CS245 Tutorial

Oct. 26, 2015

1. Give a natural deduction proof of:

$$\forall x \cdot ((P(x) \vee S(x)) \rightarrow (R(x) \wedge Q(x))),\ \neg(\forall x \cdot R(x) \wedge Q(x))\ \vdash\ \exists x \cdot \neg P(x)$$

Strategy: Proof by contradiction. From assuming $\neg \exists x \cdot \neg P(x)$, we can prove $\forall x \cdot P(x)$. Then, by working with a fresh variable, we can use premise 1 to conclude $\forall x \cdot R(x) \wedge Q(x)$, which will be a contradiction with premise 2.

1.	$\forall x \cdot (P(x) \vee S(x)) \rightarrow (R(x) \wedge Q(x))$	premise
2.	$\neg \forall x \cdot (R(x) \wedge Q(x))$	premise
3.	$\neg \exists x \cdot \neg P(x)$	assumption
4.	u fresh	
5.	$\neg P(u)$	assumption
6.	$\exists x \cdot \neg P(x)$	$\exists$i: 5
7.	$\perp$	$\neg$e: 3, 6
8.	$\neg\neg P(u)$	$\neg$i: 5–7
9.	$P(u)$	$\neg$e: 8
10.	$\forall x \cdot P(x)$	$\forall$i: 4–9
11.	v fresh	
12.	$P(v)$	$\forall$e: 10
13.	$P(v) \vee S(v)$	$\vee$i: 12
14.	$(P(v) \vee S(v)) \rightarrow (R(v) \wedge Q(v))$	$\forall$e: 1
15.	$R(v) \wedge Q(v)$	$\rightarrow$e: 13, 14
16.	$\forall x \cdot (R(x) \wedge Q(x))$	$\forall$i: 11–15
17.	$\perp$	$\neg$e: 2, 16
18.	$\neg\neg \exists x \cdot \neg P(x)$	$\neg$i: 3–17
19.	$\exists x \cdot \neg P(x)$	$\neg$e: 18

2. Show that

$$\forall x \cdot ((P(x) \vee S(x)) \rightarrow (R(x) \wedge Q(x))),\ \neg(\forall x \cdot (R(x) \wedge Q(x))) \ \not\models\ \neg \exists x \cdot \neg P(x)$$

Strategy: We know this is true because we proved $\exists x \cdot \neg P(x)$ in the previous question. But let us give a counter-example, an interpretation that satisfies each of,

$$\forall x \cdot ((P(x) \vee S(x)) \rightarrow (R(x) \wedge Q(x))),\ \neg(\forall x \cdot (R(x) \wedge Q(x)))$$

but does not satisfy,

$$\neg \exists x \cdot \neg P(x)$$

Consider the interpretation $\mathcal{M}$:

- domain: Let the domain be {a, b}.
- predicates: Let the predicate symbols P, Q, R, and S all be the relation {a}.

We can verify that $\neg \exists x \cdot \neg P(x)$ is false in this interpretation (i.e., $\forall x \cdot P(x)$ is false). We can verify that $\forall x \cdot ((P(x) \vee S(x)) \rightarrow (R(x) \wedge Q(x)))$ is true in this interpretation and that $\neg(\forall x \cdot (R(x) \wedge Q(x)))$ is true in this interpretation.

3. Give a natural deduction proof of:

$$\forall x \cdot ((P(x) \vee S(x)) \rightarrow R(x)),\ \exists y \cdot \neg(P(y) \wedge \neg S(y)),\ \neg(\forall z \cdot (Q(z) \vee \neg S(z)))\ \vdash\ \exists x \cdot R(x)$$

Strategy: To get $R(x)$ for a fresh variable, we'll need to use premise 1 and modus ponens, so we need to show $\exists x \cdot P(x) \ \vee \ S(x)$. Premise 3 is helpful because we know using quantifier equivalences that $\neg(\forall z \cdot Q(z) \vee \neg S(z))$ is logically equivalent to $\exists z \cdot \neg Q(z) \wedge S(z)$. If we get $S(z)$ for an unknown variable, we can get $\exists x \cdot P(x) \vee S(x)$. So we start by proving $\exists x \cdot S(x)$. Notice that premise 2 is not useful.

1.	$\forall x \cdot ((P(x) \vee S(x)) \rightarrow R(x))$	premise
2.	$\exists y \cdot \neg(P(y) \wedge \neg S(y))$	premise
3.	$\neg(\forall z \cdot (Q(z) \vee \neg S(z)))$	premise
4.	$\neg(\exists x \cdot S(x))$	assumption
5.	u fresh	
6.	$S(u)$	assumption
7.	$\exists x \cdot S(x)$	∃i: 6
8.	$\perp$	¬e: 4, 7
9.	$\neg S(u)$	¬i: 6–8
10.	$Q(u) \vee \neg S(u)$	∨i: 9
11.	$\forall z \cdot (Q(z) \vee \neg S(z))$	∀i: 5–10
12.	$\perp$	¬e: 3, 11
13.	$\neg\neg \exists x \cdot S(x)$	¬i: 4–12
14.	$\exists x \cdot S(x)$	¬e: 13
15.	v fresh, $S(v)$	assumption
16.	$P(v) \vee S(v)$	∨i: 15
17.	$(P(v) \vee S(v)) \rightarrow R(v)$	∀e: 1
18.	$R(v)$	→e: 16, 17
19.	$\exists x \cdot R(x)$	∃i: 18
20.	$\exists x \cdot R(x)$	∃e: 14, 15–19

