Portfolio Risk: Analytical Methods

6

Learning Objectives

After completing this reading you should be able to:

- Define, calculate, and distinguish between the following portfolio VaR measures: individual VaR, incremental VaR, marginal VaR, component VaR, undiversified portfolio VaR, and diversified portfolio VaR.
- Explain the role of correlation on portfolio risk.
- Describe the challenges associated with VaR measurement as portfolio size increases.
- Apply the concept of marginal VaR to guide decisions about portfolio VaR.
- Explain the risk-minimizing position and the risk and return-optimizing position of a portfolio.
- Explain the difference between risk management and portfolio management, and describe how to use marginal VaR in portfolio management.

Excerpt is Chapter 7 of Value at Risk: The New Benchmark for Managing Financial Risk, *Third Edition, by Philippe Jorion.*

Trust not all your goods to one ship.

—Erasmus

Absent any insight into the future, prudent investors should diversify across sources of financial risk. This was the message of portfolio analysis laid out by Harry Markowitz in 1952. Thus the concept of value-at-risk (VaR), or portfolio risk, is not new. What is new is the systematic application of VaR to many sources of financial risk, or portfolio risk. VaR explicitly accounts for leverage and portfolio diversification and provides a simple, single measure of risk based on current positions.

There are many approaches to measuring VaR. The shortest road assumes that asset payoffs are linear (or delta) functions of normally distributed risk factors. Indeed, the *delta-normal method* is a direct application of traditional portfolio analysis based on variances and covariances, which is why it is sometimes called the *covariance matrix approach.*

This approach is *analytical* because VaR is derived from closed- form solutions. The analytical method developed in this chapter is very useful because it creates a more intuitive understanding of the drivers of risk within a portfolio. It also lends itself to a simple decomposition of the portfolio VaR.

This chapter shows how to measure and manage portfolio VaR. The first section details the construction of VaR using information on positions and the covariance matrix of its constituent components.

The fact that portfolio risk is not cumulative provides great diversification benefits. To manage risk, however, we also need to understand what will reduce it. The section that follows provides a detailed analysis of VaR tools that are essential to control portfolio risk. These include marginal VaR, incremental VaR, and component VaR. These VaR tools allow users to identify the asset that contributes most to their total risk, to pick the best hedge, to rank trades, or in general, to select the asset that provides the best risk-return trade-off. Then, a fully worked out example of VaR computations for a global equity portfolio and for Barings' fatal positions will be presented.

The advantage of analytical models is that they provide closed-form solutions that help our intuition. The methods presented here, however, are quite general. We will show how to build these VaR tools in a nonparametric environment. This applies to simulations, for example.

Finally, we will be taken toward portfolio optimization, which should be the ultimate purpose of VaR. We first show how the passive measurement of risk can be extended to the management of risk, in particular, risk minimization. We then integrate risk with expected returns and show how VaR tools can be use to move the portfolio toward the best combination of risk and return.

PORTFOLIO VaR

A portfolio can be characterized by positions on a certain number of constituent assets, expressed in the base currency, say, dollars. If the positions are fixed over the selected horizon, the portfolio rate of return is a *linear* combination of the returns on underlying assets, where the weights are given by the relative amounts invested at the beginning of the period. Therefore, the VaR of a portfolio can be constructed from a combination of the risks of underlying securities.

Define the portfolio rate of return from t to $t + 1$ as

$$R_{p,t+1} = \sum_{i=1}^{N} w_i R_{i,t+1} \quad \textbf{(6.1)}$$

where N is the number of assets, $R_{i,t+1}$ is the rate of return on asset i, and w_i is the weight. The *rate of return* is defined as the change in the dollar value, or dollar return, scaled by the initial investment. This is a unitless measure.

Weights are constructed to sum to unity by scaling the dollar positions in each asset W_i by the portfolio total market value W. This immediately rules out portfolios that have zero net investment $W = 0$, such as some derivatives positions. But we could have positive and negative weights w_i, including values much larger than 1, as with a highly leveraged hedge fund. If the net portfolio value is zero, we could use another measure, such as the sum of the gross positions or absolute value of all dollar positions W^*. All weights then would be defined in relation to this benchmark. Alternatively, we could express returns in dollar terms, defining a dollar amount invested in asset i as $W_i = w_i W$. We will be using x as representing the vector of dollar amount invested in each asset so as to avoid confusion with the total dollar amount W.

It is important to note that in traditional mean-variance analysis, each constituent asset is a security. In contrast, VaR defines the component as a *risk factor* and w_i as the linear exposure to this risk factor. Whether dealing with assets or risk factors, the mathematics of portfolio VaR are equivalent, however.

To shorten notation, the portfolio return can be written using *matrix notation,* replacing a string of numbers by a single vector:

$$R_p = w_1R_1 + w_2R_2 + \cdots + w_NR_N = [w_1w_2 \cdots w_N]\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix} = w'R \quad \textbf{(6.2)}$$

where w' represents the transposed vector (i.e., horizontal) of weights, and R is the vertical vector containing individual asset returns.

The portfolio expected return is

$$E(R_p) = \mu_p = \sum_{i=1}^{N} w_i\mu_i \quad \textbf{(6.3)}$$

and the variance is

$$V(R_p) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} w_iw_j\sigma_{ij} = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_iw_j\sigma_{ij} \quad \textbf{(6.4)}$$

This sum accounts not only for the risk of the individual securities σ_i^2 but also for all covariances, which add up to a total of $N(N-1)/2$ different terms.

As the number of assets increases, it becomes difficult to keep track of all covariance terms, which is why it is more convenient to use matrix notation. The variance can be written as

$$\sigma_p^2 = [w_1 \cdots w_N]\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1N} \\ \vdots & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \cdots & \sigma_{N2} \end{bmatrix}\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

Defining Σ as the covariance matrix, the variance of the portfolio rate of return can be written more compactly as

$$s_p^2 = w'\Sigma w \quad \textbf{(6.5)}$$

where w are weights, which have no units. This also can be written in terms of dollar exposures x as

$$s_p^2W^2 = x'\Sigma x \quad \textbf{(6.6)}$$

So far nothing has been said about the distribution of the portfolio return. Ultimately, we would like to translate the portfolio variance into a VaR measure. To do so, we need to know the distribution of the portfolio return. In the delta-normal model, all individual security returns are assumed normally distributed. This is particularly convenient because the portfolio return, a linear combination of jointly normal random variables, is also normally distributed. If so, we can translate the confidence level c into a standard normal deviate α such that the probability of observing a loss worse than $-\alpha$ is c. Defining W as the initial portfolio value, the portfolio VaR is

$$\text{Portfolio VaR} = \text{VaR}_p = \alpha\sigma_pW = a\sqrt{x'\Sigma x} \quad \textbf{(6.7)}$$

Diversified VaR The portfolio VaR, taking into account diversification benefits between components.

At this point, we also can define the individual risk of each component as

$$\text{VaR}_i = \alpha\sigma_i \mid W_i \mid = \alpha\sigma_i \mid w_i \mid W \quad \textbf{(6.8)}$$

Note that we took the absolute value of the weight w_i because it can be negative, whereas the risk measure must be positive.

