CS245 Tutorial

Nov. 2, 2015

1. Give proofs in Peano Arithmetic of each the following.

   (a) 1 is a multiplicative identity: $\forall x \cdot (s(0) \times x = x)$.

   Solution: Let $\varphi = s(0) \times x = x$. We will prove $(\forall x \cdot \varphi)$, using the PA7 axiom.

   - Base: We have to prove: $s(0) \times 0 = 0$

1.	$\forall x \cdot x \times 0 = 0$	PA5
2.	$s(0) \times 0 = 0$	$\forall$e : 1 [term: $s(0)$]

   - Induction: We have to prove: $\forall x \cdot ((s(0) \times x = x) \rightarrow (s(0) \times s(x) = s(x)))$

1.	x_0 fresh	
2.	$s(0) \times x_0 = x_0$	assumption
3.	$s(0) \times s(x_0) = s(0) \times x_0 + s(0)$	PA6 + $\forall$e (2×) [terms: $s(0), x_0$]
4.	$s(0) \times s(x_0) = x_0 + s(0)$	=e: 2, 3
5.	$s(0) \times s(x_0) = s(x_0 + 0)$	PA4 : 4
6.	$s(0) \times s(x_0) = s(x_0)$	PA3 : 5
7.	$(s(0) \times x_0 = x_0) \rightarrow (s(0) \times s(x_0) = s(x_0))$	$\rightarrow$i : 2–6
8.	$\forall x \cdot ((s(0) \times x = x) \rightarrow ((s(0) \times s(x) = s(x))))$	$\forall$i : 1–7

   This puts all the pieces in place to invoke the PA7 axiom to obtain the desired result.

(b) A sum can be zero in only one way: $(x + y = 0) \to ((x = 0) \wedge (y = 0))$.

Solution: We start by proving that every non-zero natural number is a successor.

Lemma: $\forall x \cdot (\neg(x = 0) \to (\exists z \cdot s(z) = x))$.

Let $\varphi = (\neg(x = 0) \to (\exists z \cdot s(z) = x))$. We will prove $(\forall x \cdot \varphi)$, using the PA7 axiom.

- <u>Base:</u> We have to prove: $((\neg(0 = 0)) \to (\exists z \cdot s(z) = 0))$

1.	$\neg(0 = 0)$	assumption
2.	$0 = 0$	$=$i
3.	$\bot$	$\bot$i : 1,2
4.	$\exists z \cdot (s(z) = 0)$	$\bot$e : 3
5.	$(\neg(0 = 0)) \to (\exists z \cdot s(z) = 0)$	$\to$i : 1–4

- <u>Induction:</u> We have to prove

$$\forall x \cdot ((\neg(x = 0) \to (\exists z \cdot s(z) = x)) \to (((\neg(s(x) = 0)) \to (\exists z \cdot s(z) = s(x)))))$$

1.	x_0 fresh	
2.	$\neg(x_0 = 0) \to (\exists z \cdot s(z) = x_0)$	assumption
3.	$\neg(s(x_0) = 0)$	assumption
4.	$s(x_0) = s(x_0)$	$=$i
5.	$\exists z \cdot s(z) = s(x_0)$	$\exists$i : 4
6.	$\neg(s(x_0) = 0) \to (\exists z \cdot s(z) = s(x_0))$	$\to$i : 3–5
7.	$(\neg(x_0 = 0) \to (\exists z \cdot s(z) = x_0)) \to$ $(\neg(s(x_0) = 0) \to (\exists z \cdot s(z) = s(x_0)))$	$\to$i : 2–6
8.	$\forall x \cdot ((\neg(x = 0) \to (\exists z \cdot s(z) = x)) \to$ $(\neg(s(x) = 0) \to (\exists z \cdot s(z) = s(x))))$	$\forall$i : 1–7

This puts all the pieces in place to invoke the PA7 axiom to obtain the desired result for the Lemma.

Now we use the lemmas to help prove our main result. We have to prove:

