HW 1. Yuning Wang

Problem 1.

Full SVD.

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \underline{\underline{A}}^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

①. For $\underline{\underline{V}}$ and $\underline{\underline{\Sigma}}$

Compute Eigenvalue & Eigenvectors of $\underline{\underline{A}}^* \cdot \underline{\underline{A}}$

$$\underline{\underline{X}} = \underline{\underline{A}}^* \cdot \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues : $\underline{\underline{A}}^* \cdot \underline{\underline{A}} - \lambda \cdot \underline{\underline{I}}^{3\times 3} = 0$

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$$

Corresponding Eigenvectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since the Rank of $\underline{\underline{A}}^* \cdot \underline{\underline{A}}$ is 2

Thus the Right Singular Vectors of $\underline{\underline{A}}$ are

$$\boxed{\underline{\underline{V}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

The singular values of $\underline{\underline{A}}$ are

$$\boxed{\underline{\underline{\Sigma}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}$$

② Left singular vectors $\underline{\underline{U}}$

Let $\underline{\underline{X}} = \underline{\underline{A}}\cdot\underline{\underline{A}}^* = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

The eigenvalue of $\underline{\underline{X}}$ are $\begin{cases}\lambda_1 = 1\\ \lambda_2 = 1\end{cases}$

The eigenvectors of $\underline{\underline{X}}$ are $\underline{v_1} = \begin{bmatrix}1\\0\end{bmatrix}$, $\underline{v_2} = \begin{bmatrix}0\\1\end{bmatrix}$

Thus the Left singular Vectors $\underline{\underline{U}}$ are

$$\boxed{\underline{\underline{U}} = \begin{bmatrix}1&0\\0&1\end{bmatrix}} \xRightarrow{U \text{ should } \in R^{n\times m}} \boxed{\underline{\underline{U}} = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}}$$

③ Check the result.

$$V^T = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

Since $\underline{\underline{A}} = \underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\cdot\underline{\underline{V}}^T$

$$\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\cdot\underline{\underline{V}}^T = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&1&0\\1&0&0\end{bmatrix} \checkmark$$ checked

Problem 1 Pseudo Inverse of $\underline{\underline{A}}$

Label Pseudo Inverse of $\underline{\underline{A}}$ as $\underline{\underline{A}}^+$

The pseudoinverse of $\underline{\underline{A}}$ is

$$\underline{\underline{A}}^+ = (\underline{\underline{A}}^T \cdot \underline{\underline{A}})^{-1} \cdot \underline{\underline{A}}^T$$

Since rows of $\underline{\underline{A}}$ are linearly independent

$$\underline{\underline{A}}^+ = \underline{\underline{A}}^T \cdot (\underline{\underline{A}} \cdot \underline{\underline{A}}^T)^{-1}$$

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{|AA^T|} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

check the Result. $A \cdot A^+$

$$A \cdot A^+ = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}^{2\times2}$$ checked.

Problem 2. Given the Frobenius norm of a matrix $\underline{\underline{A}} \in R^{n\times m}$

$$\|\underline{\underline{A}}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|A_{ij}|^2}$$

If the SVD of $\underline{\underline{A}}$ is $\underline{\underline{A}} = \underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\cdot\underline{\underline{V}}^T$

$\underline{\underline{U}}$ and $\underline{\underline{V}}$ are unitary matrix

$$\|\underline{\underline{A}}\|_F = \|\underline{\underline{A}}\cdot\underline{\underline{V}}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|(AV)_{ij}|^2}$$

$$\Rightarrow \|\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\cdot\underline{\underline{V}}^T\cdot\underline{\underline{V}}\|_F = \left|\underline{\underline{V}}^T\cdot\underline{\underline{V}} = \underline{\underline{I}}\right| = \|\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\|_F$$

$$\Rightarrow \|\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}|^2}$$

Since $\underline{\underline{U}}$ is an unitary matrix

thus: $\|\underline{\underline{U}}\cdot\underline{\underline{\Sigma}}\|_F = \|\underline{\underline{\Sigma}}\|_F$ , $\underline{\underline{\Sigma}}$ is a diagnol matrix contains all singular $\sigma$

$$\|\underline{\underline{A}}\|_F = \|\underline{\underline{\Sigma}}\|_F = \sqrt{\sum_{j=1}^{\min(m,n)}\sigma_j^2}$$

thus
$$\boxed{\|\underline{\underline{A}}\|_F = \sqrt{\sum_{j=1}^{\min(m,n)}\sigma_j^2}}$$
proved.

Problem 3

- The $n \times n$ matrix $\underline{M}$ should be identity Matrix in order to keep the shape and values of Dataset

- If the data is in non-uniformed mesh

  The matrix $\underline{M}$ will be a $n \times n$ diagnal matrix the values on main diagnol are the weights for non-uniform meshes.