \times 10^{-7}) \quad (0.0174) \qquad (0.0158) \end{aligned}$$

These results are standard in the literature. We show them as representative examples. For these three models, the parameters are robustly obtained by the iterated estimation procedure. Another fact is that the volatility processes in these three models are very stable.

4. Option Pricing

In this section, we analysis the option valuation formula

$$C_t = S_t \Phi_{T-t}(D_1) - K r^{-(T-t)} \Phi_{T-t}(D_2)$$

for European call options on the Standard and Poor's 500 composite stock index. The quantities $\Phi_{T-t}(D_i), i = 1, 2$, are computed by Monte Carlo method. Since

$$\Phi_{T-t}(D_i) = \int \cdots \int_{D_i} (2\pi)^{-\frac{T-t}{2}} e^{-\frac{1}{2}\sum_{j=1}^{T-t} y_j^2} dy_1 \cdots dy_{T-t},$$

by independently simulating a number of $(T-t)$-dimensional standard normal random variables, one may obtain simulated values of $\Phi_{T-t}(D_i)$.

We use the three models specified in the previous section to investigate the pricing behaviour given by the option valuation formula. The underlying asset is Standard and Poor's 500 composite stock index. The present time t is the day on December 29, 2017. We consider at-the-money options for this moment, that is, the striking price $K = S(t)$. The number of days to maturity $T - t$ varies from 1 to 60. The riskless annual continuously compounded interest rate $\widetilde{r} = 2.5\%$, i.e. $r = e^{\widetilde{r}\Delta t} = 1 + 6.85 \times 10^{-5}$ with $\Delta t = 1/365$ in our formula. The results are presented in figure 4. We also included the simulated prices calculated with the traditional Black-Scholes pricing formula (see Black and Scholes (1973)). The roughness of pricing lines in Figure 4 are caused by the error of Monte Carlo simulation. From the figure, we can see that, for these three GARCH(1,1)-M models, the simulated prices based on our option valuation formula

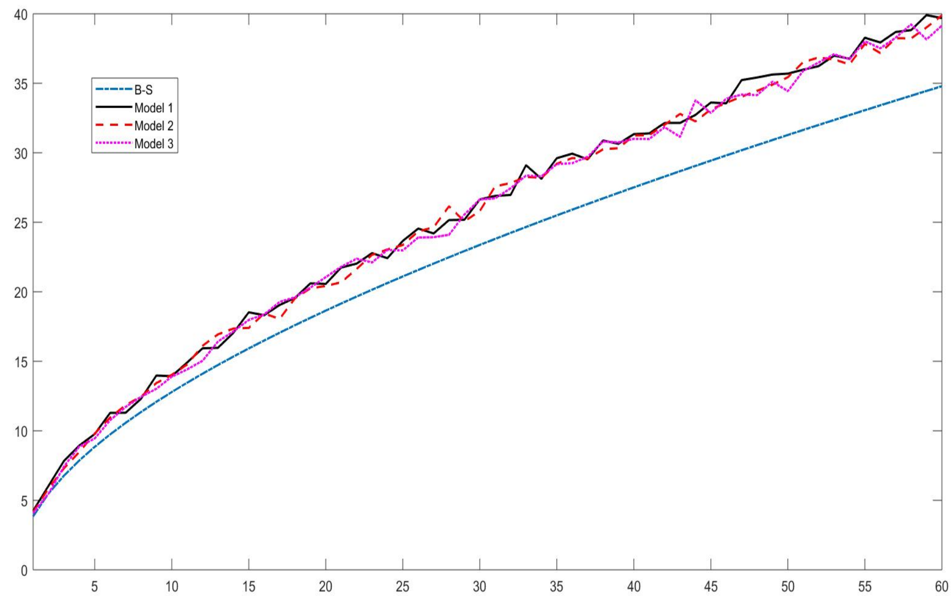

FIGURE 4. Option prices as a function of maturity by different models

are almost same and higher than the celebrated Black-Scholes prices. Since the only difference of Model 1, 2 and 3 is the return rate, it reveals to some degree that there is little sensitivity with respect to the return rate models.

5. Empirical Applications

In this section, we show the performance of our theoretical option valuation formula when applying it to Standard and Poor's (S&P) 500 index options, and SSE 50 ETF options traded on Shanghai Stock Exchange (SSE), China.

5.1. **S&P 500 Index Options.** Based on the in-sample analysis in the previous section, we shall compare our option valuation formula with the real financial market. We use GARCH-in-mean data-generating mechanism as Model 1 (with GARCH(1,1) volatility) in section §3.2 for the underlying Standard and Poor's 500 index.

We randomly pick two heavily traded SPX call options expired on 2018. One option is issued on October 23, 2017, exercised on February 15, 2018 with striking price $K = 2650$ while another one is issued on April 26, 2017, exercised on March 15, 2018 with striking price $K = 2700$. All these historical data are provided by Bloomberg. The riskless interest rate $\widetilde{r}$ over the life of the options is taken to be 2.5%, which is annual continuously compounded and approximately equals to the interest rate of US treasury bills at the same period. So the interest $r = e^{\widetilde{r}\Delta t} = 1 + 6.85 \times 10^{-5}$ with $\Delta t = 1/365$ in our theoretical valuation formula. As a remark, some informal analysis revealed little sensitivity to the choice of the riskless interest rate.