$$((x + y = 0) \to ((x = 0) \land (y = 0)))$$

1.	$x + y = 0$	assumption
2.	$\neg(x = 0)$	assumption
3.	$\exists z \cdot s(z) = x$	Lemma : 2
4.	$s(u) = x$, u fresh	assumption
5.	$s(u) + y = x + y$	EqSubs : 4
6.	$s(u + y) = x + y$	PA4 : 5
7.	$s(u + y) = 0$	=e: 1, 6
8.	$\neg(s(u + y) = 0)$	PA1
9.	$\perp$	$\perp$i : 7, 8
10.	$\perp$	$\exists$e : 3, 4–9
11.	$\neg\neg(x = 0)$	$\neg$i : 2–10
12.	$(x = 0)$	$\neg\neg$e : 10
13.	$\neg(y = 0)$	assumption
14.	$\exists z \cdot s(z) = y$	Lemma : 12
15.	$s(v) = x$, v fresh	assumption
16.	$s(v) + x = x + y$	EqSubs : 15
17.	$s(v + x) = x + y$	PA4 : 16
18.	$s(v + x) = 0$	=e: 1, 17
19.	$\neg(s(v + x) = 0)$	PA1
20.	$\perp$	$\perp$i : 18, 19
21.	$\perp$	$\exists$e : 14, 15–20
22.	$\neg\neg(y = 0)$	$\neg$i : 13–21
23.	$(y = 0)$	$\neg\neg$e : 22
24.	$(x = 0) \land (y = 0)$	$\land$i : 12, 23
25.	$(x + y = 0) \to ((x = 0) \land (y = 0))$	$\to$i : 1–24

(c) There are no **zero-divisors**: $((x \times y = 0) \to ((x = 0) \vee (y = 0)))$.

Solution: We start by proving a useful exercise, one version of a **DeMorgan Law**. We will prove, for any formulae α, β, that $\{\neg(\alpha \vee \beta)\} \vdash (\neg\alpha) \wedge (\neg\beta)$.

1.	$\neg(\alpha \vee \beta)$	Premise
2.	α	assumption
3.	$(\alpha \vee \beta)$	$\vee$i : 2
4.	$\perp$	$\perp$i : 1, 3
5.	$(\neg\alpha)$	$\neg$i : 2–4
6.	β	assumption
7.	$(\alpha \vee \beta)$	$\vee$i : 6
8.	$\perp$	$\perp$i : 1, 7
9.	$(\neg\beta)$	$\neg$i : 6–8
10.	$(\neg\alpha) \wedge (\neg\beta)$	$\wedge$i : 5, 9

Now we use the exercise to prove our main result. We have to prove:

$$((x \times y = 0) \to ((x = 0) \vee (y = 0)))$$

1.	$(x \times y = 0)$	assumption
2.	$\neg((x = 0) \vee (y = 0))$	assumption
3.	$(\neg(x = 0)) \wedge (\neg(y = 0))$	DeMorgan Law : 2
4.	$\neg(x = 0)$	$\wedge$e : 3
5.	$\exists z \cdot s(z) = x$	Lemma : 4
6.	$\neg(y = 0)$	$\wedge$e : 3
7.	$\exists z \cdot s(z) = y$	Lemma : 6
8.	$s(u) = x$, u fresh	assumption (using 5)
9.	$s(v) = y$, v fresh	assumption (using 7)
10.	$x \times y = s(u) \times s(v)$	EqSubs: 8, 9
11.	$x \times y = s(u) \times v + s(u)$	PA6 : 10
12.	$0 = s(u) \times v + s(u)$	=e: 1, 11
13.	$(s(u) \times v = 0) \wedge (s(u) = 0)$	Part (b) : 12
14.	$(s(u) = 0)$	$\wedge$e : 13
15.	$\neg(s(u) = 0)$	PA1
16.	$\perp$	$\perp$i : 14, 15
17.	$\perp$	$\exists$e : 7, 9–16
18.	$\perp$	$\exists$e : 5, 8–17
19.	$\neg\neg((x = 0) \vee (y = 0))$	$\neg$i : 2–18
20.	$((x = 0) \vee (y = 0))$	$\neg\neg$e : 19
21.	$((x \times y = 0) \to ((x = 0) \vee (y = 0)))$	$\to$i : 1–20

2. (a) Write out the angle-bracket notation for each of the following basic lists. What is the length of each?

   i. $cons(cons(e,e), cons(e,e))$.

   Solution:

$$\begin{aligned} cons(cons(e,e), cons(e,e)) &= cons(\langle e\rangle, \langle e\rangle) \\ &= \langle\langle e\rangle, e\rangle. \end{aligned}$$

   The list has length 2.

   ii. $cons(e, cons(cons(e,e), e))$.

   Solution:

$$\begin{aligned} cons(e, cons(cons(e,e), e)) &= cons(e, cons(\langle e\rangle, e)) \\ &= cons(e, \langle\langle e\rangle\rangle) \\ &= \langle e, \langle e\rangle\rangle. \end{aligned}$$

   The list has length 2.

   iii. $cons(cons(cons(e,e), e), e)$.

   Solution:

$$\begin{aligned} cons(cons(cons(e,e), e), e) &= cons(cons(\langle e\rangle, e), e) \\ &= cons(\langle\langle e\rangle\rangle, e) \\ &= \langle\langle\langle e\rangle\rangle\rangle. \end{aligned}$$

   The list has length 1.

