CSC4020 Homework 4 Programming

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Missing Data Problem

In [26]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import norm
from tqdm import tqdm
```

0. Complete Dataset

In [2]:
```python
n = 3000
d = 10
```

In [3]:
```python
mu = np.array([0,1,2,4,8,10,20,50,80,100])
sigma = np.array([[10,  4,  8,  4, 10,  4,  4,  0,  6,  8],
                  [ 4,  6,  2,  4,  6,  4,  4,  2,  2,  4],
                  [ 8,  2, 10,  4, 10,  6,  2,  0,  4,  8],
                  [ 4,  4,  4,  8,  6,  8,  4,  4,  2,  6],
                  [10,  6, 10,  6, 18, 10,  8,  4,  8, 12],
                  [ 4,  4,  6,  8, 10, 12,  6,  6,  4,  8],
                  [ 4,  4,  2,  4,  8,  6,  8,  4,  6,  4],
                  [ 0,  2,  0,  4,  4,  6,  4,  6,  2,  2],
                  [ 6,  2,  4,  2,  8,  4,  6,  2,  8,  4],
                  [ 8,  4,  8,  6, 12,  8,  4,  2,  4, 14]])
```

In [4]:
```python
np.random.seed(42)
data_complete = np.random.multivariate_normal(mu, sigma, 3000)
```

1. EM Algorithm

In [37]:
```python
class EMalgo:
    def __init__(self):
        pass

    def _EM(self):
        mu = np.zeros(self.d)
        sigma = np.zeros([self.d,self.d])

        for i, row in enumerate(self.data):
            u = self._missing[i,:]
            o = ~u
            x = row[o]
            v = self._sigma[u,:][:,u]-self._sigma[u,:][:,o]@np.linalg.pinv(self._sigma[o,:][:,o])@self._sigma[o,:][:,u
]
            m = self._mu[u]+self._sigma[u,:][:,o]@np.linalg.inv(self._sigma[o,:][:,o])@(x-self._mu[o])

            Ex = np.zeros(self.d)
            Ex[u] = m
            Ex[o] = x

            Exx =np.zeros([self.d,self.d])
            Exx[np.outer(u,u)] = (np.outer(m,m)+v).flatten()
            Exx[np.outer(o,o)] = np.outer(x,x).flatten()
            Exx[np.outer(o,u)] = np.outer(x,m).flatten()
            Exx[np.outer(u,o)] = np.outer(m,x).flatten()

            mu += Ex
            sigma += Exx

        mu = mu/self.n
        sigma = sigma/self.n-np.outer(mu,mu)
        return mu, sigma

    def fit(self,data,maxiter = 1000,delta=1e-05):
        from tqdm import tqdm
        from IPython.display import display, clear_output
        self.data = data
        self.n, self.d = data.shape
        self._missing = np.isnan(data)
        complete = data[~self._missing.any(axis=1)]
        if (len(complete)>0):
            self._mu = np.mean(complete,axis=0)
            self._sigma = np.cov(complete.T)
        else:
            filled = np.nan_to_num(data)+self._missing*np.random.standard_normal(data.shape)
            self._mu = np.mean(filled,axis=0)
            self._sigma = np.cov(filled.T)

        for i in range(maxiter):
            mu, sigma = self._EM()
            error_mu = norm(mu-self._mu)/norm(mu)
            error_sigma = norm(sigma-self._sigma)/norm(sigma)
            if (error_mu < delta) and (error_sigma < delta):
                print("iters:",i,"error_mu:",error_mu,"error_sigma:",error_sigma)
                return error_mu,error_sigma
            self._mu = mu
            self._sigma = sigma
```

2. Experiment

2.1 Missing Percentage = 0.4

In [8]:
```python
p = 0.4
errors_1 = np.zeros([10,2])
for i in tqdm(range(10)):
    np.random.seed(i)
    data_missing = data_complete.flatten()
    data_missing[np.random.choice(n*d,int(n*d*p))]=np.nan
    data_missing = data_missing.reshape(3000,10)
    em = EMalgo()
    em.fit(data_missing)
    errors_1[i] = [norm(em._mu-mu)/norm(mu),
                   norm(em._sigma-sigma)/norm(sigma)]
```

