6th Homework

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Exercise 1

2, 3

Exercise 2

3

Exercise 3

4

Exercise 4

4

Exercise 5

1

Exercise 6

1

Exercise 7

1

Exercise 8

2

Exercise 9

1,4,5,7

Exercise 10

1,3

Exercise 11

4

Exercise 12

2,3

Exercise 13

The Lagrangian function is:
$L(P_1, P_2, P_3) = \sum_{i=1}^{N} \sum_{j=1}^{3} P(j|x_i) ln P_j + \lambda(\sum_{j=1}^{3} P_j - 1)$

Solving the equation $\frac{\partial(L(P_1,P_2,P_3))}{\partial(P_j)} = 0$
$\frac{1}{P_1} \sum_{i=1}^{N} P(1|x_i) + \lambda = 0 \Rightarrow P_1 = -\frac{1}{\lambda} \sum_{i=1}^{N} P(1|x_i)$ (13.1)
the same apply for the rest of the P_j

Using the equation $\sum_{j=1}^{3} P_j = 1$
$P_1 + P_2 + P_3 = 1 \Rightarrow -\frac{1}{\lambda} \sum_{i=1}^{N} P(1|x_i) + -\frac{1}{\lambda} \sum_{i=1}^{N} P(2|x_i) + -\frac{1}{\lambda} \sum_{i=1}^{N} P(3|x_i) = 1$

$\Rightarrow \lambda = -(\sum_{i=1}^{N} P(1|x_i) + \sum_{i=1}^{N} P(2|x_i) + \sum_{i=1}^{N} P(3|x_i)) = -\sum_{i=1}^{N} \sum_{j=1}^{3} P(j|x_i)$
(13.2)

Substituting 13.2 to 13.1 we have
$P_1 = -\frac{1}{\lambda} \sum_{i=1}^{N} P(1|x_i) = \frac{1}{\sum_{i=1}^{N} \sum_{j=1}^{3} P(j|x_i)} \sum_{i=1}^{N} P(1|x_i)$

it holds that $\sum_{j=1}^{3} P(j|x_i) = 1$ thus

$\Rightarrow P_1 = \frac{1}{\sum_{i=1}^{N} 1} \sum_{i=1}^{N} P(1|x_i) = \frac{1}{N} \sum_{i=1}^{N} P(1|x_i)$ generalising this expression
$\Rightarrow P_j = \frac{1}{N} \sum_{i=1}^{N} P(j|x_i)$

Exercise 14

Since we are working with normal distributions

$$p(x_i|j;\mu_j) = \frac{1}{(2\pi)^j|\Sigma_j|^{1/2}} exp\left(-\frac{(x-\mu_j)^T\Sigma_j^{-1}(x-\mu_j)}{2}\right)$$

for diagonal covariance matrix

$$p(x_i|j;\mu_j) = \prod_{i=1}^N \left(\frac{1}{\sqrt{2\pi\sigma_j^2}} exp(-\frac{(x_i-\mu_j)^2}{2\sigma_j^2})\right) \Rightarrow$$

$$\Rightarrow lnp(x_i|j;\mu_j) = -\sum_{i=1}^N \left(ln\frac{1}{\sqrt{2\pi\sigma_j^2}} + \frac{(x_i-\mu_j)^2}{2\sigma_j^2}\right)$$

$$\frac{\partial \sum_{i=1}^N P(j|x_i) ln(p(x_i|j;m_j)}{\partial\mu_j} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \sum_{i=1}^N P(j|x_i) ln(p(x_i|j;m_j)}{\partial\mu_j} = \frac{\partial \sum_{i=1}^N P(j|x_i)\left(-ln\frac{1}{\sqrt{2\pi\sigma_j^2}} + \frac{(x_i-\mu_j)^2}{2\sigma_j^2}\right)}{\partial\mu_j} = \sum_{i=1}^N P(j|x_i)\left(\frac{(x_i-\mu_j)}{\sigma_j^2}\right) =$$

$$\Rightarrow \sum_{i=1}^N P(j|x_i)x_i - \sum_{i=1}^N P(j|x_i)\mu_j = 0 \Rightarrow \sum_{i=1}^N P(j|x_i)x_i = \sum_{i=1}^N P(j|x_i)\mu_j \Rightarrow$$