Individual VaR The VaR of one component taken in isolation.

Equation (6.4) shows that the portfolio VaR depends on variances, covariances, and the number of assets. Covariance is a measure of the extent to which two variables move linearly together. If two variables are independent, their covariance is equal to zero. A positive covariance means that the two variables tend to move in the same direction; a negative covariance means that they tend to move in opposite directions. The magnitude of covariance, however, depends on the variances of the individual components and is not easily interpreted. The *correlation coefficient* is a more convenient, scale-free measure of linear dependence:

$$\rho_{12} = \sigma_{12}/(\sigma_1\sigma_2) \quad \textbf{(6.9)}$$

The correlation coefficient ρ always lies between -1 and $+1$. When equal to unity, the two variables are said to be *perfectly correlated.* When 0, the variables are *uncorrelated.*

Lower portfolio risk can be achieved through low correlations or a large number of assets. To see the effect of N, assume that all assets have the same risk and that all correlations are the same, that equal weight is put on each asset. Figure 6-1 shows how portfolio risk decreases with the number of assets.

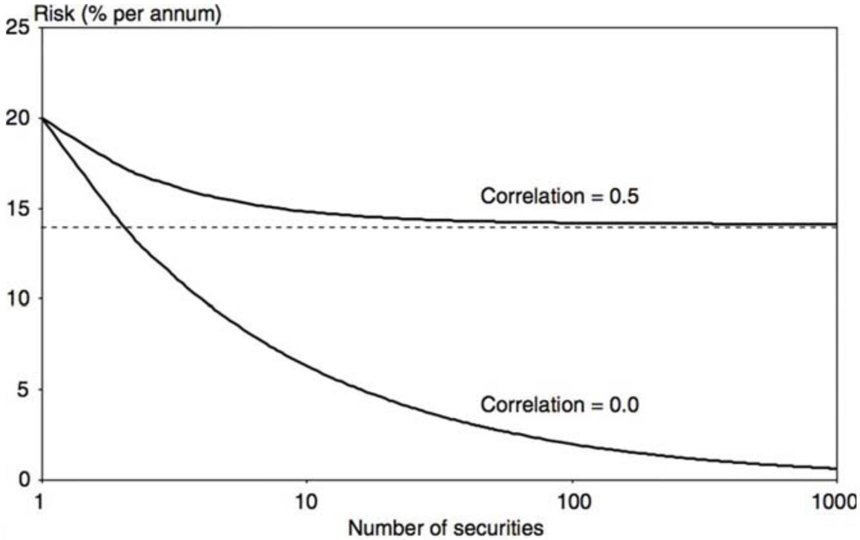

FIGURE 6-1 Risk and number of securities.

Start with the risk of one security, which is assumed to be 20 percent. When ρ is equal to zero, the risk of a 10-asset portfolio drops to 6.3 percent; increasing N to 100 drops the risk even further to 2.0 percent. Risk tends asymptotically to zero. More generally, portfolio risk is

$$\sigma_p = \sigma\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right)\rho} \tag{6.10}$$

which tends to $\sigma\sqrt{\rho}$ as N increases. Thus, when $\rho = 0.5$, risk decreases rapidly from 20 to 14.8 percent as N goes to 10 and afterward converges more slowly toward its minimum value of 14.1 percent.

Low correlations thus help to diversify portfolio risk. Take a simple example with two assets only. The "diversified" portfolio variance is

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{6.11}$$

The portfolio VaR is then

$$\text{VaR}_p = \alpha\sigma_p W = \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}\,W \tag{6.12}$$

This can be related to the individual VaR as defined in Equation (6.8).

When the correlation ρ is zero, the portfolio VaR reduces to

$$\text{VaR}_p = \sqrt{\alpha^2 w_1^2 W^2\sigma_1^2 + \alpha^2 w_2^2 W^2\sigma_2^2} = \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2} \tag{6.13}$$

The portfolio risk must be lower than the sum of the individual VaRs: $\text{VaR}_p < \text{VaR}_1 + \text{VaR}_2$. This reflects the fact that with assets that move independently, a portfolio will be less risky than either asset. Thus VaR is a *coherent* risk measure for normal and, more generally, elliptical distributions.

When the correlation is exactly unity and w_1 and w_2 are both positive, Equation (6.12) reduces to

$$\begin{aligned}\text{VaR}_p &= \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\text{VaR}_1 \times \text{VaR}_2} \\ &= \text{VaR}_1 + \text{VaR}_2\end{aligned} \tag{6.14}$$

In other words, the portfolio VaR is equal to the sum of the individual VaR measures if the two assets are perfectly correlated. In general, though, this will not be the case because correlations typically are imperfect. The benefit from diversification can be measured by the difference between the *diversified* VaR and the *undiversified* VaR, which typically is shown in VaR reporting systems.

> **Undiversified VaR** The sum of individual VaRs, or the portfolio VaR when there is no short position and all correlations are unity.

This interpretation differs when short sales are allowed. Suppose that the portfolio is long asset 1 but short asset 2 (w_1 is positive, and w_2 is negative). This could represent a hedge fund that has \$1 in capital and a \$1 billion long position in corporate bonds and a \$1 billion short position in Treasury bonds, the rationale for the position being that corporate yields are slightly higher than Treasury yields. If the correlation is exactly unity, the fund has no risk because any loss in one asset will be offset by a matching gain in the other. The portfolio VaR then is zero.

Instead, the risk will be greatest if the correlation is −1, in which case losses in one asset will be amplified by the other. Here, the *undiversified* VaR can be interpreted as the portfolio VaR when the correlation attains its worst value, which is −1. Therefore, the undiversified VaR provides an upper bound on the portfolio VaR should correlations prove unstable and all move at the same time in the wrong direction. It provides an absolute worst-case scenario for the portfolio at hand.

Example 6.1

Consider a portfolio with two foreign currencies, the Canadian dollar (CAD) and the euro (EUR). Assume that these two currencies are uncorrelated and have a volatility against the dollar of 5 and 12 percent, respectively. The first step is to mark to market the positions in the base currency. The portfolio has US$2 million invested in the CAD and US$1 million in the EUR. We seek to find the portfolio VaR at the 95 percent confidence level.

First, we will compute the variance of the portfolio dollar return. Define x as the dollar amounts allocated to each risk factor, in millions. Compute the product

$$\Sigma x = \begin{bmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{bmatrix}\begin{bmatrix} \$2 \\ \$1 \end{bmatrix} = \begin{bmatrix} 0.05^2 \times \$2 + 0 \times \$1 \\ 0 \times \$2 + 0.12^2 \times \$1 \end{bmatrix} = \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix}$$

The portfolio variance then is (in dollar units)

$$\sigma_p^2 W^2 = x'(\Sigma x) = [\$2\ \$1]\begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} = 0.0100 + 0.0144 = 0.0244$$

The dollar volatility is $\sqrt{0.0244} = \$0.156205$ million. Using $\alpha = 1.65$, we find $\text{VaR}_p = 1.65 \times 156{,}205 = \$257{,}738$.

