1. $\neg(p \vee q) \vdash \neg p \wedge \neg q$

   The conclusion is of the form $\alpha \wedge \beta$ so this suggests a proof strategy where we prove $\neg p$, prove $\neg q$, and use $\wedge$i to get the conclusion. To prove $\neg p$, we will use the $\neg$i inference rule using p as an assumption. Similarly, to prove $\neg q$, we will use the $\neg$i inference rule.

1.	$\neg(p \vee q)$	premise
2.	p	assumption
3.	$p \vee q$	$\vee$i: 2
4.	$\perp$	$\neg$e: 1, 3
5.	$\neg p$	$\neg$i: 2–4
6.	q	assumption
7.	$p \vee q$	$\vee$i: 6
8.	$\perp$	$\neg$e: 1, 7
9.	$\neg q$	$\neg$i: 6–8
10.	$\neg p \wedge \neg q$	$\wedge$i: 5, 9

2. $\neg p \wedge \neg q \vdash \neg(p \vee q)$

   The conclusion is of the form $\neg\alpha$ so this suggests a proof strategy where we use the $\neg$i inference rule using $p \vee q$ as an assumption. To arrive at a contradiction in the sub-proof we use $\vee$e using the assumption as our starting point.

1.	$\neg p \wedge \neg q$	premise
2.	$\neg p$	$\wedge$e: 1
3.	$\neg q$	$\wedge$e: 1
4.	$p \vee q$	assumption
5.	p	assumption
6.	$\perp$	$\neg$e: 2, 5
7.	q	assumption
8.	$\perp$	$\neg$e: 3, 7
9.	$\perp$	$\vee$e: 4, 5–6, 7–8
10.	$\neg(p \vee q)$	$\neg$i: 4–9

3. $\neg(p \wedge q) \vdash \neg p \vee \neg q$

The conclusion is of the form $\alpha \vee \beta$ so this suggests a proof strategy either based on $\vee$i or on $\neg$i (where we assume the negation of what we want to prove and derive a contradiction). We pursue the latter approach by deriving a contradiction to $\neg(p \wedge q)$; i.e., by deriving $p \wedge q$ using $\wedge$i.

1.	$\neg(p \wedge q)$	premise
2.	$\neg(\neg p \vee \neg q)$	assumption
3.	$\neg p$	assumption
4.	$\neg p \vee \neg q$	$\vee$i: 3
5.	$\perp$	$\neg$e: 2, 4
6.	$\neg\neg p$	$\neg$i: 3–5
7.	p	$\neg\neg$e: 6
8.	$\neg q$	assumption
9.	$\neg p \vee \neg q$	$\vee$i: 8
10.	$\perp$	$\neg$e: 2, 9
11.	$\neg\neg q$	$\neg$i: 8–10
12.	q	$\neg\neg$e: 11
13.	$p \wedge q$	$\wedge$i: 7, 12
14.	$\perp$	$\neg$e: 1, 13
15.	$\neg\neg(\neg p \vee \neg q)$	$\neg$i: 2–14
16.	$\neg p \vee \neg q$	$\neg\neg$e: 15

A shorter proof can be given if we are permitted to use the law of excluded middle (LEM) rule.

1.	$\neg(p \wedge q)$	premise
2.	$p \vee \neg p$	LEM
3.	p	assumption
4.	q	assumption
5.	$p \wedge q$	$\wedge$i: 3, 4
6.	$\perp$	$\neg$e: 1, 5
7.	$\neg q$	$\neg$i: 4–6
8.	$\neg p \vee \neg q$	$\vee$i: 7
9.	$\neg p$	assumption
10.	$\neg p \vee \neg q$	$\vee$i: 9
11.	$\neg p \vee \neg q$	$\vee$e: 2, 3–8, 9–10

4. $\neg p \vee \neg q \vdash \neg(p \wedge q)$

   The conclusion is of the form $\neg\alpha$ so this suggests a proof strategy where we use the ¬i inference rule using $p \wedge q$ as an assumption. To arrive at a contradiction in the sub-proof we use ∨e using the premise as our starting point.

