CS245 Tutorial

Oct. 5, 2015

1. Convert the formula $(p \wedge \neg r) \vee \neg(q \vee \neg r)$ to conjunctive normal form.

$(p \wedge \neg r) \vee \neg(q \vee \neg r)$	
$\equiv (p \wedge \neg r) \vee (\neg q \wedge \neg\neg r)$	De Morgan
$\equiv (p \wedge \neg r) \vee (\neg q \wedge r)$	Negation
$\equiv ((p \wedge \neg r) \vee \neg q) \wedge ((p \wedge \neg r) \vee r)$	Distributive
$\equiv (p \vee \neg q) \wedge (\neg r \vee \neg q) \wedge (p \vee r) \wedge (\neg r \vee r)$	Distributive ($2\times$)
$\equiv (p \vee \neg q) \wedge (\neg r \vee \neg q) \wedge (p \vee r) \wedge \mathtt{T}$	Excluded Middle
$\equiv (p \vee \neg q) \wedge (\neg r \vee \neg q) \wedge (p \vee r)$	Simplification I

2. Consider the set of propositional formulas: $\{a \to b, (b \wedge c) \to d, (d \wedge (e \vee f)) \to g, a, c, \neg e\}$

Convert the formulas to conjunctive normal form and for each of the following queries, either prove the query using resolution refutation or show that the query does not logically follow.

Q1. d
Q2. $f \to g$
Q3. $g \to \neg f$

Conversion of each formula into conjunctive normal form.

1.	$\neg a \vee b$
2.	$\neg b \vee \neg c \vee d$
3a.	$\neg d \vee g \vee \neg e$
3b.	$\neg d \vee g \vee \neg f$
5.	a
6.	c
7.	$\neg e$

A resolution refutation proof of Q1.

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	assumption (from negated conclusion)
8.	$\neg b \vee \neg c$	2, 7
9.	$\neg b$	5, 8
10.	$\neg a$	1, 9
11.	$\perp$	4, 10

A resolution refutation proof of Q2.

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	assumption (from negated conclusion)
8.	$\neg g$	assumption (from negated conclusion)
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9
11.	$\neg b \vee \neg c$	2, 10
12.	$\neg b$	5, 11
13.	$\neg a$	1, 12
14.	$\perp$	4, 13

Q3 does not logically follow. One can show that there is no proof; i.e., resolving all possible clauses together does not lead to the empty clause. One can also show a counter-example: where the assumptions are all true but the query $g \rightarrow \neg f$ is false.

3. Consider the following:

   > "If the dragon is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the dragon does not have scales, then it is mortal and not a mammal. The dragon is magical if it has scales."

   Represent the above in propositional logic.

   I will use the following propositions:

   y = The dragon is mythical.
   o = The dragon is mortal.
   m = The dragon is a mammal.
   s = The dragon has scales.
   g = The dragon is magical.

   The sentences are given by:

   s0. If the dragon is mythical, then it is immortal.
   $y \rightarrow \neg o$
   s1. If the dragon is not mythical, then it is a mortal mammal.
   $\neg y \rightarrow (o \wedge m)$
   s2. If the dragon does not have scales, then it is mortal and not a mammal.
   $\neg s \rightarrow (o \wedge \neg m)$
   s3. The dragon is magical if it has scales.
   $s \rightarrow g$

Conversion of each formula into conjunctive normal form.

c0.	$\neg y \vee \neg o$
c1a.	$y \vee o$
c1b.	$y \vee m$
c2a.	$s \vee o$
c2b.	$s \vee \neg m$
c3.	$\neg s \vee g$

Let Σ be the set consisting of the above clauses. Show whether or not the following possible queries logically follow from Σ. If a query logically follows, give a resolution refutation proof. If a query does not logically follow, show a counter-example.

Q1. Is the dragon mythical?

Neither y nor $\neg y$ is a logical consequence of Σ. The definition of $\Sigma \models \varphi$ says that every model of Σ must also be a model of φ. One way of showing that a given φ is not a logical consequence is to find a counter-example, an interpretation that satisfies Σ but does not satisfy φ. Consider the following interpretation.

y is false,
s is true, g is true, o is true, m is true.

We can verify that each of the formulas in Σ is satisfied, but the query y is not satisfied. Similarly, consider the following interpretation.

y is true, s is true, g is true,
o is false, m is false.

We can verify that each of the formulas in Σ is satisfied, but the query $\neg y$ is not satisfied.

Q2. Is the dragon magical?

g is a logical consequence of Σ.

Resolution refutation proof.

1.	$\neg y \vee \neg o$	assumption
2.	$y \vee o$	assumption
3.	$y \vee m$	assumption
4.	$s \vee o$	assumption
5.	$s \vee \neg m$	assumption
6.	$\neg s \vee g$	assumption
7.	$\neg g$	assumption (from negated conclusion)
8.	$\neg s$	6, 7
9.	o	4, 8
10.	$\neg y$	1, 9
11.	m	3, 10
12.	s	5, 11
13.	$\perp$	8, 12

In the above proof I list the formula followed by the two clauses which I have resolved together to derive the formula.

Q3. Does the dragon have scales?

s is a logical consequence of Σ.

Resolution refutation proof.

1.	$\neg y \vee \neg o$	assumption
2.	$y \vee o$	assumption
3.	$y \vee m$	assumption
4.	$s \vee o$	assumption
5.	$s \vee \neg m$	assumption
6.	$\neg s \vee g$	assumption
7.	$\neg s$	assumption (from negated conclusion)
8.	$\neg m$	5, 7
9.	y	3, 8
10.	$\neg o$	1, 9
11.	s	3, 10
12.	$\perp$	7, 11

Once again, in the above proof I list the formula followed by the two clauses which I have resolved together to derive the formula.