The h-Cobordism Theorem

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Morse functions

All the criticals are non-degenerated.

Criticals

$$(\mathrm{d}f)_x = 0$$

Non-degenerate

$$\det(\mathrm{Hess} f)_x \neq 0$$

Morse functions

All the criticals are non-degenerated.

Facts

Are isolated. The criticals of a Morse function are isolated.

Do exist. Morse functions form an open dense subset of $C^\infty(M, \mathbb{R})$ in C^2-topology.

Morse functions, locally

Around a critical x,

$$f = f(x) - (x^1)^2 - (x^2)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2$$

for some chart $(x^1, \cdots, x^n)$ originated at x.

Morse index

λ

Morse #

# of criticals

Morse functions, locally

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$$f = f(x) - (x^1)^2 - (x^2)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2$$

for some chart $(x^1, \cdots, x^n)$ originated at x.

Fact

One can readoff a cellular structure of the space from indexes of the criticals, namely,

$$\{\text{criticals of a Morse function}\} \overset{1:1}{\longleftrightarrow} \{\text{cells of a CW structure}\}$$

$$\text{an } \lambda\text{-ind critical} \mapsto \text{a } \lambda\text{-dim cell}$$

Gradient-like vector field X

- $X \cdot f > 0$ away from criticals;
- $X = \nabla f$ around every critical, where the gradient is induced from the Morse chart. (We have not introduced a metric on the manifold yet.)

Altering X gives a new Morse function.

Cobordism

A $(n+1)$-dim cobordism is a 5-tuple $(W^{n+1}; V, V'; h, h')$ of smooth oriented compact manifolds W, V, V' and orientation-preserving diffeomorphisms h, h' such that $\partial W \overset{\text{diff}}{\approx} V \sqcup -V'$, where the h- and h'-image is understood, also denoted as

$$V \xrightarrow{W} V'.$$

Product cobordism

$$W \overset{\text{diff}}{\approx} V \times [0,1]$$

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Product cobordism

$$W \overset{\text{diff}}{\approx} V \times [0,1]$$

Question

Under what topological conditions of W, can we identify it as a product cobordism?

Morse theory on a cobordism

Fact

Relative Morse function does exist.

$V \xrightarrow{W} V'$ admits a Morse function f, with $f(0) = V, f(1) = V'$.

Gradient-like vf does exist.

With respect to f, there admits a gradient-like vector field.

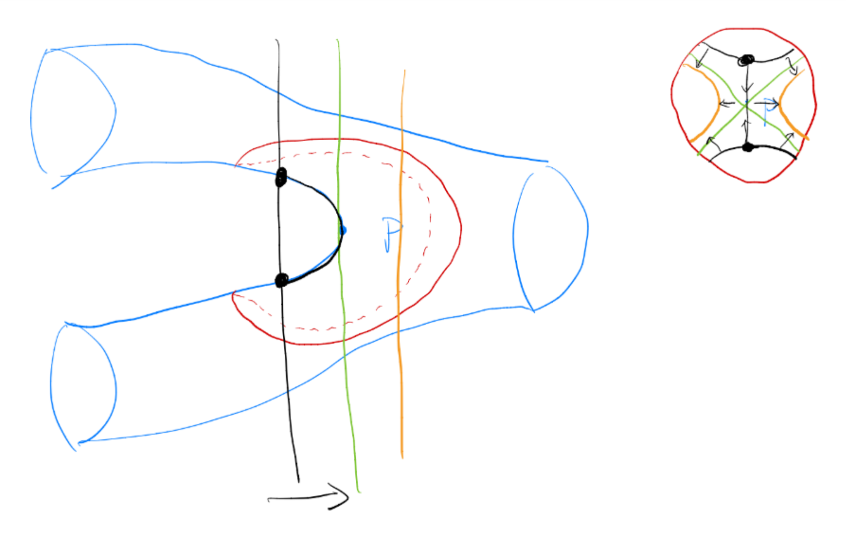
Important observation

If there are no criticals of f, then it ***is*** a product cobordism.

Question updated

Under what topological conditions of W, can we eliminate all the criticals?