$$\Rightarrow \mu_j = \frac{\sum_{i=1}^N P(j|x_i)x_i}{\sum_{i=1}^N P(j|x_i)}$$

Exercise 15

X_1 dataset Parametric approach

In [29]:
```
import scipy.io as sio
import numpy as np
import matplotlib.pyplot as plt
from sklearn.mixture import GaussianMixture
from mpl_toolkits import mplot3d
from sklearn.neighbors import KernelDensity

Dataset = sio.loadmat('Dataset.mat')
X1 = Dataset['X1']
X2 = Dataset['X2']
```

Our data looks like they are gathered around one cluster. Therefore in the Gaussian Mixture function we will set n_components equal to 1.

In [30]:
```
# Parametric approach

gm_1 = GaussianMixture(n_components=1, random_state=0).fit(X1)
mean = gm_1.means_
cov = gm_1.covariances_

x_1 = np.array([[2.01, 2.99, 3.98 , 5.02],[20.78, -15.26, 19.38, -25.02],[3.

for i, x_i in enumerate(x_1):
    cov_norm = np.linalg.norm(cov)**1/2
```

```
    cov_inv = np.linalg.inv(cov)
    x_mu = x_i - mean
    print(f"p(x{i+1}): = ", f"{(1/ ( (2*np.pi)**2 * cov_norm )) * np.exp(-0.
    (x_mu).dot(cov_inv).dot((x_mu).T))[0][0][0]:.4f}")
```

```
p(x1): =  0.0051
p(x2): =  0.0000
p(x3): =  0.0030
```

Non Parametric approach

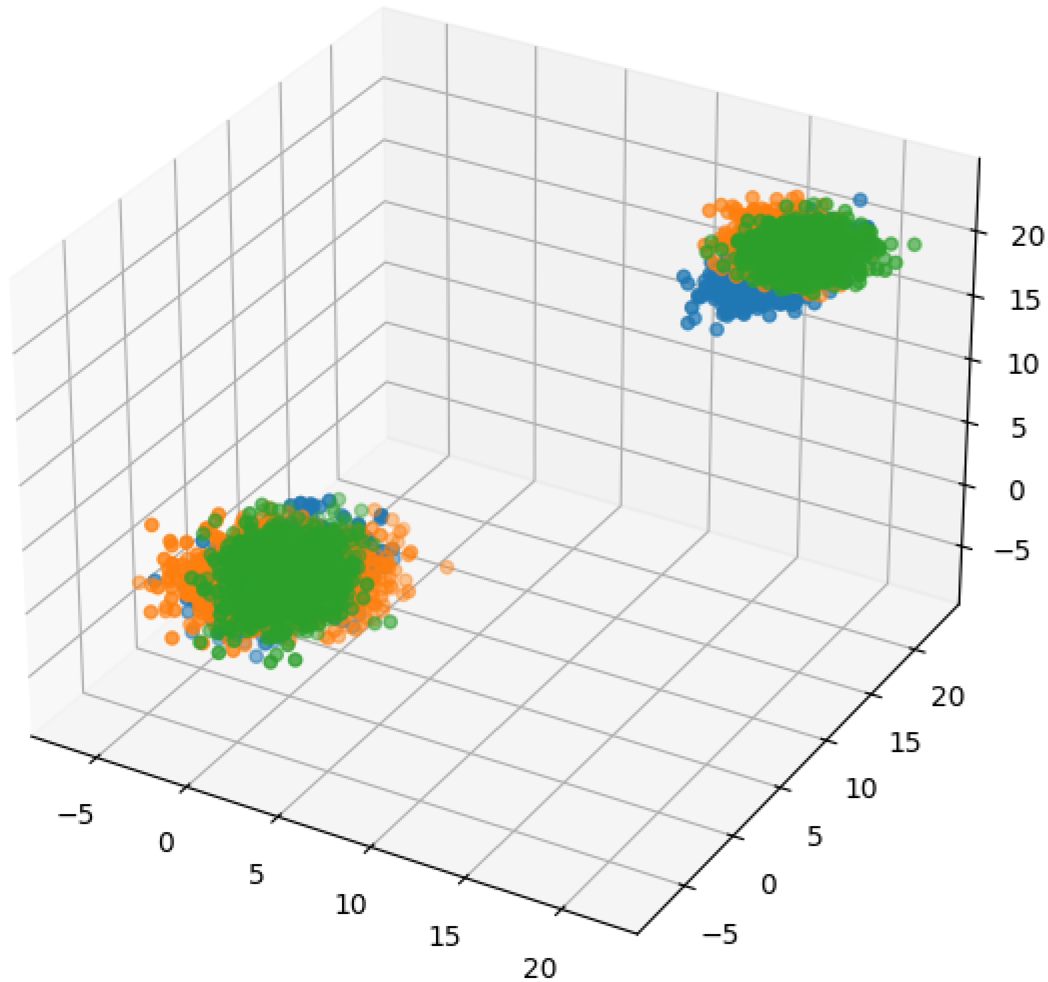
We will use the Krnel Density Method from scikit-learn library.