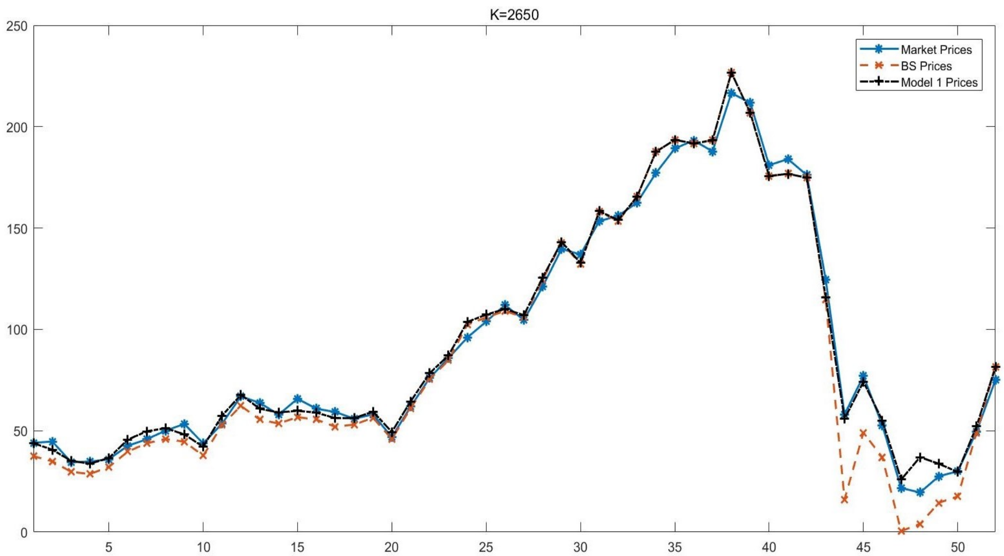

FIGURE 5. SPX option (C2650) prices from December 1, 2017 to February 15, 2018. $\widetilde{r} = 2.5\%$, $K = 2650$, issued on October 23, 2017, exercised on February 15, 2018.

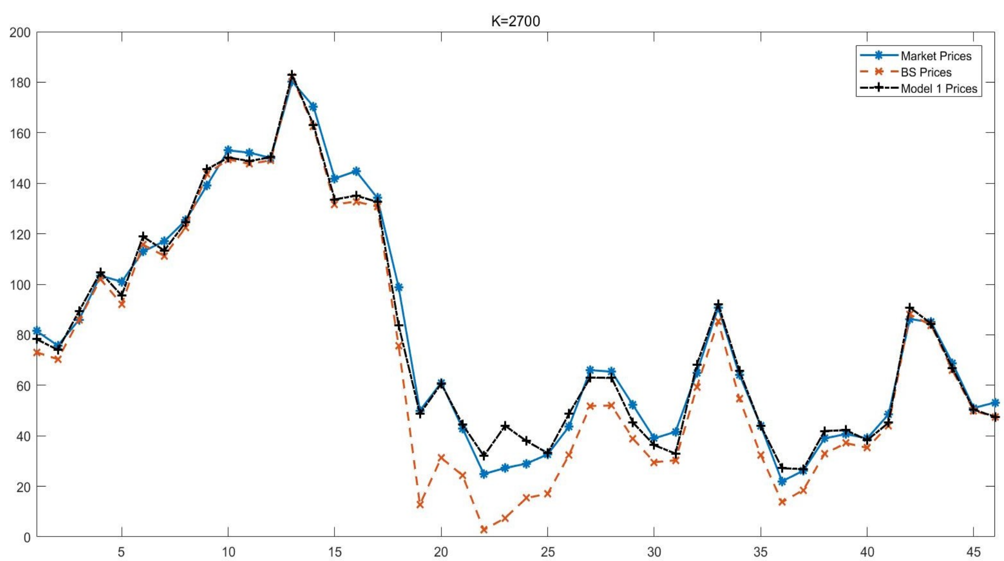

FIGURE 6. SPX option (C2700) prices from January 9, 2018 to March 15, 2018. $\widetilde{r} = 2.5\%$, $K = 2700$, issued on April 26, 2017, exercised on March 15, 2018.

For the first SPX call option C2650, we computed the model prices from December 1, 2017 to February 15, 2018 by Model 1 data-generating mechanism in section §3.2 using the rolling windows method, that is, the parameters of Model 1 are estimated by the past one year daily data of the underlying index SPX before the date we are computing. We also compared the model prices calculated by the celebrated Black-Scholes option pricing formula. We showed the results in Figure 5. In this figure, the three prices - market prices, Black-Scholes model prices and GARCH-M Model 1 prices, of SPX call option C2650 are compared. For the other SPX call option C2700, we use the same approach to obtain the model prices from January 9, 2018 to March 15, 2018, which are shown in Figure 6.

