First-Order Predicate Logic

What Propositional Logic Cannot Express

Propositional logic dealt with logical forms of compound propositions. It worked well with relationships like *not*, *and*, *or*, *if/then*.

We would like to have a way to talk about *individuals* (also called *objects*) and in addition to talk about *some* object, and *all* objects, without enumerating all objects in a set.

This requires extensions to Propositional Logic.

Some example statements:

Not all birds can fly.

Every student is younger than some instructor.

These refer to things: birds, students, instructors. They also refer to properties of things, either as individuals (ability to fly) or in combination (relative age).

We would like to make such statements in our logic and to combine them with the connectives of propositional logic.

Further Example Statements

More examples:

- For any natural number n, there is a prime number greater than n.
- 2^{100} is a natural number.
- There is a prime number greater than 2^{100}.
- There is a number c such that for every input of n characters, the program executes at most $c \cdot n^2$ operations.

First-Order Logic (FOL), also called *Predicate Logic*, gives us a language to express statements about objects and their properties.

Ingredients of FOL

FOL is expressed with the following ingredients:

- A domain of objects (individuals)
  (e.g., the set of natural numbers, people)
- Names of individuals (e.g., '0', Prime Minister)
  (Also called "constants")
- Variables (denoting "generic" objects)
- Relations (e.g., equal, younger-than, etc.)
- Functions (e.g., '+', mother-of)
- Quantifiers
- Propositional connectives

We shall discuss each informally, and later treat syntax and semantics formally.

Domains

A *domain* is a set of objects. In principle, any non-empty set can be a domain: the natural numbers, people now alive, $\{\mathtt{T}, \mathtt{F}\}$, etc.

Normally, one or more objects in the domain will have a name; e.g., 0, Stephen Harper, $\mathtt{T}$, etc. Such names are called *constant symbols*.

A *predicate*, or *relation*, represents a property that an individual, or collection of individuals, may (or may not) have. In English, we might express a predicate as

"______ is a student".

In symbolic logic, we write "$S(x)$" to mean "x has property S".

For example, if S is the property of being a student, then "Alex is a student" becomes "$S(Alex)$".

Similarly, we might use $I(Sam)$ for "Sam is an instructor"

and $Y(Alex{,}Sam)$ for "Alex is younger than Sam".

Mathematically, we represent a relation by the set of all things that have the property. If S is the set of all students, then $x \in S$ means x is a student. The only restriction on a relation is that it must be a subset of the domain.

A k-ary relation is a set of k-tuples of domain elements. For example, the binary relation less-than, over a domain $\mathcal{D}$, is represented by the set

$$\{\, \langle x, y \rangle \in \mathcal{D}^2 \mid x < y \,\} \quad .$$

(In a "relational database", the listing of such a set is called a "table".)

Variables

Variables make statements more expressive.
You may think of a variable as a "place holder", or "blank", that can be replaced by a concrete object.

Alternatively, a variable is a name without a fixed referent. Which object the name refers to can vary from time to time.

A variable lets us refer to an object, without specifying—perhaps without even knowing—which particular object it is. Thus we can express a relation "in the abstract".

$S(x)$:	x is a student
$I(x)$:	x is an instructor
$Y(x, y)$:	x is younger than y

Uses of Variables

In general, we use variables that range over the domain to make general statements, such as

$$x^2 \geq 0 \quad ,$$

and in expressing conditions which individuals may or may not satisfy, such as

$$x + x = x * x \quad .$$

This latter condition is satisfied by only two numbers: 0 and 2.

The meaning of such an expression will depend on the domain. For example, the formula $x^2 < x$ is always false over the domain of integers, but not over the domain of rational numbers.

How to handle "**Every** student x is younger than **some** professor y"?

In math-speak, we say "for all" to express "every" and "there exists" to express "some." A familiar(?) example from calculus:

> For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all y,
> if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

"For all" is denoted by '$\forall$', the *universal quantifier* symbol, and "there exists" is denoted by '$\exists$', the *existential quantifier* symbol.

In FOL, the above comes out as the formula

$$\forall \varepsilon \cdot \big(\varepsilon > 0 \to \exists \delta \cdot \big(\delta > 0 \wedge \forall y \cdot (|x - y| < \delta \to |f(x) - f(y)| < \varepsilon)\big)\big) \ .$$

Quantifiers require a variable: $\forall x$ (for all x) or $\exists z$ (there exists z).

For example, the statement "Not all birds can fly" can be written as

$$\neg(\forall x \cdot (B(x) \rightarrow F(x))) \ .$$

"Every student is younger than some instructor" can become

$$\forall x \cdot (S(x) \rightarrow (\exists y \cdot (I(y) \wedge Y(x, y)))) \ .$$

Or should that be $\exists y \cdot (I(y) \wedge \forall x \cdot (S(x) \rightarrow Y(x, y)))$?

In addition to predicates and quantifiers, first-order logic extends propositional logic by using *functions* as well. To see why, consider the following statement.

Every child is younger than its mother.

One might try to express this statement in FOL by the formula

$$\forall x \cdot \forall y \cdot ((C(x) \wedge M(y, x)) \rightarrow Y(x, y)) \quad .$$

But this allows x to have several mothers!

Functions: Example and Definition

Functions in FOL give us a way to express statements more concisely. The previous example can be expressed as

$$\forall x \cdot \big(C(x) \to Y(x, m(x))\big)$$

where m denotes the function that takes one argument and returns the mother of that argument.

Formally, we represent a k-ary function f as the $k+1$-ary relation R_f given by

$$R_f = \left\{ \langle x_1, \ldots, x_k, x_{k+1} \rangle \in \mathcal{D}^{k+1} \mid f(x_1, \ldots, x_k) = x_{k+1} \right\} \quad .$$

More examples:

- Alex and Sam have the same maternal grandmother:

$$m(m(a)) = m(m(s)) \quad .$$

- Some program computes the squaring function:

$$\exists p \cdot \forall x \cdot r(p, x) = x * x \quad .$$

These use $m(\cdot)$ as "mother-of" and $r(\cdot, \cdot)$ as "result-of".

Syntax of Predicate Logic

The Language of First-Order Logic

The seven kinds of symbols:

1.	Constant symbols.	Usually $c, d, c_1, c_2, \ldots, d_1, d_2 \ldots$
2.	Variables.	Usually $x, y, z, \ldots x_1, x_2, \ldots, y_1, y_2 \ldots$
3.	Function symbols.	Usually $f, g, h, \ldots f_1, f_2, \ldots, g_1, g_2, \ldots$
4.	Predicate symbols.	P, Q, $\ldots P_1$, P_2, $\ldots$, Q_1, Q_2, $\ldots$
5.	Connectives:	$\neg, \wedge, \vee, \rightarrow$
6.	Quantifiers:	$\forall$ and $\exists$
7.	Punctuation:	'(', ')', '$\cdot$', and ','

The last three kinds of symbols—connectives, quantifiers, and punctuation—will have their meaning fixed by the syntax and semantics.

The first four kinds—constants, variables, functions, and predicates—are not restricted. They may be assigned any meaning, consistent with their kind and arity.

In FOL, we need to consider two kinds of expressions:

- those that can have a truth value, called *formulas*, and
- those that refer to an object of the domain, called *terms*.

We start with terms.

Definition. The set of terms is defined inductively as follows.

1. Each constant symbol is a term, and each variable is a term. Such terms are called *atomic* terms.
2. If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term. If $n = 2$ (a binary function symbol), we may write $(t_1 \; f \; t_2)$ instead of $f(t_1, t_2)$.
3. Nothing else is a term.

Example 1. If 0 is a constant symbol, x and y are variables, and $s^{(1)}$ and $+^{(2)}$ are function symbols, then 0, x, and y are terms, as are $s(0)$ and $+\big(x, s(y)\big)$.

The expressions $s(x, y)$ and $s + x$ are not terms.

Example 2. Suppose f is a unary function symbol, g is a binary function symbol, and a is a constant symbol.

Then $g(f(a), a)$ and $f\left(g\left(a, f(a)\right)\right)$ are terms.

