Solving the Incomplete Markets Model using the Krusell-Smith and Reiter Methods

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1 Outline

This note aims to provide a simple introduction to solving the incomplete markets model using the methods of both Krusell & Smith (1998) and Reiter (2009). In particular, I have been unable to find a simple introduction to the latter.

To solve the model with these methods, I have built on the framework provided by Mongey (2015) which heavily utilizes the CompEcon toolbox of Miranda & Fackler, but concentrates on solution methods for models without aggregate uncertainty. I have provided Matlab code that solves the model using both methods on GitHub.

I start by solving for the stationary equilibrium of the model, primarily as it provides a key input for the Reiter method when aggregate shocks are introduced. I then show how to solve the full model using both the Krusell-Smith and Reiter methods. As there exist fewer introductions to the latter, I explain explicitly how to set up the "finite representation" of the model so that, when it is linearized, it is in the form considered by Sims (2001), and I can solve it in Matlab using gensys.

2 Model

I will consider the simple incomplete markets model from the User Guide accompanying Winberry (2016). There is a continuum of households with preferences:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t \log c_{it}$$

Each household supplies ϵ_{it} efficiency units of labor inelastically. ϵ_{it} follows a two-state Markov process: $\epsilon_{it} \in \{0, 1\}$. Households with $\epsilon_{it} = 1$ receive after-tax earnings $w_t(1-\tau)$, while households with $\epsilon_{it} = 0$ receive unemployment benefits bw_t. Asset markets are incomplete: households can only trade in capital subject to a no-borrowing constraint. Consequently, the household faces the following two constraints:

$$
\begin{aligned}
c_{it} + a_{i,t+1} &= w_t(1-\tau)\epsilon_{it} + bw_t(1-\epsilon_{it}) + R_t a_{it} \\
a_{i,t+1} &\geq 0
\end{aligned}
$$

The government's budget is balanced each period, implying $\tau = \frac{b(1-L)}{L}$ where L is the mass of households with $\epsilon_{jt} = 1$.

There is a representative firm with the production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

The wage and return on capital are:

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$R_t = \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} + 1 - \delta$$

Aggregate productivity follows an AR(1) process in logs:

$$\log A_{t+1} = \rho_A \log A_t + \epsilon^A_{t+1}\ ,\ \epsilon^A_{t+1} \sim N(0, \sigma_A^2)$$

3 Stationary Equilibrium

First, consider a version of the model without aggregate uncertainty (i.e. $A_t = 1\ \forall t$). In this case, the individual state variables for the household are (a, ϵ). The recursive form of the household's problem is:

$$
\begin{aligned}
V(a, \epsilon) = \max_{a'} &\{u(w(K)(1-\tau)\epsilon + bw(K)(1-\epsilon) + R(K)a - a') + \beta \mathbb{E} V(a', \epsilon')\} \\
\text{s.t. } & a' \geq 0
\end{aligned}
$$

The algorithm for solving for a stationary equilibrium is as follows:

1. Guess K_0

2. Use firm FOCs and $L_t = L$ to calculate $w(K_0)$ and $R(K_0)$
3. Given prices, solve the household's problem for the value function and policy functions.
4. Given policy functions, find the invariant distribution of household's over individual states.
5. Use invariant distribution to calculate aggregate capital. If different from K_0, update guess (using bisection) and return to step 1.

3.1 Solving the household's problem

To compute the stationary equilibrium, I will split the value function into two:

$$\begin{aligned} V(a,\epsilon) &= \max_{a'} \{u(w(K)(1-\tau)\epsilon + bw(K)(1-\epsilon) + R(K)a - a') + \beta V^e(a',\epsilon)\} \\ V^e(a,\epsilon) &= \sum_{\epsilon'} \pi(\epsilon'|\epsilon)V(a,\epsilon') \end{aligned}$$