The GARCH-M Model 1 prices are clearly closer to the market prices than the Black-Scholes model prices for most of the trading days. It means that the forecast of future volatility works better with our method. The performance of Model 1 is much better than the traditional method especially when the market volatility is not pleasant, e.g. February 2018. For better performance, one can try to look for a better forecast of future volatility, for example, employing FIGARCH, EGARCH process to model the volatility rather than just using GARCH(1,1) model.

5.2. **Chinese Financial Market.** In this subsection, we show results by applying our option valuation formula to China financial market. Options were introduced to the Chinese financial market in February 9, 2015. Chinese SSE 50 ETF option is the first and only standardized option traded on Chinese market. The underlying asset is the SSE 50 ETF. All 50 ETF options are European options and are traded on the Shanghai Stock Exchange (SSE). For the underlying asset SSE 50 ETF, it is also traded on the Shanghai Stock Exchange and was the first ETF traded in China. This ETF tracks the SSE 50 index, which includes 50 of the most active and reputable stocks listed on the Shanghai Stock Exchange. It is one of the most heavily traded ETFs in China.

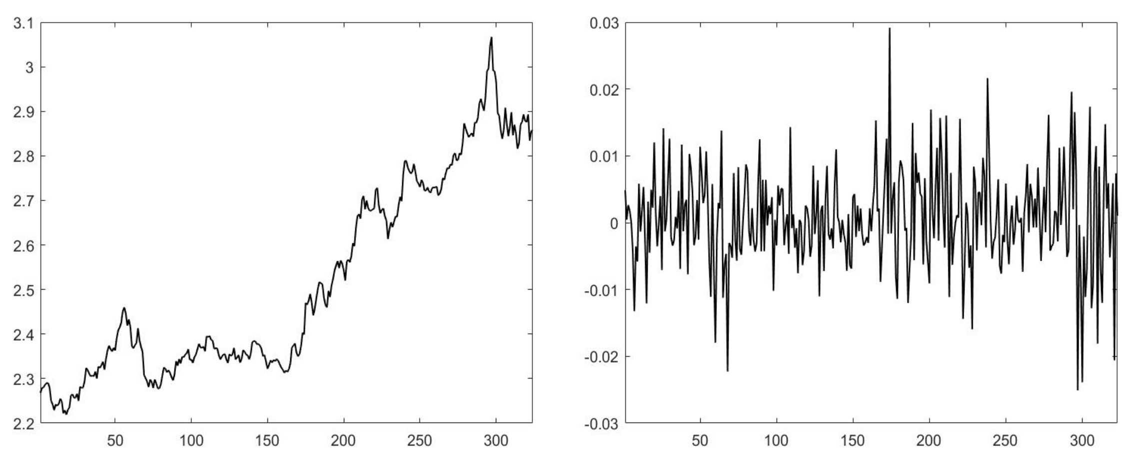

Figure 7. Prices and log-Return of SSE 50 ETF (9/1/2016-12/29/2017)

The data of the underlying asset we use are the daily close prices of SSE 50 ETF from September 1, 2016 through December 29, 2017. There are 324 prices in total. The dataset was provided by Wind Info, Inc. The Figure 7 plots the price process and the associates logarithm return of SSE 50 ETF for this dataset.

Our aim in this section is to compute the market prices with our option valuation formula for Chinese financial market. We also compared the prices with the celebrated Black-Scholes option pricing formula as previous subsection.

The market prices of these SSE 50 ETF options at the end of each day from September 1, 2017 to September 29, 2017 are available. There are 21 trading days in this month, and there are 9 striking prices for these options: 1 at-the-money, 4 out-of-the-money and 4 in-the-money. We use three actively traded options, whose symbols are 10000993.SH, 10000994.SH and 10000995.SH. Both started on August 24, 2017, and exercised on October 25, 2017. The exercise prices K are RMB 2.70, 2.75 and 2.80, respectively. We calculated the model prices of these options for every trading day of this whole September by using estimates of the parameters of GARCH-M models based on past data of underlying asset SSE 50 ETF. The riskless annual continuously compound interest rate $\widetilde{r}$ is taken to be 3.9%, which is the constant interest rate of Chinese bond at the same period.

Standing on the date September 1, 2017, on which the price of SSE 50 ETF is RMB 2.76, we want to decide the option model price on September 1 by using the past one year daily prices of SSE 50 ETF, i.e. the data from September 1, 2016 to September 1, 2017. By using these past one year data, we first obtain estimated parameters for GARCH-M models as Model 1 in section §3.2, then we can work out a model price with our option pricing formula. For the next trading day, we use the rolling windows method, we can compute a new option price for it. The results are shown in Figure 8.

It is apparent to see that the GARCH-M Model 1 prices are between the market prices and the Black-Scholes model prices for most of the time.

6. Conclusion

We present option pricing formulas for general GARCH-M models based on risk-neutral arguments in this paper. These formulas are not only beautiful but also realistic for applications. We propose a parameter estimation procedure in section §3 and employ Monte Carlo simulation to evaluate the prices. The pricing behaviour of these theoretical formulas for S&P 500 index options are shown in section §4 based on three data-generating mechanism. Empirical evidence suggests that both in U.S. stock market and Chinese financial market the performances of these theoretical pricing formulas are very good compared with real market prices, and better than the celebrated Black-Scholes pricing formula.

