Homework V - George Triantis

Exercise 1: 2 → True
3 → True

Exercise 2: 1 → True

Exercise 3: 3 → True

Exercise 4: 2 → True

Exercise 5: 4 → Wrong

Exercise 6: 1, 3, 4 → True

Exercise 7: 1, 2 → True

Exercise 8 → 2

Exercise 9 → 4

Exercise 10: 2 → True

Exercise 11 → 2, 3 → True

Exercise 12: $p(x) = \theta^2 x e^{-\theta x} u(x)$, $u(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}$

Log-Likelihood: $L(\theta) = \sum_{i=1}^{N} \ln(p(x_i;\theta)) =$

$$= \sum_{i=1}^{N} \ln(\theta^2 x e^{-\theta x} u(x)) =$$

$$= \sum_{i=1}^{N} \left[2\ln\theta + \ln(x_i) - \theta x_i + \ln(u(x_i)) \right]$$

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I consider $\theta > 0$

$$\frac{\partial L(\theta)}{\partial \theta} = 0$$

$$\frac{\partial}{\partial \theta}\left[\sum_{i=1}^{N} 2\ln\theta + \cancel{\ln(x_i)}^{0} - \theta x_i + \cancel{\ln(u(x_i))}^{0}\right] = 0$$

$$\sum_{i=1}^{N}\left(2\frac{\partial}{\partial\theta}\ln\theta - x_i\frac{\partial}{\partial\theta}\theta\right) = 0$$

$$\overset{\theta>0}{\Rightarrow} \sum_{i=1}^{N}\frac{2}{\theta} = \sum_{i=1}^{N} x_i$$

$$\frac{1}{\hat{\theta}_{ML}} \cdot 2N = \sum_{i=1}^{N} x_i$$

$$\boxed{\hat{\theta}_{ML} = \frac{2N}{\sum_{i=1}^{N} x_i}}$$

b) $N=5$, $\hat{\theta}_{ML} = \frac{2N}{\sum_{i=1}^{N} x_i} = \frac{10}{2+2,2+2,7+2,4+2,6} \simeq 0,84$

For $\theta = \hat{\theta}_{ML}$: $\hat{p}(x) = \begin{cases} (0,84)^2 \cdot 2,3\, e^{-(0,84\cdot 2,3)} \simeq 0,235, & x=2,3 \\ (0,84)^2 \cdot 2,9\, e^{-(0,84\cdot 2,9)} \simeq 0,179, & x=2,9 \end{cases}$

Exercise 13: $p(x;\theta) = 2\theta x e^{-\theta x^2} u(x)$, $u(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 0 \end{cases}$

a) $\hat{\theta}_{MAP} = \operatorname{argmax}[p(\theta)\cdot p(Y|\theta)] = \operatorname{argmax}[\ln(p(\theta)p(Y|\theta))]$

$$p(\theta)\,p(Y|\theta) = \ln\left[\frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\left(\frac{(\theta-\theta_0)^2}{2\sigma_0^2}\right)} \prod_{i=1}^{N} 2\theta x e^{-\theta x^2} u(x)\right]$$

$$p(\theta)p(Y|\theta) = \ln\frac{1}{\sqrt{2\pi}\sigma_0} - \frac{(\theta-\theta_0)^2}{2\sigma_0^2} + \sum_{i=1}^{N} \ln\left(2\theta x e^{-\theta x^2} u(x)\right) =$$

$$= \ln\frac{1}{\sqrt{2\pi}\sigma_0} - \frac{(\theta-\theta_0)^2}{2\sigma_0^2} + N\ln 2\theta + \sum_{i=1}^{N} \ln x_i - \theta\sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{N} \ln($$

I have $\ln u(x)$, so I assumed that $x > 0$, so $\ln(u(x)) = 1$

$$\ln\left(p(\theta)p(Y|\theta)\right) = \ln\frac{1}{\sqrt{2\pi}\sigma_0} - \frac{(\theta-\theta_0)^2}{2\sigma_0^2} + N\ln 2\theta + \sum_{i=1}^{N} \ln(x_i) - \theta\sum_{i=1}^{N} (x_i^2)$$

$$\frac{\partial}{\partial\theta}\ln\left(p(\theta)p(Y|\theta)\right) = -\frac{\theta-\theta_0}{\sigma_0^2} + \frac{N}{\theta} - \sum_{i=1}^{N} x_i^2 = 0$$

$$\frac{-\theta^2\cancel{\sigma_0^2} - \theta_0\cancel{\sigma_0^2}\theta}{\sigma_0^2} + N\sigma_0^2 - \theta\sigma_0^2\sum_{i=1}^{N} x_i^2 = 0$$

$$-\theta^2 + \theta_0\theta - \left(\sigma_0^2\sum_{i=1}^{N} x_i^2\right)\theta + N\sigma_0^2 = 0$$

$$\theta^2 + \left[\sigma_0^2\sum_{i=1}^{N} x_i^2 - \theta_0\right]\theta - \frac{N}{\sigma_0^2} = 0$$

