CS245 Tutorial

Sept. 21, 2015

1. In this question, use the following propositional variables.

$$s: \text{ I study for exams} \qquad g: \text{I get a good grades}$$
$$p: \text{I will pass the class} \qquad h: \text{I eat healthy food}$$

(a) If I study for exams, then I get good grades.
(b) I do not eat healthy food whether or not I study for exams.
(c) I will pass the class only if I get a good grades.
(d) If I do not study for exams, then I get good grades only if I eat healthy food.
(e) I will either pass the class or eat healthy food, but not both.

Solutions:

(a) $s \to g$
(b) $(s \vee \neg s) \to \neg h$
(c) $p \to g$
(d) $\neg s \to (g \to h)$
(e) $(p \vee h) \wedge \neg(p \wedge h)$

2. Given the set of connectives $C = \{\neg, \wedge, \vee, \to, \leftrightarrow\}$ and any set P of propositional variables,

(a) What is an example of a well-formed formula?
(b) What is an example an expression that is **not** a well-formed formula?
(c) Briefly review inductive definition of a well-formed formula.

3. Use structural induction to prove the following:

> Suppose that the number of occurrences of atoms in a formula is m and the number of occurrences of binary connectives (any of $\wedge$, $\vee$, $\rightarrow$, and/or $\leftrightarrow$) is n. Then $m = n + 1$.

Proof. Start by proving this is true for the base case—atomic formulas. An atomic formula has one variable and no connective; that is, $m = 1$ and $n = 0$. Therefore $m = n + 1$, as required.

For the inductive step, suppose two formulas φ and ψ satisfy the requirement. That is, defining

m_φ: the number of occurrences of atoms in φ,
m_ψ: the number of occurrences of atoms in ψ,
n_φ: the number of occurrences of binary connectives in φ,
n_ψ: the number of occurrences of binary connectives in ψ,

we assume (as the induction hypothesis) that $m_\varphi = n_\varphi + 1$ and $m_\psi = n_\psi + 1$.

There are five possible cases; one for each of the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.
(Note: we have to consider all connectives—not just the binary ones!)

- The formula $(\neg\varphi)$ has $m = m_\varphi$ and $n = n_\varphi$, and by hypothesis $m = n + 1$.
- The formula $(\varphi \wedge \psi)$ has $m = m_\varphi + m_\psi$ and $n = n_\varphi + n_\psi + 1$. Thus

$$m = m_\varphi + m_\psi = (n_\varphi + 1) + (n_\psi + 1) = n_\varphi + n_\psi + 1 + 1 = n + 1$$

- The formula $(\varphi \vee \psi)$ has $m = m_\varphi + m_\psi$ and $n = n_\varphi + n_\psi + 1$. Thus

$$m = m_\varphi + m_\psi = (n_\varphi + 1) + (n_\psi + 1) = n_\varphi + n_\psi + 1 + 1 = n + 1$$

- The formula $(\varphi \rightarrow \psi)$ has $m = m_\varphi + m_\psi$ and $n = n_\varphi + n_\psi + 1$. Thus

$$m = m_\varphi + m_\psi = (n_\varphi + 1) + (n_\psi + 1) = n_\varphi + n_\psi + 1 + 1 = n + 1$$

- The formula $(\varphi \leftrightarrow \psi)$ has $m = m_\varphi + m_\psi$ and $n = n_\varphi + n_\psi + 1$. Thus

$$m = m_\varphi + m_\psi = (n_\varphi + 1) + (n_\psi + 1) = n_\varphi + n_\psi + 1 + 1 = n + 1$$

In all cases, a formula formed from φ and ψ has $m = n + 1$, as required. This completes the inductive step.

By the principle of structural induction, every formula has the required property.

[Note: students may ask about using induction on an integer. This is fine; the integer can be the length of the formula, its number of connectives, etc. Do emphasize that the property needs to be stated carefully: "Every formula of length k *or less*...," or the like.]