It would be interesting to explore more properties and implications of these theoretical pricing formulas, and extend the empirical analysis to other stocks with appropriate GARCH type volatility. A more formal and detailed empirical investigation of these issues would be an important work both for research and real applications.

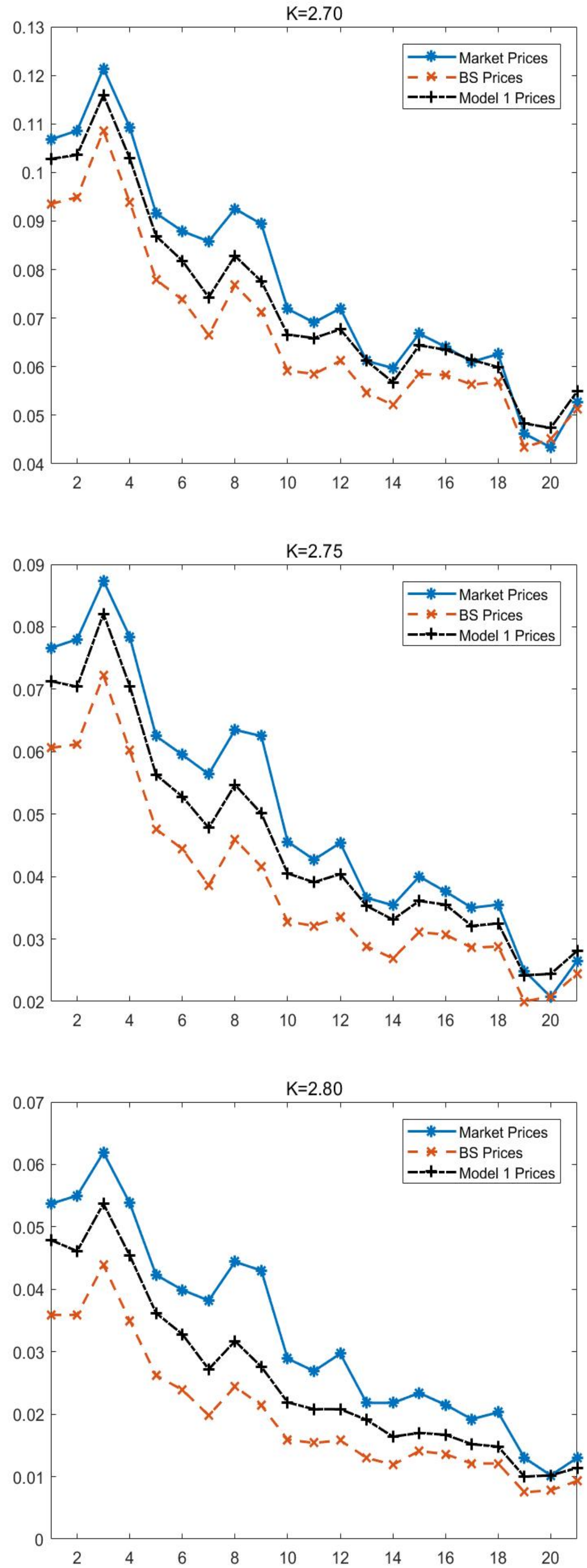

FIGURE 8. SSE 50 ETF options' prices from September 1, 2017 to September 29, 2017. $\widetilde{r} = 3.9\%$, $K = 2.70, 2.75, 2.80$, issued on August 24, 2017, exercised on October 25, 2017.

Appendix A. Proof of Theorem 2.1

Assume that the riskless interest rate is constant, and let interest r denote one plus the riskless interest rate over one period. In risk-neutral world, the expected rate of return on the asset would be the riskless interest rate, so we want to find a risk-neutral measure $\mathbf{Q}$ such that

$$\mathbb{E}^{\mathbf{Q}}\left[S_{t+k}|\mathcal{F}_t\right] = r^k S_t, \ \forall k \geq 1, \ t \geq 0. \tag{A.1}$$

The price of an option for this underlying asset is then

$$C_t = r^{-(T-t)}\mathbb{E}^{\mathbf{Q}}[C_T|\mathcal{F}_t]. \tag{A.2}$$

For European call option, $C_T = \max\{0, S_T - K\}$. Actually, under measure $\mathbf{Q}$, the two processes $\{\frac{S_t}{r^t}, t \in \mathbb{Z}_+\}$ and $\{\frac{C_t}{r^t}, t = 0, 1, 2, \cdots, T\}$ are discrete-time martingales. The question left is how to construct the risk-neutral measure $\mathbf{Q}$.

