1st Homework

Excercise 1

(a)

To determine the parameters involved in the equations we will express them in the form $y = \begin{bmatrix}\theta_0 & \theta\end{bmatrix}\begin{bmatrix}1\\x\end{bmatrix}$ where the array $\begin{bmatrix}\theta_0 & \theta\end{bmatrix}$ include the parameters

$(i)\ \ y = \theta_0 + theta_1 x_1 + theta_2 x_2 + theta_3 x_1 x_2 + x_1^2 = \begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3 & 1\end{bmatrix}\begin{bmatrix}1\\x_1\\x_2\\x_1x_2\\x_1^2\end{bmatrix}$

$(ii)\ \ y = sign\left(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 x_2 + \theta_4 x_1^2\right) = sign\left(\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4\end{bmatrix}\begin{bmatrix}1\\x_1\\x_2\\x_1x_2\\x_1^2\end{bmatrix}\right)$

$(iii)\ \ y = 2x_1 + sign(3-7)x_2 + ReLU(3)x_1x_2 = 2x_1 + 0 + 3x_1x_2 = \begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_1x_2\end{bmatrix}$

$(iv)\ \ y = \theta + \theta x_1 + \theta x_2 + \theta x_1 x_2 = \begin{bmatrix}\theta & \theta & \theta & \theta\end{bmatrix}\begin{bmatrix}1\\x_1\\x_2\\x_1x_2\end{bmatrix}$

Thus the parameters for each equation are:

(i) 4 parameters (1 fixed constant): $\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3 & 1\end{bmatrix}$
(ii) 5 parameters : $\begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4\end{bmatrix}$
(iii) 0 parameters (2 fixed constants) : $\begin{bmatrix}2 & 3\end{bmatrix}$
(iv) 4 parameters : $\begin{bmatrix}\theta & \theta & \theta & \theta\end{bmatrix}$

(b)

Only the (iv) equation y is linear to θ as it can be written in the form

$$y = \theta\ \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\x_1\\x_2\\x_1x_2\end{bmatrix} = \theta\ (1 + x_1 + x_2 + x_1x_2)$$

Excercise 2

Based on the definition that a parametric model involves a set of parameters, whose number is fixed while non parametric models either involve no parameters or involves parameters, whose number depends on the size of the data set X:

- 1 and 3 are parametric models
- 2 has no parameters and 4 is not parametric as the parameter's number depend on the size of the data

Excercise 3

(a)

The general form of the qudratic equation is $f_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 = \begin{bmatrix}\theta_0 & \theta_1 & \theta_2\end{bmatrix} \begin{bmatrix}1 \\ x \\ x^2\end{bmatrix}$

Instance 1: $f_\theta(x) = 1 + 2x - 3x^2$, in this case $\theta = [1, 2, -3]$
Instance 2: $f_\theta(x) = 0 + 4x - 1.5x^2$, in this case $\theta = [0, 4, -1.5]$
The dimensionality of $\boldsymbol{\theta}$ is 3

(b)

The general form of the 3rd degree polynomials $f_\theta : R^2 \to R$ is
$f_\theta(x_1, x_2) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_1 x_2 + \theta_5 x_2^2 + \theta_6 x_1^3 + \theta_7 x_1^2 x_2 + \theta_8 x_1 x_2^2 + \theta_9 x_2^3$

$$= \begin{bmatrix}\theta_0 & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 & \theta_9\end{bmatrix} \begin{bmatrix}1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3\end{bmatrix}$$

Instance 1: $f_\theta(x) = 1 + 2x_1 + 3x_2 + 4x_1^2 + 5x_1 x_2 + 6x_2^2 + 7x_1^3 + 8x_1^2 x_2 + 9x_1 x_2^2 + 10x_2^3$, in this case $\theta = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$
Instance 2: $f_\theta(x) = 0 + 2x_1 - 3x_2 + 2x_1^2 + 5x_1 x_2 + x_2^2 - x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 + 3x_2^3$, in this case $\theta = [0, 2, -3, 2, 5, 1, -1, 2, -1, 3]$ The dimensionality of of $\boldsymbol{\theta}$ is 10

(c)