Next, the individual (undiversified) VaR is found simply as $\text{VaR}_i = \alpha\sigma_i x_i$, that is,

$$\begin{bmatrix} \text{VaR}_1 \\ \text{VaR}_2 \end{bmatrix} = \begin{bmatrix} 1.65 \times 0.05 \times \$2 \text{ million} \\ 1.65 \times 0.12 \times \$1 \text{ million} \end{bmatrix} = \begin{bmatrix} \$165{,}000 \\ \$198{,}000 \end{bmatrix}$$

Note that these numbers sum to an undiversified VaR of \$363,000, which is greater than the portfolio VaR of \$257,738 owing to diversification effects.

VaR TOOLS

Initially, VaR was developed as a methodology to measure portfolio risk. There is much more to VaR than simply reporting a single number, however. Over time, risk managers have discovered that they could use the VaR process for active risk management. A typical question may be, "Which position should I alter to modify my VaR most effectively?" Such information is quite useful because portfolios typically are traded incrementally owing to transaction costs. This is the purpose of VaR tools, which include marginal, incremental, and component VaR.

Marginal VaR

To measure the effect of changing positions on portfolio risk, individual VaRs are not sufficient. Volatility measures the uncertainty in the return of an asset, taken in isolation. When this asset belongs to a portfolio, however, what matters is the contribution to portfolio risk.

We start from the existing portfolio, which is made up of N securities, numbered as $j = 1, \ldots, N$. A new portfolio is obtained by adding one unit of security i. To assess the impact of this trade, we measure its "marginal" contribution to risk by increasing w by a small amount or differentiating Equation (6.4) with respect to w_i, that is,

$$\frac{\partial \sigma_p^2}{\partial w_i} = 2w_i\sigma_i^2 + 2\sum_{j=1, j\neq i}^{N} w_j\sigma_{ij}$$
$$= 2\text{cov}(R_i, w_iR_i + \sum_{j\neq i}^{N} w_jR_j) = 2\text{cov}(R_i, R_p) \qquad \textbf{(6.15)}$$

Instead of the derivative of the variance, we need that of the volatility. Noting that $\partial\sigma_p^2/\partial w_i = 2\sigma_p\partial\sigma_p/\partial w_i$, the sensitivity of the portfolio volatility to a change in the weight is then

$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\text{cov}(R_i, R_p)}{\sigma_p} \qquad \textbf{(6.16)}$$

Converting into a VaR number, we find an expression for the *marginal VaR*, which is a vector with component

$$\Delta\text{VaR}_i = \frac{\partial \text{VaR}}{\partial x_i} = \frac{\partial \text{VaR}}{\partial w_i W} = \alpha\frac{\partial \sigma_p}{\partial w_i} = \alpha\frac{\text{cov}(R_i, R_p)}{\sigma_p} \qquad \textbf{(6.17)}$$

Since this was defined as a ratio of the dollar amounts, this marginal VaR measure is unitless.

> **Marginal VaR** The change in portfolio VaR resulting from taking an additional dollar of exposure to a given component. It is also the partial (or linear) derivative with respect to the component position.

This marginal VaR is closely related to the *beta*, defined as

$$\beta_i = \frac{\text{cov}(R_i, R_p)}{\sigma_p^2} = \frac{\sigma_{ip}}{\sigma_p^2} = \frac{\rho_{ip}\sigma_i\sigma_p}{\sigma_p^2} = \rho_{ip}\frac{\sigma_i}{\sigma_p} \qquad \textbf{(6.18)}$$

which measures the contribution of one security to total portfolio risk. Beta is also called the *systematic risk* of security i vis-à-vis portfolio p and can be measured from the slope coefficient in a regression of R_i on R_p, that is,

$$R_{i,t} = \alpha_i + \beta_i R_{p,t} + \epsilon_{i,t} \quad t = 1, \ldots, T \tag{6.19}$$

Using matrix notation, we can write the vector β, including all assets, as

$$\beta = \frac{\Sigma w}{(w'\Sigma w)}$$

Note that we already computed the vector Σw as an intermediate step in the calculation of VaR. Therefore, β and the marginal VaR can be derived easily once VaR has been calculated.

Beta risk is the basis for capital asset pricing model (CAPM) developed by Sharpe (1964). According to the CAPM, well-diversified investors only need to be compensated for the systematic risk of securities relative to the market. In other words, the risk premium on all assets should depend on beta only. Whether this is an appropriate description of capital markets has been the subject of much of finance research in the last decades. Even though this proposition is still debated hotly, the fact remains that systematic risk is a useful statistical measure of marginal portfolio risk.

To summarize, the relationship between the ΔVaR and β is

$$\Delta \mathrm{VaR}_i = \frac{\partial \mathrm{VaR}}{\partial x_i} = \alpha(\beta_i \times \sigma_p) = \frac{\mathrm{VaR}}{W} \times \beta_i \tag{6.20}$$

The marginal VaR can be used for a variety of risk management purposes. Suppose that an investor wants to lower the portfolio VaR and has the choice to reduce all positions by a fixed amount, say, \$100,000. The investor should rank all marginal VaR numbers and pick the asset with the largest ΔVaR because it will have the greatest hedging effect.

Incremental VaR

This methodology can be extended to evaluate the total impact of a proposed trade on portfolio p. The new trade is represented by position a, which is a vector of additional exposures to our risk factors, measured in dollars.

Ideally, we should measure the portfolio VaR at the initial position VaR_p and then again at the new position VaR_{p+a}.

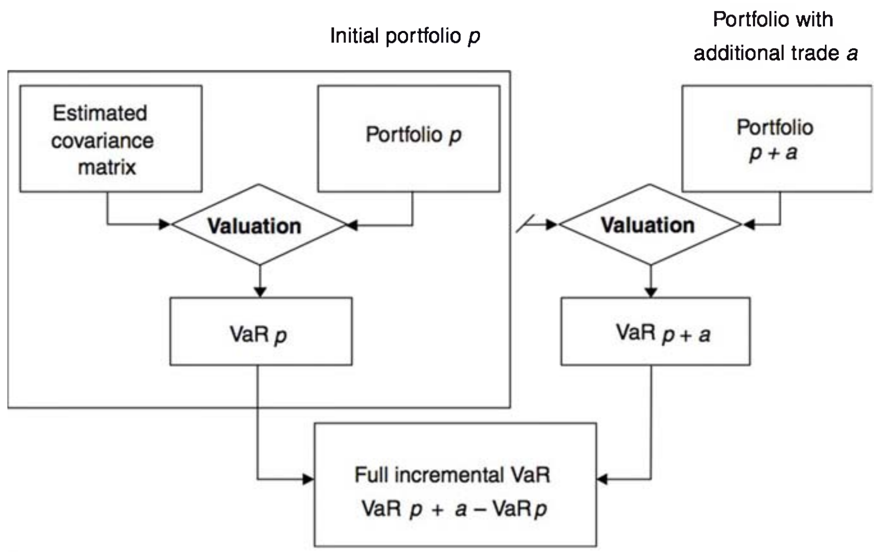

FIGURE 6-2 The impact of a proposed trade with full revaluation.