For GARCH-M models, by equation (A.1) we have

$$\mathbb{E}^{\mathbf{Q}}\left[e^{\sum_{i=t+1}^{t+k} R_i}|\mathcal{F}_t\right] = r^k, \ \forall k \geq 1, \ t \geq 0. \tag{A.3}$$

Then

$$\mathbb{E}^{\mathbf{Q}}\left[e^{\sum_{i=t+1}^{t+k}\sigma_i z_i + k(\mu - \log r) - \frac{1}{2}\sum_{i=t+1}^{t+k}\sigma_i^2}|\mathcal{F}_t\right] = 1, \ \forall k \geq 1, \ t \geq 0. \tag{A.4}$$

In the following, we are going to construct the risk-neutral measure $\mathbf{Q}$ satisfying equation (A.4). Define measure $\mathbb{P}$ as the distribution of Gaussian white noise $\{z_i, i \geq 0\}$, and

$$M_t \triangleq M_0 + \sum_{i=1}^{t}\sigma_i z_i, \tag{A.5}$$

$$Z_t \triangleq \exp\left\{\sum_{i=1}^{t}\frac{\varrho}{\sigma_i}z_i - \frac{1}{2}\sum_{i=1}^{t}\frac{\varrho^2}{\sigma_i^2}\right\}, \quad \varrho = -(\mu - \log r). \tag{A.6}$$

We can verify that M_t and Z_t are martingales under measure $\mathbb{P}$.

Let measure $\mathbf{Q}$ be

$$\mathbf{Q}(A) \triangleq \mathbb{E}^{\mathbb{P}}\left(\chi_A Z_t\right), \quad \forall A \in \mathcal{F}_t, \tag{A.7}$$

with $\chi_A(\omega) = 1$, if $\omega \in A$, otherwise $\chi_A(\omega) = 0$, and

$$\widehat{M_t} \triangleq M_t - \varrho t = M_0 + \sum_{i=1}^{t}\sigma_i z_i + (\mu - \log r)t. \tag{A.8}$$

Lemma A.1. *For any $t, k \in \mathbb{Z}_+$, if any random variable X is measurable in $\mathcal{F}_{t+k}$, then*

$$\mathbb{E}^{\mathbf{Q}}\left[X|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}^{\mathbb{P}}\left[XZ_{t+k}|\mathcal{F}_t\right]. \tag{A.9}$$

Proof. For any $A \in \mathcal{F}_t$, by the definition of conditional expectation, we have

$$\mathbb{E}^{\mathbf{Q}}\left[\chi_A\frac{1}{Z_t}\mathbb{E}^{\mathbb{P}}\left[XZ_{t+k}|\mathcal{F}_t\right]\right] = \mathbb{E}^{\mathbb{P}}\left[\chi_A\mathbb{E}^{\mathbb{P}}\left[XZ_{t+k}|\mathcal{F}_t\right]\right] = \mathbb{E}^{\mathbb{P}}\left[\chi_A X Z_{t+k}\right] = \mathbb{E}^{\mathbf{Q}}\left[\chi_A X\right].$$

This concluded this lemma. *Q.E.D.*

Theorem A.2. *Under measure* $\mathbf{Q}$*, stochastic processes* $\widehat{M}_t$ *and*

$$\exp\left\{\widehat{M}_t - \frac{1}{2}\sum_{i=1}^{t}\sigma_i^2\right\}$$

are $\mathcal{F}_t$*-martingales.*

Proof. (i) We show that $\widehat{M}_t$ is a $\mathbf{Q}$-martingale below. We first prove that $\widehat{M}_t Z_t$ is a $\mathbb{P}$-martingale. Since

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+1}Z_{t+1}|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{P}}\left[\left(\widehat{M}_t + \sigma_{t+1}z_{t+1} - \varrho\right) Z_t e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right] \\ = \widehat{M}_t Z_t \mathbb{E}^{\mathbb{P}}\left[e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right] &+ Z_t\mathbb{E}^{\mathbb{P}}\left[\left(\sigma_{t+1}z_{t+1} - \varrho\right) e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right].\end{aligned}$$

For the first term on the right hand side, as $\sigma_{t+1}^2 \in \mathcal{F}_t$, and z_{t+1} is independent of $\mathcal{F}_t$ under $\mathbb{P}$, thus

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[e^{\frac{\varrho}{y}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{y^2}}\right]_{y=\sigma_{t+1}} = 1.$$

For the second term, we have

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\left(\sigma_{t+1}z_{t+1} - \varrho\right) e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(yz_{t+1} - \varrho\right) e^{\frac{\varrho}{y}z_{t+1} - \frac{1}{2}\frac{\varrho^2}{y^2}}\right]_{y=\sigma_{t+1}} \\ &= \int_{-\infty}^{\infty}\left(\sigma_{t+1}x - \varrho\right) e^{\frac{\varrho}{\sigma_{t+1}}x - \frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= \int_{-\infty}^{\infty}\left(\sigma_{t+1}x - \varrho\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma_{t+1}^2}\left(\sigma_{t+1}x - \varrho\right)^2}dx \\ &= \int_{-\infty}^{\infty}u\frac{1}{\sqrt{2\pi\sigma_{t+1}^2}}e^{-\frac{u^2}{2\sigma_{t+1}^2}}du = 0.\end{aligned}$$

Thus we have shown that

$$\mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+1}Z_{t+1}|\mathcal{F}_t\right] = \widehat{M}_t Z_t, \ \forall\ t \in \mathbb{Z}_+.$$

For any $k \in \mathbb{Z}_+$,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+k}Z_{t+k}\middle|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+k}Z_{t+k}\middle|\mathcal{F}_{t+k-1}\right]\middle|\mathcal{F}_t\right) \\ = \mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+k-1}Z_{t+k-1}\middle|\mathcal{F}_t\right] &= \cdots = \widehat{M}_t Z_t.\end{aligned}$$

This means that $\widehat{M}_t Z_t$ is a $\mathbb{P}$-martingale.