I will approximate both value functions with cubic splines for each level of employment and solve for the value function coefficients using collocation. Let s be the set of collocation nodes for the approximation: s is an $N_k \times 2$ matrix, where N_k is the number of grid points for capital. Denote the first column (capital) as s_1 and the second column (employment) as s_2. Then we can stack the two functional equations as:

$$\begin{aligned} \Phi(s)c &= \max_{a' \in B(s)} \{u(w(K)(1-\tau)s_2 + bw(K)(1-s_2) + R(K)s_1 - a') + \beta\Phi([a',s_2])c^e\} \\ \Phi(s)c^e &= (P \otimes I_{N_k})\Phi(s)c \end{aligned}$$

where $\Phi(s)$ is the basis matrix, evaluated at s.[1] We can actually get away with only approximating the expected value function, as we can substitute the first equation into the second:

$$\Phi(s)c^e = (P \otimes I_{N_k}) \max_{a' \in B(s)} \left\{u(w(K)(1-\tau)s_2 + bw(K)(1-s_2) + R(K)s_1 - a') + \beta\Phi([a',s_2])c^e\right\}$$

As the above value function is linear in the coefficients, the household's problem can be solved quickly using an updating scheme based on Newton's method, giving us $c^e(K)$.

[1] If either of the above equations is unclear, see Mongey (2015).

3.2 Constructing the invariant distribution

To solve for the stationary distribution over idiosyncratic states, we solve for the policy functions again, this time using a finer grid (with N_{kf} points) over individual capital holdings. We can then use the Young (2010) non-stochastic simulation method to create a transition matrix, Q. Initializing the distribution over the finer set of idiosyncratic states s^f at λ_0, we can then find the stationary distribution by iterating on the following equation until convergence:

$$\lambda_{t+1} = Q'\lambda_t$$

(a similar transitional equation for the distribution will be one of the key inputs for the Reiter method).

Once we have found the stationary distribution we can proceed with step 5 of the above algorithm for finding the stationary distribution. I have solved for the stationary distribution using the parameters in Table 1 (the same as those used in Winberry (2016)). Figure 1 plots the consumption and saving decision rules for both employed and unemployed workers, as well as the distribution of households over asset levels for each employment level.

4 Aggregate Uncertainty

Now lets turn on aggregate uncertainty. To the individual states used above, we need to add the aggregate states (A, λ) (where λ is the measure of households across individual states).

The household recursive problem is:

$$\begin{aligned} V(a, \epsilon; A, \lambda) &= \max_{c,a'} \left\{ u(c) + \beta \mathbb{E} V(a', \epsilon'; A', \lambda') \right\} \\ &s.t. \\ c + a' &= w(A, \lambda)(1-\tau)\epsilon + bw(A, \lambda)(1-\epsilon) + R(A, \lambda)a \\ a' &\geq 0 \\ \lambda' &= \Psi(A, \lambda) \end{aligned}$$

where the final equation is the law of motion of the distribution.

This law of motion is required for the household's problem as they need to be able to forecast λ' in order to forecast the next period's prices. The well known computational issue with solving this model is that λ is an infinite-dimensional object.

5 Krusell-Smith Method

The method proposed by Krusell & Smith (1998) is to assume a degree of "bounded rationality" with regards to how agents perceive the evolution of λ over time. In practice, this means reducing the distribution to a small set of moments $\mathbf{m}$ and then specifying a functional form for the new law of motion:

$$\mathbf{m}' = \Psi(A, \mathbf{m})$$

Krusell-Smith show that using just the first moment (i.e. aggregate capital) is enough. In their paper, aggregate productivity follows a Markov process. Here I will leave aggregate productivity as a continuous variable, in order to facilitate comparison of the Reiter and Krusell-Smith solutions. Consequently, I will use a forecasting equation of the following form:

$$\log K' = b^0 + b^1 \log K + b^2 \log A + b^3 \log K \log A$$

Given this forecasting rule, the household's "boundedly-rational" problem is:

$$\begin{aligned}
V(a, \epsilon; A, K) &= \max_{c,a'} \left\{ u(c) + \beta \mathbb{E} V(a', \epsilon'; A', K') \right\} \\
&\quad s.t. \\
c + a' &= w(A, K)(1 - \tau)\epsilon + bw(A, K)(1 - \epsilon) + R(A, K)a \\
a' &\geq 0 \\
\log K' &= b^0 + b^1 \log K + b^2 \log A + b^3 \log K \log A
\end{aligned}$$