4. Give a natural deduction proof of:

$$\forall x \cdot ((P(x) \vee R(x)) \to \neg Q(x)),\ \exists x \cdot \neg(\neg P(x) \wedge \neg R(x)) \quad \vdash \quad \exists x \cdot \neg Q(x)$$

1.	$\forall x \cdot (P(x) \vee R(x)) \to \neg Q(x)$	premise
2.	$\exists x \cdot \neg(\neg P(x) \wedge \neg R(x))$	premise
3.	u fresh, $\neg(\neg P(u) \wedge \neg R(u))$	assumption
4.	$\neg(P(u) \vee R(u))$	assumption
5.	$P(u)$	assumption
6.	$P(u) \vee R(u)$	$\vee$i: 5
7.	$\perp$	$\neg$e: 4, 6
8.	$\neg P(u)$	$\neg$i: 5–7
9.	$R(u)$	assumption
10.	$P(u) \vee R(u)$	$\vee$i: 9
11.	$\perp$	$\neg$e: 4, 10
12.	$\neg R(u)$	$\neg$i: 9–11
13.	$\neg P(u) \wedge \neg R(u)$	$\wedge$i: 8, 12
14.	$\perp$	$\neg$e: 3, 13
15.	$\neg\neg(P(u) \vee R(u))$	$\neg$i: 4–14
16.	$P(u) \vee R(u)$	$\neg$e: 15
17.	$(P(u) \vee R(u)) \to \neg Q(u)$	$\forall$e: 1
18.	$\neg Q(u)$	$\to$e: 16, 17
19.	$\exists x \cdot \neg Q(x)$	$\exists$i: 18
20.	$\exists x \cdot \neg Q(x)$	$\exists$e: 3–19

5. Show that

$$\forall x \cdot ((P(x) \vee R(x)) \to \neg Q(x)),\ \exists x \cdot \neg(\neg P(x) \wedge \neg R(x)) \quad \models \quad \exists x \cdot \neg Q(x)$$

Use semantic arguments (about possible interpretations/models).

Strategy: Suppose, by way of contradiction, that there is some interpretation such that $\exists x \cdot \neg Q(x)$ is false in the interpretation but the premises are all true in the interpretation. This means, of course, that $\neg\exists x \cdot \neg Q(x)$ is true in the interpretation; i.e., $\forall x \cdot Q(x)$ is true. Consider the premise $\forall x \cdot ((P(x) \vee R(x)) \to \neg Q(x))$. Since $\forall x \cdot \neg Q(x)$ is false, it must be the case that $\forall x \cdot (P(x) \vee R(x))$ is false (otherwise the premise would not be true). But this contradicts the premise, $\exists x \cdot \neg(\neg P(x) \wedge \neg R(x))$; i.e., $\exists x \cdot (P(x) \vee R(x))$. Hence, such an interpretation cannot exist.

6. Give a natural deduction proof of:

$$\neg(\exists y \cdot \forall x \cdot (P(x) \lor Q(y))) \;\vdash\; \forall y \cdot \neg Q(y)$$

1.	$\neg(\exists y \cdot \forall x \cdot (P(x) \lor Q(y)))$	premise
2.	u fresh	
3.	$Q(u)$	assumption
4.	v fresh	
5.	$P(v) \lor Q(u)$	$\lor$i: 3
6.	$\forall x \cdot (P(x) \lor Q(u))$	$\forall$i: 4–5
7.	$\exists y \cdot \forall x \cdot (P(x) \lor Q(y))$	$\exists$i: 6
8.	$\bot$	$\neg$e: 1, 7
9.	$\neg Q(u)$	$\neg$i: 3–8
10.	$\forall y \cdot \neg Q(y)$	$\forall$i: 2–9

7. Give a natural deduction proof of:

$$\begin{array}{ll} & \forall x \cdot \forall y \cdot ((P(x) \wedge Q(y)) \rightarrow (x = y)), \\ & \exists x \cdot (P(x) \wedge R(x)), \\ & \exists x \cdot (Q(x) \wedge S(x)) \\ \vdash & \exists x \cdot (R(x) \wedge S(x)) \end{array}$$

1.	$\forall x \cdot \forall y \cdot ((P(x) \wedge Q(y)) \rightarrow (x = y))$	premise
2.	$\exists x \cdot (P(x) \wedge R(x))$	premise
3.	$\exists x \cdot (Q(x) \wedge S(x))$	premise
4.	u fresh, $P(u) \wedge R(u)$	assumption
5.	v fresh, $Q(v) \wedge S(v)$	assumption
6.	$\forall y \cdot ((P(u) \wedge Q(y)) \rightarrow (u = y))$	$\forall$e: 1
7.	$P(u) \wedge Q(v) \rightarrow (u = v)$	$\forall$e: 6
8.	$P(u)$	$\wedge$e: 4
9.	$Q(v)$	$\wedge$e: 5
10.	$P(u) \wedge Q(v)$	$\wedge$i: 8, 9
11.	$u = v$	$\rightarrow$e: 7, 10
12.	$R(u)$	$\wedge$e: 4
13.	$S(v)$	$\wedge$e: 5
14.	$S(u)$	$=$e: 11, 13
15.	$R(u) \wedge S(u)$	$\wedge$i: 12, 14
16.	$\exists x \cdot (R(x) \wedge S(x))$	$\exists$i: 15
17.	$\exists x \cdot (R(x) \wedge S(x))$	$\exists$e: 3, 5–16
18.	$\exists x \cdot (R(x) \wedge S(x))$	$\exists$e: 2, 4–17