The general form of the 3rd degree polynomials $f_\theta : R^3 \to R$ is
$f_\theta(x_1, x_2, x_3) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_1^2 + \theta_5 x_1 x_2 + \theta_6 x_1 x_3 + \theta_7 x_2^2 + \theta_8 x_2 x_3 + \theta_9 x_3^2$
$+ \theta_{10} x_1^3 + \theta_{11} x_1^2 x_2 + \theta_{12} x_1^2 x_3 + \theta_{13} x_1 x_2^2 + \theta_{14} x_1 x_2 x_3 + \theta_{15} x_1 x_3^2 + \theta_{16} x_2^3 + \theta_{17} x_2^2 x_3 + \theta_{18} x_2 x_3^2$
$+ \theta_{19} x_3^3$

Instance 1:
$f_\theta(x_1, x_2, x_3) = 1 + 2x_1 + 3x_2 + 4x_3 + 5x_1^2 + 6x_1 x_2 + 7x_1 x_3 + 8x_2^2 + 9x_2 x_3 + 10x_3^2 + 11x_1^3$
$+ 12x_1^2 x_2 + 13x_1^2 x_3 + 14x_1 x_2^2 + 15x_1 x_2 x_3 + 16x_1 x_3^2 + 17x_2^3 + 18x_2^2 x_3 + 19x_2 x_3^2 + 20x_3^3$

Instance 2:
$f_\theta(x_1, x_2, x_3) = 0 + 2x_1 - 3x_2 + 2x_3 + 2x_1^2 + 5x_1 x_2 + x_1 x_3 - x_1^3 + 2x_1^2 x_2 + 3x_1^2 x_3 + x_1 x_2^2$
$+ 2x_1 x_2 x_3 + 3x_1 x_3^2 + x_2^3 + 2x_2^2 x_3 + x_2 x_3^2 + 3x_3^3$

The dimensionality of $\boldsymbol{\theta}$ is 20

(d)

The input space $\boldsymbol{x}$ is a vector in $\mathbb{R}^5$ thus a 5-dimensional vector. As such the vector $\boldsymbol{\theta}$ should be a 5-dimensional vector as well.

$\boldsymbol{x} = [\, x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T$
$\boldsymbol{\theta} = [\, \theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \,]^T$

Two instances of the parametric set could be:
$\boldsymbol{\theta} = [\, 1 \quad 2 \quad 3 \quad 4 \quad 5 \,]^T$
$\boldsymbol{\theta} = [\, 0.5 \quad -1 \quad 2 \quad 3 \quad -0.5 \,]^T$

The dimensionality of $\boldsymbol{\theta}$ is 5

(e)

(a), (b) and (c) can be written in the general form of $f_\theta(x) = (\theta)^T \boldsymbol{x}$ where $\boldsymbol{\theta} = [\, \theta_0 \quad \theta_1 \quad \theta_2 \quad \dots \quad \theta_N \,]^T$ and as such $f_\theta(x)$ is linear with respect to $\boldsymbol{\theta}$

Excercise 4

$\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_l]^T$
$\boldsymbol{x} = [x_1, x_2, \dots, x_l]^T$

$$(\boldsymbol{\theta}^T \boldsymbol{x})\boldsymbol{x} = (\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l)\boldsymbol{x} = \begin{bmatrix} x_1(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \\ x_2(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \\ \vdots \\ x_l(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \end{bmatrix}$$

$$(\boldsymbol{x}\boldsymbol{x}^T)\boldsymbol{\theta} = \begin{bmatrix} x_1^2 & x_1 x_2 & \dots & x_1 x_l \\ x_2 x_1 & x_2^2 & \dots & x_2 x_l \\ \vdots & \vdots & \ddots & \\ x_l x_1 & x_l x_2 & \dots & x_l^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_l \end{bmatrix} = \begin{bmatrix} x_1^2\theta_1 + x_1 x_2 \theta_2 + \dots + x_1 x_l \theta_l \\ x_2 x_1 \theta_1 + x_2^2 \theta_2 + \dots + x_2 x_l \theta_l \\ \vdots \\ x_l x_1 \theta_1 + x_l x_2 \theta_2 + \dots + x_l^2 \theta_l \end{bmatrix}$$

$$= \begin{bmatrix} x_1(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \\ x_2(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \\ \vdots \\ x_l(\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_l x_l) \end{bmatrix}$$

Thus $(\boldsymbol{\theta}^T \boldsymbol{x})\boldsymbol{x} = (\boldsymbol{x}\boldsymbol{x}^T)\boldsymbol{\theta}$

Excercise 5

(a)

