K-Theory and The Index Theorem

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Lecture 1-3 are trivial. So starts from Lecture 4.

1 Lecture 1

Linear algebra.

Topology.

Differential geometry.

2 Lecture 2

Transition function.

Theorem 2.1 (Reconstruction Theorem.)**.**

Examples.

3 Lecture 3

Operations on vector bundles.

1. Dual bundle.
2. Direct sum.
3. Tensor product.
4. Pullback.
5. Subbundles & quotient bundles.

4 Lecture 4

Classifying vector bundles.

Lemma 4.1. *Given a vector bundle E over a compact Hausdorff space X with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ for a finite open covering $\{U_\alpha\}_{\alpha=1}^N$, then there is a continuous map (called **classifying map**)*

$$f_E : X \to \mathrm{Gr}_k(\mathbb{R}^{k\times N})$$

such that $f^*\xi_k \cong E$.

Proof. Choose a PoU $\{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$. Define

$$F : E \to \prod_{\alpha=1}^N \varphi_\alpha(\pi^{-1}(U_\alpha)) \xrightarrow{\prod \pi^\alpha} \mathbb{R}^{k\times N}$$

by $(x, v) \mapsto \prod_\alpha \rho_\alpha(\pi^\alpha \circ \varphi_\alpha(x, v))_\alpha$, where $\pi^\alpha : \varphi_\alpha(\pi^{-1}(U_\alpha)) \cong U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection. Note that if $(x, v) \notin \pi^{-1}(U_\alpha)$, then $\rho_\alpha(x) = 0$, hence $\rho_\alpha(\pi \circ \varphi_\alpha(x, v)) = 0$. For all x,

$$E_x \xrightarrow{F} \mathbb{R}^{k\times N}$$

is injective and linear.

Define $f_E(x) := F(E_x) \in \mathrm{Gr}_k(\mathbb{R}^{k\times N})$. We need to check $E \cong f_E^*\xi_k$, i.e.

$$\begin{tikzcd} E \arrow[r, "\tilde{F}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi_k \arrow[d] \\ X \arrow[r] & \mathrm{Gr}_k(\mathbb{R}^{k\times N}) \end{tikzcd}$$

is a pullback. Note that $\tilde{F}(x, v) = (f_E(x), F(v))$, so the commutivity is obvious. $\square$

Lemma 4.2. *If $f_{E_0} \simeq f_{E_1} : X \to Y$, then $E_0 \cong E_1$.*

More generally we have

Lemma 4.3. *If $f_0 \simeq f_1 : X \to Y$, then $f_0^*\xi \cong f_1^*\xi$.*

Proof. Let $F : [0, 1] \times X \to Y$ be the homotopy, $E := F^*\xi$ the pullback bundle. We want to show $E|_{0\times X} \cong E|_{1\times X}$.

Note that for all t,

$$\begin{tikzcd} \mathrm{Hom}(E_t, E_t) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}(E, [0,1] \times E_t) \arrow[d] \\ t \times X \arrow[r, hook] & [0,1] \times X \end{tikzcd}$$

where the horizon arrows are closed embeddings. Recall that

Lemma 4.4 (Tietze extension theorem for section)**.**

$$\begin{tikzcd} \xi|_A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi \arrow[d] \\ A \arrow[r, hook] \arrow[u, bend left, "s_A"] & X \arrow[u, dashed, bend right, "\exists s"'] \end{tikzcd}$$

such that $s|_A = s_A$.

For a section $(\mathrm{id}_{E_t} : E_x \cong E_x) \in \mathrm{Hom}(E_t, E_t)$, we have a section s which extends id_{E_t}. Since isomorphism of vector bundle is an open condition, there is a neighborhood Δ_t of t such that $s|_{\Delta_t \times X} : E|_{\Delta_t \times X} \cong \Delta_t \times X$. By compactness of $[0,1]$, we derived $E|_{0\times X} \cong E|_{1\times X}$, completing the proof. □

Proof of the Tietze extension theorem for section. Recall the original version of Tietze extension theorem:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & X \arrow[d, dashed, "\exists \tilde{f}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathbb{R} \end{tikzcd}$$

such that $\tilde{f}|_A = f$.