(b) Give the explicit list corresponding to each of the following, in terms of *cons* and e only.

i. $\langle e, \langle e, e\rangle, \langle e\rangle\rangle$

Solution:

$$\begin{aligned}
\langle e, \langle e, e\rangle, \langle e\rangle\rangle &= \langle e, cons(e, \langle e\rangle), cons(e, e)\rangle \\
&= \langle e, cons(e, cons(e, e)), cons(e, e)\rangle \\
&= cons(e, cons(cons(e, cons(e, e)), cons(cons(e, e), e))).
\end{aligned}$$

ii. $\langle\langle\langle\rangle, \langle\rangle\rangle, \langle\rangle, \langle\langle\rangle\rangle\rangle$

Solution:

$$\begin{aligned}
\langle\langle\langle\rangle, \langle\rangle\rangle, \langle\rangle, \langle\langle\rangle\rangle\rangle &= \langle\langle e, e\rangle, e, \langle e\rangle\rangle \\
&= \langle cons(e, cons(e, e)), e, cons(e, e)\rangle \\
&= cons(cons(e, cons(e, e)), cons(e, cons(cons(e, e)))).
\end{aligned}$$

iii. $\langle\langle\langle e\rangle, e, \langle e\rangle\rangle\rangle$

Solution:

$$\begin{aligned}
\langle\langle\langle e\rangle, e, \langle e\rangle\rangle\rangle &= \langle\langle cons(e, e), e, cons(e, e)\rangle\rangle \\
&= \langle cons(cons(e, e), cons(e, cons(cons(e, e), e)))\rangle \\
&= cons(cons(cons(e, e), cons(cons(e, cons(cons(e, e), e))), e)).
\end{aligned}$$

3. In the domain of Basic Lists, prove that every non-e object is a cons. In detail, give a Natural Deduction proof demonstrating that:

$$\vdash_{BL} (\forall x \cdot ((x \neq e) \rightarrow (\exists y \cdot (\exists z \cdot (cons(y, z) = x))))) :$$

(Recall that "$\vdash_{BL}$" means that your proof may use the axioms BL1, BL2 and BL3.) Write the formulas and explanations of your proof in the following order, and identify the parts.

- Write out the instance of BL3 that you will use.
- Prove the base case.
- Prove the inductive case.
- Complete the proof.

Solution: Let

$$\varphi = ((x \neq e) \rightarrow (\exists y \cdot (\exists z \cdot (cons(y, z) = x)))).$$

We will prove using the BL3 axiom for this choice of φ.

- Base: We have to prove: $((e \neq e) \rightarrow (\exists y \cdot (\exists z \cdot (cons(y, z) = e))))$

1.	$e \neq e$	assumption
2.	$e = e$	=i
3.	$\bot$	$\bot$i : 1,2
4.	$\exists y \cdot \exists z \cdot (cons(y, z) = e)$	$\bot$e : 3
5.	$(e \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y, z) = e))$	$\rightarrow$i : 1–4

- Induction: We have to prove

$$\forall x \cdot (((x \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y,z) = x))) \\ \rightarrow (\forall w \cdot ((cons(w,x) \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y,z) = cons(w,x))))))$$

1.	x_0 fresh	
2.	$(x_0 \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y,z) = x_0))$	assumption
3.	w_0 fresh	
4.	$cons(w_0, x_0) \neq e$	assumption
5.	$cons(w_0, x_0) = cons(w_0, x_0)$	=i
6.	$\exists z \cdot (cons(w_0, z) = cons(w_0, x_0))$	∃i : 5
7.	$\exists y \cdot \exists z \cdot (cons(y,z) = cons(w_0, x_0))$	∃i : 6
8.	$(cons(w_0, x_0) \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y,z) = cons(w_0, x_0)))$	→i : 4–7
9.	$\forall w \cdot ((cons(w, x_0) \neq e) \rightarrow (\exists y \cdot \exists z \cdot (cons(y,z) = cons(w_0, x_0))))$	∀i : 3–8
10.	$(((x \neq e) \rightarrow (\exists y \cdot (\exists z \cdot (cons(y,z) = x)))) \rightarrow$ $(\forall w \cdot ((cons(w,x) \neq e) \rightarrow$ $(\exists y \cdot (\exists z \cdot (cons(y,z) = cons(w,x)))))))$	→i : 2-9
11.	$(\forall x \cdot (((x \neq e) \rightarrow$ $(\exists y \cdot (\exists z \cdot (cons(y,z) = x)))) \rightarrow$ $(\forall w \cdot ((cons(w,x) \neq e) \rightarrow$ $(\exists y \cdot (\exists z \cdot (cons(y,z) = cons(w,x))))))))$	∀i : 1–10

This puts all the pieces in place to invoke the BL3 axiom to obtain the desired result.