```
 10%|█          | 1/10 [00:38<05:48, 38.77s/it]
iters: 36 error_mu: 3.640315919039465e-07 error_sigma: 9.060808267696697e-06
 20%|██         | 2/10 [01:20<05:16, 39.62s/it]
iters: 41 error_mu: 3.4986670155535905e-07 error_sigma: 9.529439701690773e-06
 30%|███        | 3/10 [01:58<04:34, 39.24s/it]
iters: 39 error_mu: 5.060482578930686e-07 error_sigma: 9.689290789327931e-06
 40%|████       | 4/10 [02:45<04:08, 41.40s/it]
iters: 50 error_mu: 3.598751819178609e-07 error_sigma: 9.493131462099589e-06
 50%|█████      | 5/10 [03:23<03:21, 40.40s/it]
iters: 41 error_mu: 2.5991734063778293e-07 error_sigma: 9.348625339804081e-06
 60%|██████     | 6/10 [03:59<02:36, 39.20s/it]
iters: 40 error_mu: 4.705526102821601e-07 error_sigma: 9.648470698089334e-06
 70%|███████    | 7/10 [04:40<01:59, 39.73s/it]
iters: 42 error_mu: 3.828509291838099e-07 error_sigma: 9.591493560714903e-06
 80%|████████   | 8/10 [05:22<01:20, 40.32s/it]
iters: 42 error_mu: 2.2083792785599205e-07 error_sigma: 9.335492654287573e-06
 90%|█████████  | 9/10 [06:09<00:42, 42.46s/it]
iters: 47 error_mu: 3.557985660397522e-07 error_sigma: 9.6845272508968e-06
100%|██████████| 10/10 [06:59<00:00, 41.91s/it]
iters: 52 error_mu: 1.0987413942707271e-07 error_sigma: 9.938298440878376e-06
```

Mean of relative errors for mu and sigma:

In [9]:
```python
np.mean(errors_1,axis=0)
```

Out[9]:
```
array([0.00082065, 0.05704103])
```

Standard deviaton of relative errors for mu and sigma:

In [10]:
```python
np.std(errors_1,axis=0)
```

Out[10]:
```
array([0.0001107 , 0.00314384])
```

2.2 Missing Percentage = 0.3

In [11]:
```python
p = 0.3
errors_2 = np.zeros([10,2])
for i in tqdm(range(10)):
    np.random.seed(i)
    data_missing = data_complete.flatten()
    data_missing[np.random.choice(n*d,int(n*d*p))]=np.nan
    data_missing = data_missing.reshape(3000,10)
    em = EMalgo()
    em.fit(data_missing)
    errors_2[i] = [norm(em._mu-mu)/norm(mu),
                   norm(em._sigma-sigma)/norm(sigma)]
```

```
 10%|█          | 1/10 [00:25<03:46, 25.17s/it]
iters: 23 error_mu: 7.675575856480842e-07 error_sigma: 9.27112748155055e-06
 20%|██         | 2/10 [00:48<03:15, 24.50s/it]
iters: 22 error_mu: 5.15732936352488e-07 error_sigma: 9.220586993710647e-06
 30%|███        | 3/10 [01:12<02:50, 24.42s/it]
iters: 23 error_mu: 3.167135995709506e-07 error_sigma: 9.351348390844278e-06
 40%|████       | 4/10 [01:35<02:23, 23.91s/it]
iters: 21 error_mu: 6.27737926695363e-07 error_sigma: 8.239615452044519e-06
 50%|█████      | 5/10 [01:58<01:58, 23.75s/it]
iters: 23 error_mu: 7.752765415915142e-08 error_sigma: 9.834125658957431e-06
 60%|██████     | 6/10 [02:29<01:44, 26.08s/it]
iters: 30 error_mu: 2.1536035714073477e-08 error_sigma: 9.114888136518008e-06
 70%|███████    | 7/10 [03:02<01:24, 28.08s/it]
iters: 30 error_mu: 6.073355363666473e-08 error_sigma: 9.15841382765311e-06
 80%|████████   | 8/10 [03:28<00:54, 27.36s/it]
iters: 24 error_mu: 9.800317265044921e-08 error_sigma: 9.145360605064797e-06
 90%|█████████  | 9/10 [03:54<00:27, 27.06s/it]
iters: 26 error_mu: 2.1126476925261428e-07 error_sigma: 9.896601543103648e-06
100%|██████████| 10/10 [04:18<00:00, 25.84s/it]
iters: 23 error_mu: 9.06198360024603e-08 error_sigma: 9.56313120395823e-06
```

Mean of relative errors for mu and sigma:

In [12]:
```python
np.mean(errors_2,axis=0)
```

Out[12]:
```
array([0.00078913, 0.05556101])
```

Standard deviaton of relative errors for mu and sigma:

In [13]:
```python
np.std(errors_2,axis=0)
```

Out[13]:
```
array([0.00010946, 0.00354356])
```

2.3 Missing Percentage = 0.2

In [14]:
```python
p = 0.2
errors_3 = np.zeros([10,2])
for i in tqdm(range(10)):
    np.random.seed(i)
    data_missing = data_complete.flatten()
    data_missing[np.random.choice(n*d,int(n*d*p))]=np.nan
    data_missing = data_missing.reshape(3000,10)
    em = EMalgo()
    em.fit(data_missing)
    errors_3[i] = [norm(em._mu-mu)/norm(mu),
                   norm(em._sigma-sigma)/norm(sigma)]
```