The incremental VaR then is obtained, as described in Figure 6-2, as

$$\text{Incremental VaR} = \mathrm{VaR}_{p+a} - \mathrm{VaR}_p \tag{6.21}$$

This "before and after" comparison is quite informative. If VaR is decreased, the new trade is risk-reducing or is a hedge; otherwise, the new trade is risk-increasing. Note that a may represent a change in a single component or a more complex trade with changes in multiple components. Hence, in general, a represents a vector of new positions.

> **Incremental VaR** The change in VaR owing to a new position. It differs from the marginal VaR in that the amount added or subtracted can be large, in which case VaR changes in a nonlinear fashion.

The main drawback of this approach is that it requires a full revaluation of the portfolio VaR with the new trade. This can be quite time-consuming for large portfolios. Suppose, for instance, that an institution has 100,000 trades on its books and that it takes 10 minutes to do a VaR calculation. The bank has measured its VaR at some point during the day.

Then a client comes with a proposed trade. Evaluating the effect of this trade on the bank's portfolio again would require 10 minutes using the incremental-VaR approach.

Most likely, this will be too long to wait to take action. If we are willing to accept an approximation, however, we can take a shortcut.[1]

Expanding VaR_{p+a} in series around the original point,

$$\text{VaR}_{p+a} = \text{VaR}_p + (\Delta\text{VaR})' \times a + \ldots \quad \textbf{(6.22)}$$

where we ignored second-order terms if the deviations a are small. Hence the incremental VaR can be reported as, approximately,

$$\text{Incremental VaR} \approx (\Delta\text{VaR})' \times a \quad \textbf{(6.23)}$$

This measure is much faster to implement because the ΔVaR vector is a by-product of the initial VaR_p computation. The new process is described in Figure 6-3.

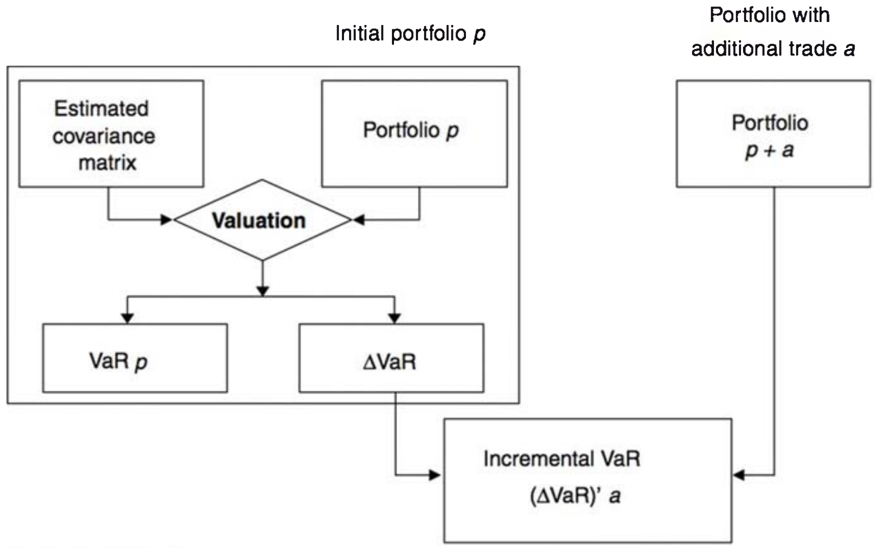

FIGURE 6-3 The impact of a proposed trade with marginal VaR.

Here we are trading off faster computation time against accuracy. How much of an improvement is this shortcut relative to the full incremental VaR method? The shortcut will be especially useful for large portfolios where a full revaluation requires a large number of computations. Indeed, the number of operations increases with the square of the number of risk factors. In addition, the shortcut will prove to be a good approximation for large portfolios where a proposed trade is likely to be small relative to the outstanding portfolio. Thus the simplified VaR method allows real-time trading limits.

The incremental VaR method applies to the general case where a trade involves a set of new exposures on the risk factors. Consider instead the particular case where a new trade involves a position in one risk factor only (or asset). The portfolio value changes from the old value of W to the new value of $W_{p+a} = W + a$, where a is the amount invested in asset i. We can write the variance of the dollar returns on the new portfolio as

$$\sigma^2_{p+a} W^2_{p+a} = \sigma^2_p W^2 + 2aW\sigma_{ip} + a^2\sigma^2_i \quad \textbf{(6.24)}$$

An interesting question for portfolio managers is to find the size of the new trade that leads to the lowest portfolio risk. Differentiating with respect to a,

$$\frac{\partial \sigma^2_{p+a} W^2_{p+a}}{\partial a} = 2W\sigma_{ip+2a\sigma_i^2} \quad \textbf{(6.25)}$$

which attains a zero value for

$$a^* = -W\frac{\sigma_{ip}}{\sigma^2_i} = -W\beta_i \frac{\sigma^2_p}{\sigma^2_i} \quad \textbf{(6.26)}$$

This is the variance-minimizing position, also known as best hedge.

Best hedge Additional amount to invest in an asset so as to minimize the risk of the total portfolio.

Example 6.1 (continued)

Going back to the previous two-currency example, we are now considering increasing the CAD position by US$10,000.

First, we use the marginal-VaR method. We note that β can be obtained from a previous intermediate step. Because we used dollar amounts, this should be adjusted so that β is unitless, that is,

$$\beta = \frac{\Sigma w}{w'\Sigma w} = W \times \frac{\Sigma x}{x'\Sigma x}$$

We have

$$\beta = \$3 \times \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} / (\$0.156^2) = \$3 \begin{bmatrix} 0.205 \\ 0.590 \end{bmatrix} = \begin{bmatrix} 0.615 \\ 1.770 \end{bmatrix}$$

[1] See also Garman (1996 and 1997).

The marginal VaR is now

$$\Delta\text{VaR} = \alpha\frac{\text{cov}(R,R_p)}{\sigma_p} = 1.65 \times \begin{bmatrix} \$0.0050 \\ \$0.0144 \end{bmatrix} / \$0.156 = \begin{bmatrix} 0.0528 \\ 0.1521 \end{bmatrix}$$

As we increase the first position by \$10,000, the incremental VaR is

$$(\Delta\text{VaR})' \times a = [0.0528\ 0.1521]\begin{bmatrix} \$10{,}000 \\ 0 \end{bmatrix}$$
$$= 0.0528 \times \$10{,}000 + 0.1521 \times 0 = \$528$$

Next, we compare this with the incremental VaR obtained from a full revaluation of the portfolio risk. Adding \$0.01 million to the first position, we find

$$\sigma^2_{p+a}W^2_{p+a} = [\$2.01\ \$1]\begin{bmatrix} 0.05^2 & 0 \\ 0 & 0.12^2 \end{bmatrix}\begin{bmatrix} \$2.01 \\ \$1 \end{bmatrix}$$

which gives VaR_{p+a} = \$258,267. Relative to the initial VaR_p = \$257,738, the exact increment is \$529. Note how close the ΔVaR approximation of \$528 comes to the true value. The linear approximation is excellent because the change in the position is very small.