4. Soundness and Completeness of Natural Deduction in Predicate Logic

   Let

$$\begin{aligned}
\Sigma &= \{(\forall x \cdot (P(x) \to Q(x)))\} \\
\varphi &= (\exists x \cdot P(x)) \to (\exists x \cdot Q(x)) \\
\alpha &= (\forall x \cdot (Q(x) \to P(x))) \\
\beta &= (\forall x \cdot ((P(x) \to Q(x)) \to (\neg P(x) \vee Q(x))))
\end{aligned}$$

   (a) Give a model (an interpretation) $\mathcal{M}$ and an environment θ such that $\mathcal{M} \models_\theta \Sigma$.

   Suppose that we have shown that $\Sigma \vdash \varphi$ using natural deduction. What, if anything, can we conclude about $\varphi^{(\mathcal{M},\theta)}$?

   Solution: Let the domain be $\mathbb{N}$, the natural numbers. Let $P^{\mathcal{M}} = \{5, 10, 15, \ldots\}$, the positive multiples of 5. Let $Q^{\mathcal{M}} = \{5, 6, 7, 8, 9, \ldots\}$, the natural numbers strictly greater than 4. There are no free variables, so there is nothing to specify for the environment. It is clear from construction that $\mathcal{M} \models_\theta \Sigma$; i.e., the interpretation and the environment satisfy each of the formulas in Σ.

   By soundness, we have that $\Sigma \models \varphi$, which implies that $\mathcal{M} \models_\theta \varphi$, in other words $\varphi^{(\mathcal{M},\theta)} = T$.

   (b) Give a model $\mathcal{M}$ and an environment θ such that $\mathcal{M} \models_\theta \Sigma$ and $\mathcal{M} \not\models_\theta \alpha$.

   Suppose that you are asked to show that $\Sigma \vdash \alpha$ using natural deduction. Can you do it? Why or why not?

   Solution: The model $\mathcal{M}$ and environment θ from the previous part satisfy Σ but do not satisfy α.

   This model and environment show that $\Sigma \not\models \alpha$. Then by the contrapositive of soundness, we conclude that $\Sigma \nvdash \alpha$.

Let

$$
\begin{aligned}
\Sigma &= \{(\forall x \cdot (P(x) \to Q(x)))\} \\
\varphi &= (\exists x \cdot P(x)) \to (\exists x \cdot Q(x)) \\
\alpha &= (\forall x \cdot (Q(x) \to P(x))) \\
\beta &= (\forall x \cdot ((P(x) \to Q(x)) \to (\neg P(x) \vee Q(x))))
\end{aligned}
$$

(c) Show that $\emptyset \models \beta$, where $\emptyset$ is the empty set (i.e. show that β is a **valid** formula). Use semantic arguments (about possible interpretations/models).

Solution: Let $\mathcal{M}$ be any model and θ any environment. Let $\mathcal{D} = dom(\mathcal{M})$ be the domain, and let $a \in \mathcal{D}$ be an arbitrary element of the domain. There are two cases to consider.

(a) $\underline{\text{If } ((\neg P(x)) \vee Q(x))^{(\mathcal{M},\theta[x\mapsto a])} = T}$, then

$$\mathcal{M} \models_{\theta[x\mapsto a]} ((P(x) \to Q(x)) \to ((\neg P(x)) \vee Q(x))),$$

by properties of implication.

(b) $\underline{\text{If } ((\neg P(x)) \vee Q(x))^{(\mathcal{M},\theta[x\mapsto a])} = F}$, then $P(x)^{(\mathcal{M},\theta[x\mapsto a])} = T$ and $Q(x)^{(\mathcal{M},\theta[x\mapsto a])} = F$. By properties of implication, $(P(x) \to Q(x))^{(\mathcal{M},\theta[x\mapsto a])} = F$. Again by properties of implication, we have

$$\mathcal{M} \models_{\theta[x\mapsto a]} ((P(x) \to Q(x)) \to ((\neg P(x)) \vee Q(x))).$$

In both cases, we conclude that

$$\mathcal{M} \models_{\theta[x\mapsto a]} ((P(x) \to Q(x)) \to ((\neg P(x)) \vee Q(x))).$$

Since $\mathcal{M}, \theta$ and a were arbitrary (and since $\mathcal{M} \models_{\theta[x\mapsto a]} \emptyset$), therefore $\emptyset \models \beta$, as required.

(d) Suppose that you are asked to show that $\emptyset \vdash \beta$ using natural deduction. Can you do it? Why or why not?

Solution: From part (c) we know that $\emptyset \models \beta$. By completeness, we conclude that $\emptyset \vdash \beta$. In other words, there does exist a natural deduction proof of β.