In [39]:
```
def parzen_window_est(x_samples, h, center):
    dimensions = x_samples.shape[1]

    assert (len(center) == dimensions), 'Number of center coordinates have t
    k = 0
    for x in x_samples:
        is_inside = 1
        for axis,center_point in zip(x, center):
            if np.abs(axis-center_point) > (h/2):
                is_inside = 0
        k += is_inside
    return f"{(k / len(x_samples)) / (h**dimensions):.4f}"

for i, x_i in enumerate(x_1):
    # print('p(x) =', parzen_window_est(X1, h=1))
    print(f"p(x{i+1}) = ", parzen_window_est(X1, 1, x_i))
```

```
p(x1) =  0.0060
p(x2) =  0.0000
p(x3) =  0.0020
```

X_2 dataset Parametric approach

In [26]:
```
fig = plt.figure(figsize = (10, 7))
ax = plt.axes(projection ="3d")
ax.scatter3D(X2[:,0], X2[:,1], X2[:,2])
ax.scatter3D(X2[:,0], X2[:,2], X2[:,3])
_ = ax.scatter3D(X2[:,1], X2[:,2], X2[:,3])
```



In [35]:
```
gm_2 = GaussianMixture(n_components=2, random_state=0).fit(X2)
mean = gm_2.means_
cov = gm_2.covariances_

x_2 = np.array([[0.05,0.15,-0.12,-0.08],[7.18,7.98,9.12,9.94],[3.48,4.01,4.5

for i, x_i in enumerate(x_2):
    p1 = gm_2.predict_proba([x_2[0]])[0][0]
    p2 = gm_2.predict_proba([x_2[0]])[0][1]

    cov_norm_1, cov_norm_2 = np.linalg.norm(cov[0])**1/2, np.linalg.norm(cov
    cov_inv_1, cov_inv_2 = np.linalg.inv(cov[0]), np.linalg.inv(cov[1])

    x_mu_1, x_mu_2 = x_i - mean[0], x_i - mean[1]
    p_x_1 = 1/ ( (2*np.pi)**2 * cov_norm_1 ) * np.exp(-0.5 * (x_mu_1).dot(co

    p_x_2 = 1/ ( (2*np.pi)**2 * cov_norm_2 ) * np.exp(-0.5 * (x_mu_2).dot(co

    P1_x = p1 * p_x_1 / (p1 * p_x_1 + p2 *p_x_2)
    P2_x = p2 * p_x_2 / (p1 * p_x_1 + p2 *p_x_2)
    print(f"p(x{i+1}): ", f"{p1 * p_x_1 + p2 *p_x_2:.4f}")
```

```
p(x1):  0.0050
p(x2):  0.0000
p(x3):  0.0000
p(x4):  0.0000
```

Non Parametric approach

We will use the Krnel Density Method from scikit-learn library.

In [40]:
```
for i, x_i in enumerate(x_2):
    print(f"p(x{i+1}) = ", parzen_window_est(X2, 1, x_i))
```

```
p(x1) =  0.0005
p(x2) =  0.0000
p(x3) =  0.0000
p(x4) =  0.0000
```