Reference08

10. $\int_0^1 dx\int_0^1 dy\int_y^1 \frac{e^{-z^2}}{x^2+1}dz$ 对 y,z 局部看 $0\le y\le 1,\ y\le z\le 1$ $\Downarrow$ $0\le z\le 1\ \ 0\le y\le z$

$= \int_0^1 \frac{1}{x^2+1}dx\int_0^1 dz\int_0^z e^{-z^2}dy$

$= \int_0^1 \frac{1}{x^2+1}dx\int_0^1 e^{-z^2}\cdot z\,dz$ (二者无关)

$= \int_0^1 \frac{1}{x^2+1}dx\cdot\int_0^1 e^{-z^2}z\,dz$

$= \arctan x\Big|_0^1\cdot\frac{1}{2}\int_0^1 e^{-z^2}dz^2$

$= \frac{\pi}{4}\cdot\frac{1}{2}\cdot(-e^{-z^2})\Big|_0^1$

$= \frac{\pi}{8}\cdot(-e^{-1}+1) = \frac{\pi(e-1)}{8e}$

11. 设重心坐标 $(\bar{x},\bar{y},\bar{z})$.

由对称性可知, $\bar{x}=\bar{y}=0$.

而其质量 $m=\iint_S c\,dS = c\iint_S dS$

设球壳在 xOy 面上的投影区域为 D. 有 $D:\{(x,y)\mid x^2+y^2\le 2\}$

又 $z'_x = x,\ z'_y = y$

则 $m = c\iint_D\sqrt{1+x^2+y^2}\,dxdy = c\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\sqrt{1+r^2}\,r\,dr$

$= c\cdot 2\pi\cdot\frac{1}{2}\int_0^{\sqrt{2}}\sqrt{1+r^2}\,d(r^2+1) = \pi c\cdot\frac{2}{3}(r^2+1)^{\frac{3}{2}}\Big|_0^{\sqrt{2}}$

$= \pi c\cdot\frac{2}{3}(3^{\frac{3}{2}}-1) = \pi c\,(2\sqrt{3}-\frac{2}{3})$

又 $\iint_S cz\,dS = c\iint_D\frac{1}{2}(x^2+y^2)\cdot\sqrt{1+x^2+y^2}\,dxdy$

$= \frac{1}{2}c\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r^2\cdot\sqrt{1+r^2}\cdot r\,dr$

$= c\pi\cdot\frac{1}{2}\int_0^{\sqrt{2}}r^2\sqrt{1+r^2}\,dr^2 \overset{u=r^2+1}{=\!=} \frac{1}{2}c\pi\int_1^3\sqrt{u}(u-1)\,d(u-1)$

$= \frac{1}{2}c\pi\int_1^3 u^{\frac{3}{2}}-u^{\frac{1}{2}}\,du$

$= \frac{1}{2}c\pi\cdot\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\Big|_1^3\right) = \frac{1}{2}c\pi\left(\frac{2}{5}\cdot 3^{\frac{5}{2}}-\frac{2}{3}\cdot 3^{\frac{3}{2}}-\left(\frac{2}{5}-\frac{2}{3}\right)\right)$

$= \frac{1}{2}c\pi\left(\frac{18}{5}\sqrt{3}-2\sqrt{3}+\frac{4}{15}\right)$

$= c\pi\left(\frac{4\sqrt{3}}{5}+\frac{2}{15}\right) = \frac{2+12\sqrt{3}}{15}c\pi$

则 $\bar{z} = \frac{\iint_S cz\,dS}{m} = \frac{\frac{2+12\sqrt{3}}{15}c\pi}{(2\sqrt{3}-\frac{2}{3})c\pi} = \frac{2+12\sqrt{3}}{30\sqrt{3}-10} = \frac{2(6\sqrt{3}+1)(3\sqrt{3}+1)}{10(3\sqrt{3}-1)(3\sqrt{3}+1)}$

$= \frac{18\times 3+6\sqrt{3}+3\sqrt{3}+1}{5\times 26} = \frac{55+9\sqrt{3}}{130}$

则重心坐标为 $(0,0,\frac{55+9\sqrt{3}}{130})$

12.