To prove that $\widehat{M}_t$ is a $\mathbf{Q}$-martingale, we apply Lemma A.1 and the results above, then get

$$\mathbb{E}^{\mathbf{Q}}\left[\widehat{M}_{t+k}|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}^{\mathbb{P}}\left[\widehat{M}_{t+k}Z_{t+k}|\mathcal{F}_t\right] = \widehat{M}_t. \tag{A.10}$$

We concluded the first statement of this theorem.

(ii) $\exp\left\{\widehat{M}_t - \frac{1}{2}\sum_{i=1}^t \sigma_i^2\right\}$ is also a $\mathbf{Q}$-martingale. In fact, by Lemma A.1,

$$\begin{aligned}
&\mathbb{E}^{\mathbf{Q}}\left[e^{\widehat{M}_{t+1}-\frac{1}{2}\sum_{i=1}^{t+1}\sigma_i^2}\middle|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}^{\mathbb{P}}\left[e^{\widehat{M}_{t+1}-\frac{1}{2}\sum_{i=1}^{t+1}\sigma_i^2}Z_{t+1}\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{P}}\left[e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2+\sigma_{t+1}z_{t+1}-\varrho-\frac{1}{2}\sigma_{t+1}^2}e^{\frac{\varrho}{\sigma_{t+1}}z_{t+1}-\frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\middle|\mathcal{F}_t\right]\\
&= e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}\mathbb{E}^{\mathbb{P}}\left[e^{yz_{t+1}-\varrho-\frac{1}{2}y^2}e^{\frac{\varrho}{y}z_{t+1}-\frac{1}{2}\frac{\varrho^2}{y^2}}\right]_{y=\sigma_{t+1}}\\
&= e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}\int_{-\infty}^{+\infty} e^{\sigma_{t+1}x-\varrho-\frac{1}{2}\sigma_{t+1}^2}e^{\frac{\varrho}{\sigma_{t+1}}x-\frac{1}{2}\frac{\varrho^2}{\sigma_{t+1}^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\\
&= e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}e^{-\frac{1}{2}\sigma_{t+1}^2}\int_{-\infty}^{+\infty} e^{\sigma_{t+1}x-\varrho}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma_{t+1}^2}(\sigma_{t+1}x-\varrho)^2}dx\\
&= e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}e^{-\frac{1}{2}\sigma_{t+1}^2}\int_{-\infty}^{+\infty} e^{u}\frac{1}{\sqrt{2\pi\sigma_{t+1}^2}}e^{-\frac{u^2}{2\sigma_{t+1}^2}}du = e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}.
\end{aligned}$$

For any $k \in \mathbb{Z}_+$,

$$\mathbb{E}^{\mathbf{Q}}\left[e^{\widehat{M}_{t+k}-\frac{1}{2}\sum_{i=1}^{t+k}\sigma_i^2}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbf{Q}}\left(\mathbb{E}^{\mathbf{Q}}\left[e^{\widehat{M}_{t+k}-\frac{1}{2}\sum_{i=1}^{t+k}\sigma_i^2}\middle|\mathcal{F}_{t+k-1}\right]\middle|\mathcal{F}_t\right) = \cdots = e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}.$$

Thus we have proved that $e^{\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2}$ is a $\mathbf{Q}$-martingale. *Q.E.D.*

Therefore,

$$\mathbb{E}^{\mathbf{Q}}\left[e^{\sum_{i=t+1}^{t+k}\sigma_i z_i+k(\mu-\log r)-\frac{1}{2}\sum_{i=t+1}^{t+k}\sigma_i^2}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbf{Q}}\left[e^{(\widehat{M}_{t+k}-\sum_{i=1}^{t+k}\frac{1}{2}\sigma_i^2)-(\widehat{M}_t-\frac{1}{2}\sum_{i=1}^{t}\sigma_i^2)}|\mathcal{F}_t\right] = 1.$$

That is, we have constructed a risk-neutral measure $\mathbf{Q}$ satisfying equation (A.4) for these GARCH-M models.