The algorithm to solve the model then involves iterating on the guessed coefficients for the forecasting rule for capital:

1. Guess $\{b^0, b^1, b^2, b^3\}$
2. Given forecasting rule, solve the household's problem to get policy function $a'(a, \epsilon; A, K)$ and value function $V(a, \epsilon; A, K)$
3. Simulate the economy using non-stochastic simulation for T periods. Discard the first T^0 periods. Using the remaining sequence, run the regression:

   $$\log K' = \beta^0 + \beta^1 \log K + \beta^2 \log A + \beta^3 \log K \log A$$

4. Compare $\{b^0, b^1, b^2, b^3\}$ and $\{\beta^0, \beta^1, \beta^2, \beta^3\}$. If close enough, stop. If not, update coefficients and return to step 1.

5.1 Solving the household's problem

To ensure that the solution using the KS method is comparable to the solution using the Reiter method, I use a quadrature method to approximate the distribution of aggregate productivity shocks. CompEcon's qnwnorm provides nodes $\boldsymbol{\eta}$ and weights $\boldsymbol{f}$ to approximate a normal distribution. I will approximate both the value function and the expected value function, where the latter now takes expectations over *both* the idiosyncratic and aggregate productivity shocks.

The size of our matrix s of collocation nodes is now $(N_k \times 2 \times N_K \times N_A)$ by 4. The relevant functional equations are now [2]:

$$\Phi(s)c = \max_{a'} \left\{ u(w(s_3, s_4)(1-\tau)s_2 + bw(s_3, s_4)(1-s_2) + R(s_3, s_4)s_1 - a') + \beta\Phi([a', s_2, K', s_4])c^e \right\}$$

$$\Phi(s)c^e = (I_{N_A} \otimes I_{N_K} \otimes P \otimes I_{N_k})(I_N \otimes f')\Phi([s_1 \otimes i_{N_\epsilon}, s_2 \otimes i_{N_\epsilon}, s_3 \otimes i_{N_\epsilon}, g(s_4 \otimes i_{N_\epsilon}, i_N \otimes \eta)])c$$

The relevance of the guessed coefficients of the law of motion for $\log K$ is seen in the determination of K' at the end of the first equation, where:

$$K' = \exp(b^0 + b^1 \log s_3 + b^2 \log s_4 + b^3 \log s_3 \log s_4)$$

The $g(\cdot, \cdot)$ function returns the next periods aggregate productivity level, given a productivity level today, A, and a productivity shock, η:

$$\tilde{g}(A, \eta) = \exp(\rho_A \log(A) + \eta)$$
$$g(A, \eta) = \max(\min(\tilde{g}(A, \eta), \bar{A}), \underline{A})$$

(The second equation above ensures that A remains on the chosen grid.)

We solve the household's problem using Bellman/Newton iterations to find $c(b)$ and $c^e(b)$.

5.2 Non-Stochastic Simulation

To simulate the model, initialize the idiosyncratic distribution over s^f at λ_0 (the simplest initialization is a uniform distribution, but using the stationary distribution is better as we can get away with a shorter burn-in period and a tighter grid for K). From λ_0 we can find K_0, using:

$$K_0 = \lambda_0' s_1^f$$

We then solve for the policy functions on this fine grid:

$$\max_{a'} \left\{ u(w(K_t, A_t)(1-\tau)s_2^f + bw(K_t, A_t)(1-s_2^f) + R(K_t, A_t)s_1^f - a') + \beta\Phi([a', s_2^f, \mathbb{E}_t[K_{t+1}], A_t])c^e \right\}$$

This gives us $a'(s_1^f, s_2^f, K_t, A_t)$. Note that, while we are using a finer grid for idiosyncratic capital levels, we are only solving for the policy function at one particular level of capital and aggregate productivity.

[2] If these are unclear, consult Mongey (2015).