(1) 由球面坐标

$V(\Omega)=\iiint\limits_{\Omega} dxdydz$

$=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{\rho(\theta,\varphi)}\rho^2\sin\varphi\, d\rho$

$=\frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi}\left(\rho(\theta,\varphi)\right)^3\sin\varphi\, d\varphi$

(直接对最后部分积分即可)

(2) 封闭曲面，球坐标方程转化为

$\rho\sin\varphi\sin\theta=(\rho^2\sin^2\varphi)^2+\rho^4\cos^4\varphi$

此时 $\rho=\left(\frac{\sin\varphi\sin\theta}{\sin^4\varphi+\cos^4\varphi}\right)^{\frac{1}{3}}$

但注意到，此时对范围有 $\begin{cases}\sin\varphi\ge 0\\ \sin\theta\ge 0\end{cases}\Rightarrow\begin{cases}0\le\varphi\le\pi\\ 0\le\theta\le\pi\end{cases}$

将上述式子代入(1)中结果

有 $V(\Omega)=\frac{1}{3}\int_0^{\pi}d\theta\int_0^{\pi}\frac{\sin\varphi\sin\theta}{\sin^4\varphi+\cos^4\varphi}\sin\varphi\, d\varphi$ (相当于其在 $(\pi,2\pi)$ 上取0)

$=\frac{1}{3}\int_0^{\pi}\sin\theta\, d\theta\cdot\int_0^{\pi}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi$

$=\frac{2}{3}\int_0^{\pi}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi$

$=\frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi=\frac{4}{3}\int_0^{+\infty}\frac{t^2}{1+t^4}dt$

$=\frac{\sqrt{2}\pi}{3}$

① $\int_0^{\pi}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi=2\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi$

对 $\int_{\frac{\pi}{2}}^{\pi}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi\overset{t=\pi-\varphi}{=\!=\!=}\int_{\frac{\pi}{2}}^{0}\frac{\sin^2(\pi-t)}{\sin^4(\pi-t)+\cos^4(\pi-t)}d(\pi-t)$

$=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{\sin^4 t+\cos^4 t}dt$ $\quad(\sin(\pi-t)=\sin t,\ \cos(\pi-t)=-\cos t)$

② 对 $\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\sin^4\varphi+\cos^4\varphi}d\varphi\overset{t=\tan\varphi}{=\!=\!=}\int_0^{+\infty}\frac{t^2}{1+t^4}dt$

$=\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi(\cos^2\varphi+\sin^2\varphi\tan^2\varphi)}d\varphi=\int_0^{\frac{\pi}{2}}\frac{\tan^2\varphi}{\cos^2\varphi+\sin^2\varphi\tan^2\varphi}d\varphi$

$=\int_0^{\frac{\pi}{2}}\frac{\tan^2\varphi}{1+\tan^4\varphi}\frac{1}{\cos^2\varphi}d\varphi=\int_0^{\frac{\pi}{2}}\frac{\tan^2\varphi}{1+\tan^4\varphi}d\tan\varphi$

③ $\int_0^{+\infty}\frac{t^2}{1+t^4}dt$

对 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt\overset{u=\frac{1}{t}}{=\!=}\int_{+\infty}^{0}\frac{\frac{1}{u^2}}{1+\frac{1}{u^4}}d\frac{1}{u}=\int_0^{+\infty}\frac{u^2}{u^4+1}\cdot\frac{1}{u^2}du$

$=\int_0^{+\infty}\frac{1}{u^4+1}du$ $\quad$ 则 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt=\frac{1}{2}\int_0^{+\infty}\frac{t^2+1}{t^4+1}dt$

而 $\int_0^{+\infty}\frac{t^2+1}{t^4+1}dt=\int_0^{+\infty}\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=\int_0^{+\infty}\frac{dt-\frac{1}{t}}{(t-\frac{1}{t})^2+2}\overset{x=t-\frac{1}{t}}{=\!=\!=}\int_{-\infty}^{+\infty}\frac{dx}{x^2+2}$