$X^T X = \sum_{n=1}^{N} (\boldsymbol{x}_n \boldsymbol{x}_n^T)$

$$X^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{l1} \\ x_{12} & x_{22} & \dots & x_{l2} \\ \vdots & \vdots & \ddots & \\ x_{1N} & x_{N2} & \dots & x_{lN} \end{bmatrix} = [\, \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \dots \quad \boldsymbol{x}_N \,]$$

$$X^T X = [\, \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \dots \quad \boldsymbol{x}_N \,] \begin{bmatrix} \boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T \\ \vdots \\ \boldsymbol{x}_N^T \end{bmatrix} = \boldsymbol{x}_1\boldsymbol{x}_1^T + \boldsymbol{x}_2\boldsymbol{x}_2^T + \ldots + \boldsymbol{x}_N\boldsymbol{x}_N^T = \sum_{n=1}^{N}(\boldsymbol{x}_n\boldsymbol{x}_n^T)$$

$$X^T \boldsymbol{y} = [\, \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \dots \quad \boldsymbol{x}_N \,] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \boldsymbol{x}_1 y_1 + \boldsymbol{x}_2 + y_2 + \ldots + \boldsymbol{x}_N y_n =$$

since y_n is a scalar we can use the commutative property

$= y_1\boldsymbol{x}_1 + y_2\boldsymbol{x}_2 + \ldots + y_N\boldsymbol{x}_N = \sum_{n=1}^{N}(y_n\boldsymbol{x}_n)$

(b)

X size is (N x l)

$\boldsymbol{y}$ size is (N x 1)

$X^T X$ size is (l x N) x (N x l) = (l x l)

$X^T \boldsymbol{y}$ size is (l x N) x (N x 1) = (l x 1)

(c)

Since a column vector of 1's is added in from of the matrix X the new X table will be:

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1l} \\ 1 & x_{21} & x_{22} & \dots & x_{2l} \\ & & & & \\ 1 & \vdots & \vdots & \ddots & \\ 1 & x_{N1} & x_{N2} & \dots & x_{Nl} \end{bmatrix}$$ with dimensions ((N+1) x l)

(c-i)

thus the dimensionality of the b) will become

X size is ((N+1) x l)

$\boldsymbol{y}$ size is (N x 1)

$X^T X$ size is (l x (N+1)) x ((N+1) x l) = (l x l)

$X^T \boldsymbol{y}$ size is (l x N+1) x (N x 1) this multiplication is not valid!

(c-ii)

No the identities given in (a) do not hold. In detail

$X^T X = [\, \boldsymbol{x}_0 \quad \boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \dots \quad \boldsymbol{x}_N \,] = \sum_{n=0}^{N}(\boldsymbol{x}_n\boldsymbol{x}_n^T)$ assuming $\boldsymbol{x}_0 = [1, 1, \ldots, 1]$

The multiplication $X^T \boldsymbol{y}$ is not valid because of dimension missmatch (l x N+1) cannot be multiplied with (Nx1)

Excercise 5

(a)

The equation describing the velocity is $v = v_0 + at$.

We can express this set of equations using matrices as:

$y = \boldsymbol{\theta}^T \boldsymbol{x}$ or $\boldsymbol{y} = X\boldsymbol{\theta}$ where $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} v_0 \\ a \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} 5.1 \\ 6.8 \\ 9.2 \\ 10.9 \\ 13.1 \end{bmatrix}$

We need to solve the system:

$X^T X\boldsymbol{\theta} = X^T(y)$

$X^T X = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$

$X^T y = \begin{bmatrix} 45.1 \\ 172.1 \end{bmatrix}$

$\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} v_0 \\ a \end{bmatrix} = \begin{bmatrix} 45.1 \\ 155.4 \end{bmatrix}$

1st equation: $5v_0 + 15a = 45.1$
2nd equation: $15v_0 + 55a = 155.4$

Thus : $v_0 = 2.99$ and $a = 2.01$

(b)

$v = 2.99 + 2.01t$

(c)