5.2.1 Envelope Condition Method

If we had to do this maximisation step using golden section search at every t, the KS algorithm would be slow. We can avoid this by using the Envelope Condition Method:

The FOCs of the household's problem imply that

$$V_1(a, \epsilon; A_t, K_t) = u'(c)R_t$$

We can use CompEcon's funeval function to find the derivative of the value function, and then rearrange this equation to solve for $c(a, \epsilon; A_t, K_t)$. We can then find $a'(a, \epsilon; A_t, K_t)$ from the budget constraint. This finds the household's policy function while avoiding a more computationally costly maximization step.[3]

We can compute our transition matrix Q_t using $a'(s_1^f, s_2, K_t, A_t)$, and then update λ_{t+1} and consequently K_{t+1}.

Once we have simulated the model for T periods, we discard the first T^0 as a "burn-in" and then run our regression on the aggregate capital and productivity series from the remaining periods.

6 Reiter Method

The Reiter method combines elements of projection and perturbation methods. Specifically, the idea is to solve for the stationary equilibrium of the model without aggregate shocks as above, and then to take a first-order perturbation with respect to the aggregate shocks.

While the model being considered here is simple enough to solve fairly quickly using the KS method, the Reiter approach can deal with a large number of aggregate shocks, and can bring the model within reach of formal estimation (e.g. Mongey & Williams (2017)).

An overview of the Reiter method is as follows:

1. Set up finite representation of the model. In particular this means approximating value or policy functions of the households, as well as the distribution (and its transition) over idiosyncratic states.
2. Solve for the steady-state of the model without aggregate uncertainty.
3. Calculate a linear perturbation of the finite representation with respect to aggregate shocks.

[3] Another alternative here whould be to approximate the policy function directly, after solving the household's problem.

I will construct a finite representation using the same solution method as used in Section 2. In particular, I will approximate the expected value function and use a histogram method to approximate the distribution. Note that policy functions can then be constructed from the approximated value function.

An important point to realize is that the coefficients of the value function and the distribution over individual states will be included as variables in the perturbation stage of the solution.

6.1 Finite approximation

The finite approximation resembles the key equations of the stationary version of the model, but now incorporates aggregate uncertainty through a dependence on t.

I have written the model in the form required for the Sims (2001) method for solving rational expectations models[4] This requires models of the form:

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi \eta(t)$$

where $z(t)$ are exogenous disturbances and $\eta(t)$ are expectational errors satisfying $\mathbb{E}_{t-1}\eta(t) = 0$. To get the model in this form, the expectation at $t-1$ of the value function coefficients at period t must be included in the $y(t-1)$ vector, and we then add equations defining the "expectational error".

Given this, the finite approximation of the model I will use is:

$$\begin{aligned}
\Phi(s)c_{t-1} &= (P \otimes I_{N_k}) \max_{a' \in B(s)} \left\{ u(w_{t-1}(1-\tau)s_2 + bw_{t-1}(1-s_2) + R_{t-1}s_1 - a') + \beta\Phi([a', s_2])\mathbb{E}_{t-1}c_t \right\} \\
\lambda_t &= Q'_{t-1}\lambda_{t-1} \\
\log A_t &= \rho_A \log A_{t-1} + \epsilon_t^A \\
c_t &= \mathbb{E}_{t-1}c_t + \eta_t
\end{aligned}$$

At first glance, this might seem to be not enough equations. However, remember that the distribution λ_{t-1} determines K_{t-1}, and that K_{t-1} implies R_{t-1} and w_{t-1} from the FOCs of the representative firm.

I do not need to include these equations as separate inputs for the finite approximation, and consequently do not need to linearize them, as long as when perturbing the model with respect to λ_{t-1}, I take into account the effect that this has on w_{t-1} and R_{t-1}.

[4] There are a number of other solution methods for linear rational expectations models that also could have been used, e.g. Klein (2000).

The above system is then $N_s = 2N_k + 2N_{kf} + 1 + 2N_k$ equations in the same number of unknowns.