$=\frac{\sqrt{2}}{2}\int_{-\infty}^{+\infty}\frac{d\frac{x}{\sqrt{2}}}{1+(\frac{x}{\sqrt{2}})^2}=\frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}}\Big|_{-\infty}^{+\infty}=\frac{\sqrt{2}}{2}\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{\sqrt{2}}{2}\pi$

则 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt=\frac{\sqrt{2}}{4}\pi$

③ $\int_0^{+\infty}\frac{t^2}{1+t^4}dt$

对 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt \xlongequal{u=\frac{1}{t}} \int_{+\infty}^{0}\frac{\frac{1}{u^2}}{1+\frac{1}{u^4}}d\frac{1}{u}=\int_0^{+\infty}\frac{u^2}{u^4+1}\cdot\frac{1}{u^2}du$

$=\int_0^{+\infty}\frac{1}{u^4+1}du$ 则 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt=\frac{1}{2}\int_0^{+\infty}\frac{t^2+1}{t^4+1}dt$

而 $\int_0^{+\infty}\frac{t^2+1}{t^4+1}dt=\int_0^{+\infty}\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=\int_0^{+\infty}\frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2}\xlongequal{x=t-\frac{1}{t}}\int_{-\infty}^{+\infty}\frac{dx}{x^2+2}$

$=\frac{\sqrt{2}}{2}\int_{-\infty}^{+\infty}\frac{d\frac{x}{\sqrt{2}}}{1+\left(\frac{x}{\sqrt{2}}\right)^2}=\frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}}\Big|_{-\infty}^{+\infty}=\frac{\sqrt{2}}{2}\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{\sqrt{2}}{2}\pi$

则 $\int_0^{+\infty}\frac{t^2}{1+t^4}dt=\frac{\sqrt{2}}{4}\pi$

【解】 $F(t)=\iiint\limits_{\Omega}[z^2+f(x^2+y^2)]dv=\int_0^h dz\int_0^{2\pi}d\theta\int_0^t[z^2+f(r^2)]r\,dr$

$$=\frac{\pi h^3}{3}t^2+2\pi h\int_0^t rf(r^2)dr,$$

$$\lim_{t\to0^+}\frac{F(t)}{t^2}=\lim_{t\to0^+}\frac{\frac{\pi h^3}{3}t^2+2\pi h\int_0^t rf(r^2)dr}{t^2}=\frac{\pi h^3}{3}+2\pi h\lim_{t\to0^+}\frac{\int_0^t rf(r^2)dr}{t^2}$$

$$=\frac{\pi h^3}{3}+2\pi h\lim_{t\to0^+}\frac{tf(t^2)}{2t}=\frac{\pi h^3}{3}+\pi hf(0).$$

例题 22.5.4 若直线 $x=0$, $x=a$, $y=0$ 与正连续曲线 $y=f(x)$ 围成的区域的质心的 x 坐标是 $g(a)$, 证明

$$f(x)=\frac{Ag'(x)}{[x-g(x)]^2}\exp\left(\int\frac{dx}{x-g(x)}\right),$$

其中 A 为正常数, a 是参数.

证 见图 22.10,

$$g(a)=\frac{M_x(1)}{M(0)}=\frac{\int_0^a xf(x)\,dx}{\int_0^a f(x)\,dx},$$

即

$$g(a)\int_0^a f(x)\,dx=\int_0^a xf(x)\,dx.$$

两边对 a 求导得

$$g(a)f(a)+g'(a)\int_0^a f(x)\,dx=af(a).$$

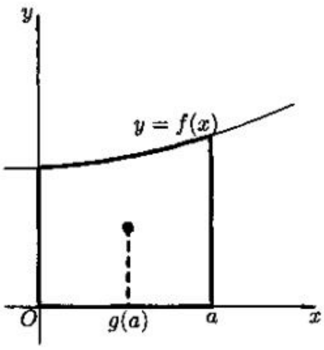

图 22.10

令 $F(a)=\int_0^a f(x)\,dx$, 注意到 $a-g(a)\neq0$, 则

$$\frac{F'(a)}{F(a)}=\frac{g'(a)}{a-g(a)}.$$

两边对 a 积分, 得

$$\ln F(a)=\int\frac{g'(a)}{a-g(a)}\,da+C.$$

所以

$$\int_0^a f(x)\,\mathrm{d}x = F(a) = A\exp\left(\int \frac{g'(a)}{a-g(a)}\,\mathrm{d}a\right).$$