Component VaR

In order to manage risk, it would be extremely useful to have a *risk decomposition* of the current portfolio. This is not straightforward because the portfolio volatility is a highly nonlinear function of its components. Taking all individual VaRs, adding them up, and computing their percentage, for instance, is not useful because it completely ignores diversification effects. Instead, what we need is an additive decomposition of VaR that recognizes the power of diversification.

This is why we turn to marginal VaR as a tool to help us measure the contribution of each asset to the existing portfolio risk. Multiply the marginal VaR by the current dollar position in asset or risk factor *i*, that is,

$$\text{Component VaR}_i = (\Delta\text{VaR}_i) \times w_i W = \frac{\text{VaR}\beta_i}{W} \times w_i W = \text{VaR}\beta_i w_i \quad \textbf{(6.27)}$$

Thus the component VaR indicates how the portfolio VaR would change approximately if the component was deleted from the portfolio. We should note, however, that the quality of this linear approximation improves when the VaR components are small. Hence this decomposition is more useful with large portfolios, which tend to have many small positions.

We now show that these component VaRs precisely add up to the total portfolio VaR. The sum is

$$\text{CVaR}_1 + \text{CVaR}_2 + \ldots + \text{CVaR}_N = \text{VaR}\left(\sum_{i=1}^{N} w_i\beta_i\right) = \text{VaR} \quad \textbf{(6.28)}$$

because the term between parentheses is simply the beta of the portfolio with itself, which is unity.[2] Thus we established that these *component* VaR measures add up to the total VaR. We have an additive measure of portfolio risk that reflects correlations. Components with a negative sign act as a hedge against the remainder of the portfolio. In contrast, components with a positive sign increase the risk of the portfolio.

> **Component VaR** A partition of the portfolio VaR that indicates how much the portfolio VaR would change approximately if the given component was deleted. By construction, component VaRs sum to the portfolio VaR.

The component VaR can be simplified further. Taking into account the fact that β_i is equal to the correlation ρ_i times σ_i divided by the portfolio σ_p, we can write

$$\text{CVaR}_i = \text{VaR}w_i\beta_i = (\alpha\sigma_p W)w_i\beta_i = (\alpha\sigma_i w_i W)\rho_i = \text{VaR}_i\rho_i \quad \textbf{(6.29)}$$

This conveniently transforms the individual VaR into its contribution to the total portfolio simply by multiplying it by the correlation coefficient.

Finally, we can normalize by the total portfolio VaR and report

Percent contribution to VaR of component

$$i = \frac{\text{CVaR}_i}{\text{VaR}} = w_i\beta_i \quad \textbf{(6.30)}$$

VaR systems can provide a breakdown of the contribution to risk using any desired criterion. For large portfolios, component VaR may be shown by type of currency, by type of asset class, by geographic location, or by business unit. Such detail is invaluable for drill-down exercises, which enable users to control their VaR.

---

[2] This can be proved by expanding the portfolio variance into $\sigma_p^2 = w_1 \text{ cov}(R_1,R_p) + w_2 \text{ cov}(R_2,R_p) + \cdots = w_1(\beta_1\sigma_p^2) + w_2(\beta_2\sigma_p^2) + \cdots = \sigma_p^2(\sum_{i=1}^{N} w_i\beta_i)$. Therefore, the term between parentheses must be equal to 1.

Example 6.1 (continued)

Continuing with the previous two-currency example, we find the component VaR for the portfolio using $\text{CVaR}_i = \Delta\text{VaR}_i\, x_i$, that is,

$$\begin{bmatrix}\text{CVaR}_1\\ \text{CVaR}_2\end{bmatrix} = \begin{bmatrix}0.0528 \times \$2\text{ million}\\ 0.1521 \times \$1\text{ million}\end{bmatrix} = \begin{bmatrix}\$105{,}630\\ \$152{,}108\end{bmatrix} = \text{VaR} \times \begin{bmatrix}41.0\%\\ 59.0\%\end{bmatrix}$$

We verify that these two components indeed sum to the total VaR of \$257,738. The largest component is due to the EUR, which has the highest volatility. Both numbers are positive, indicating that neither position serves as a net hedge for the portfolio. Note that the percentage contribution to VaR also could have been obtained as

$$\begin{bmatrix}\text{CVaR}_1 / \text{VaR}\\ \text{CVaR}_2 / \text{VaR}\end{bmatrix} = \begin{bmatrix}w_1\beta_1\\ w_2\beta_2\end{bmatrix} = \begin{bmatrix}0.667 \times 0.615\\ 0.333 \times 1.770\end{bmatrix} = \begin{bmatrix}41.0\%\\ 59.0\%\end{bmatrix}$$

Next, we can compute the change in the VaR if the euro position is set to zero and compare with the preceding result. Since the portfolio has only two assets, the new VaR without the EUR position is simply the VaR of the CAD component, VaR_1 = \$165,000. The incremental VaR of the EUR position is (\$257,738 − \$165,000) = \$92,738. The component VaR of \$152,108 is higher, although of the same order of magnitude. The approximation is not as good as before because there are only two assets in the portfolio, which individually account for a large proportion of the total VaR. We would expect a better approximation if the VaR components are small relative to the total VaR.

Summary

Figure 6-4 presents a graphic summary of VaR tools for our two-currency portfolio. The graph plots the portfolio VaR as a function of the amount invested in this asset, the euro. At the current position of \$1 million, the portfolio VaR is \$257,738.

The marginal VaR is the change in VaR owing to an addition of \$1 in EUR, or 0.0528; this represents the slope of the straight line that is tangent to the VaR curve at the current value.

The incremental VaR is the change in VaR owing to the deletion of the euro position, which is \$92,738 and is measured along the curve. This is approximated by the component VaR, which is simply the marginal VaR times the current position of \$1 million, or \$152,108. The latter is measured along the straight line that is tangent to the VaR curve. The graph illustrates that the component VaR is only an approximation of the incremental VaR. These component VaR measures add up to the total portfolio VaR, which gives a quick decomposition of the total risk.

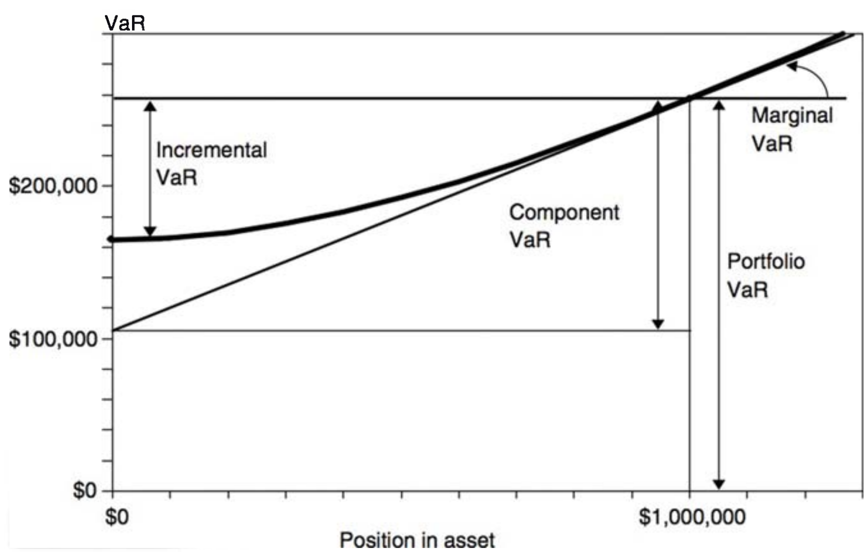

FIGURE 6-4 VaR decomposition.