I will write this non-linear rational expectations system as

$$F(X_t, X_{t-1}, \epsilon_t, \eta_t) = 0$$

where $F(\cdot)$ is the LHS minus the RHS of each of the above equations, and

$$X_t = \{c_t, \lambda_t, \log A_t, \mathbb{E}_t c_{t+1}\}$$

F is satistfied exactly when $X_t = X_{t-1} = X_{SS}$ and $\eta_t = \epsilon_t = 0$.

The next step is to linearize the system, by differentiating F with respect to each of its arguments, evaluated at the steady-state. The linear approximation of the system is then:

$$F_1(X_t - X_{SS}) + F_2(X_{t-1} - X_{SS}) + F_3\epsilon_t + F_4\eta_t = 0$$

where $F_1 = \frac{\partial F}{\partial X_t}$, $F_2 = \frac{\partial F}{\partial X_{t-1}}$, $F_3 = \frac{\partial F}{\partial \epsilon_t}$, $F_4 = \frac{\partial F}{\partial \eta_t}$.

F_3 and F_4 can be calculated exactly, due to the simple way in which ϵ and η enter F. For F_1 and F_2, I use forward differentiation from the steady-state with relative step size $1e^{-6}$. We can then map these matrices into those required by gensys:

$$\begin{aligned}
\Gamma_0 &= -F_1 \\
\Gamma_1 &= F_2 \\
C &= 0 \\
\Psi &= F_3 \\
\Pi &= F_4
\end{aligned}$$

The key outputs from gensys are matrices A and B such that:

$$X_t = AX_{t-1} + B\epsilon_t$$

Note that as X_t includes λ_t (which implies K_t), all aggregate variables can be easily constructed from a simulation of X_t.

7 Comparison of Solution Methods

To compare the Reiter and KS methods, Figure 2 plots the IRFs to an aggregate TFP shock in both models. As the IRFs are so similar, Figure 3 plots the difference between them.

Consistent with the similarity of the IRFs, Figure 4 shows that simulating the model using both methods produces very similar time series for K_t, albeit at a slightly higher level for the Krusell-Smith method.

The business cycle statistics from both methods are almost identical (Table 2), and very similar to those reported in Winberry (2016).

References

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[4] Simon Mongey. A set of tools for computing recursive equilibria. 2015.

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Table 1: Parameters

Parameter	Description	Value
β	Discount Factor	0.96
α	Capital Share	0.36
δ	Depreciation Rate	0.1
b	UI replacement rate	0.1
π_{01}	U to E probability	0.5
π_{10}	E to U probability	0.038
ρ_A	Aggregate TFP Persistence	0.859
σ_A	Aggregate TFP Volatility	0.014

Table 2: Business Cycle Statistics

Variable	SD (relative to Output)		Correlation with Output	
	KS	Reiter	KS	Reiter
Output	1.32%	1.32%	1	1
Consumption	0.487	0.497	0.912	0.918
Investment	2.67	2.64	0.976	0.975
Real Wage	1	1	1	1
Real interest Rate	0.149	0.149	0.903	0.905