1.	$\neg p \vee \neg q$	premise
2.	$p \wedge q$	assumption
3.	p	∧e: 2
4.	q	∧e: 2
5.	$\neg p$	assumption
6.	$\bot$	¬e: 3, 5
7.	$\neg q$	assumption
8.	$\bot$	¬e: 4, 7
9.	$\bot$	∨e: 1, 5–6, 7–8
10.	$\neg(p \wedge q)$	¬i: 2–9

5. $\vdash p \rightarrow (q \rightarrow p)$

   Note that there are no premises. This means that the formula we are proving is a tautology. The conclusion is of the form $\alpha \rightarrow \beta$ so this suggests a proof strategy based on →i.

1.	p	assumption
2.	q	assumption
3.	p	copy of 1
4.	$q \rightarrow p$	→i: 2–3
5.	$p \rightarrow (q \rightarrow p)$	→i: 1–4

6. $\neg p \vee q \vdash p \rightarrow q$

   The conclusion is of the form $\alpha \rightarrow \beta$ so this suggests a proof strategy where we use the →i inference rule using p as an assumption and deriving q. To derive q we use ∨e using the premise as our starting point.

1.	$\neg p \vee q$	premise
2.	p	assumption
3.	$\neg p$	assumption
4.	$\bot$	¬e: 2, 3
5.	q	⊥e: 4
6.	q	assumption
7.	q	copy of 6
8.	q	∨e: 1, 3–5, 6–7
9.	$p \rightarrow q$	→i: 2–8

7. $p \to q \vdash \neg p \lor q$

The conclusion is of the form $\alpha \lor \beta$ so this suggests a proof strategy either based on $\lor$i or on $\neg$i (where we assume the negation of what we want to prove and derive a contradiction). We pursue the latter approach by assuming $\neg(\neg p \lor q)$ and deriving a contradiction to this assumption. Of course, to derive a contradiction to $\neg(\neg p \lor q)$ we must derive $\neg p \lor q$. We will use $\lor$i to derive $\neg p \lor q$.

1.	$p \to q$	premise
2.	$\neg(\neg p \lor q)$	assumption
3.	p	assumption
4.	q	$\to$e: 1, 3
5.	$\neg p \lor q$	$\lor$i: 4
6.	$\bot$	$\neg$e: 2, 5
7.	$\neg p$	$\neg$i: 3–6
8.	$\neg p \lor q$	$\lor$i: 7
9.	$\bot$	$\neg$e: 2, 8
10.	$\neg\neg(\neg p \lor q)$	$\neg$i: 2–9
11.	$\neg p \lor q$	$\neg\neg$e: 10

An alternative proof.

1.	$p \to q$	premise
2.	$\neg(\neg p \lor q)$	assumption
3.	$\neg p$	assumption
4.	$\neg p \lor q$	$\lor$i: 3
5.	$\bot$	$\neg$e: 2, 4
6.	$\neg\neg p$	$\neg$i: 3–5
7.	p	$\neg\neg$e: 6
8.	q	$\to$e: 1, 7
9.	$\neg p \lor q$	$\lor$i: 8
10.	$\bot$	$\neg$e: 2, 9
11.	$\neg\neg(\neg p \lor q)$	$\neg$i: 2–9
12.	$\neg p \lor q$	$\neg\neg$e: 10

8. $\alpha \vee p, \neg p \vdash \alpha$

The unit resolution inference rule is not available to us in natural deduction. This example and the next shows how any proof that would use unit resolution can be redone using just the basic inference rules in natural deduction.