TABLE 6-1 VaR Decomposition for Sample Portfolio

Currency	Current Position, x_i or w_iW	Individual VaR, $\text{VaR}_i = \alpha\sigma_i w_i W$	Marginal VaR, $\Delta\text{VaR}_i = \text{VaR}\,\beta_i/W$	Component VaR, $\text{CVaR}_i = \Delta\text{VaR}_i x_i$	Percent Contribution, CVaR_i/VaR
CAD	\$2 million	\$165,000	0.0528	\$105,630	41.0%
EUR	\$1 million	\$198,000	0.1521	\$152,108	59.0%
Total	\$3 million				
Undiversified VaR		\$363,000			
Diversified VaR				\$257,738	100.0%

The graph also shows that the best hedge is a net zero position in the euro. Indeed, the VaR function attains a minimum when the position in the euro is zero.

The results are summarized in Table 6-1. This report gives not only the portfolio VaR but also a wealth of information for risk managers. For instance, the marginal VaR column can be used to determine how to reduce risk. Since the marginal VaR for the EUR is three times as large as that for the CAD, cutting the position in the EUR will be much more effective than cutting the CAD position by the same amount.

EXAMPLES

This section provides a number of applications of VaR measures. The first example illustrates a risk report for a global equity portfolio. The second shows how VaR could have been used to dissect the Barings portfolio.

A Global Portfolio Equity Report

To further illustrate the use of our VaR tools, Table 6-2 displays a risk management report for a global equity portfolio. Here, risk is measured in relative terms, that is, relative to the benchmark portfolio. The current portfolio has an annualized tracking error volatility σ_p, of 1.82 percent per annum. This number can be translated easily into a VaR number using $\text{VaR} = \alpha\sigma_p W$. Hence we can deal with VaR or more directly with σ_p.

Positions are reported as deviations in percent from the benchmark in the second column. Since the weights of the benchmark and of the current portfolio must sum to one, the deviations must sum to zero. Traditional portfolio reporting systems only provide information about current positions for the portfolio. The position, data, however, could be used to provide detailed information about risk.

The next columns report the individual risk, marginal risk, and percentage contribution to total risk. Positions contributing to more than 5 percent of the total are called *Hot Spots.*[3] The table shows that two countries, Japan and Brazil, account for more than 50 percent of the risk. This is an important but not intuitive result because the positions in these markets, displayed in the first column, are not the largest in terms of weights.

In fact, the United States and United Kingdom, which have the largest deviations from the index, contribute to only 20 percent of the risk. The contributions of Japan and Brazil are high because of their high volatility and correlations with the portfolio.

To control risk, we turn to the "Best Hedge" column. The table shows that the 4.5 percent overweight position in Japan should be decreased to lower risk. The optimal change is a decrease of 4.93 percent, after which the new volatility will have decreased from the original value of 1.82 to 1.48 percent. In contrast, the 4.0 percent overweight position in Canada has little impact on the portfolio risk.

This type of report is invaluable to control risk. In the end, of course, portfolio managers add value by judicious bets on markets, currencies, or securities. Such VaR tools are useful, however, because analysts now can balance their return forecasts against risk explicitly.

Barings: An Example in Risks

Barings' collapse provides an interesting application of the VaR methodology. Leeson was reported to be long about \$7.7 billion worth of Japanese stock index (Nikkei) futures and short \$16 billion worth of Japanese government bond (JGB) futures. Unfortunately, official reports to Barings showed "nil" risk because the positions were fraudulent.

If a proper VaR system had been in place, the parent company could have answered the following questions: What

[3] Hot Spots is a trademark of Goldman Sachs.

TABLE 6-2 Global Equity Portfolio Report

Country	Current Position (%) w_i	Individual Risk $w_i\sigma_i$	Marginal Risk β_i	Percent Contribution to Risk $w_i\beta_i$	Best Hedge (%)	Volatility at Best Hedge
Japan	4.5	0.96%	0.068	**31.2**	−4.93	1.48%
Brazil	2.0	1.02%	0.118	**22.9**	−1.50	1.66%
U.S.	−7.0	0.89%	−0.019	**13.6**	3.80	1.75%
Thailand	2.0	0.55%	0.052	**10.2**	−2.30	1.71%
U.K.	−6.0	0.46%	0.035	**7.0**	2.10	1.80%
Italy	2.0	0.79%	−0.011	**6.8**	−2.18	1.75%
Germany	2.0	0.35%	0.019	3.7	−2.06	1.79%
France	−3.5	0.57%	−0.009	3.4	1.18	1.81%
Switzerland	2.5	0.39%	0.011	2.6	−1.45	1.81%
Canada	4.0	0.49%	0.001	1.5	−0.11	1.82%
South Africa	−1.0	0.20%	0.008	−0.7	−0.65	1.82%
Australia	−1.5	0.24%	0.014	−2.0	−1.89	1.80%
Total	0.0			100.0		
Undiversified risk		6.91%				
Diversified risk	1.82%					

Source: Adapted from Litterman (1996).

TABLE 6-3 Barings' Risks

	Risk % σ	Correlation Matrix R		Covariance Matrix Σ		Positions ($ millions) x	Individual VaR $\alpha\sigma x$
10-year JGB	1.18	1	−0.114	0.000139	−0.000078	($16,000)	$310.88
Nikkei	5.83	−0.114	1	−0.000078	0.003397	$7,700	$740.51
Total						$8,300	$1051.39

	Total VaR Computation		Marginal VaR β_i for $1 million			
Asset i	$(\Sigma x)_i$	$x_i'(\Sigma x)_i$	$(\Sigma x)_i/\sigma_p^2$	β_iVaR	Component VaR $\beta_i x_i$VaR	Percent Contribution
10-yr JGB	−2.82	45138.8	−0.0000110	($0.00920)	$147.15	17.6%
Nikkei	27.41	211055.1	0.0001070	$0.08935	$688.01	82.4%
Total		256193.8			$835.16	100.0%
Risk $=\sigma_p$		506.16				
VaR $=\alpha\sigma_p$		$835.16				

was Leeson's actual VaR? Which component contributed most to VaR? Were the positions hedging each other or adding to the risk?

The top panel of Table 6-3 displays monthly volatility measures and correlations for positions in the 10-year zero JGB and the Nikkei Index. The correlation between Japanese stocks and bonds is negative, indicating that increases in stock prices are associated with decreases in bond prices or increases in interest rates. The next column displays positions that are reported in millions of dollar equivalents.