The expressions $g(a)$ and $f\left(f(a), a\right)$ are not terms.

Atomic Formulas

As in propositional logic, a formula represents a proposition (a true/false statement). The relation symbols produce propositions.

Definition: An *atomic formula* (or atom) is an expression of the form

$$P(t_1, \ldots, t_n)$$

where P is an n-ary relation symbol and each t_i is a term $(1 \leq i \leq n)$.

If P has arity 2, the atom $P(t_1, t_2)$ may alternatively be written $(t_1 \; P \; t_2)$.

General Formulas

We define the set of formulas of first-order logic inductively as follows.

1. An atomic formula is a formula.
2. If α is a formula, then $(\neg\alpha)$ is a formula.
3. If α and β are formulas, and $\star$ is a binary connective symbol, then $(\alpha \star \beta)$ is a formula.
4. If α is a formula and x is a variable, then each of $(\forall x \cdot \alpha)$ and $(\exists x \cdot \alpha)$ is a formula.
5. Nothing else is a formula.

In case 4, the formula α is called the *scope* of the quantifier. The quantifier keeps the same scope if it is included in a larger formula.

Parse Trees

Parse trees for FOL formulas are similar to parse trees for propositional formulas.

- Quantifiers $\forall x$ and $\exists y$ form nodes is the same way as negation (i.e., only one sub-tree).
- A predicate $P(t_1, t_2, \ldots, t_n)$ has a node labelled P with a sub-tree for each of the terms $t_1, t_2, \ldots, t_n$.

Example: $(\forall x \cdot ((P(x) \rightarrow Q(x)) \wedge S(x, y)))$.

Example: $(\forall x \cdot ((P(x) \to Q(x)) \land S(x, y)))$.

Example: $(\forall x \cdot (F(b) \to (\exists y \cdot (\forall z \cdot (G(y, z) \lor H(u, x, y))))))$

Example: $(\forall x \cdot ((P(x) \rightarrow Q(x)) \wedge S(x, y)))$.

Example: $(\forall x \cdot (F(b) \rightarrow (\exists y \cdot (\forall z \cdot (G(y, z) \vee H(u, x, y))))))$

Ordinarily, one would omit many of the parentheses in the second formula, and write simply

$$\forall x \cdot \Big(F(b) \rightarrow \exists y \cdot \forall z \cdot \big(G(y, z) \vee H(u, x, y)\big)\Big) \ .$$

We shall cover more about syntax later, but we first start the discussion of semantics.

Definition: Fix a set $\mathcal{L}$ of constant symbols, function symbols, and relation symbols.

An *interpretation* $\mathcal{M}$ (for the set $\mathcal{L}$) consists of

- A non-empty set $dom(\mathcal{M})$, called the domain (or universe) of $\mathcal{M}$.
- For each constant symbol c, a member $c^{\mathcal{M}}$ of $dom(\mathcal{M})$.
- For each function symbol $f^{(i)}$, an i-ary function $f^{\mathcal{M}}$.
- For each relation symbol $R^{(i)}$, an i-ary relation $R^{\mathcal{M}}$.

An interpretation is also called a *model*.

For terms and formulas that contain no variables or quantifiers, an interpretation suffices to specify their meaning. The meaning arises in the obvious(?) fashion from the syntax of the term or formula.

Definition: Fix an interpretation $\mathcal{M}$. For each term t containing no variables, the value of t under interpretation $\mathcal{M}$, denoted $t^{\mathcal{M}}$, is as follows.

- If t is a constant c, the value $t^{\mathcal{M}}$ is $c^{\mathcal{M}}$.
- If t is $f(t_1, \ldots, t_n)$, the value $t^{\mathcal{M}}$ is $f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$.

The value of a term is always a member of the domain of $\mathcal{M}$.

Formulas with Variable-Free Terms

Formulas get values in much the same fashion as terms, except that values of formulas lie in $\{\mathtt{F}, \mathtt{T}\}$.

Definition: Fix an interpretation $\mathcal{M}$. For each formula α containing no variables, the value of α under interpretation $\mathcal{M}$, denoted $\alpha^{\mathcal{M}}$, is as follows.

- If α is $R(t_1, \ldots, t_n)$, then

$$\alpha^{\mathcal{M}} = \begin{cases} \mathtt{T} & \text{if } \langle t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}} \rangle \in R^{\mathcal{M}} \\ \mathtt{F} & \text{otherwise.} \end{cases}$$

- If α is $(\neg\beta)$ or $(\beta \star \gamma)$, then $\alpha^{\mathcal{M}}$ is determined by $\beta^{\mathcal{M}}$ and $\gamma^{\mathcal{M}}$ in the same way as for propositional logic.

Examples

Let 0 be a constant symbol, $f^{(1)}$ a function symbol and $E^{(1)}$ a relation symbol. Thus $E\big(f(0)\big)$ and $E\Big(f\big(f(0)\big)\Big)$ are both formulas.

Consider an interpretation $\mathcal{M}$ with

Domain:	$\mathbb{N}$, the natural numbers
$0^{\mathcal{M}}$:	zero
$f^{\mathcal{M}}$:	successor; $\big\{\langle x, x+1\rangle \mid x \in \mathbb{N}\big\}$
$E^{\mathcal{M}}$:	"is even"; $\big\{2y \mid y \in \mathbb{N}\big\}$

Terms get numerical values: $f(0)^{\mathcal{M}}$ is 1 and $f\big(f(0)\big)^{\mathcal{M}}$ is 2.

Formula $E\big(f(0)\big)$ means "1 is even", and $E\big(f(0)\big)^{\mathcal{M}} = \mathtt{F}$.
Formula $E\Big(f\big(f(0)\big)\Big)$ means "2 is even", and $E\Big(f\big(f(0)\big)\Big)^{\mathcal{M}} = \mathtt{T}$.

What about some other interpretation?

Example, Continued

Let $\mathcal{N}$ be the interpretation with

Domain:	$\mathbb{Q}$, the rational numbers
$0^{\mathcal{N}}$:	two
$f^{\mathcal{N}}$:	halving; $\{\langle x, x/2\rangle \mid x \in \mathbb{Q}\}$
$E^{\mathcal{N}}$:	"is an integer"; $\{x \mid x \in \mathbb{Z}\}$

$E(f(0))$ means "1 is an integer", and $E(f(0))^{\mathcal{N}}$ is $\mathtt{T}$.
$E(f(f(0)))$ means "1/2 is an integer", and $E(f(f(0)))^{\mathcal{N}}$ is $\mathtt{F}$.

Exercise: in both $\mathcal{M}$ and $\mathcal{N}$, the formula $E(f(f(0))) \wedge E(f(0))$ receives value $\mathtt{F}$. Find another interpretation which gives it the value $\mathtt{T}$.

"Gotchas"

Two often-overlooked points about interpretations.

1. There is NO default meaning for relation, function or constant symbols.

   "$1 + 2 = 3$" might mean that one plus two equals three—but only if we specify that interpretation. Any interpretation of constants 1, 2, and 3, function symbol $+^{(2)}$ and relation symbol $=^{(2)}$ is possible.

2. Functions must be defined at every point in the domain. (I.e., they must be *total*.)

   If we have language with a binary function symbol "$-$", we cannot specify an interpretation with domain $\mathbb{N}$ and subtraction for "$-$". Subtraction is not total on $\mathbb{N}$.

Variables

To discuss the evaluation of formulas that contain variables, we need a few more concepts from syntax.

We shall discuss

- "bound" and "free" variables,
- substitution of terms for variables.

Free and Bound Variables

Recall: the *scope* of a quantifier in a sub-formula $\forall x \cdot \alpha$ or $\exists x \cdot \alpha$ is the formula α.

An occurrence of a variable in a formula is *bound* if it lies in the scope of some quantifier of the same variable; otherwise it is *free*. In other words, a quantifier *binds* its variable within its scope.

Example. In formula $\forall x \cdot \exists y \cdot (x + y = z)$, x is bound (by $\forall x$), y is bound (by $\exists y$), and z is free.