```
 10%|█          | 1/10 [00:12<01:51, 12.42s/it]
iters: 11 error_mu: 5.670971907698353e-07 error_sigma: 9.258974628187916e-06
 20%|██         | 2/10 [00:28<01:49, 13.65s/it]
iters: 13 error_mu: 2.001840477884286e-07 error_sigma: 8.021381015403547e-06
 30%|███        | 3/10 [00:44<01:38, 14.11s/it]
iters: 12 error_mu: 7.922352874520702e-08 error_sigma: 9.897756622414724e-06
 40%|████       | 4/10 [00:57<01:23, 13.96s/it]
iters: 12 error_mu: 3.946350011699897e-08 error_sigma: 8.291863994035686e-06
 50%|█████      | 5/10 [01:08<01:05, 13.10s/it]
iters: 10 error_mu: 3.972554122787903e-07 error_sigma: 9.673664162793171e-06
 60%|██████     | 6/10 [01:24<00:55, 13.91s/it]
iters: 15 error_mu: 9.606800061286919e-08 error_sigma: 7.852820038777334e-06
 70%|███████    | 7/10 [01:39<00:42, 14.06s/it]
iters: 14 error_mu: 1.4202127390952366e-07 error_sigma: 8.068087605161223e-06
 80%|████████   | 8/10 [01:51<00:27, 13.67s/it]
iters: 12 error_mu: 2.9953798406114814e-07 error_sigma: 7.97098724337098e-06
 90%|█████████  | 9/10 [02:06<00:14, 14.01s/it]
iters: 14 error_mu: 1.924511795024197e-07 error_sigma: 8.539393139697052e-06
100%|██████████| 10/10 [02:20<00:00, 14.07s/it]
iters: 12 error_mu: 1.9463487799237215e-07 error_sigma: 8.538694911525374e-06
```

Mean of relative errors for mu and sigma:

In [15]:
```python
np.mean(abs(errors_3),axis=0)
```

Out[15]:
```
array([0.00075147, 0.05388891])
```

Standard deviaton of relative errors for mu and sigma:

In [16]:
```python
np.std(errors_3,axis=0)
```

Out[16]:
```
array([5.59634957e-05, 2.50730301e-03])
```

2.4 Missing Percentage = 0.1

In [17]:
```python
p = 0.1
errors_4 = np.zeros([10,2])
for i in tqdm(range(10)):
    np.random.seed(i)
    data_missing = data_complete.flatten()
    data_missing[np.random.choice(n*d,int(n*d*p))]=np.nan
    data_missing = data_missing.reshape(3000,10)
    em = EMalgo()
    em.fit(data_missing)
    errors_4[i] = [norm(em._mu-mu)/norm(mu),
                   norm(em._sigma-sigma)/norm(sigma)]
```