Now we present the discrete-time option pricing formula for European call. Since $\{\frac{C_t}{r^t}, t = 0, 1, 2, \cdots, T\}$ is a martingale under risk-neutral measure $\mathbf{Q}$, then one pricing formula is

$$C_t = \mathbb{E}^{\mathbf{Q}}\left[\left(S_t e^{\sum_{i=t+1}^{T}\sigma_i z_i-\varrho(T-t)-\frac{1}{2}\sum_{i=t+1}^{T}\sigma_i^2} - Kr^{-(T-t)}\right)^+\middle|\mathcal{F}_t\right]. \tag{A.11}$$

The option pricing formula can also be calculated under measure $\mathbb{P}$, that is, by Lemma A.1,

(A.12)

$$C_t = \mathbb{E}^{\mathbb{P}}\left[\left(S_t e^{\sum_{i=t+1}^{T}\sigma_i z_i-\varrho(T-t)-\frac{1}{2}\sum_{i=t+1}^{T}\sigma_i^2} - Kr^{-(T-t)}\right)^+ e^{\sum_{i=t+1}^{T}\frac{\varrho}{\sigma_i}z_i-\frac{1}{2}\sum_{i=t+1}^{T}\frac{\varrho^2}{\sigma_i^2}}\middle|\mathcal{F}_t\right].$$

Since

$$\sigma_{t+j} = \sigma_{t+j}(z_{t+j-1}, \cdots, z_{t+1}, z_t, \cdots), \tag{A.13}$$

they are correlated with $\{z_t, t \in \mathbb{Z}_+\}$. The price of option C_t depends on attributes of the forecasting of the path followed by $\{\sigma_{t+1}^2, \sigma_{t+2}^2, \cdots, \sigma_T^2\}$.

In the following, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Actually, from the pricing formula in equation (A.12), we have the following representation:

$$C_t = S_t N_1 - K r^{-(T-t)} N_2, \tag{A.14}$$

where

$$\begin{aligned} N_1 = &\int \cdots \int_D e^{\sum_{j=1}^{T-t} \sigma_{t+j} x_j - \varrho(T-t) - \frac{1}{2}\sum_{j=1}^{T-t} \sigma_{t+j}^2} \\ &\times e^{\sum_{j=1}^{T-t} \frac{\varrho}{\sigma_{t+j}} x_j - \frac{1}{2}\sum_{j=1}^{T-t} \frac{\varrho^2}{\sigma_{t+j}^2}} (2\pi)^{-\frac{T-t}{2}} e^{-\frac{1}{2}\sum_{j=1}^{T-t} x_j^2} dx_1 \cdots dx_{T-t}, \end{aligned} \tag{A.15}$$

$$N_2 = \int \cdots \int_D e^{\sum_{j=1}^{T-t} \frac{\varrho}{\sigma_{t+j}} x_j - \frac{1}{2}\sum_{j=1}^{T-t} \frac{\varrho^2}{\sigma_{t+j}^2}} (2\pi)^{-\frac{T-t}{2}} e^{-\frac{1}{2}\sum_{j=1}^{T-t} x_j^2} dx_1 \cdots dx_{T-t}, \tag{A.16}$$

and

$$\begin{aligned} &\sigma_{t+1} = \sigma_{t+1}(z_t, z_{t-1}, \cdots), \\ &\sigma_{t+j} = \sigma_{t+j}(x_{j-1}, \cdots, x_1; z_t, z_{t-1}, \cdots), \ j = 2, \cdots, T-t, \end{aligned}$$

and the domain

$$D = \left\{ \sum_{j=1}^{T-t} \sigma_{t+j} x_j - \varrho(T-t) > -\left(\log \frac{S_t}{K r^{-(T-t)}} - \frac{1}{2}\sum_{j=1}^{T-t} \sigma_{t+j}^2 \right) \right\}.$$

By some calculations, we further have

$$\begin{aligned} N_1 = &\int \cdots \int_D e^{\sum_{j=1}^{T-t} (\sigma_{t+j} x_j - \varrho) - \frac{1}{2}\sum_{j=1}^{T-t} \sigma_{t+j}^2} \\ &\times (2\pi)^{-\frac{T-t}{2}} e^{-\sum_{j=1}^{T-t} \frac{(\sigma_{t+j} x_j - \varrho)^2}{2\sigma_{t+j}^2}} dx_1 \cdots dx_{T-t} \\ = &\int \cdots \int_D (2\pi)^{-\frac{T-t}{2}} e^{-\sum_{j=1}^{T-t} \frac{(\sigma_{t+j} x_j - \varrho - \sigma_{t+j}^2)^2}{2\sigma_{t+j}^2}} dx_1 \cdots dx_{T-t} \end{aligned}$$

Define

$$u_j = x_j - \frac{\varrho}{\sigma_{t+j}(x)} - \sigma_{t+j}(x),$$

we get

$$N_1 = \int \cdots \int_{D_1} (2\pi)^{-\frac{T-t}{2}} e^{-\sum_{j=1}^{T-t} \frac{u_j^2}{2}} du_1 \cdots du_{T-t} = \Phi_{T-t}(D_1),$$

with the domain D_1 as stated in the theorem.