Example. In formula $P(x) \wedge \forall x \cdot \neg Q(x)$, the first occurrence of x is free and the last occurrence of x is bound.

(The variable symbol immediately after $\exists$ or $\forall$ is neither free nor bound.)

Formally, a variable occurs free in a formula α if and only if it is a member of the set $\mathsf{FV}(\alpha)$ defined as follows.

1. If α is $P(t_1, \ldots, t_k)$, then $\mathsf{FV}(\alpha) = \left\{ x \mid x \text{ appears in some } t_i \right\}$.
2. If α is $(\neg\beta)$, then $\mathsf{FV}(\alpha) = \mathsf{FV}(\beta)$
3. If α is $(\beta \star \gamma)$, then $\mathsf{FV}(\alpha) = \mathsf{FV}(\beta) \cup \mathsf{FV}(\gamma)$.
4. If α is $\mathsf{Q}x \cdot \beta$ (for $\mathsf{Q} \in \{\forall, \exists\}$), then $\mathsf{FV}(\alpha) = \mathsf{FV}(\beta) - \{x\}$

A formula has the same free variables as its parts, except that a quantified variable becomes bound.

The notation $\alpha[t/x]$, for a variable x, a term t, and a formula α, denotes the formula obtained from α by replacing each *free* occurrence of x with t. Intuitively, it is the formula that answers the question,

> "What happens to α if x has the value specified by term t?"

Examples.

- If α is the formula $E\big(f(x)\big)$, then $\alpha[y+y/x]$ is $E\big(f(y+y)\big)$.
- $\alpha[f(x)/x]$ is $E\Big(f\big(f(x)\big)\Big)$.
- $E\big(f(x+y)\big)[y/x]$ is $E\big(f(y+y)\big)$.

Substitution does NOT affect bound occurrences of the variable.

- If β is $\forall x \cdot \big(E(f(x)) \wedge S(x,y)\big)$, then $\beta[g(x,y)/x]$ is β, because β has no free occurrence of x.

Example. Let β be $P(x) \wedge \exists x \cdot Q(x)$. What is $\beta[y/x]$?

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Examples: Substitution

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Example. What about $\beta[(y-1)/z]$, where β is $\forall x \cdot \exists y \cdot x + y = z$?

At first thought, we might say $\forall x \cdot \exists y \cdot x + y = y - 1$. But there's a problem—the free variable y in the term $y-1$ got "captured" by the quantifier $\exists y$.

We want to avoid this capture.

Avoiding Capture

Example. Formula $\alpha = S(x) \wedge \forall y \cdot \big(P(x) \to Q(y)\big)$; term $t = f(y, y)$.

The leftmost x can be substituted by t since it is not in the scope of any quantifier, but substituting in $P(x)$ puts the variable y into the scope of $\forall y$.

We can prevent capture of variables in two ways.

- Declare that a substitution is undefined in cases where capture would occur.
  One can often evade problems by a different choice of variable.
  (Above, we might be able to substitute $f(z, z)$ instead of $f(y, y)$.
  Or alter α to quantify some other variable.)
- Write the definition of substitution carefully, to prevent capture.

Huth and Ryan opt for the first method. We shall use the second.

Let x be a variable and t a term.

For a term u, the term $u[t/x]$ is u with each occurrence of the variable x replaced by the term t.

For a formula α,

1. If α is $P(t_1, \ldots, t_k)$, then $\alpha[t/x]$ is $P\big(t_1[t/x], \ldots, t_k[t/x]\big)$.
2. If α is $(\neg\beta)$, then $\alpha[t/x]$ is $(\neg\beta[t/x])$.
3. If α is $(\beta \star \gamma)$, then $\alpha[t/x]$ is $(\beta[t/x] \star \gamma[t/x])$.
4. $\ldots$

(Continued next page...)

Substitution—Formal Definition (2)

For variable x, term t and formula α:

$\vdots$

4. If α is $(Qx \cdot \beta)$, then $\alpha[t/x]$ is α.
5. If α is $(Qy \cdot \beta)$ for some other variable y, then
   (a) If y does not occur in t, then $\alpha[t/x]$ is $(Qy \cdot \beta[t/x])$.
   (b) Otherwise, select a variable z that occurs in neither α nor t; then $\alpha[t/x]$ is $(Qz \cdot (\beta[z/y])[t/x])$.

The last case prevents capture by renaming the quantified variable to something harmless.

(Huth and Ryan specify that the substitution is undefined if capture would occur—case 5(b) above. With this more complex definition, one never has to add a condition regarding undefined substitutions. Substitution always behaves "the way it should".)

Example. If α is $\forall x \cdot \exists y \cdot x + y = z$, what is $\alpha[(y-1)/z]$?

This falls under case 5(b): the term to be substituted, namely $y-1$, contains a variable y quantified in formula α.

Let β be $x + y = z$; thus α is $\forall x \cdot \exists y \cdot \beta$.

Example. If α is $\forall x \cdot \exists y \cdot x + y = z$, what is $\alpha[(y-1)/z]$?

This falls under case 5(b): the term to be substituted, namely $y-1$, contains a variable y quantified in formula α.

Let β be $x + y = z$; thus α is $\forall x \cdot \exists y \cdot \beta$.

Select a new variable, say w. Then

$$\beta[w/y] \quad \text{is} \quad x + w = z,$$

and

$$\beta[w/y][(y-1)/z] \quad \text{is} \quad x + w = y - 1.$$

Thus the required formula $\alpha[(y-1)/z]$ is

$$\forall x \cdot \exists w \cdot x + w = y - 1 \ .$$

Semantics of Predicate Logic

FOL Adds to Propositional Logic

In propositional logic, semantics was described in terms of valuations to propositional atoms.

FOL includes more ingredients (i.e., predicates, functions, variables, terms, constants, etc.) and, hence, the semantics for FOL must account for all of the ingredients.

We already saw the concept of an interpretation, which specifies the domain and the identities of the constants, relations and functions.

Formulas that include variables, and perhaps quantifiers, require additional information, known as an *environment* (or *assignment*).

A first-order *environment* is a function that assigns a value in the domain to each variable.

Example. With the domain $\mathbb{N}$, we might have environment θ_1 given by $\theta_1(x) = 9$ and $\theta_1(y) = 2$.

If the interpretation specifies $<$ is less-than, then $x < y$ gets value false.

Example. With the domain of fictional animals, we might have $\theta_2(x) = \mathit{Tweety}$ and $\theta_2(y) = \mathit{Nemo}$.

If the interpretation specifies $<$ is "was created before", then $x < y$ gets value true.

Example: Let α_1 be $P(c)$ (c a constant), and let α_2 be $P(x)$ (x a variable).

Let $\mathcal{M}$ be the interpretation with domain $\mathbb{N}$, $c^{\mathcal{M}} = 2$ and $P^{\mathcal{M}} =$ "is even". Then $\alpha_1^{\mathcal{M}} = \mathtt{T}$, but $\alpha_2^{\mathcal{M}}$ is undefined.

To give α_2 a value, we must also specify an environment. For example, if $\theta(x) = 2$, then $\alpha_2^{(\mathcal{M},\theta)} = \mathtt{T}$.

If we wish, we can consider a formula such as α_2 that contains a free variable x as expressing a function: the function that maps $\theta(x)$ to $\alpha_2^{(\mathcal{M},\theta)}$.

The combination of an interpretation and an environment supplies a value for every term.

Definition: Fix an interpretation $\mathcal{M}$ and environment θ. For each term t, the value of t under $\mathcal{M}$ and θ, denoted $t^{(\mathcal{M},\theta)}$, is as follows.

- If t is a constant c, the value $t^{(\mathcal{M},\theta)}$ is $c^{\mathcal{M}}$.
- If t is a variable x, the value $t^{(\mathcal{M},\theta)}$ is x^{θ}.
- If t is $f(t_1,\ldots,t_n)$, the value $t^{(\mathcal{M},\theta)}$ is $f^{\mathcal{M}}(t_1^{(\mathcal{M},\theta)},\ldots,t_n^{(\mathcal{M},\theta)})$.

To extend this definition to formulas, we must consider quantifiers.