```
 10%|█          | 1/10 [00:07<01:09,  7.73s/it]
iters: 6 error_mu: 1.368886651415144e-07 error_sigma: 5.290474603031804e-06
 20%|██         | 2/10 [00:14<00:59,  7.41s/it]
iters: 6 error_mu: 2.469319366367871e-07 error_sigma: 6.012402580168153e-06
 30%|███        | 3/10 [00:21<00:52,  7.46s/it]
iters: 6 error_mu: 2.907888996910723e-08 error_sigma: 5.334128347374957e-06
 40%|████       | 4/10 [00:27<00:41,  6.89s/it]
iters: 5 error_mu: 2.7393456853690556e-07 error_sigma: 8.836870747351823e-06
 50%|█████      | 5/10 [00:33<00:33,  6.69s/it]
iters: 5 error_mu: 7.294694356987802e-07 error_sigma: 7.057057842278393e-06
 60%|██████     | 6/10 [00:42<00:28,  7.18s/it]
iters: 7 error_mu: 6.71738247518029e-08 error_sigma: 4.607691774316012e-06
 70%|███████    | 7/10 [00:49<00:21,  7.11s/it]
iters: 6 error_mu: 1.1615226499092396e-07 error_sigma: 6.887163925876548e-06
 80%|████████   | 8/10 [00:56<00:14,  7.25s/it]
iters: 7 error_mu: 3.6769084546937866e-08 error_sigma: 5.971284918809178e-06
 90%|█████████  | 9/10 [01:04<00:07,  7.40s/it]
iters: 7 error_mu: 7.892981451171109e-08 error_sigma: 5.321436227441244e-06
100%|██████████| 10/10 [01:10<00:00,  7.08s/it]
iters: 5 error_mu: 4.886694717143275e-07 error_sigma: 8.53507212537467e-06
```

Mean of relative errors for mu and sigma:

In [18]:
```python
np.mean(errors_4,axis=0)
```

Out[18]:
```
array([0.00069346, 0.0543418 ])
```

Standard deviaton of relative errors for mu and sigma:

In [19]:
```python
np.std(errors_4,axis=0)
```

Out[19]:
```
array([3.92619434e-05, 1.79644689e-03])
```

2.5 Missing Percentage = 0.05

In [20]:
```python
p = 0.05
errors_5 = np.zeros([10,2])
for i in tqdm(range(10)):
    np.random.seed(i)
    data_missing = data_complete.flatten()
    data_missing[np.random.choice(n*d,int(n*d*p))]=np.nan
    data_missing = data_missing.reshape(3000,10)
    em = EMalgo()
    em.fit(data_missing)
    errors_5[i] = [norm(em._mu-mu)/norm(mu),
                   norm(em._sigma-sigma)/norm(sigma)]
```

