Comparison of different solutions methods on K&S model

by

Weimin Zhou

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Abstract

Since the distribution of agents is an infinite dimension object (many state variables), one method is approximating the distribution using only few moments, i.e. Krusell&Smith simulation, etc. And others such as using a perturbation method are more popular recently, i.e. LeGrand-Ragot who truncates the history of idiosyncratic shock; Reiter who solves the individual problem with a projection method, and approximate the law of motion of aggregate variables and the distribution with a perturbation method.

Thus, I'm going to learn about such methods and try to implement them on Krusell&Smith Model.

1 Basic Model Set up

The key technical things for Heterogeneous Agent Models are distribution of agents, which relates to individual choices and aggregate variables is an infinite dimensional object(many state variables).

For every type i agent we have the general formula of equilibrium:

$$E_t f(x_{t-1}^i, x_t^i, x_{t+1}^i, u_t^i, u_{t+1}^i, x_{t-1}, x_t, x_{t+1}, u_t, u_{t+1}) = 0,$$

where, x is endogenous variables, u denotes exogenous.

And for aggregate and individual endogenous variables, we have:

$$x_t \equiv \int \gamma(x_t^i) d_t(i) di \quad \text{with} \quad \int d_t(i) di = 1$$

Law of motion of the density function (share of each type i agent):

$$d_t(\cdot) = \mathcal{P}(d_{t-1}(\cdot), u_t)$$

1.1 Krusell&Smith Model

Individual endogenous variables: $x_t^i = \{c_t^i, a_t^i\}$;

Aggregate endogenous variables: $x_t = W_t, R_t, K_t, Y_t$;

Exogenous variables: idiosyncratic shocks s_t^i and TFP shocks z_t.

HH foc:

$$\begin{aligned} (c_t^i)^{-\sigma} &= \beta E_t R_{t+1} (c_{t+1}^i)^{-\sigma} \quad \text{if } a_t^i > a_{min} \\ (c_t^i)^{-\sigma} &> \beta E_t R_{t+1} (c_{t+1}^i)^{-\sigma} \quad \text{if } a_t^i = a_{min} \\ c_t^i + a_t^i &= s_t^i W_t + a_{t-1}^i R_{t-1} \end{aligned} \tag{1}$$

by Firm Maximization problem:

$$\begin{aligned} W_t &= (1-\alpha) Y_t / L_t \\ R_t &= \alpha Y_t / K_{t-1} + (1-\delta) \end{aligned} \tag{2}$$

And market clearing condition for assets: $\int a_t^i d_t(i) di = K_t$

Above is a way of describing the models very generally. In PS1 and PS2, we solve

KS model by using **Krusell-Smith Algorithm with conventional VFI**, in the Literature later on, there's some extension based on Krusell-Smith Algorithm also. In this Final Project, I want to implement another which is frequently used recently in the research, the projection and perturbation methods.

2 Literature Review

One strand of literature approximates the cross-sectional distribution using a parametric family, like Algan, Allais and Den Haan (2008)[1] in which they solve for the dynamics of the distribution using a globally accurate projection technique.

Another strand of literature uses a mix of globally accurate and locally accurate approximations to solve for the dynamics of heterogeneous agent models like Reiter (2009)[2], he solves the individual problem with a projection method, and approximates the law of motion of aggregate variables and the distribution with a perturbation method.

However, approximates the distribution with a fine histogram, which requires many parameters to achieve acceptable accuracy. This limits Reiter (2009)[2] approach to problems which have a low-dimensional individual state space because the size of the histogram grows exponentially in the number of individual states. More advanced things have been done by Winberry, Thomas (2018)[3], where he approximates the distribution with a flexible parametric family, reducing its dimensionality to a finite set of endogenous parameters, and solve for the dynamics of these endogenous parameters by perturbation. This part of work can be found in Winberry's website[1]

3 Projection and Perturbation: Reiter Method

Now, we only discretized individual state space into grids, and also the density function such that $\sum_{i=1}^{N} d_t^i = 1$ which is like discretization the continuous distribution with "histograms". And then, we could approximate individual policy function using a projection method:

$$x_t^i \approx \tilde{g}(x_{t-1}^i, u_t^i, \theta_t) \quad \text{for } x_t^i \equiv g(x_{t-1}^i, u_t^i, x_{t-1}, u_t, d_{t-1}),$$

where θ_t is the vector of coefficients.