But first, a few examples.

Example. Suppose a language has constant symbol 0, a unary function s, and a binary function $+$. We shall write $+$ in infix position: $x+y$ instead of $+(x,y)$.

The expressions $s\big(s(0)+s(x)\big)$ and $s\Big(x+s\big(x+s(0)\big)\Big)$ are both terms.

The following are examples of interpretations and environments.

- $dom\{\mathcal{I}\} = \{0,1,2,\ldots\}$, $0^{\mathcal{I}} = 0$, $s^{\mathcal{I}}$ is the successor function and $+^{\mathcal{I}}$ is the addition operation. Then, if $\theta(x) = 3$, the terms get values $\Big(s\big(s(0)+s(x)\big)\Big)^{(\mathcal{I},\theta)} = 6$ and $\Bigg(s\Big(x+s\big(x+s(0)\big)\Big)\Bigg)^{(\mathcal{I},\theta)} = 9$.

- $dom\{\mathcal{J}\}$ is the collection of all words over the alphabet $\{a, b\}$,
  $0^{\mathcal{J}} = a$,
  $s^{\mathcal{J}}$ appends a to the end of a string, and
  $+^{\mathcal{J}}$ is concatenation.

  Let $\theta(x) = aba$. Then

  $$\left(s\left(s(0) + s(x)\right)\right)^{(\mathcal{J},\theta)} = aaabaaa$$

  and

  $$\left(s\left(x + s\left(x + s(0)\right)\right)\right)^{(\mathcal{J},\theta)} = abaabaaaaa \quad .$$

Quantified Formulas

To evaluate the truthfulness of a formula $\forall x \cdot \alpha$ (or $\exists x \cdot \alpha$), we should check whether α holds for every (respectively, for some) value a in the domain.

How can we express this precisely?

Definition: For any environment θ and domain element d, the environment "θ with x re-assigned to d", denoted $\theta[x \mapsto \mathrm{d}]$, is given by

$$\theta[x \mapsto \mathrm{d}](y) = \begin{cases} \mathrm{d} & \text{if } y \text{ is } x \\ \theta(y) & \text{if } y \text{ is not } x. \end{cases}$$

Values of Quantified Formulas

Definition: The values of $\forall x \cdot \alpha$ and $\exists x \cdot \alpha$ are given by

- $(\forall x \cdot \alpha)^{(\mathcal{M},\theta)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\mathcal{M},\theta[x \mapsto \mathrm{d}])} = \mathtt{T} \text{ for every } \mathrm{d} \text{ in } dom(\mathcal{M}) \\ \mathtt{F} & \text{otherwise} \end{cases}$
- $(\exists x \cdot \alpha)^{(\mathcal{M},\theta)} = \begin{cases} \mathtt{T} & \text{if } \alpha^{(\mathcal{M},\theta[x \mapsto \mathrm{d}])} = \mathtt{T} \text{ for some } \mathrm{d} \text{ in } dom(\mathcal{M}) \\ \mathtt{F} & \text{otherwise} \end{cases}$

Note: The values of $(\forall x \cdot \alpha)^{(\mathcal{M},\theta)}$ and $(\exists x \cdot \alpha)^{(\mathcal{M},\theta)}$ do not depend on the value of $\theta(x)$.

The value $\theta(x)$ only matters for free occurrences of x.

Examples: Value of a Quantified Formula

Example. Let $dom(M) = \{a, b\}$ and $R^{\mathcal{M}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$.

Let $\theta(x) = a$ and $\theta(y) = b$. We have

- $R(x, x)^{(\mathcal{M},\theta)} = \mathtt{T}$, since $\left\langle \theta(x), \theta(x)\right\rangle = \left\langle a, a\right\rangle \in R^{\mathcal{M}}$.
- $R(y, x)^{(\mathcal{M},\theta)} = \mathtt{F}$, since $\left\langle \theta(y), \theta(x)\right\rangle = \left\langle b, a\right\rangle \notin R^{\mathcal{M}}$.
- $\left(\exists y \cdot R(y, x)\right)^{(\mathcal{M},\theta)} = \mathtt{T}$, since $R(y, x)^{(\mathcal{M},\theta[y \mapsto a])} = \mathtt{T}$.

  (That is, $\left\langle \theta[y \mapsto a](y), \theta[y \mapsto a](x)\right\rangle = \left\langle a, a\right\rangle \in R^{\mathcal{M}}$).
- What is $\left(\forall x \cdot \forall y \cdot R(x, y)\right)^{(\mathcal{M},\theta)}$?

Examples: Continued

Example. Let $dom(M) = \{a, b\}$ and $R^{\mathcal{M}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$.

Let $\theta(x) = a$ and $\theta(y) = b$.

- What is $\left(\forall x \cdot \forall y \cdot R(x, y)\right)^{(\mathcal{M},\theta)}$?

  Since $\langle b, a\rangle \notin R^{\mathcal{M}}$, we have

  $$R(x, y)^{(\mathcal{M},\theta[x \mapsto b][y \mapsto a])} = \mathtt{F} \enspace ,$$

  and thus

  $$\left(\forall x \cdot \forall y \cdot R(x, y)\right)^{(\mathcal{M},\theta)} = \mathtt{F} \enspace .$$

Example. Let $dom(M) = \{a, b\}$ and $R^{\mathcal{M}} = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$.

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  Since $\langle b, a\rangle \notin R^{\mathcal{M}}$, we have

  $$R(x, y)^{(\mathcal{M},\theta[x\mapsto b][y\mapsto a])} = \mathtt{F} \quad ,$$

  and thus

  $$\left(\forall x \cdot \forall y \cdot R(x, y)\right)^{(\mathcal{M},\theta)} = \mathtt{F} \quad .$$

- What about $\left(\forall x \cdot \exists y \cdot R(x, y)\right)^{(\mathcal{M},\theta)}$?

A Question of Syntax

In the previous example, we wrote

$$R(x,y)^{(\mathcal{M},\theta[x\mapsto b][y\mapsto a])} = \mathtt{F} \ .$$

Why did we not write simply

$$R(b,a) = \mathtt{F}$$

or perhaps

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$$R(b,a) = \mathtt{F}$$

or perhaps

$$R(b,a)^{(\mathcal{M},\theta)} = \mathtt{F} \ ?$$

Because "$R(b,a)$" is not a formula. The elements a and b of $dom(\mathcal{M})$ are not symbols in the language; they cannot appear in a formula.

Satisfaction of Formulas

An interpretation $\mathcal{M}$ and environment θ *satisfy* a formula α, denoted $\mathcal{M} \models_\theta \alpha$, if $\alpha^{(\mathcal{M},\theta)} = \mathtt{T}$;
they do not satisfy α, denoted $\mathcal{M} \not\models_\theta \alpha$, if $\alpha^{(\mathcal{M},\theta)} = \mathtt{F}$.

Form of α	*Condition for $\mathcal{M} \models_\theta \alpha$*
$R(t_1, \ldots, t_k)$	$\left\langle t_1^{(\mathcal{M},\theta)}, \ldots, t_k^{(\mathcal{M},\theta)} \right\rangle \in R^{\mathcal{M}}$
$\neg\beta$	$\mathcal{M} \not\models_\theta \beta$
$\beta \wedge \gamma$	both $\mathcal{M} \models_\theta \beta$ and $\mathcal{M} \models_\theta \gamma$
$\beta \vee \gamma$	either $\mathcal{M} \models_\theta \beta$ or $\mathcal{M} \models_\theta \gamma$ (or both)
$\beta \to \gamma$	either $\mathcal{M} \not\models_\theta \beta$ or $\mathcal{M} \models_\theta \gamma$ (or both)
$\forall x \cdot \beta$	for every $\mathrm{a} \in dom(\mathcal{M})$, $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \beta$
$\exists x \cdot \beta$	there is some $\mathrm{a} \in dom(\mathcal{M})$ such that $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \beta$

If $\mathcal{M} \models_\theta \alpha$ for every θ, then $\mathcal{M}$ *satisfies* α, denoted $\mathcal{M} \models \alpha$.

