Problem 3.

a) False

counter example:

Suppose $f(n) = n$ and $g(n) = \begin{cases} 0 & \forall n \text{ is odd} \\ n^2 & \forall n \text{ is even} \end{cases}$

part a) to show $f(n) \in O(g(n))$

then there $\exists$ constans $c > 0$ and $n_0 > 0$

s.t. $0 \le n \le c \cdot g(n)$ for $\forall n \ge n_0$

However, when $n$ is an odd number, ~~and~~.

$\Rightarrow n > c \cdot g(n)$, where $c g(n) = 0 \Rightarrow f(n) > c g(n)$

Therefore, $f(n) \notin O(g(n))$

Part b) to show $f(n) \in \Omega(g(n))$

then there $\exists$ constants $c > 0$ and $n_0 > 0$

s.t. $0 \le c g(n) \le f(n)$ for $\forall n \ge n_0$

Howevever, when $n$ is an even number.

$\Rightarrow n \le c n^2$ since $n \le n^2$ and $1 \le c$.

$\Rightarrow f(n) \le c g(n)$

Therefore. $f(n) \notin \Omega(g(n))$

from part a) and b) we can show that $\begin{cases} f(n) \notin O(g(n)) \\ f(n) \notin \Omega(g(n)) \end{cases}$

which shows that $f(n) \notin \Theta(g(n))$ by order notations

Thus, the statement is false

b) $f(n) \in \Theta(g(n))$ and $h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{h(n)} \in \Theta(1)$

Proof: Assume $f(n) \in \Theta(g(n))$ and $h(n) \in \Theta(g(n))$

then there $\exists$ constants $c_1, c_2 > 0$ and $n_1 > 0$

s.t $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for $\forall n \ge n_1$

and there $\exists$ constants $c_3, c_4 > 0$ and $n_2 > 0$

s.t. $0 \le c_3 g(n) \le h(n) \le c_4 g(n)$ for $\forall n \ge n_2$

Then, we got

$$\frac{1}{c_3 g(n)} \ge \frac{1}{h(n)} \ge \frac{1}{c_4 g(n)} \text{ for } \forall n \ge n_2$$

let $n_3 = \max\{n_1, n_2\}$

Thus we have

$$\frac{c_2 g(n)}{c_3 g(n)} \ge \frac{f(n)}{h(n)} \ge \frac{c_1 g(n)}{c_4 g(n)} \text{ for } \forall n \ge n_3$$

$$\Rightarrow \frac{c_2}{c_3} \ge \frac{f(n)}{h(n)} \ge \frac{c_1}{c_4} \text{ for } \forall n \ge n_3$$

if we want to show $\frac{f(n)}{h(n)} \in \Theta(1)$

then there $\exists$ constants $c_5, c_6 > 0$ and $n_4 > 0$

s.t $0 \le c_5 \le \frac{f(n)}{h(n)} \le c_6$ for $\forall n \ge n_4$

let $c_5 = \frac{c_1}{c_4}$, $c_6 = \frac{c_2}{c_3}$

Thus, $c_5 \le \frac{f(n)}{h(n)} \le c_6$ for $\forall n \ge n_4$

Therefore, we can show that $\frac{f(n)}{h(n)} \in \Theta(1)$

$\Rightarrow$ the statement is true

c) $f(n) \in \Theta(g(n)) \Rightarrow 2^{f(n)} \in \Theta(2^{g(n)})$

Proof: the statement is false

counter example: let $f(n) = \log n$. $g(n) = 2\log n$

then $f(n) \le c_2 g(n)$ for $c_2 = 1$. $\forall n \ge n_0$

$f(n) \ge c_1 g(n)$ for $c_1 = \frac{1}{24}$ $\forall n \ge n_0$

Thus $f(n) \in \Theta(g(n))$

$\because 2^{f(n)} = 2^{\log n} = n$

$2^{g(n)} = 2^{2\log n} = n^2$

However $f(n) \notin \Omega(g(n))$

$\Rightarrow f(n) \notin \Theta(g(n))$.

Thus, the statement is false

d) $\min(f(n), g(n)) \in \Theta\left(\frac{f(n)g(n)}{f(n)+g(n)}\right)$

Proof: Suppose there $\exists$ constants $c_1\ c_2 > 0$ and $n_0 > 0$

s.t. $0 \le c_1 \cdot \frac{f(n)g(n)}{f(n)+g(n)} \le \min(f(n), g(n)) \le c_2 \frac{f(n)g(n)}{f(n)+g(n)}$ for $\forall n \ge n_0$.

since $f(n) \cdot g(n) = \min(f(n), g(n)) \cdot \max(f(n), g(n))$

therefore $\frac{f(n) \cdot g(n)}{f(n)+g(n)} = \frac{\min(f(n), g(n)) \cdot \max(f(n), g(n))}{f(n)+g(n)}$

since $\frac{\max(f(n), g(n))}{f(n)+g(n)} < 1$

$\Rightarrow \frac{f(n) \cdot g(n)}{f(n)+g(n)} < \min(f(n), g(n))$ $c_1 = 1$ $\overset{n_0=1}{\checkmark}$ $\forall n \ge n_0$

and we can got $\min(f(n), g(n)) \le c_2 \frac{f(n)g(n)}{f(n)+g(n)}$ by choose $c_2 = 2$, $n_0 = 1$, then

Thus we can show that

$\min(f(n), g(n)) \in \Theta\left(\frac{f(n) \cdot g(n)}{f(n)+g(n)}\right)$