Surname:	Personal name:	ID #:

Circle the time/room of your tutorial:		8:30, MC 2035 (TUT 101)
11:30, RCH 204 (TUT 107)	12:30, MC 2034 (TUT 104)	12:30, RCH 305 (TUT 106)
1:30, RCH 308 (TUT 105)	4:30, MC 4060 (TUT 102)	4:30, MC 2034 (TUT 103)

Due Wednesday, Oct. 21, to the drop-box near MC 4065.
Attach this sheet as a cover sheet on your submission.

Question 1 (15 marks).

For each part, give a well-formed formula of (First-Order) Predicate Logic that expresses the condition given.

Use the following relation and constant symbols in your formulas, with the given meaning (interpretation). You should not need any others.

$P(x)$**:** x is a program.

$U(w, x, y)$**:** Program w with input x produces output y.

$N(x)$**:** x is a natural number.

$E(x)$**:** x is an even number.

$A(x, y, z)$**:** Adding x and y produces z.

$M(x, y, z)$**:** Multiplying x and y produces z.

$=$**,** $<$**:** Equality and less-than, as usual.

$0, 1$**:** The numbers zero and one, as usual.

(The domain must include both programs and natural numbers. It may also contain other things. Your formulas should work in any such domain.)

(a) Every even number is divisible by two.

(b) Some natural number is less than its square.

(c) Program y always produces output, whatever its input.

(d) Program z can produce any output, if given an appropriate input.

(e) Some program computes square roots of natural numbers that have a square root.

Marks:	1		2		3		Total:	

Question 2 (8 marks).

Let φ be the first-order logic formula $(\forall x \cdot (\exists y \cdot P(f(x), y)))$. where $f^{(1)}$ is a function symbol and $P^{(2)}$ is a predicate symbol.

(a) Give a model $\mathcal{M}_1$ such that $\mathcal{M}_1 \models \varphi$. Explain why your choice of $\mathcal{M}_1$ is correct.

(b) Give a model $\mathcal{M}_2$ such that $\mathcal{M}_2 \not\models \varphi$. Explain why your choice of $\mathcal{M}_2$ is correct.

Question 3 (9 marks). For this question, we use the symbols '+', '$\cdot$' and '=' with their normal mathematical meanings: addition, multiplication and equality, respectively. Consider the formula

$$\forall y \cdot \forall z \cdot \big(x = y \cdot z \rightarrow (y = x \vee z = x)\big) \ ,$$

which might be written to mean "x is a prime number".

For each of the domains below, explain what it actually means; that is, describe the set of x in the domain that make the formula true. If the set is finite, list its elements; if the set is infinite, describe in one English phrase or sentence the elements of the set.

(a) Domain $\mathbb{N}$, the natural numbers (including zero).
(b) Domain $\mathbb{Z}$, the integers.
(c) Domain $\mathbb{Q}$, the rational numbers.

Study Exercises

SE 1. Huth and Ryan, Exercises 2.1, Problem 1

SE 2. Huth and Ryan, Exercises 2.1, Problem 3

SE 3. Huth and Ryan, Exercises 2.2, Problem 1

SE 4. Huth and Ryan, Exercises 2.2, Problem 3

SE 5. On the overheads, you saw the example formula

$$\forall \varepsilon \cdot \Big(\varepsilon > 0 \rightarrow \exists \delta \cdot \big(\delta > 0 \wedge \forall y \cdot (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)\big)\Big) \ ,$$

which expresses that function f is continuous at the point x. To express that f is continuous everywhere, we can add "$\forall x$" in front of that formula. What about "f is uniformly continuous"? Consider the following formulas—which one(s), if any, express uniform continuity? What do the others express?

$$\forall \varepsilon \cdot \forall x \cdot \Big(\varepsilon > 0 \rightarrow \exists \delta \cdot \big(\delta > 0 \wedge \forall y \cdot (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)\big)\Big) \qquad (1)$$

$$\forall \varepsilon \cdot \Big(\varepsilon > 0 \rightarrow \forall x \cdot \exists \delta \cdot \big(\delta > 0 \wedge \forall y \cdot (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)\big)\Big) \qquad (2)$$

$$\forall \varepsilon \cdot \Big(\varepsilon > 0 \rightarrow \exists \delta \cdot \forall x \cdot \big(\delta > 0 \wedge \forall y \cdot (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)\big)\Big) \qquad (3)$$

$$\forall \varepsilon \cdot \Big(\varepsilon > 0 \rightarrow \exists \delta \cdot \big(\delta > 0 \wedge \forall x \cdot \forall y \cdot (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)\big)\Big) \qquad (4)$$

Pay particular attention to the quantifiers and their placement in the formulas.