[1]Thomas also provide codes for Benchmark RBC model with Firm Heterogeneity

3.1 The Algorithm

Step 1: Solve for SS

By discretization, we can solve individual policy function $x_t^i \approx \tilde{g}(x_{t-1}^i, u_t^i, \theta_t)$ with a projection method[2], and use this policy function to compute the stationary distribution of agents. Then, we check market clearing condition $K = \sum_{i=1}^N a^i d^i$, and update K until convergence

Another thing worth to notice is that during computing the evolution of distribution, we need next period asset a_{next}^i, previous, I use interpolation once (faster) but mostly straightly rely on the grids of assets. Here, we use approximation based on interpolation weights W_{ij} since next period assets are separated among different types on the grids:

$$W_{ij} = \begin{cases} 1 - \frac{a_{next}^i - a_j}{a_{j+1} - a_j} & \text{if } a_{next}^i \in [a_j, a_{j+1}] \\ \frac{a_{next}^i - a_{j-1}}{a_j - a_{j-1}} & \text{if } a_{next}^i \in [a_{j-1}, a_j] \\ 0 & \text{otherwise} \end{cases}.$$

Up to now, we have girds, predetermined transition probabilities, and obtained individual policy function, we can compute the evolution of distribution:

$$\mathcal{P}_{i'|i} = w_{ij} \times Prob(s'|s^i),$$

where distribution of agents evolves accroding to $d_t = \mathcal{P}' d_{t-1}$.

Now, we check market clearing condition: $residuals \equiv K - \sum_{i=1}^N a^i d^i$

Recap about the selection on $\mathcal{P}_{i'|i}$: if we assume H=2 histories of agents are the same, then we have:

$$\mathcal{P}_{i'|i} = \begin{bmatrix} & BB & BG & GB & GG \\ BB & p_B & p_B & 0 & 0 \\ BG & 0 & 0 & 1-p_G & 1-p_G \\ GB & 1-p_B & 1-p_B & 0 & 0 \\ GG & 0 & 0 & p_G & p_G \end{bmatrix}$$

Then, Household problem in Equation 1 becomes[3]:

[2] Setting aggregate shocks to zero, guess SS of capital, then we can solve for policy function coefficients θ

[3] for S state spaces power of H same-histories type of agents: $I = S^H$

$$
\begin{aligned}
(c_t^i)^{-\sigma} &= \beta R_t E_t \sum_{j=1}^{I} \mathcal{P}_{ji} (c_{t+1}^i)^{-\sigma} \quad \text{if } a_t^i > a_{min} \\
c_t^i + a_t^i &= s_t^i W_t + R_{t-1} \sum_{j=1}^{I} \mathcal{P}_{ji} a_{t-1}^j \frac{d_{t-1}^j}{d_t^i} \\
d_t^i &= \sum_{j=1}^{N} \mathcal{P}_{ji} d_{t-1}^j
\end{aligned}
\tag{3}
$$

Step 2: compute First Order Approximation of the model

Step 3: Solve the linearized system of equation

Now, by ”histogram” the density function, aggregate endogenous variable x_t can be approximated as $\tilde{x}_t = (\theta_t, x_t, d_t)$, and update by:

$$
\tilde{x}_t \equiv G_x \tilde{x}_{t-1} + G_u u_t
$$

4 Quantitative Performance

References

[1] Yann Algan, Olivier Allais, and Wouter J Den Haan. Solving heterogeneous-agent models with parameterized cross-sectional distributions. *Journal of Economic Dynamics and Control*, 32(3):875–908, 2008.

[2] Michael Reiter. Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665, 2009.

[3] Thomas Winberry. A method for solving and estimating heterogeneous agent macro models. 2018.