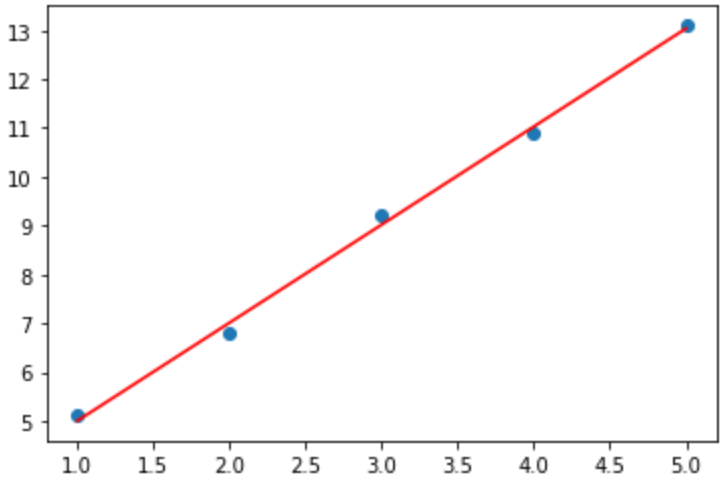
$v = 2.99 + 2.01 \cdot 2.3 = 7.613$

In [46]:

```
import matplotlib.pyplot as plt
x = np.array([1,2,3,4,5])
y = np.array([5.1,6.8,9.2,10.9,13.1])

y_2 = 2.99 + 2.01*x

plt.scatter(x, y)
_=plt.plot(x, y_2, color='red', linestyle='-', marker='', label='Line of Best Fit')
```



In [49]:

```
pip install nbconvert
```

```
Requirement already satisfied: nbconvert in /Users/fanis/opt/anaconda3/lib/python3.9/site-
packages (6.1.0)
Requirement already satisfied: pygments>=2.4.1 in /Users/fanis/opt/anaconda3/lib/python3.
9/site-packages (from nbconvert) (2.10.0)
Requirement already satisfied: bleach in /Users/fanis/opt/anaconda3/lib/python3.9/site-pac
kages (from nbconvert) (4.0.0)
Requirement already satisfied: jupyterlab-pygments in /Users/fanis/opt/anaconda3/lib/pytho
n3.9/site-packages (from nbconvert) (0.1.2)
Requirement already satisfied: traitlets>=5.0 in /Users/fanis/opt/anaconda3/lib/python3.9/
site-packages (from nbconvert) (5.1.0)
Requirement already satisfied: mistune<2,>=0.8.1 in /Users/fanis/opt/anaconda3/lib/python
3.9/site-packages (from nbconvert) (0.8.4)
Requirement already satisfied: entrypoints>=0.2.2 in /Users/fanis/opt/anaconda3/lib/python
3.9/site-packages (from nbconvert) (0.3)
Requirement already satisfied: nbformat>=4.4 in /Users/fanis/opt/anaconda3/lib/python3.9/s
ite-packages (from nbconvert) (5.1.3)
Requirement already satisfied: jinja2>=2.4 in /Users/fanis/opt/anaconda3/lib/python3.9/sit
e-packages (from nbconvert) (2.11.3)
Requirement already satisfied: testpath in /Users/fanis/opt/anaconda3/lib/python3.9/site-p
ackages (from nbconvert) (0.5.0)
Requirement already satisfied: pandocfilters>=1.4.1 in /Users/fanis/opt/anaconda3/lib/pyth
on3.9/site-packages (from nbconvert) (1.4.3)
Requirement already satisfied: defusedxml in /Users/fanis/opt/anaconda3/lib/python3.9/site
-packages (from nbconvert) (0.7.1)
Requirement already satisfied: nbclient<0.6.0,>=0.5.0 in /Users/fanis/opt/anaconda3/lib/py
thon3.9/site-packages (from nbconvert) (0.5.3)
Requirement already satisfied: jupyter-core in /Users/fanis/opt/anaconda3/lib/python3.9/si
te-packages (from nbconvert) (4.8.1)
Requirement already satisfied: MarkupSafe>=0.23 in /Users/fanis/opt/anaconda3/lib/python3.
9/site-packages (from jinja2>=2.4->nbconvert) (1.1.1)
Requirement already satisfied: async-generator in /Users/fanis/opt/anaconda3/lib/python3.
9/site-packages (from nbclient<0.6.0,>=0.5.0->nbconvert) (1.10)
Requirement already satisfied: nest-asyncio in /Users/fanis/opt/anaconda3/lib/python3.9/si
te-packages (from nbclient<0.6.0,>=0.5.0->nbconvert) (1.5.1)
Requirement already satisfied: jupyter-client>=6.1.5 in /Users/fanis/opt/anaconda3/lib/pyt
hon3.9/site-packages (from nbclient<0.6.0,>=0.5.0->nbconvert) (6.1.12)
Requirement already satisfied: python-dateutil>=2.1 in /Users/fanis/opt/anaconda3/lib/pyth
on3.9/site-packages (from jupyter-client>=6.1.5->nbclient<0.6.0,>=0.5.0->nbconvert) (2.8.
2)
Requirement already satisfied: tornado>=4.1 in /Users/fanis/opt/anaconda3/lib/python3.9/si
te-packages (from jupyter-client>=6.1.5->nbclient<0.6.0,>=0.5.0->nbconvert) (6.1)
Requirement already satisfied: pyzmq>=13 in /Users/fanis/opt/anaconda3/lib/python3.9/site-
packages (from jupyter-client>=6.1.5->nbclient<0.6.0,>=0.5.0->nbconvert) (22.2.1)
Requirement already satisfied: ipython-genutils in /Users/fanis/opt/anaconda3/lib/python3.
9/site-packages (from nbformat>=4.4->nbconvert) (0.2.0)
Requirement already satisfied: jsonschema!=2.5.0,>=2.4 in /Users/fanis/opt/anaconda3/lib/p
ython3.9/site-packages (from nbformat>=4.4->nbconvert) (3.2.0)
Requirement already satisfied: six>=1.11.0 in /Users/fanis/opt/anaconda3/lib/python3.9/sit
e-packages (from jsonschema!=2.5.0,>=2.4->nbformat>=4.4->nbconvert) (1.16.0)
Requirement already satisfied: attrs>=17.4.0 in /Users/fanis/opt/anaconda3/lib/python3.9/s
ite-packages (from jsonschema!=2.5.0,>=2.4->nbformat>=4.4->nbconvert) (21.2.0)
Requirement already satisfied: pyrsistent>=0.14.0 in /Users/fanis/opt/anaconda3/lib/python
3.9/site-packages (from jsonschema!=2.5.0,>=2.4->nbformat>=4.4->nbconvert) (0.18.0)
Requirement already satisfied: setuptools in /Users/fanis/opt/anaconda3/lib/python3.9/site
-packages (from jsonschema!=2.5.0,>=2.4->nbformat>=4.4->nbconvert) (58.0.4)
Requirement already satisfied: webencodings in /Users/fanis/opt/anaconda3/lib/python3.9/si
te-packages (from bleach->nbconvert) (0.5.1)
Requirement already satisfied: packaging in /Users/fanis/opt/anaconda3/lib/python3.9/site-
packages (from bleach->nbconvert) (23.2)
Note: you may need to restart the kernel to use updated packages.
```