$$\Delta = \left(\sigma_0^2\sum_{i=1}^{N} x_i^2 - \theta\right)^2 - 4N\sigma_0^2$$

$$\theta_{1,2} = \frac{\theta_0 - \sigma_0^2\sum_{i=1}^{N} x_i^2 \pm \sqrt{\left(\sigma_0^2\sum_{i=1}^{N} x_i - \theta\right)^2 - 4N\sigma_0^2}}{2}$$

b) i) $\theta_{1,2} = \dfrac{-\dfrac{\sigma_0^2 \sum_{i=1}^{N} x_i^2}{N} + \dfrac{\theta_0}{N} \pm \sqrt{\left[\dfrac{\sigma_0^2 \sum_{i=1}^{N} x_i - \theta_0}{N}\right]^2 + \dfrac{4\sigma_0^2}{N^2}}}{\dfrac{2}{N}}$

$N \to \infty$, $\theta_{MAP} \to 0$

ii) $\sigma_0^2 >> 1$ $\quad \theta_{1,2} = \dfrac{\dfrac{\theta_0}{\sigma_0^2} - \sum_{i=1}^{N} x_i^2 \pm \sqrt{\left(\sum_{i=1}^{N} x_i - \dfrac{\theta_0}{\sigma_0^2}\right)^2 + 4N}}{\dfrac{\theta_0}{\sigma_0^2}}$

$\hat{\theta}_{MAP} \to \infty$

iii) $\sigma_0^2 << 1$ $\quad \hat{\theta}_{MAP} \to 0$

c) Which exercise 1?

d) $\theta_{MAP} = \begin{cases} \theta_1 \simeq -28.82 \\ \theta_2 \simeq 0.17 \end{cases}$

Exercise 14 a) $p(X|\lambda) = \prod_{i=1}^{N} p(x_1, \dots x_N|\lambda) = \prod_{i=1}^{N} p(x_i;\lambda) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i}$

$p(X|\lambda) = \begin{cases} \lambda^N e^{-\sum_{i=1}^{N} \lambda x_i} & x \geq 0 \\ 0 & , x < 0 \end{cases}$

b) $x, \lambda \geq 0$

$p(\lambda)\, p(X|\lambda) = \lambda^N e^{-\sum_{i=1}^{N} \lambda x_i} \dfrac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$

$$\ln\left(p(Y|\lambda)p(\lambda)\right) = \ln\left[\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\lambda^N e^{-\sum_{i=1}^{N}\lambda x_i}\right] =$$

$$= a\ln b - \ln\Gamma(a) + (a-1)\ln\lambda - b\lambda + N\ln\lambda - \lambda\sum_{i=1}^{N} x_i$$

$$\frac{\partial}{\partial\lambda}\left(\ln\left(p(Y|\lambda)p(\lambda)\right)\right) = \frac{a-1}{\lambda} - b + \frac{N}{\lambda} - \sum_{i=1}^{N} x_i = 0$$

$$\frac{a-1+N}{\lambda} = \sum_{i=1}^{N} x_i + b$$

$$\hat{\lambda}_{MAP} = \lambda = \frac{a-1+N}{\sum_{i=1}^{N} x_i + b}$$

c) $$\hat{p}(x) = \begin{cases} \hat{\lambda}_{MAP}\, e^{-\hat{\lambda}_{MAP} x} & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} \frac{a-1+N}{\sum_{i=1}^{N} x_i + b}\, e^{-\frac{a-1+N}{\sum x_i + b}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

d) $$p(\lambda|Y) = \frac{p(\lambda)\,p(Y|\lambda)}{P(Y)} = \underbrace{\frac{b^a}{P(Y)\Gamma(a)}}_{=C}\,\lambda^{\overbrace{a-1+N}^{r}}\, e^{-\underbrace{\left(\sum_{i=1}^{N} x_i + b\right)}_{s}\lambda}$$

Hint ii) Gamma distribution: $C\lambda^r e^{-s\lambda}$

$$p(\lambda|Y) = \frac{s^r}{\Gamma(r)}\lambda^{r-1}e^{-s\lambda}$$

Exercise 15 :

$$L(\theta) = \sum_{n=1}^{N} (x_n - \theta)^2 + \lambda\left((\theta - \theta_0)^2 - \rho\right)$$

$$\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\sum_{n=1}^{N} (x_n - \theta)^2 + \lambda\left((\theta - \theta_0)^2 - \rho\right)\right) =$$

$$= -2 \sum_{n=1}^{N} (x_n - \theta) + 2\lambda(\theta - \theta_0) =$$

$$= \cancel{2}\lambda\theta - \cancel{2}\lambda\theta_0 + \cancel{2}N\theta - \cancel{2}\sum_{n=1}^{N} x_n = 0$$

$$\theta(N + \lambda) = \lambda\theta_0 + \sum_{n=1}^{N} x_n$$

$$\hat{\theta}_{RR} = \theta = \frac{\lambda\theta_0 + \sum_{n=1}^{N} x_n}{N + \lambda}$$