Example: Satisfaction

Example. Consider the formula $\exists y \cdot R(x, y \oplus y)$.

(For R a binary relation and $\oplus$ a binary function.)

Suppose $dom(\mathcal{M}) = \{1, 2, 3, \ldots\}$,
$\oplus^{\mathcal{M}}$ is the addition operation, and
$R^{\mathcal{M}}$ is the equality relation.

Then $\mathcal{M} \models_\theta \exists y \cdot R(x, y \oplus y)$ iff $\theta(x)$ is an even number.

Validity and Satisfiability

Validity and satisfiability of formulas have definitions analogous to the ones for propositional logic.

Definition: A formula α is

- *valid* if every interpretation and environment satisfy α; that is, if $\mathcal{M} \models_E \alpha$ for every $\mathcal{M}$ and E,
- *satisfiable* if some interpretation and environment satisfy α; that is, if $\mathcal{M} \models_E \alpha$ for some $\mathcal{M}$ and E, and
- *unsatisfiable* if no interpretation and environment satisfy α; that is, if $\mathcal{M} \not\models_E \alpha$ for every $\mathcal{M}$ and E.

(The term "tautology" is not used in predicate logic.)

Example: Satisfiability and Validity

Let α be the formula $P\Big(f\big(g(x), g(y)\big), g(z)\Big)$. The formula is satisfiable:

- $dom(\mathcal{M})$: $\mathbb{N}$
- $f^{\mathcal{M}}$: summation
- $g^{\mathcal{M}}$: squaring
- $P^{\mathcal{M}}$: equality
- $\theta(x) = 3$, $\theta(y) = 4$ and $\theta(z) = 5$.

α is not valid. (Why?)

Quantifiers Over Finite Domains

The universal and existential quantifiers may be understood respectively as generalizations of conjunction and disjunction. If the domain $D = \{a_1, \ldots, a_k\}$ is finite then:

For all x, $R(x)$ iff $R(a_1)$ and ... and $R(a_k)$

There exists x, $R(x)$ iff $R(a_1)$ or ... or $R(a_k)$

where R is a property.

Lemma:

Let α be a first-order formula, $\mathcal{M}$ be an interpretation, and θ_1 and θ_2 be two environments such that

$$\theta_1(x) = \theta_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\mathcal{M} \models_{\theta_1} \alpha \text{ if and only if } \mathcal{M} \models_{\theta_2} \alpha \ .$$

Proof by induction on the structure of α.

Logical Consequence

Suppose Σ is a set of formulas and α is a formula. We say that α is a *logical consequence* of Σ, written as $\Sigma \models \alpha$, iff for any interpretation $\mathcal{M}$ and environment θ, we have $\mathcal{M} \models_\theta \Sigma$ implies $\mathcal{M} \models_\theta \alpha$.

$\models \alpha$ means that α is valid.

Example: Show that $\models \big(\forall x \cdot (\alpha \to \beta)\big) \to \big((\forall x \cdot \alpha) \to (\forall x \cdot \beta)\big)$.

Proof by contradiction. Suppose there are $\mathcal{M}$ and θ such that

$$\mathcal{M} \not\models_\theta \big(\forall x \cdot (\alpha \to \beta)\big) \to \big((\forall x \cdot \alpha) \to (\forall x \cdot \beta)\big) \ .$$

Then we must have $\mathcal{M} \models_\theta \forall x \cdot (\alpha \to \beta)$ and $\mathcal{M} \not\models_\theta (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$;

the second gives $\mathcal{M} \models_\theta \forall x \cdot \alpha$ and $\mathcal{M} \not\models_\theta \forall x \cdot \beta$.

Using the definition of $\models$ for formulas with $\forall$, we have
for every a $\in dom(\mathcal{M})$, $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \alpha \to \beta$ and $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \alpha$.
Thus also $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \beta$ for every a $\in dom(\mathcal{M})$.

Thus $\mathcal{M} \models_\theta \forall x \cdot \beta$, a contradiction.

Example. Show that $\forall x \cdot \neg\gamma \models \neg\exists x \cdot \gamma$.

Example. Show that $\forall x \cdot \neg\gamma \models \neg\exists x \cdot \gamma$.

Suppose that $\mathcal{M} \models_\theta \forall x \cdot \neg\gamma$. By definition, this means

for every $\mathrm{a} \in dom(\mathcal{M})$, $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \neg\gamma$.

Again by definition (for a formula with $\neg$), this is equivalent to

for every $\mathrm{a} \in dom(\mathcal{M})$, $\mathcal{M} \not\models_{\theta[x \mapsto \mathrm{a}]} \gamma$

and also

there is no $\mathrm{a} \in dom(\mathcal{M})$ such that $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \gamma$.

This last is the definition of $\mathcal{M} \models_\theta \neg\exists x \cdot \gamma$, as required.

Example: Show that, in general,

$$(\forall x \cdot \alpha) \rightarrow (\forall x \cdot \beta) \not\models \forall x \cdot (\alpha \rightarrow \beta) \ .$$

(That is, find α and β such that consequence does not hold.)

Example: Show that, in general,

$$(\forall x \cdot \alpha) \to (\forall x \cdot \beta) \not\models \forall x \cdot (\alpha \to \beta) \ .$$

(That is, find α and β such that consequence does not hold.)

Key idea: $\alpha \to \beta$ yields true whenever α is false.

Let α be $R(x)$. Let $\mathcal{M}$ have domain $\{a, b\}$ and $R^{\mathcal{M}} = \{a\}$. Then $\mathcal{M} \models (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$ for any β. (Why?)

Example: Show that, in general,

$$(\forall x \cdot \alpha) \to (\forall x \cdot \beta) \not\models \forall x \cdot (\alpha \to \beta) \ .$$

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Key idea: $\alpha \to \beta$ yields true whenever α is false.

Let α be $R(x)$. Let $\mathcal{M}$ have domain $\{a, b\}$ and $R^{\mathcal{M}} = \{a\}$. Then $\mathcal{M} \models (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$ for any β. (Why?)

To obtain $M \not\models \forall x \cdot (\alpha \to \beta)$, we can use $\neg R(x)$ for β. (Why?)

Thus $\big((\forall x \cdot \alpha) \to (\forall x \cdot \beta)\big) \not\models \forall x \cdot (\alpha \to \beta)$, as required. (Why?)

Example: for any formula α and term t,

$$\models (\forall x \cdot \alpha) \rightarrow \alpha[t/x] \ .$$

Proofs in First-Order Logic Using Natural Deduction

Natural Deduction for FOL extends Natural Deduction for propositional logic by including rules for introduction and elimination of quantifiers.

Other proof techniques and tricks remain the same as natural deduction for propositional logic.

∀e and ∃i

Elimination of ∀ and introduction of ∃ are fairly straightforward.

Name	⊢-notation	inference notation
∀-elimination (∀e)	If $\Sigma \vdash \forall x \cdot \alpha$ then $\Sigma \vdash \alpha[t/x]$	$\dfrac{\forall x \cdot \alpha}{\alpha[t/x]}$
∃-introduction (∃i)	If $\Sigma \vdash \alpha[t/x]$, then $\Sigma \vdash \exists x \cdot \alpha$	$\dfrac{\alpha[t/x]}{\exists x \cdot \alpha}$

Given that a formula is true for every value of x,
conclude it is true for any particular value, such as that of t.

Given that a formula is true for a particular value (of t),
conclude it is true for some value.

Example: $\forall$e

"All fish can swim. Nemo is a fish. Therefore, Nemo can swim."

In FOL: show that $\forall x \cdot (F(x) \rightarrow S(x))$, $F(\mathit{Nemo}) \vdash S(\mathit{Nemo})$..

Proof:

1.	$\forall x \cdot (F(x) \rightarrow S(x))$	Premise
2.	$F(\mathit{Nemo})$	Premise
3.	$F(\mathit{Nemo}) \rightarrow S(\mathit{Nemo})$	$\forall$e: 1
4.	$S(\mathit{Nemo})$	$\rightarrow$i: 2, 3

The proof doesn't care what F and S mean. Fishiness and swimming ability really have nothing to do with the argument.

Example. Show $\neg P(y) \vdash \exists x \cdot \big(P(x) \to Q(y)\big)$.

1.	$\neg P(y)$	Premise
2.	$P(y)$	Assumption
3.	$\bot$	$\neg$e: 2, 1
4.	$Q(y)$	$\bot$e: 3
5.	$P(y) \to Q(y)$	$\to$i: 2–4
6.	$\exists x \cdot \big(P(x) \to Q(y)\big)$	∃i: 5

(The last step could have produced $\exists x \cdot \big(P(x) \to Q(x)\big)$, if desired.)

Soundness of ∀-Elimination and ∃-Introduction

Claim: For any formula φ, variable x and term t,

$$\forall x \cdot \varphi \models \varphi[t/x] \qquad \text{and} \qquad \varphi[t/x] \models \exists x \varphi \ .$$

Proof: Suppose $\mathcal{M} \models_E \mathrm{Q}x \cdot \varphi$; i.e., for (every/some) $\mathrm{d} \in dom(\mathcal{M})$,

$$\varphi^{(\mathcal{M}, E[x \mapsto \mathrm{d}])} = \mathtt{T} \ .$$

Since $\mathrm{d} = t^{(\mathcal{M},E)}$ is a domain value, it suffices to show

Claim II: For every formula φ, variable x and term t,

$$\varphi[t/x]^{(\mathcal{M},E)} = \varphi^{(\mathcal{M}, E[x \mapsto t^{(\mathcal{M},E)}])} \ .$$

To prove this second claim, use the definition of substitution.

(Left to you. Cases 4, 5(a) and 5(b) of the definition make it work.)

Proving a Universal

Our next rule is $\forall$-introduction, but we start with an example. To prove:

> Sam is less than three meters tall.

How could you prove this?

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> The person named "Sam" is less than three meters tall.

If we have no information about who might have the name "Sam", this is essentially the same as

> A person who might be referred to as "Sam" (or might not) is less than three meters tall.

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How could you prove this? Let's re-phrase it:

> The person named "Sam" is less than three meters tall.

If we have no information about who might have the name "Sam", this is essentially the same as

> A person who might be referred to as "Sam" (or might not) is less than three meters tall.

More simply put,

> Every person is less than three meters tall.

Rule ∀-Introduction

Definition: a variable is *fresh* in a subproof if it occurs nowhere outside the box of the subproof.

Freshness captures the notion of "no information available".

Name	$\vdash$-notation	inference notation
∀-introduction (∀i)	If $\Sigma \vdash \alpha[y/x]$ and y not free in Σ or α, then $\Sigma \vdash \forall x \cdot \alpha$	$\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{\forall x \cdot \alpha}$

In words: in order to prove $\forall x \cdot \alpha(x)$, prove $\alpha(y)$ for arbitrary y.

To further clarify the rule ∀i, we show that it is sound. That is,

> Suppose that $\Sigma \models \alpha[y/x]$ and y is not free in Σ or α.
> Then $\Sigma \models \forall x \cdot \alpha$.

Proof: Fix an arbitrary $\mathcal{M}$ and θ with $\mathcal{M} \models_\theta \Sigma$.
The supposition $\Sigma \models \alpha[y/x]$ thus requires $\mathcal{M} \models_\theta \alpha[y/x]$.

We need to show that $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \alpha$ for every $\mathrm{a} \in dom(\mathcal{M})$.

Consider an arbitrary $\mathrm{a} \in dom(\mathcal{M})$.
Since y is not free in Σ, the Relevance Lemma yields $\mathcal{M} \models_{\theta[y \mapsto \mathrm{a}]} \Sigma$.

Since y is not free in α, we have $\alpha[y/x]^{(\mathcal{M},\theta[y \mapsto \mathrm{a}])} = \alpha^{(\mathcal{M},\theta[x \mapsto \mathrm{a}])}$.

Therefore $\mathcal{M} \models_{\theta[x \mapsto \mathrm{a}]} \alpha$ for every a, and thus $\mathcal{M} \models_\theta \forall x \cdot \alpha$ as required.

Example: Use of ∀i

Example. Show that $\neg\exists x \cdot \alpha \vdash \forall x \cdot \neg\alpha$, for any α.

1.	$\neg\exists x \cdot \alpha$	Premise
2.	u fresh	
	$\neg\alpha[u/x]$	??
n.	$\forall x \cdot \neg\alpha$	∀i: 2–6

Note: "u fresh" means we choose any variable not in α (and not x).

Example: Use of ∀i

Example. Show that $\neg \exists x \cdot \alpha \vdash \forall x \cdot \neg\alpha$, for any α.

1.	$\neg \exists x \cdot \alpha$	Premise
2.	u fresh	
3.	$\alpha[u/x]$	Assumption
4.	$\exists x \cdot \alpha$	∃i: 3
5.	$\bot$	¬e: 1, 4
6.	$\neg\alpha[u/x]$	¬i: 3–5
7.	$\forall x \cdot \neg\alpha$	∀i: 2–6

Note: "u fresh" means we choose any variable not in α (and not x).

Show that $\forall x \cdot (\alpha \to \beta) \vdash (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$.

1.	$\forall x \cdot (\alpha \to \beta)$	Premise
	$(\forall x \cdot \alpha) \to (\forall x \cdot \beta)$	→i??

Note: do not apply rule ∀e until you know which term to use.

Show that $\forall x \cdot (\alpha \to \beta) \vdash (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$.

1.	$\forall x \cdot (\alpha \to \beta)$	Premise
2.	$\forall x \cdot \alpha$	Assumption
	$\forall x \cdot \beta$	∀i???
8.	$(\forall x \cdot \alpha) \to (\forall x \cdot \beta)$	→i??

Note: do not apply rule ∀e until you know which term to use.

Show that $\forall x \cdot (\alpha \to \beta) \vdash (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$.

1.	$\forall x \cdot (\alpha \to \beta)$	Premise
2.	$\forall x \cdot \alpha$	Assumption
3.	u fresh	
6.	$\beta[u/x]$	??
7.	$\forall x \cdot \beta$	∀i???
8.	$(\forall x \cdot \alpha) \to (\forall x \cdot \beta)$	→i??

Note: do not apply rule ∀e until you know which term to use.

Example: Another use of ∀i

Show that $\forall x \cdot (\alpha \to \beta) \vdash (\forall x \cdot \alpha) \to (\forall x \cdot \beta)$.

1.	$\forall x \cdot (\alpha \to \beta)$	Premise
2.	$\forall x \cdot \alpha$	Assumption
3.	u fresh	
4.	$\alpha[u/x] \to \beta[u/x]$	∀e: 1
5.	$\alpha[u/x]$	∀e: 2
6.	$\beta[u/x]$	→e: 4, 5
7.	$\forall x \cdot \beta$	∀i: 3–6
8.	$(\forall x \cdot \alpha) \to (\forall x \cdot \beta)$	→i: 2–7

Note: do not apply rule ∀e until you know which term to use.

Elimination of an Existential Quantifier

Name	$\vdash$-notation	inference notation
$\exists$-elimination ($\exists$e)	If $\Sigma, \alpha[u/x] \vdash \beta$, with u fresh, then $\Sigma, \exists x \cdot \alpha \vdash \beta$	$$\dfrac{\exists x \cdot \alpha \qquad \boxed{\begin{array}{c}\alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta\end{array}}}{\beta}$$

In $\exists$e, the variable u should not occur free in Σ, α, or β.

(Of course, u will normally be free in $\alpha[u/x]$.)

Rule ∃e Is Sound

The rule ∃e is sound. That is,

> Suppose that $\Sigma, \alpha[u/x] \models \beta$ and u is not free in Σ, α, or β.
> Then $\Sigma, \exists x \cdot \alpha \models \beta$.

Proof: Exercise. Follow the proof of soundness of ∀i.

Example. Show that $\exists x \cdot R(x) \vdash \exists y \cdot R(y)$.