To compute the VaR, we first construct the covariance matrix Σ from the correlations. Next, we compute the vector Σx, which is in the first column of the bottom panel. For instance, the −2.82 entry is found from $\sigma_1^2 x_1 + \sigma_{12} x_2 = 0.000139 \times (-\$16{,}000) + (-0.000078) \times \$7700 = -2.82$. The next column reports $x_1(\Sigma x)_1$ and $x_2(\Sigma x)_2$, which sum to the total portfolio variance of 256,193.8, for a portfolio volatility of $\sqrt{256{,}194} = \$506$ million. At the 95 percent confidence level, Barings' VaR was 1.65 × $506, or $835 million.

This represents the worst monthly loss at the 95 percent confidence level under normal market conditions. In fact, Leeson's total loss was reported at $1.3 billion, which is comparable to the VaR reported here. The difference is because the position was changed over the course of the 2 months, there were other positions (such as short options), and also bad luck. In particular, on January 23, 1995, one week after the Kobe earthquake, the Nikkei Index lost 6.4 percent. Based on a monthly volatility of 5.83 percent, the daily VaR of Japanese stocks at the 95 percent confidence level should be 2.5 percent. Therefore, this was a very unusual move—even though we expect to exceed VaR in 5 percent of situations.

The marginal risk of each leg is also revealing. With a negative correlation between bonds and stocks, a hedged position typically would be long the two assets. Instead, Leeson was short the bond market, which market observers were at a loss to explain. A trader said, "This does not work as a hedge. It would have to be the other way round."[4] Thus Leeson was increasing his risk from the two legs of the position.

[4] *Financial Times,* March 1, 1995.

This is formalized in the table, which displays the marginal VaR computation. The β column is obtained by dividing each element of Σx by $x'\Sigma x$, for instance, -2.82 by 256,194 to obtain -0.000011. Multiplying by the VaR, we obtain the marginal change in VaR from increasing the bond position by \$1 million, which is $-\$0.00920$ million. Similarly, increasing the stock position by \$1 million increased the VaR by \$0.08935.

Overall, the component VaR owing to the total bond position is \$147.15 million; that owing to the stock position is \$688.01 million. By construction, these two numbers add up to the total VaR of \$835.16 million. This analysis shows that most of the risk was due to the Nikkei exposure and that the bond position, instead of hedging, made things even worse. As Box 6-1 shows, however, Leeson was able to hide his positions from the bank's VaR system.

BOX 6-1 Barings' Risk Management

The Barings case is a case in point of lack of trader controls. A good risk management system might have raised the alarm early and possibly avoided most of the \$1.3 billion loss.

Barings had installed in London a credit-risk management system in the 1980s. The bank was installing a market-risk management system in its London offices. The system, developed by California-based Infinity Financial Technology, has the capability to price derivatives and to support VaR reports. Barings' technology, however, was far more advanced in London than in its foreign branches. Big systems are expensive to install and support for small operations, which is why the bank relied heavily on local management.

The damning factor in the Barings affair was Leeson's joint responsibility for front- and back-office functions, which allowed him to hide trading losses. In July 1992, he created a special "error" account, numbered 88888, that was hidden from the trade file, price file, and London gross file. Losing trades and unmatched trades were parked in this account. Daily reports to Barings' Asset and Liability Committee showed Leeson's trading positions on the Nikkei 225 as fully matched. Reports to London therefore showed no risk. Had Barings used internal audits to provide independent checks on inputs, the company might have survived.

VaR TOOLS FOR GENERAL DISTRIBUTIONS

So far we have derived analytical expressions for these VaR tools assuming a normal distribution. These results can be generalized. In Equation (6.1), the portfolio return is a function of the positions on the individual components $R_p = f(w_1, \dots, w_N)$. Multiplying all positions by a constant k will enlarge the portfolio return by the same amount, that is,

$$kR_p = f(kw_1, \dots, kw_N) \qquad \textbf{(6.31)}$$

Such function is said to be *homogeneous of degree one,* in which case we can apply *Euler's theorem,* which states that

$$R_p = f(w_1,\dots,w_N) = \sum_{i=1}^{N} \frac{\partial f}{\partial w_i} w_i \qquad \textbf{(6.32)}$$

The portfolio VaR is simply a realization of a large dollar loss. Setting R_p to the portfolio VaR gives:

$$\begin{aligned} \text{VaR} &= \sum_{i=1}^{N} \frac{\partial \text{VaR}}{\partial w_i} \times w_i = \sum_{i=1}^{N} \frac{\partial \text{VaR}}{\partial x_i} \times x_i \\ &= \sum_{i=1}^{N} (\Delta \text{VaR}_i) \times x_i \end{aligned} \qquad \textbf{(6.33)}$$

This shows that the decomposition in Equation (6.28) is totally general. With a normal distribution, the marginal VaR is $\Delta\text{VaR}_i = \beta_i(\alpha\sigma_p)$, which is proportional to β_i. This analytical result also holds for *elliptical distributions.* In these cases, marginal VaR can be estimated using the sample beta coefficient, which uses all the sample information, such as the portfolio standard deviation, and as a result should be precisely measured.

Consider now another situation where the risk manager has generated a distribution of returns $R_{p,1}, \dots, R_{p,T}$, and cannot to approximate it by an elliptical distribution perhaps because of an irregular shape owing to option positions. VaR is estimated from the observation R_p^*. One can show that applying Euler's theorem gives

$$R_p^* = \sum_{i=1}^{N} E(R_i \mid R_p = R_p^*) w_i \qquad \textbf{(6.34)}$$

where the $E(\cdot)$ term is the expectation of the risk factor conditional on the portfolio having a return equal to VaR.[5]

[5] For proofs, see Tasche (2000) or Hallerbach (2003).

Thus CVaR_i could be estimated from the decomposition of R^* into the realized value of each component.

Such estimates, however, are less reliable because they are based on one data point only. Another solution is to examine a window of observations around R^* and to average the realized values of each component over this window.

FROM VaR TO PORTFOLIO MANAGEMENT

From Risk Measurement to Risk Management

Marginal VaR and component VaR are useful tools, best suited to small changes in the portfolio. This can help the portfolio manager to decrease the risk of the portfolio. Positions should be cut first where the marginal VaR is the greatest, keeping portfolio constraints satisfied. For example, if the portfolio needs to be fully invested, some other position, with the lowest marginal VaR, should be added to make up for the first change.

This process can be repeated up to the point where the portfolio risk has reached a global minimum. At this point, all the marginal VaRs, or the portfolio betas, must be equal:

$$\Delta \text{VaR}_i = \frac{\text{VaR}}{W} \times \beta_i = \text{constant} \qquad \textbf{(6.35)}$$

Table 6-4 illustrates this process with the previous two-currency portfolio. The original position of $2 million in CAD and $1 million in EUR created a VaR of $257,738, or portfolio volatility of 15.62 percent. The marginal VaR is 0.1521 for the EUR, which is higher than for the CAD.

As a result, the EUR position should be cut first while adding to the CAD position. The table shows the final risk-minimizing position. The weight on the EUR has decreased from 33.33 to 14.79 percent. The portfolio volatility has been lowered from 15.62 to 13.85 percent, which is a substantial drop. We also verify that the betas of all positions are equal when risk is minimized.