In [50]:

```
pip install pyppeteer
```

```
Collecting pyppeteer
  Downloading pyppeteer-1.0.2-py3-none-any.whl (83 kB)
```

```
     |████████████████████████████████| 83 kB 1.2 MB/s eta 0:00:01
Collecting pyee<9.0.0,>=8.1.0
  Downloading pyee-8.2.2-py2.py3-none-any.whl (12 kB)
Requirement already satisfied: importlib-metadata>=1.4 in /Users/fanis/opt/anaconda3/lib/p
ython3.9/site-packages (from pyppeteer) (4.8.1)
Requirement already satisfied: appdirs<2.0.0,>=1.4.3 in /Users/fanis/opt/anaconda3/lib/pyt
hon3.9/site-packages (from pyppeteer) (1.4.4)
Collecting websockets<11.0,>=10.0
  Downloading websockets-10.4-cp39-cp39-macosx_10_9_x86_64.whl (97 kB)
     |████████████████████████████████| 97 kB 5.8 MB/s eta 0:00:01
Requirement already satisfied: tqdm<5.0.0,>=4.42.1 in /Users/fanis/opt/anaconda3/lib/pytho
n3.9/site-packages (from pyppeteer) (4.62.3)
Requirement already satisfied: urllib3<2.0.0,>=1.25.8 in /Users/fanis/opt/anaconda3/lib/py
thon3.9/site-packages (from pyppeteer) (1.26.7)
Requirement already satisfied: certifi>=2021 in /Users/fanis/opt/anaconda3/lib/python3.9/s
ite-packages (from pyppeteer) (2021.10.8)
Requirement already satisfied: zipp>=0.5 in /Users/fanis/opt/anaconda3/lib/python3.9/site-
packages (from importlib-metadata>=1.4->pyppeteer) (3.6.0)
Installing collected packages: websockets, pyee, pyppeteer
Successfully installed pyee-8.2.2 pyppeteer-1.0.2 websockets-10.4
Note: you may need to restart the kernel to use updated packages.
```

In [ ]: