SM2001/FSM3001 DATA-DRIVEN METHODS IN ENGINEERING

Homework #1

Problem 1

Find by hand the full, non-reduced singular-value decomposition (SVD) and the pseudoinverse of the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Problem 2

Given the Frobenius norm of a matrix $\boldsymbol{A}$ with n rows and m columns:

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} = |A_{ij}|^2},$$

show that this norm can also be written as:

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{j=1}^{\min(m,n)} \sigma_j^2},$$

where σ_j^2 are the singular values of $\boldsymbol{A}$.
Hint: You may use the fact that multiplication (on the left or right) by a unitary matrix does not change the Frobenius norm of the original matrix, *i.e.* $\|\boldsymbol{AV}\|_F = \|\boldsymbol{A}\|_F$ if $\boldsymbol{V}$ is unitary.

Problem 3

Suppose that we want to define an inner product on $\mathbb{R}^n$ by:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T \boldsymbol{M} \boldsymbol{x},$$

where $\boldsymbol{M}$ is an $n \times n$ matrix. Which properties must $\boldsymbol{M}$ have for this to be an inner product? Note that a non-standard inner product such as this, where $\boldsymbol{M}$ is not the identity matrix, can be used in situations where the data is in a non-uniform mesh and/or it is unevenly sampled in time (see discussion on POD at the end of Chapter 1).

Problem 4

Import the data provided in the file `HW1Q4.csv`. Considering this data as a matrix, compute its (reduced/economy) singular-value decomposition. Plot the singular values, and the first three left and right singular vectors. Verify that the square of the Frobenius norm of the data projected onto the first three singular vectors is equal to the sum of the squares of the first three singular values.

Problem 5

Imagine that you are a coin dealer who is interested in acquiring a set of 100 gold coins. The seller is offering you a good deal, but you suspect that some of these coins are fraudulent, and are not actually solid gold. This can be most easily tested by assessing whether any coins have a different weight than that of the pure-gold coins. Upon you expressing your suspicions, the seller will not let you weigh each coin individually, but says as a compromise that they will allow you to perform 20 weighings only. You agree with these terms, on the condition that if you are able to tell exactly which coins are fake using only these weighings, you can have all of the coins for free. The seller accepts this offer, thinking that it will be very unlikely to detect all the fake coins with such few weighings. However, the seller does not know that you have knowledge of the dark art of compressed sensing.

You proceed to make 20 weighings, each of a randomly-selected set of coins. You express your set of weighings as:

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{x}$ is an unknown column vector of length 100 representing the difference between the weight of each coin and the expected weight of a real gold coin; $\boldsymbol{C}$ is a 20×100 matrix, the rows of which indicate which coins are included in each weighing; and $\boldsymbol{b}$ is a column vector of length 20 representing the difference between each weighing and the expected weight for real gold coins. The matrix $\boldsymbol{C}$ and the vector $\boldsymbol{b}$ are available in the files `hw1Q5C.dat` and `hw1Q5b.dat`. From these results, determine which of the coins (if any) are fraudulent, *i.e.* find which are the non-zero entries of $\boldsymbol{x}$.

Problem 6

Consider the data generated by the file `hw1Q6.m` (in Matlab) or `hw1Q6.ipynb` (Python notebook), which is assembled into the matrix $\boldsymbol{X}$, and answer the following questions:

a) Compute the SVD of $\boldsymbol{X}$, and plot the first two left singular vectors against the spatial coordinate y. Why are the first two singular values much larger than the rest, which are essentially zero?

b) Compute the dynamic-mode decomposition (DMD) of $\boldsymbol{X}$ (*i.e.*, find a matrix $\boldsymbol{A}$ that maps each column of $\boldsymbol{X}$ one step forward in time, and compute the eigendecomposition of this matrix). Plot the eigenvectors of $\boldsymbol{A}$ that correspond to non-zero eigenvalues. How do they compare with the singular vectors from part (a)?

c) How do the non-zero eigenvalues of $\boldsymbol{A}$ relate to the parameters used to define the data? *Hint:* you may want to convert the discrete-time eigenvalues λ_d to continuous-time eigenvalues λ_c via $\lambda_c = \log(\lambda_d)/\mathrm{d}t$.

d) Using the QR pivoting method (Chapter 2), determine the best two spatial locations to measure this system in order to reconstruct the data.