From Risk Management to Portfolio Management

The next step is to consider the portfolio expected return as well as its risk. Indeed, the role of the portfolio manager is to choose a portfolio that represents the best combination of expected return and risk. Thus we are moving from *risk management* to *portfolio management.* We will consider each portfolio in a graph that plots its expected return against its risk, as shown in Figure 6-5.

Define E_p as the expected return on the portfolio. This is a linear combination of the expected returns on the component positions, that is,

$$E_p = \sum_{i=1}^{N} w_i E_i \qquad \textbf{(6.36)}$$

For simplicity, all returns are defined in excess of the risk-free rate. In the figure, this translates all the points down by the same amount so that the risk-free asset is at the origin.

We then can define the best portfolio combinations as the portfolios that minimize risk for varying levels of expected return. This defines the *efficient frontier,* which is shown as a solid line in Figure 6-5.

TABLE 6-4 Risk-Minimizing Position

Asset	Original Position, w_i	Marginal VaR, ΔVaR_i	Final Position, w_i	Marginal VaR, ΔVaR_i	Beta β_i
CAD	66.67%	0.0528	85.21%	0.0762	1.000
EUR	33.33%	0.1521	14.79%	0.0762	1.000
Total	100.00%		100.00%		
Diversified VaR	$257,738		$228,462		
Standard deviation	15.62%		13.85%		

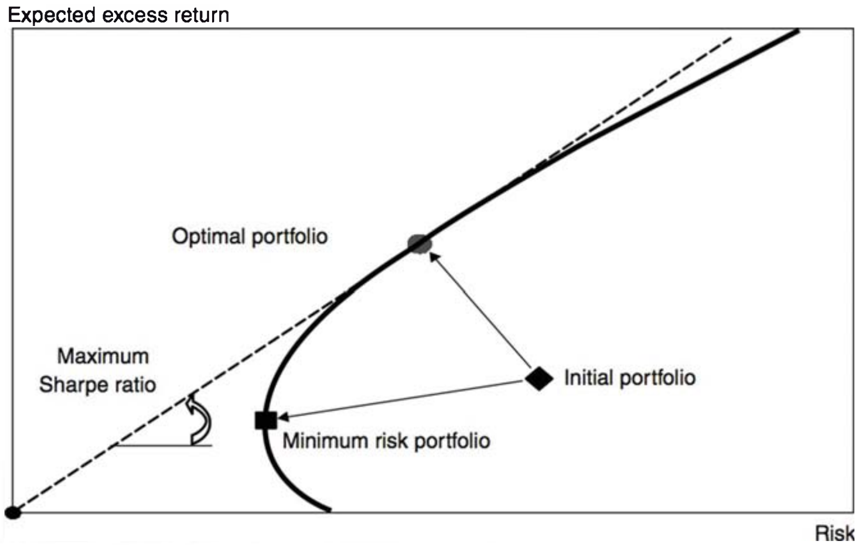

FIGURE 6-5 From VaR to portfolio management.

TABLE 6-5 Risk and Return-Optimizing Position

Asset	Expected Return E_i	Original Position w_i	Beta β_i	Ratio E_i/β_i	Final Position w_i	Beta β_i	Ratio E_i/β_i
CAD	8.00%	66.67%	0.615	0.1301	90.21%	1.038	0.0771
EUR	5.00%	33.33%	1.770	0.0282	9.79%	0.649	0.0771
Total		100.00%			100.00%		
Diversified VaR		$257,738			$230,720		
Standard deviation		15.62%			13.98%		
Expected return		7.00%			7.71%		
Sharpe ratio		0.448			0.551		

Suppose now that the objective function is to maximize the ratio of expected return to risk. This *Sharpe ratio* is

$$SR_p = \frac{E_p}{\sigma_p} \tag{6.37}$$

More generally, this could be written with VaR in the denominator.

How do we move from the current position to this optimal portfolio? The preceding section showed how to move the portfolio from its original position to the *global minimum-risk* portfolio. This portfolio, however, does not take expected returns into account.

We now wish to increase the portfolio expected return as well, moving to the portfolio with the highest Sharpe ratio. This portfolio is on the efficient set and maximizes the slope of the tangent from the origin. We call this portfolio the *optimal portfolio.* At this point, the ratio of all expected returns to marginal VaRs must be equal. This also can be written in terms of the excess expected return for each asset divided by its beta relative to the optimized portfolio. At the optimum,

$$\frac{E_i}{\Delta \text{VaR}_i} = \frac{E_i}{\beta_i} = \text{constant} \tag{6.38}$$

Note that this is simply a restatement of the *capital asset pricing model,* which states that the market portfolio must be mean-variance efficient. Roll (1977) showed that the efficiency of any portfolio implies that the expected return on any component asset must be proportional to its beta relative to this portfolio, that is,

$$E_i = E_m \beta_i \tag{6.39}$$

Thus, for each asset, the ratio between the excess return E_i and the beta must be constant.

Table 6-5 shows our two-currency portfolio, for which we assumed that E_1=8 percent and E_2=5 percent. The original position has a Sharpe ratio of 0.448. The ratio of E_i/β_i is 0.1301 for CAD, which is greater than the 0.0282 value for EUR. This implies that the CAD position should be increased to improve portfolio performance. Indeed, at the optimum, the CAD weight has increased from 66.67 to 90.21 percent. The portfolio Sharpe ratio has increased substantially from 0.448 to 0.551. We verify that the ratios E_i/β_i are identical for the two assets at the optimum. The same values of 0.0771 indicate that there is no reason to deviate from the final allocation.

CONCLUSIONS

This chapter has shown how to measure and manage risk using analytical methods based on the standard deviation. Such methods apply when risk factors have distributions that are jointly normal or, more generally, elliptical.

Analytical methods are particularly convenient because they lead to closed-form solutions that are easy to interpret. This is akin to the Black-Scholes model, an analytical model to price options. This model is used widely because it yields powerful insights that can be applied to all options, including those that are computed using numerical methods. Thus the VaR tools developed here for parametric VaR also can be used with nonparametric, simulation-based VaR models.

We have seen that the VaR approach is much richer than the computation of a single risk measure. It provides a framework for managing risk using VaR tools such as marginal VaR and component VaR. These measures can be used to analyze the effect of marginal changes in portfolio composition.

A typical situation is that of a bank trader who has to evaluate whether a proposed trade with a client will increase or decrease the risk of the existing portfolio. Marginal VaR provides useful information to control the risk profile throughout the day. If the trade is risk-decreasing, then the trader should adjust the bid-offer spread to increase the probability that the client will do the trade. On the other hand, a trade that increases risk should be discouraged.

At the end, however, risk is only one component of the portfolio management process. Expected returns must be considered as well. The role of the portfolio manager is to balance increasing risk against increasing expected returns.

This is where VaR methods prove their usefulness. Combining expected profits into a portfolio is an intuitive process because expected returns are additive. In contrast, risk is not additive and is a complicated function of the portfolio positions and risk-factor characteristics. This explains why the battery of VaR tools is useful to manage portfolios better.