Similarly, by defining

$$v_j = x_j - \frac{\varrho}{\sigma_{t+j}(x)},$$

we have

$$N_2 = \int \cdots \int_D (2\pi)^{-\frac{T-t}{2}} e^{-\sum_{j=1}^{T-t} \frac{(\sigma_{t+j} x_j - \varrho)^2}{2\sigma_{t+j}^2}} dx_1 \cdots dx_{T-t}$$
$$= \int \cdots \int_{D_2} (2\pi)^{-\frac{T-t}{2}} e^{-\sum_{j=1}^{T-t} \frac{v_j^2}{2}} dv_1 \cdots dv_{T-t} = \Phi_{T-t}(D_2).$$

Thus, we concluded this theorem. *Q.E.D.*

Appendix B. Proof of Theorem 2.3

As the above case in which the return rate μ is constant, we can prove

(B.1) $$M_t \triangleq M_0 + \sum_{i=1}^{t} \sigma_i z_i,$$

(B.2) $$Z_t \triangleq \exp\left\{ \sum_{i=1}^{t} \frac{\varrho_i}{\sigma_i} z_i - \frac{1}{2} \sum_{i=1}^{t} \frac{\varrho_i^2}{\sigma_i^2} \right\}, \quad \varrho_i = -(\mu_i - \log r_i),$$

are martingales under measure $\mathbb{P}$.

Let measure $\mathbf{Q}$ be

(B.3) $$\mathbf{Q}(A) \triangleq \mathbb{E}^{\mathbb{P}}\left(\chi_A Z_t\right), \quad \forall A \in \mathcal{F}_t,$$

and

(B.4) $$\widehat{M_t} \triangleq M_t - \sum_{i=1}^{t} \varrho_i = M_0 + \sum_{i=1}^{t} \sigma_i z_i + \sum_{i=1}^{t} (\mu_i - \log r_i).$$

Proving line by line as the constant return rate case, we know that the stochastic processes $\widehat{M_t}$ and

$$\exp\left\{ \widehat{M_t} - \frac{1}{2} \sum_{i=1}^{t} \sigma_i^2 \right\}$$

are $\mathcal{F}_t$-martingales under measure $\mathbf{Q}$.

For the proof of Theorem 2.3, it is almost the same line-by-line with Theorem 2.1. So we omit the proofs of these theorems here. *Q.E.D.*

References

Aït-Sahalia Y., and J. Jacod (2014). High-Frequency Financial Econometrics. *Princeton University Press.*

Anyfantaki, S., and A. Demos (2016). Estimation and Properties of a Time-Varying EGARCH(1,1) in Mean Model. *Econometric Reviews*, **35**, 2, 293-310.

Black, F., and M. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81**, 3, 637-659.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, **31**, 3, 307-327.

Bollerslev, T., and H. Mikkelsen (1996). Modeling and Pricing Long Memory in Stock Market Volatility. *Journal of Econometrics*, **73**, 1, 151-184.

Brennan, M. (1979). The Pricing of Contingent Claims in Discrete Time Models. *Journal of Finance*, *34*, 1, 53-68.

Christensen, B., C. Dahl, and E. Iglesias (2012). Semiparametric Inference in a GARCH-in-mean Model. *Journal of Econometrics*, **167**, 2, 458-472.

Conrad, C., and E. Mammen (2016). Asymptotics for Parametric GARCH-in-Mean Models. *Journal of Econometrics*, **194**, 2, 319-329.

Cox, J., S. Ross, and M. Rubinstein (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, **7**, 3, 229-263.

Dias, G. (2017). The Time-varying GARCH-in-mean Model. *Economics Letters*, **157**, 129-132.

Duan, J.-C. (1995). The GARCH Option Pricing Model. *Mathematical Finance*, **5**, 1, 13-32.

Engle, R. F. (1982). Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation. *Econometrica*, **50**, 4, 987-1008.

Engle, R. F. (2002). New Frontiers for ARCH Models. *Journal of Applied Econometrics*, **17**, 425-446.

Engle, R. F., D. M. Lilien, and R. P. Robins (1987). Estimating the Time Varying Risk Premia in the Term Structure: The ARCH-M Model. *Econometrica*, **55**, 2, 391-407.

Engle, R. F., and C. Mustafa (1992). Implied ARCH Models from Options Prices. *Journal of Econometrics*, **52**, 289-311.

Fiorentini, G., E. Sentana, and N. Shephard (2004). Likelihood-Based Estimation of Latent Generalized ARCH Structures. *Econometrica*, **72**, 5, 1481-1517.

Francq, C., and J.-M. Zakoian (2010). GARCH Models: Structure, Statistical Inference and Financial Applications. *Wiley*.

Hull, J., and A. White (1987). The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance*, **42**, 2, 281-300.

Merton, R. C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, **4**, 1, 141-183.

Nelson, D. B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica*, **59**, 2, 347-370.

Noureldin, D., N. Shephard, and K. Sheppard (2012). Multivariate High-Frequency-Based Volatility (HEAVY) Models, *Journal of Applied Econometrics*, **27**, 6, 907-933.

Papadopoulos, Y., and A. Lewis (2018). A First Option Calibration of the GARCH Diffusion Model by a PDE Method, arXiv: 1801.06141.

Shephard, N., and K. Sheppard (2010). Realising the Future: Forecasting with High-Frequency-Based Volatility (HEAVY) Models, *Journal of Applied Econometrics*, **25**, 2, 197-231.

Zumbach, G., and L. Fernández (2013). Fast and Realistic European ARCH Option Pricing and Hedging. *Quantitative Finance*, **13**, 5, 713-728.

Zumbach, G., and L. Fernández (2014). Option Pricing with Realistic ARCH Processes. *Quantitative Finance*, **14**, 1, 143-170.