1.	$\alpha \vee p$	premise
2.	$\neg p$	premise
3.	α	assumption
4.	α	copy of 3
5.	p	assumption
6.	$\bot$	$\neg$e: 2, 5
7.	α	$\bot$e: 6
8.	α	$\vee$e: 1, 3–4, 5–7

9. $p, \neg p \vee \beta \vdash \beta$

1.	p	premise
2.	$\neg p \vee \beta$	premise
3.	$\neg p$	assumption
4.	$\bot$	$\neg$e: 1, 3
5.	β	$\bot$e: 4
6.	β	assumption
7.	β	copy of 6
8.	β	$\vee$e: 1, 3–5, 6–7

10. $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

1.	$p \rightarrow (q \rightarrow r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge$e: 2
4.	q	$\wedge$e: 2
5.	$q \rightarrow r$	$\rightarrow$e: 1, 3
6.	r	$\rightarrow$e: 4, 5
7.	$(p \wedge q) \rightarrow r$	$\rightarrow$i: 2–6

11. $p \to r, q \to s \vdash (p \to (r \vee s)) \wedge (q \to (r \vee s))$

1.	$p \to r$	premise
2.	$q \to s$	premise
3.	p	assumption
4.	r	$\to$e: 1, 3
5.	$r \vee s$	$\vee$i: 4
6.	$p \to (r \vee s)$	$\to$i: 3–5
7.	q	assumption
8.	s	$\to$e: 2, 7
9.	$r \vee s$	$\vee$i: 8
10.	$q \to (r \vee s)$	$\to$i: 7–9
11.	$(p \to (r \vee s)) \wedge (q \to (r \vee s))$	$\wedge$i: 6, 10

12. $(p \wedge q) \vee r \vdash \neg p \to r$

1.	$(p \wedge q) \vee r$	premise
2.	$\neg p$	assumption
3.	$p \wedge q$	assumption
4.	p	$\wedge$e: 3
5.	$\bot$	$\neg$e: 2, 4
6.	r	$\bot$e: 5
7.	r	assumption
8.	r	copy of 7
9.	r	$\vee$e: 1, 3–6, 7–8
10.	$\neg p \to r$	$\to$i: 2–9

13. $p \to q, r \to s, (q \vee s) \to t, \neg t \vdash \neg p \wedge \neg r$

1.	$p \to q$	premise
2.	$r \to s$	premise
3.	$(q \vee s) \to t$	premise
4.	$\neg t$	premise
5.	$\neg(q \vee s)$	$\to$e: 3, 4
6.	p	assumption
7.	q	$\to$e: 1, 6
8.	$q \vee s$	$\vee$i: 7
9.	$\bot$	$\neg$e: 5, 8
10.	$\neg p$	$\neg$i: 6–9
11.	r	assumption
12.	s	$\to$e: 2, 11
13.	$q \vee s$	$\vee$i: 12
14.	$\bot$	$\neg$e: 5, 13
15.	$\neg r$	$\neg$i: 11–14
16.	$\neg p \wedge \neg r$	$\wedge$i: 10, 15

14. $p \to \neg q,\ q \vee r \vee s,\ (\neg r \vee s) \to p,\ \neg r\ \vdash\ s$

Notice in this example that we are doing ∨e on the formula $q \vee r \vee s$ using the bracketing $q \vee (r \vee s)$.

1.	$p \to \neg q$	premise
2.	$q \vee r \vee s$	premise
3.	$(\neg r \vee s) \to p$	premise
4.	$\neg r$	premise
5.	$\neg r \vee s$	∨i: 4
6.	p	→e: 3, 5
7.	$\neg q$	→e: 1, 6
8.	q	premise
9.	$\perp$	¬e: 7, 8
10.	$r \vee s$	⊥e: 9
11.	$r \vee s$	premise
12.	$r \vee s$	copy of 11
13.	$r \vee s$	∨e: 2, 8–10, 11–12
14.	r	premise
15.	$\perp$	¬e: 4, 14
16.	s	⊥e: 15
17.	s	premise
18.	s	copy of 17
19.	s	4, 7, ∨e: 13, 14–16, 17–18