1.	$\exists x \cdot R(x)$	Premise
2.	$R(u)$, u fresh	Assumption
3.	$\exists y \cdot R(y)$	∃i: 2 (term u)
4.	$\exists y \cdot R(y)$	∃e: 1, 2–3

Extending the example?

Clearly, the previous proof did not depend on the particular relation R that we used. Can we do the same proof for arbitrary formulas?

Does $\exists x \cdot \alpha \vdash \exists y \cdot \alpha[y/x]$ hold?

1.	$\exists x \cdot \alpha$	Premise
2.	$\alpha[u/x]$, u fresh	Assumption
3.	$\alpha[y/x][u/y]$	????
4.	$\exists y \cdot \alpha[y/x]$	$\exists$i: 3 (term u)
5.	$\exists y \cdot \alpha[y/x]$	$\exists$e: 1, 2–4

Is the formula on line 2 the same as the one on line 3?

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Clearly, the previous proof did not depend on the particular relation R that we used. Can we do the same proof for arbitrary formulas?

Does $\exists x \cdot \alpha \vdash \exists y \cdot \alpha[y/x]$ hold?

1.	$\exists x \cdot \alpha$	Premise
2.	$\alpha[u/x]$, u fresh	Assumption
3.	$\alpha[y/x][u/y]$	????
4.	$\exists y \cdot \alpha[y/x]$	$\exists$i: 3 (term u)
5.	$\exists y \cdot \alpha[y/x]$	$\exists$e: 1, 2–4

Is the formula on line 2 the same as the one on line 3?

If y is free in α, then no — the derivation fails.
But otherwise, it works.

Example. Show that $\exists x \cdot \neg\alpha \vdash \neg\forall x \cdot \alpha$.

1.	$\exists x \cdot \neg\alpha$	Premise
	$\neg\forall x \cdot \alpha$	∃e ??

Example. Show that $\exists x \cdot \neg\alpha \vdash \neg\forall x \cdot \alpha$.

1.	$\exists x \cdot \neg\alpha$	Premise
2.	$\neg\alpha[u/x]$, u fresh	Assumption
	$\neg\forall x \cdot \alpha$	¬i ??
7.	$\neg\forall x \cdot \alpha$	∃e ??

Example. Show that $\exists x \cdot \neg \alpha \vdash \neg \forall x \cdot \alpha$.

1.	$\exists x \cdot \neg \alpha$	Premise
2.	$\neg \alpha[u/x]$, u fresh	Assumption
3.	$\forall x \cdot \alpha$	Assumption
4.	$\alpha[u/x]$	∀e: 3
5.	$\perp$	¬e: 4, 2
6.	$\neg \forall x \cdot \alpha$	¬i: 3–5
7.	$\neg \forall x \cdot \alpha$	∃e: 1, 2–6

We can interchange the quantifiers in the previous deduction.

Example. Show $\forall x \cdot \neg\alpha \vdash \neg\exists x \cdot \alpha$.

1.	$\forall x \cdot \neg\alpha$	Assumption
2.	$\exists x \cdot \alpha$	Assumption
3.	$\alpha[u/x]$ (u fresh)	Assumption
4.	$\neg\alpha[u/x]$	∀e: 1
5.	$\perp$	¬e: 3, 4
6.	$\perp$	∃e: 2, 3–5
7.	$\neg\exists x \cdot \alpha$	¬i: 2–6

So far, we have shown $\neg\exists x \cdot \alpha \vdash \forall x \cdot \neg\alpha$,
$\forall x \cdot \neg\alpha \vdash \neg\exists x \cdot \alpha$, and
$\exists x \cdot \neg\alpha \vdash \neg\forall x \cdot \alpha$.

Example. Show that $\neg\forall x \cdot \alpha \vdash \exists x \cdot \neg\alpha$.

1.	$\neg\forall x \cdot \alpha$	Premise
	$\neg\alpha[t/x]$	??
	$\exists x \cdot \neg\alpha$	∃i: ??

Quantifiers and Negation: The final case

So far, we have shown $\neg\exists x \cdot \alpha \vdash \forall x \cdot \neg\alpha$,
$\forall x \cdot \neg\alpha \vdash \neg\exists x \cdot \alpha$, and
$\exists x \cdot \neg\alpha \vdash \neg\forall x \cdot \alpha$.

Example. Show that $\neg\forall x \cdot \alpha \vdash \exists x \cdot \neg\alpha$.

1.	$\neg\forall x \cdot \alpha$	Premise
	$\neg\alpha[t/x]$	??
	$\exists x \cdot \neg\alpha$	$\exists$i: ??

For what term t can we prove $\neg\alpha[t/x]$?

Quantifiers and Negation: The final case

So far, we have shown $\neg\exists x \cdot \alpha \vdash \forall x \cdot \neg\alpha$,
$\forall x \cdot \neg\alpha \vdash \neg\exists x \cdot \alpha$, and
$\exists x \cdot \neg\alpha \vdash \neg\forall x \cdot \alpha$.

Example. Show that $\neg\forall x \cdot \alpha \vdash \exists x \cdot \neg\alpha$.

1.	$\neg\forall x \cdot \alpha$	Premise
	$\neg\alpha[t/x]$	??
	$\exists x \cdot \neg\alpha$	∃i: ??

For what term t can we prove $\neg\alpha[t/x]$?

There is no such t!

We need to try something cleverer. . . .

Example. Show that $\neg \forall x \cdot \alpha \vdash \exists x \cdot \neg\alpha$.

1.	$\neg \forall x \cdot \alpha$	Premise
2.	$\neg \exists x \cdot \neg\alpha$	Assumption
3.	u fresh	
4.	$\neg\alpha[u/x]$	Assumption
5.	$\exists x \cdot \neg\alpha$	∃i: 4
6.	$\perp$	¬e: 5, 2
7.	$\neg\neg\alpha[u/x]$	¬i: 4–6
8.	$\alpha[u/x]$	¬¬e: 7
9.	$\forall x \cdot \alpha$	∀i: 3–8
10.	$\perp$	¬e: 9, 1
11.	$\neg\neg \exists x \cdot \neg\alpha$	¬i: 2–10
12.	$\exists x \cdot \neg\alpha$	¬¬e: 11

The rules for elimination and introduction of quantifiers can be generalized to multiple quantifiers.

Let $x_1, \ldots, x_n$ be n distinct variables.

- If $\Sigma \vdash \forall x_1 \cdots \forall x_n \cdot \alpha$, then $\Sigma \vdash \alpha[t_1/x_1]\cdots[t_n/x_n]$.
- If $\Sigma \vdash \alpha[t_1/x_1]\cdots[t_n/x_n]$, for terms $t_1, \ldots, t_n$, then $\Sigma \vdash \exists x_1 \cdots \exists x_n \cdot \alpha$.
- If $\Sigma \vdash \alpha[u_1/x_1]\cdots[u_n/x_n]$, with variables $u_1, \ldots, u_n$ fresh, then $\Sigma \vdash \forall x_1 \cdots \forall x_n \cdot \alpha$.
- If $\Sigma \vdash \exists x_1 \cdots \exists x_n \cdot \alpha$ and $\Sigma \cup \{\alpha[u_1/x_1]\cdots[u_n/x_n] \vdash \beta$, with $u_1, \ldots, u_n$ fresh, then $\Sigma \vdash \beta$.

Example: Repeated universal quantifiers

Example. Show that $\forall x \cdot \forall y \cdot A(x, y) \vdash \forall y \cdot \forall x \cdot A(x, y)$.

1.	$\forall x \cdot \forall y \cdot A(x, y)$	Premise
2.	u, v fresh	
3.	$A(u, v)$	$\forall$e$(\times 2)$: 1
4.	$\forall y \cdot \forall x \cdot A(x, y)$	$\forall$i$(\times 2)$: 3

Exercise on Quantifier Rules

Exercise. Show that

$$\{\,\forall x \cdot \big(Q(x) \rightarrow R(x)\big),\ \exists x \cdot \big(P(x) \wedge Q(x)\big)\,\} \vdash \exists x \cdot \big(P(x) \wedge R(x)\big)\ .$$

Left to you.