```
 10%|█          | 1/10 [00:04<00:37,  4.14s/it]
iters: 3 error_mu: 1.8378317779895777e-07 error_sigma: 8.592029913896554e-06
 20%|██         | 2/10 [00:09<00:35,  4.40s/it]
iters: 4 error_mu: 1.536298353034614e-07 error_sigma: 3.1188569682007045e-06
 30%|███        | 3/10 [00:13<00:30,  4.36s/it]
iters: 3 error_mu: 3.3312370789669116e-07 error_sigma: 7.15605153297962e-06
 40%|████       | 4/10 [00:18<00:27,  4.64s/it]
iters: 4 error_mu: 6.084765690548946e-08 error_sigma: 3.068334789531943e-06
 50%|█████      | 5/10 [00:23<00:23,  4.73s/it]
iters: 4 error_mu: 8.347108136177117e-08 error_sigma: 3.3175774075004606e-06
 60%|██████     | 6/10 [00:28<00:19,  4.83s/it]
iters: 4 error_mu: 1.2364946655417403e-07 error_sigma: 3.74221525304771e-06
 70%|███████    | 7/10 [00:33<00:14,  4.91s/it]
iters: 4 error_mu: 4.087132234834519e-08 error_sigma: 4.8225076635723126e-06
 80%|████████   | 8/10 [00:38<00:09,  4.98s/it]
iters: 4 error_mu: 7.03767803911442e-08 error_sigma: 4.6140075432764285e-06
 90%|█████████  | 9/10 [00:43<00:04,  4.73s/it]
iters: 3 error_mu: 2.983339951099137e-07 error_sigma: 6.82302547193699e-06
100%|██████████| 10/10 [00:47<00:00,  4.71s/it]
iters: 3 error_mu: 2.1625385615962575e-07 error_sigma: 6.497366647544994e-06
```

Mean of relative errors for mu and sigma:

In [21]:
```python
np.mean(errors_5,axis=0)
```

Out[21]:
```
array([0.00068113, 0.05320922])
```

Standard deviaton of relative errors for mu and sigma:

In [22]:
```python
np.std(errors_5,axis=0)
```

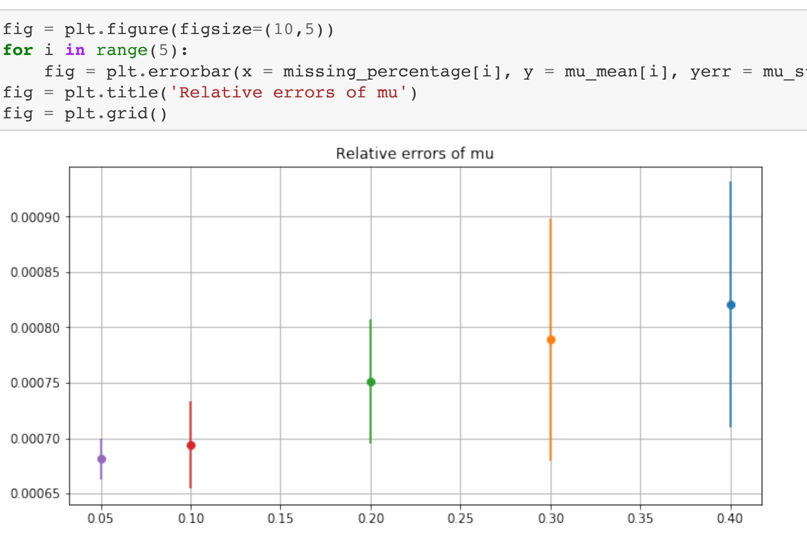
Out[22]:
```
array([1.84617223e-05, 1.21177838e-03])
```

3. Analysis

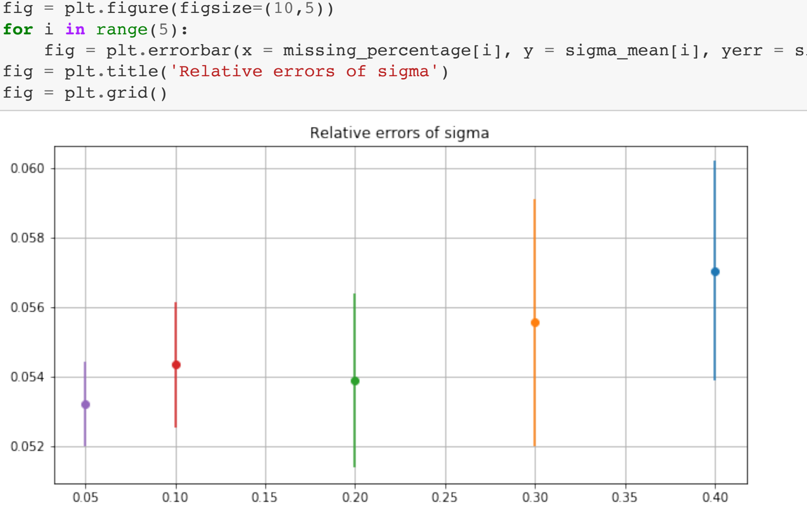
3.1 Vertical Bar Plot

In [24]:
```python
missing_percentage = [0.4, 0.3, 0.2, 0.1, 0.05]
mu_mean = [np.mean(errors_1,axis=0)[0],
           np.mean(errors_2,axis=0)[0],
           np.mean(errors_3,axis=0)[0],
           np.mean(errors_4,axis=0)[0],
           np.mean(errors_5,axis=0)[0]]
mu_std = [np.std(errors_1,axis=0)[0],
          np.std(errors_2,axis=0)[0],
          np.std(errors_3,axis=0)[0],
          np.std(errors_4,axis=0)[0],
          np.std(errors_5,axis=0)[0]]
sigma_mean = [np.mean(errors_1,axis=0)[1],
              np.mean(errors_2,axis=0)[1],
              np.mean(errors_3,axis=0)[1],
              np.mean(errors_4,axis=0)[1],
              np.mean(errors_5,axis=0)[1]]
sigma_std = [np.std(errors_1,axis=0)[1],
             np.std(errors_2,axis=0)[1],
             np.std(errors_3,axis=0)[1],
             np.std(errors_4,axis=0)[1],
             np.std(errors_5,axis=0)[1]]
```

In [34]:
```python
fig = plt.figure(figsize=(10,5))
for i in range(5):
    fig = plt.errorbar(x = missing_percentage[i], y = mu_mean[i], yerr = mu_std[i], fmt='o')
fig = plt.title('Relative errors of mu')
fig = plt.grid()
```



In [33]:
```python
fig = plt.figure(figsize=(10,5))
for i in range(5):
    fig = plt.errorbar(x = missing_percentage[i], y = sigma_mean[i], yerr = sigma_std[i], fmt='o')
fig = plt.title('Relative errors of sigma')
fig = plt.grid()
```



3.2 Summary

From the plot, it is clear that the mean and standard deviation of relative errors increases as the missing percentage increase. This is intuitive to explain, with fewer data missing, the estimator generated from EM algorithm is more accurate and is closer to the true parameters.

Interestingly, the iteration steps needed to converge also decreases as the missing percenatage decreases.

By comparing the plot of mu and sigma, we can find that the relative error of sigma is larger. This is because sigma has more parameters to estimate than mu.