Anatomy of criticals



Anatomy of criticals

Let $g : OD^n \overset{\text{diff}}{\approx} U$ open around p. Let V_0, V_1 be level sets near p with $f(V_0) < f(p) < f(V_1)$.

characteristic embedding

$$\varphi_L : S^{\lambda-1} \times OD^{n-\lambda} \to V_0 : (u, \theta v) \mapsto g(u \cosh\theta, v \sinh\theta)$$

$$\varphi_R : OD^{\lambda-1} \times S^{n-\lambda} \to V_1 : (\theta u, v) \mapsto g(v \sinh\theta, u \cosh\theta)$$

left-hand sphere S_L / right-hand sphere S_R

$\varphi_L(S^{\lambda-1} \times 0), \varphi_R(0 \times S^{n-\lambda-1})$.

left-hand disk D_L / right-hand disk D_R

The union of integral curves of X ending at p (resp. starting at p).

Surgery of type $(\lambda, n-\lambda)$

We can use surgery to describe the change of characteristic embedding from the left-hand to the right-hand.

$$\chi(V,\varphi) = (V - \varphi(S^{\lambda-1} \times 0)) \sqcup (OD^{\lambda} \times S^{n-\lambda-1})/\varphi(u,\theta v) \sim (\theta u, v)$$

And use cobordism to trace this surgery.

Theorem

There exists an elementary cobordism (i.e. Morse # = 1) W such that

$$V \xrightarrow{W} \chi(V,\varphi).$$

Lesson A critical is essentially a cobordism. Can we split them?

Rearrangment of criticals

For criticals p, p' indexd $\lambda \geq \lambda'$ with $f(p) < f(p')$, there is enough room to alter X to move $S_R(p)$ out of the way of $S_L(p')$ at any level set V_c between $f(p)$ and $f(p')$, since $\dim S_R(p) + \dim S_L(p') = (n - \lambda - 1) + (\lambda' - 1) < n - 1 = \dim V_c$.

Theorem

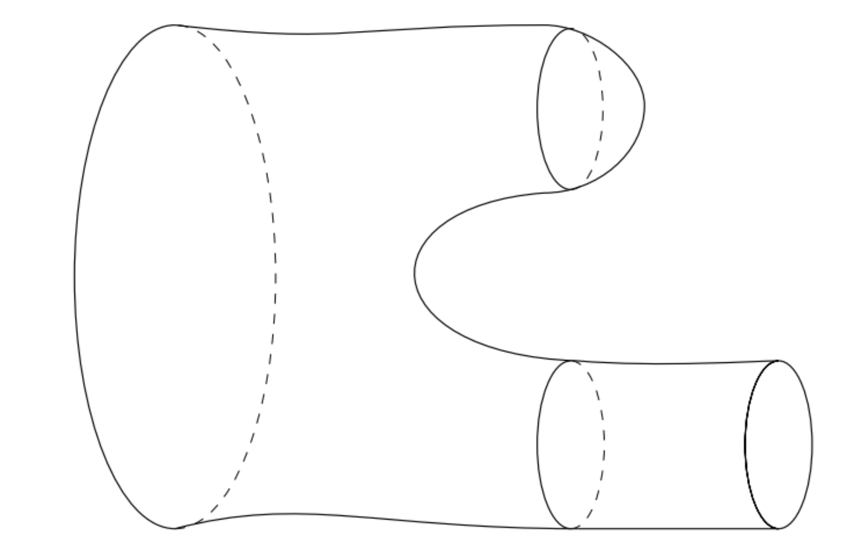
There exists a Morse function on the cobordism $V \xrightarrow{W} V'$ such that

- $f(V) = -\frac{1}{2}, f(V') = n + \frac{1}{2}$
- $f(p) = \text{index}(p), \forall \text{critical } p$

These functions are called **self-indexing**, and we can split it as compositions of cobordisms $c_0 \circ \cdots \circ c_\lambda \circ \cdots \circ c_n$, with only λ-index's in c_λ.

Suppose in the following that all Morse functions are self-indexing.

Cancellation of criticals



Cancellation of criticals

Always alter X to make the intersection transverse. We can cancel a pair (p, p') of criticals of index $\lambda, \lambda + 1$, provided that ...

Single-point intersection

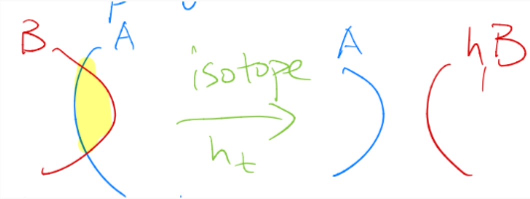
If S_R is transversely interesected with S'_L with a single point, then the cobordism is a product cobordism.

Some hard differential topology works are required.