Generally, relation symbols have no mandated interpretation.
Sometimes, however, one makes an exception for the symbol $=$.

Definition: First-Order Logic with Equality is First-Order Logic with the restriction that the symbol "$=$" must be interpreted as equality on the domain:

$$(=)^{\mathcal{I}} = \{ \langle d, d \rangle \mid d \in dom(\mathcal{I}) \} \quad .$$

Symbol $=$ gets its pair of deduction rules:

Equals-Introduction: $$\frac{}{t = t} =\mathrm{i}$$

Equals-Elimination: $$\frac{t_1 = t_2 \quad \alpha[t_1/x]}{\alpha[t_2/x]} =\mathrm{e}$$

Axioms for Equality

As an alternative to taking deduction rules for $=$, one can intead define *axioms* for equality. An axiom is a premise that is always taken; it need not be listed explicitly.

EQ1: $\forall x \cdot x = x$ is an axiom.

EQ2: For each formula α and variable z,

$$\forall x \cdot \forall y \cdot \big(x = y \to (\alpha[x/z] \to \alpha[y/z])\big)$$

is an axiom.

These axioms imply

- Symmetry of $=$: $\vdash \forall x \cdot \forall y \cdot \big(x = y \to y = x\big)$.
- Transitivity of $=$: $\vdash \forall x \cdot \forall y \cdot \forall w \cdot \big(x = y \to (y = w \to x = w)\big)$.

Lemma. $\vdash \forall x \cdot \forall y \cdot (x = y \to y = x)$.

1.

$\forall x \cdot \forall y \cdot (x = y \to y = x)$ $\quad$ $\forall$i ($\times$2): 1–?

Lemma. $\vdash \forall x \cdot \forall y \cdot \big(x = y \to y = x\big)$.

1.	u, v fresh	
	$u = v \to v = u$	???
	$\forall x \cdot \forall y \cdot \big(x = y \to y = x\big)$	$\forall$i ($\times 2$): 1–?

Lemma. $\vdash \forall x \cdot \forall y \cdot \big(x = y \rightarrow y = x\big)$.

1.	u, v fresh	
2.	$u = v$	Assumption
	$v = u$	??
	$u = v \rightarrow v = u$	$\rightarrow$i: 2–?
	$\forall x \cdot \forall y \cdot \big(x = y \rightarrow y = x\big)$	$\forall$i$(\times 2)$: 1–?

Lemma. $\vdash \forall x \cdot \forall y \cdot \big(x = y \rightarrow y = x\big)$.

1.	u, v fresh	
2.	$u = v$	Assumption
3.	$u = u$	=i
4.	$v = u$	=e: 2, 3 $[x = u]$
5.	$u = v \rightarrow v = u$	→i: 2–4
6.	$\forall x \cdot \forall y \cdot \big(x = y \rightarrow y = x\big)$	∀i (×2): 1–5

Lemma. $\vdash \forall x \cdot \forall y \cdot \forall w \cdot \big(x = y \rightarrow (y = w \rightarrow x = w)\big)$

1.	u, v, w fresh	
2.	$u = v$	Assumption
3.	$u = v \rightarrow v = u$	Symmetry of $=$, $+ \forall$e
4.	$v = u$	$\rightarrow$e: 2, 3
5.	$v = w$	Assumption
6.	$u = w$	$=$e: 3, 5 $[x = w]$
7.	$v = w \rightarrow u = w$	$\rightarrow$i: 5–6
8.	$u = v \rightarrow (v = w \rightarrow u = w)$	$\rightarrow$i: 3–7
9.	$\forall x \cdot \forall y \cdot \forall w \cdot \big(x = y \rightarrow (y = w \rightarrow x = w)\big)$	$\forall$i($\times 3$): 1–8

Equality satisfies the following derived rules.

k-Way Transitivity:

EQtrans(k): $\dfrac{t_1 = t_2 \quad t_2 = t_3 \quad \cdots \quad t_k = t_{k+1}}{t_1 = t_{k+1}}$ for any $t_1, \ldots, t_{k+1}$.

EQtrans(k) results from $k-1$ uses of transitivity.

Substitution of Equals:

EQsubs(r): $\dfrac{t_1 = t_2}{r[t_1/z] = r[t_2/z]}$ for any variable z and terms t_1 and t_2.

Appling rule =e with formulas $t_1 = t_2$ and $r[t_1/z] = r[t_1/z]$ (from =i) yields the conclusion.

Theorem.

- Natural Deduction is sound for FOL: if $\Sigma \vdash \alpha$, then $\Sigma \models \alpha$.
- Natural Deduction is complete for FOL: if $\Sigma \models \alpha$, then $\Sigma \vdash \alpha$.

Proof outline:

Soundness: Each application of a rule is sound. By induction, any finite number of rule applications is sound.

Completeness: We shall show the contrapositive:

$$\text{if } \Sigma \nvdash \alpha, \text{ then } \Sigma \not\models \alpha \ .$$

We shall not give the full proof, but we will sketch the main points.

Completeness of ND for FOL: Getting started

To show: if $\Sigma \nvdash \alpha$, then $\Sigma \not\models \alpha$.

Lemma I: If $\Sigma \nvdash \alpha$, then $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

By rule →i, if $\Sigma \cup \{\neg\alpha\} \vdash \alpha$, then $\Sigma \vdash \neg\alpha \rightarrow \alpha$. Thus $\Sigma \vdash \alpha$.

Lemma II: If there are $\mathcal{M}$ and E s.t. $\mathcal{M} \models_E \Sigma \cup \{\neg\alpha\}$, then $\Sigma \not\models \alpha$.

$\mathcal{M}$ and E satisfy Σ but not α.

Lemma III (the big one):

If $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$, then there are $\mathcal{M}$ and E such that $\mathcal{M} \models_E \Sigma \cup \{\neg\alpha\}$.

Whither a Domain?

Given: $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

Required: interpretation $\mathcal{D}$ and environment E that satisfy $\Sigma \cup \{\neg\alpha\}$.

To start, we need a domain. Where can we get one?

Whither a Domain?

Given: $\Sigma \cup \{\neg\alpha\} \nvdash \alpha$.

Required: interpretation $\mathcal{D}$ and environment E that satisfy $\Sigma \cup \{\neg\alpha\}$.

To start, we need a domain. Where can we get one?

Use terms as values! That is, let the domain be

$$\left\{ \ulcorner t \urcorner \,|\, t \text{ is a term} \right\} \ .$$

We use the notation '$\ulcorner\ \urcorner$' to indicate that we refer to the domain element, rather than to the expression.

Interpretation of Terms

For a set Σ of premises, we want an interpretation $\mathcal{I}$ and an environment E, over the domain of terms.

Constants, variables, and functions are easy to handle.

- For a constant symbol c, we define $c^{\mathcal{I}} = \ulcorner c \urcorner$.
- For a variable x, we define $x^E = \ulcorner x \urcorner$.
- For a k-ary function symbol f, we define $f^{\mathcal{I}}(\ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner) = \ulcorner f(t_1, \ldots, t_k) \urcorner$.

Relations pose a problem, since they depend on Σ. For a relation symbol $R^{(k)}$, we must determine, for each tuple $\langle t_1, \ldots, t_k \rangle$, whether to put $\left\langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_k \urcorner \right\rangle$ into the set $R^{\mathcal{I}}$.

The basic idea is to consider each possible tuple, one by one.

We suppress the details.

One "Piece of the Puzzle": Listing all formulas

As one part of the construction, we require a list of all possible formulas. Since we may have arbitrarily many constant, variable, functions and relation symbols, of any arity, we must take care that everything gets onto the list at some point. For example, if we take the ith formula to be $R(c_i)$, then many formulas ($R(x)$, $Q_4(f_7(y_{66}))$, etc.) never appear on the list.

We do the listing "in stages", starting from stage 1. At stage j, consider the first j constants, variables, and function symbols. Form all terms that combine these, using at most j applications of a function. Apply each of the first j relation symbols to each of these terms.

The set of formulas formed this way is large, but finite. After all have been listed, continue to stage $j+1$.