两边对 a 求导得

$$f(a) = \frac{Ag'(a)}{a-g(a)}\exp\left(\int \frac{g'(a)}{a-g(a)}\,\mathrm{d}a\right).$$

考虑到

$$\begin{aligned}\int \frac{g'(a)}{a-g(a)}\,\mathrm{d}a &= \int \frac{g'(a)-1}{a-g(a)}\,\mathrm{d}a + \int \frac{\mathrm{d}a}{a-g(a)}\\ &= -\ln(a-g(a)) + \int \frac{\mathrm{d}a}{a-g(a)},\end{aligned}$$

则

$$f(a) = \frac{Ag'(a)}{[a-g(a)]^2}\exp\left(\int \frac{\mathrm{d}a}{a-g(a)}\right). \quad \square$$

分析 题目给出了 $f(x,t)$ 在 x 方向上的性质: $\left|\dfrac{\partial f}{\partial x}\right| \leqslant 1$. 由此证明在 t 方向上的性质. 可用的条件是 f 在 x 方向和 t 方向之间的关系: $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$. 我们通过交换累次积分次序来转换.

证 (1) 由题设

$$f(x,t_1) - f(x,t_2) = \int_{t_1}^{t_2} \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\partial^2 f}{\partial x^2}(x,t)\,\mathrm{d}t.$$

从而, 对任何 $\overline{x} \in [0,1]$, 由累次积分次序可交换, 成立

$$\begin{aligned}\int_x^{\overline{x}} [f(x,t_1) - f(x,t_2)]\,\mathrm{d}x &= \int_x^{\overline{x}} \left(\int_{t_1}^{t_2} \frac{\partial^2 f}{\partial x^2}(x,t)\,\mathrm{d}t\right)\mathrm{d}x\\ = \int_{t_1}^{t_2} \left(\int_x^{\overline{x}} \frac{\partial^2 f}{\partial x^2}(x,t)\,\mathrm{d}x\right)\mathrm{d}t &= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}(t,\overline{x}) - \frac{\partial f}{\partial x}(x,t)\right)\mathrm{d}t.\end{aligned}$$

对上式左端应用积分中值定理, 右端利用已知条件 $\left|\dfrac{\partial f}{\partial x}\right| \leqslant 1$, 得

$$|f(\xi,t_1) - f(\xi,t_2)| \cdot |x - \overline{x}| \leqslant 2|t_1 - t_2|,$$

其中 ξ 在 x 和 $\overline{x}$ 之间. 对任何 x, t_1 和 $t_2 \in [0,1]$ 总可找到某个 $\overline{x} \in [0,1]$, 使得

$$|x - \overline{x}| = \frac{1}{2}|t_1 - t_2|^{\frac{1}{2}},$$

代入前式即得

$$|f(\xi,t_1) - f(\xi,t_2)| \leqslant 4|t_1 - t_2|^{\frac{1}{2}}.$$

(2) 利用 (1) 得

$$\begin{aligned}|f(x,t_1) - f(x,t_2)| &\leqslant |f(x,t_1) - f(\xi,t_1)| + |f(\xi,t_1) - f(\xi,t_2)| + |f(x,t_2) - f(\xi,t_2)|\\ &\leqslant 1\cdot|x-\xi| + 4|t_1 - t_2|^{\frac{1}{2}} + 1\cdot|x-\xi|\\ &\leqslant |x-\overline{x}| + 4|t_1 - t_2|^{\frac{1}{2}} + |x-\overline{x}| = 5|t_1 - t_2|^{\frac{1}{2}}. \quad \square\end{aligned}$$