Cancellation of criticals

Always alter X to make the intersection transverse. We can cancel a pair (p, p') of criticals of index $\lambda, \lambda + 1$, provided that .,,

Whitney's trick, locally



Globally

Requires $\pi_1 M = 0, \pi_1 V_{a-1}(\mathbb{R}^{a+b-2}) = 0 (\Longleftarrow \dim M = a + b - 2 \geq 6)$.

Eliminating middle index criticals

Let $W_\lambda = c_0 \circ \cdots \circ c_\lambda$, $V = W_{-1} \subset W_0 \subset \cdots \subset W_n$. We can interpret the homology generators as the left-hand disks, namely,

$$H_\lambda(W, V) \cong \bigoplus_{\text{left-hand disks}} \mathbb{Z}\langle D_L^\lambda \rangle \cong H_\lambda(W_\lambda, W_{\lambda-1}),$$

Let $C_\lambda := H_\lambda(W_\lambda, W_{\lambda-1})$ and

$$\partial_\lambda : C_\lambda \to C_{\lambda-1} : [D_L] \mapsto \sum S_R' \cdot S_L[D_L^{\lambda-1}]$$

is given by the induced differential of long exact sequence of the triad $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$.

Fact

$$H_*(C_\bullet) \cong H_*(W, V)$$

Eliminating middle index criticals

$$H_\lambda(W,V) \cong \bigoplus_{\text{left-hand disks}} \mathbb{Z}\langle D_L^\lambda \rangle \cong H_\lambda(W_\lambda, W_{\lambda-1}),$$

- If W is a product cobordism, then $H_*(W,V)=0$.
- Conversely, if $H_*(W,V)=0$, then $H_*(C_\bullet)=0$ and therefore $C_\bullet$ is exact, which means every critical is pairing up with another adjacently indexed critical.

Lesson

The criticals' data is essentially the homology data.

Question twice updated

Can $H_*(W,V)=0, \pi_1 W = 0, \dim W \geq 6$ imply that W is a product cobordism?

Facts that the elimination works generally

- All 0-indexed criticals can be canceled with equal number of 1-indexed's.
- All 1-index's can be traded for equal number of 3-index's, provided $\pi_1 V = 0$.
- Replace f by $-f$ to eliminate 0-coindex's and 1-coindex's.

Yes! from *h*-cobordism theorem

For $V \xrightarrow{W} V'$, if

1) $\pi_1 W = \pi_1 V = 0$;
2) h-cobordism;
3) $\dim W \geq 6$;

then $W \overset{\text{diff}}{\approx} V \times [0, 1]$.

h-cobordism

V, V' are deformation retract of W.

Remark

Under condition 1),
h-cob $\iff$ $H_*(W, V) = 0$.

Application

Characteristic of $D^{\geq 6}$

$\pi_1 W = \pi_1 \partial W = 0$, W compact. TFAE:

1) $W \overset{\text{diff}}{\approx} D$
2) $W \overset{\text{homeo}}{\approx} D$
3) W retractable
4) $H_* W \cong H_* pt$

1) $\implies$ 2) $\implies$ 3) $\implies$ 4) is obvious. For 4) $\implies$ 1):
Let $D \hookrightarrow W$ be an embedding of a disk, Since

$$H_*(W - intD, \partial D) \cong H_*(W, D) = 0,$$

by the h-cobordism theorem, $\varnothing \xrightarrow{W - intD} \partial D$ is a product cobordism. Combing with $\partial D \xrightarrow{D} \varnothing$, we have $W \overset{\text{diff}}{\approx} D$.

Application

Generalized Poincaré Conjecture

If $\pi_1 V^n = 0, H_* V \cong H_* S^n, n \geq 5$, then $V \overset{\text{homeo}}{\approx} S^n$.

Let $D^n \hookrightarrow V$ be an embedding of a n-disk.

$$H_*(V - intD, \varnothing) \cong H^{n-*}(V - intD, \partial D) \cong H^{n-*}(V, D) \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases} \cong H_* pt$$

By the characteristic of $D^{\geq 6}$, we have $V - intD \overset{\text{diff}}{\approx} D_0^n$, that is $V^n \overset{\text{diff}}{\approx} D^n \cup_h D_0^n$, a so-called "twisted sphere", where $h : \partial D \to \partial D_0$ is the attaching map. Let $g_0 : D_0 \hookrightarrow S^n$ be a embedding, then $g(u) = \begin{cases} g_0(u), & u \in D_0 \\ \sin \frac{\pi t}{2} g_0(h^{-1} u) + \cos \frac{\pi t}{2} e_{n+1}, & u \in D \end{cases}$ is a homeomorphism.

Thank you for your attention!

Questions?