Figure 1: Stationary Equilibrium of Incomplete Markets Model

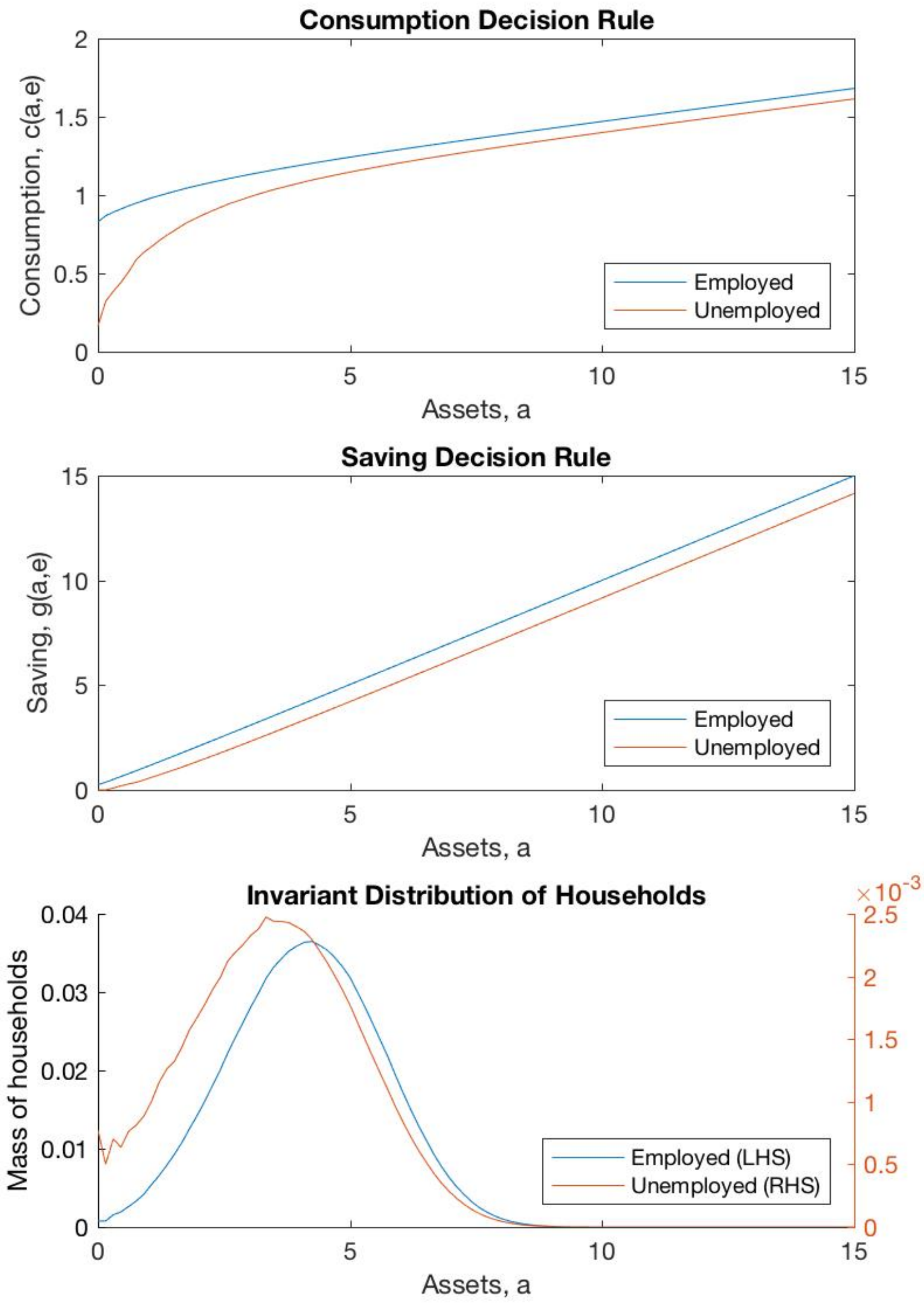

Figure 2: Impulse Response Functions to Aggregate TFP Shock

