CS 240 Assignment 1

Problem 1

a) $27n^7 + 17n^3 \log n + 2016$ is $O(n^9)$

Proof: if $27n^7 + 17n^3 \log n + 2016$ is $O(n^9)$

then, $\exists$ constant $c > 0$, $n_0 > 0$.

s.t. $27n^7 + 17n^3 \log n + 2016 \leq cn^9 \quad \forall n \geq n_0$

find some c, n.

note that $27n^7 \leq 27n^9 \quad \forall n > 0$

and $2016 \leq 2016n^9 \quad \forall n > 0$

$\because \log n < n \quad \forall n \geq 1$

and $17n^3 \leq 17n^8 \quad \forall n > 0$

$17n^3 \cdot \log n \leq 17n^8 \cdot n = 17n$

then $27n^7 + 17n^3 \log n + 2016 \leq 27n^9 + 17n^9 + 2016n^9$

$27n^7 + 17n^3 \log n + 2016 \leq 2060n^9$

so let $n_0 = 1$ $c = 2060$

$\Rightarrow 27n^7 + 17n^3 \log n + 2016 = 27 + 2016 = 2048$

$\Rightarrow 2060n^9 = 2060$

Thus $0 \leq 2048 \leq 2060$

Therefore, $0 \leq 27n^7 + 17n^3 \log n + 2016 \leq cn^9$ for $\forall n \geq n_0$

which shows $27n^7 + 17n^3 \log n + 2016 \in O(n^9)$

b) $n^2(\log n)^{1.0001}$ is $\Omega(n^2)$

Proof: by Order notation. $\exists$ constans $c>0$ and $n_0>0$.

s.t $0 \le cn^2 \le n^2(\log n)^{1.0001}$ for $\forall n \ge n_0$

since $\log(n) \ge 1$ for $\forall n > 2$, then $\log(n) \ge 1$, $\forall n \ge 2$

$$\Rightarrow \log(n)^{1.0001} \ge 1^{1.0001}$$

$$\log(n)^{1.0001} \ge 1$$

$$n^2 \log(n)^{1.0001} \ge n^2 \quad \forall n \ge 2.$$

let $n_0 = 2$ $c_1 = 1$

$$\Rightarrow 1^2 \log(1)^{1.0001} = 1 \ge 1 \quad \forall$$

Thus $n^2 \log(n)^{1.0001} \in \Omega(n^2)$.

c) $\frac{n^2}{n+\log n}$ is $\Theta(n)$

Proof: by order notation,

Θ-notation: if there exist constants $c_1, c_2 > 0$ and $n_0 > 0$

s.t. $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for $\forall n \ge n_0$

then we can devide it in to two parts:

use O-notation to show $0 \le f(n) \le c_1 g(n)$ for $\forall n \ge n_0$

and Ω-notation to show $0 \le c_2 g(n) \le f(n)$ for $\forall n \ge n_0$.

part a) show $\frac{n^2}{n+\log n} \in O(n)$.

Since $\log(n) > 0$ for all $n > 1$.

$$\Rightarrow n \le n + \log(n) \quad \forall n > 1.$$

$$\Rightarrow \frac{1}{n} \ge \frac{1}{n+\log(n)} \quad \forall n > 1$$

$$\Rightarrow \frac{n^2}{n} \ge \frac{n^2}{n+\log(n)} \quad \forall n > 1$$

$$\Rightarrow \frac{n^2}{n+\log(n)} \le c_1 n \quad \forall n > 1$$

let $n_0 = 2$ $c_1 = 1$

then $\frac{n^2}{n+\log(n)} = \frac{4}{2+\log(2)} \approx 1.738$

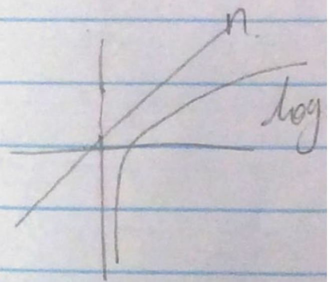
$c_1 n = 1 \cdot 2 = 2$

$\Rightarrow 1.738 \leq 2$

Thus $\frac{n^2}{n+\log(n)} \leq c_1 n$

we can show that $\frac{n^2}{n+\log(n)} \in O(n)$

part b). show $\frac{n^2}{n+\log(n)} \in \Omega(n)$



since $\log(n) \leq n$ for $\forall n > 1$

$\Rightarrow n + \log(n) \leq 2n$ for $\forall n > 1$

$\Rightarrow \frac{1}{n+\log(n)} \geq \frac{1}{2n}$ $\forall n > 1$

$\Rightarrow \frac{n^2}{n+\log(n)} \geq \frac{n^2}{2n}$ $\forall n > 1$

$\Rightarrow \frac{n^2}{n+\log(n)} \geq \frac{1}{2} n$ $\forall n > 1$

let $n_0 = 2$. $c_2 = \frac{1}{2}$

$\Rightarrow \frac{n^2}{n+\log n} = \frac{4}{2+\log 2} \approx 1.738$ $\frac{1}{2} n = \frac{1}{2} \cdot 2 = 1$

$\Rightarrow 1.738 \geq 1 \Rightarrow \frac{n^2}{n+\log(n)} \geq \frac{1}{2} n$

Thus $\frac{n^2}{n+\log(n)} \in \Omega(n)$

from the results of part a) and b), $\frac{n^2}{n+\log(n)} \in O(n)$, and $\frac{n^2}{n+\log(n)} \in \Omega(n)$, we can show that $\frac{n^2}{n+\log(n)} \in \Theta(n)$.

d) n^n is $\omega(n^{20})$.

Proof: if $n^n \in \omega(n^{20})$.

then $\forall$ constans $c>0$, $\exists$ constant $n_0>0$ s.t

$0 \le cn^{20} < n^n$). for all $n \geq n_0$.

since $n^n = n^{20} \cdot n^{n-20}$.

we want $c < n^{n-20}$

let $c>0$, choose $c=1$.

$\Rightarrow 1 < n_0^{n_0-20}$

Thus $\exists n_0 > 0$ s.t. $c < n^{n-20}$, eg $n_0 = 30$.

There fore $c < n^{n-20}$ $\exists n_0 > 0$.

$cn^{20} < n^{20} \cdot n^{n-20}$

$cn^2 < n^n$ $\exists n_0 > 0$.

let. $c=1$, $\exists . n = 30$.

$cn^2 = 1 \cdot 30^2 = 900 < 30^{30} \approx 2.06 \times 10^{44}$.

Then. $\forall$ constans $c>$. $\exists$ constant $n_0 > 0$

s.t. $0 \le cn^{20} < n^n$

Thus, it shows $n^n \in \omega(n^{20})$