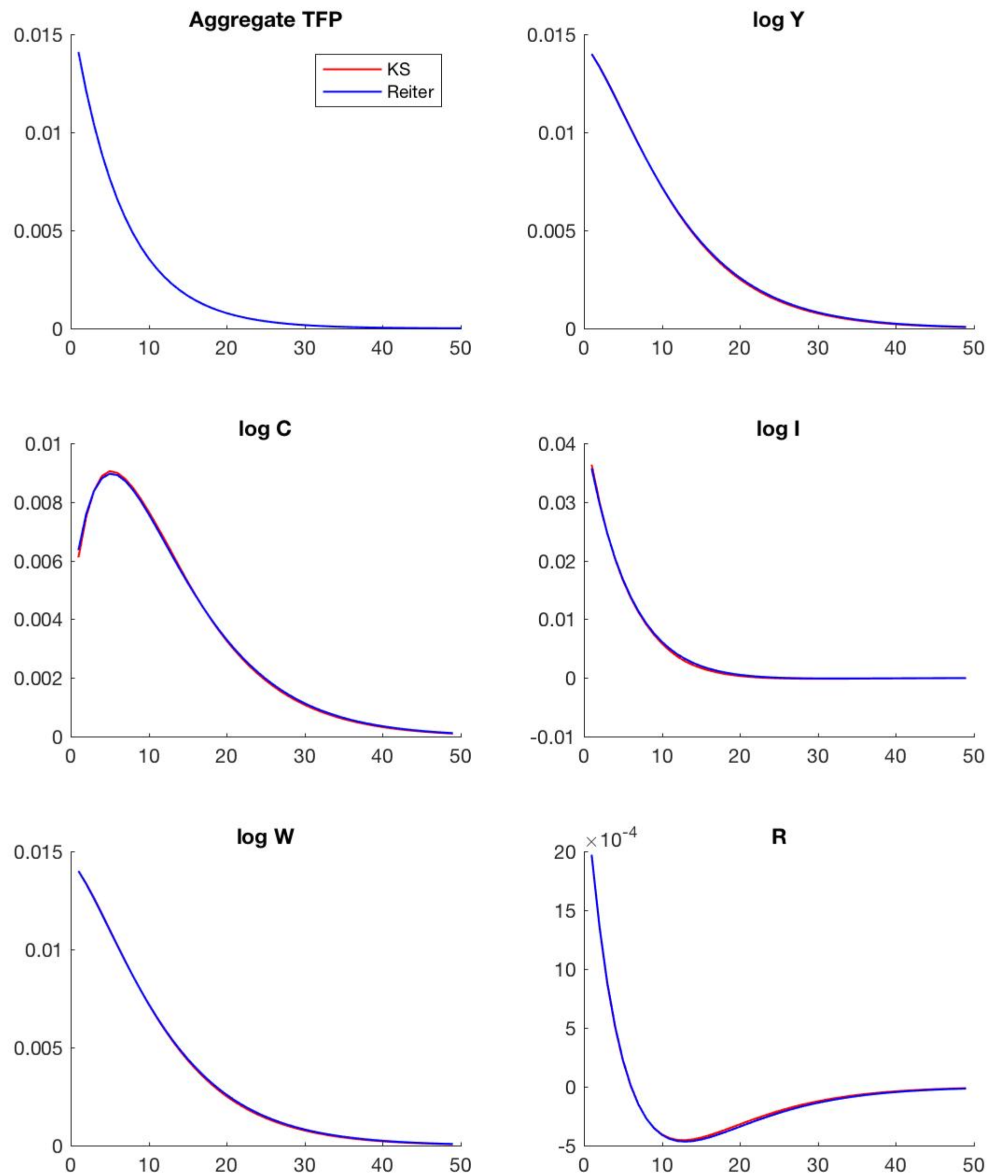

Figure 3: Difference Between KS and Reiter IRFs

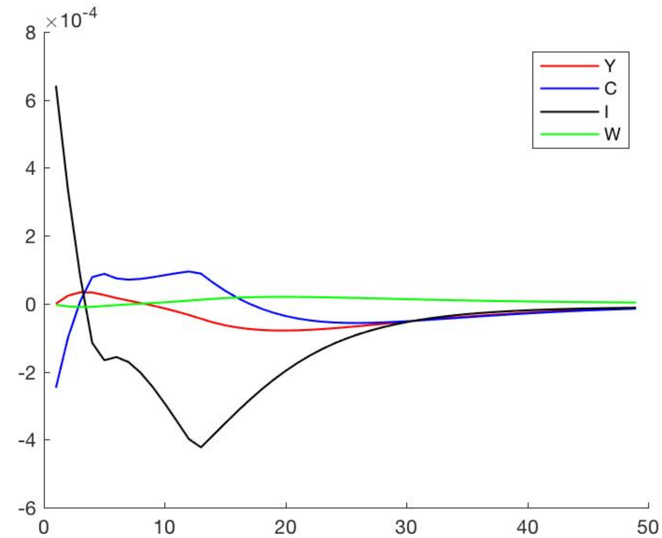

Figure 4: Representative Simulation of K_t