Locally, the diagram of the section version turns into

$$\begin{tikzcd} (A \cap U_\alpha) \times \mathbb{R}^k \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & U_\alpha \times \mathbb{R}^k \arrow[d] \\ A \cap U_\alpha \arrow[r, hook] \arrow[u, bend left, "s_A^\alpha"] & U_\alpha \arrow[u, dashed, bend right, "\exists s^\alpha"'] \end{tikzcd}$$

the existence of dashed arrow is by the original version of Tietze extension theorem. Since $s_A^\alpha = g_{\alpha\beta} s_A^\beta$ leads to $s^\alpha = g_{\alpha\beta} s^\beta$ on $U_{\alpha\beta}$, it *is* a global section. □

Then we can come to the proof that homotopical transition functions give rise to isomorphic vector bundle.

Theorem 4.5. *If* $\{g_{\alpha\beta}^0\}$ *and* $\{g_{\alpha\beta}^1\}$ *are two homotopic transition functions of two vector bundles* E_0 *and* E_1*, then* $E_0 \cong E_1$.

Proof. Let $g_{\alpha\beta}^t(x) : [0,1] \times U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})$ be the homotopy,

$$\tilde{E} = \frac{\bigsqcup_\alpha [0,1] \times U_\alpha \times \mathbb{R}^k}{(t,x,v)_\beta \sim (t,x,g_{\alpha\beta}^t(x)v)_\alpha}$$

the bundle over $[0,1] \times X$. Then the classifying map $f_{\tilde{E}}$ gives the homotopy between f_{E_0} and f_{E_1}, therefore $E_0 \cong f_{E_0}^* \xi_k \cong f_{E_1}^* \xi_k \cong E_1$ as bundle isomorphism from above *lemmata*. □

<u>Some applications.</u>

Example 4.1 (Real vector bundle over $\mathbb{S}^1$.)**.** Let $\mathbb{S}^1 = U_0 \cap U_1$ be the canonical decomposition. $g_{01} : U_{01} \to \mathrm{GL}(k, \mathbb{R})$ is homotopic to $\tilde{g}_{01} : \{\pm 1\} \to \mathrm{GL}(k, \mathbb{R})$. Since $\mathrm{GL}(k, \mathbb{R})$ has two connected component, we write $\mathrm{GL}_{\pm}(k, \mathbb{R}) := \{g | \det g > 0 \text{ or } < 0\}$.

Case I. If $g_{01}(-1)$ and $g_{01}(1)$ are in the same component, say $\mathrm{GL}_+(k, \mathbb{R})$, then $\tilde{g}_{01}$ is homotopic to a constant map to I_k. Hence $E \cong \mathbb{S}^1 \times \mathbb{R}^k$.

Case II. If $g_{01}(1) \in \mathrm{GL}_+(k, \mathbb{R}), g_{01}(-1) \in \mathrm{GL}_-(k, \mathbb{R})$, then g_{01} is homotopic to $\tilde{g}_{01} : 1 \mapsto I_k, -1 \mapsto \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$. Hence

$$E \cong (\mathbb{S}^1 \times \mathbb{R}^{k-1}) \oplus \frac{[0,1] \times \mathbb{R}}{(0, v) \sim (1, -v)} \cong \underline{\mathbb{R}}^{k-1} \oplus \underline{\mathbb{R}}_{-1}.$$

Since $\underline{\mathbb{R}}_{-1} \oplus \underline{\mathbb{R}}_{-1} \cong \underline{\mathbb{R}}^2$, above are all the cases. The reason for the isomorphism is that $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ can be connected by a path.

Example 4.2 (Complex vector bundle over $\mathbb{S}^1$.)**.** Since $g_{01} : \{\pm 1\} \to \mathrm{GL}(k, \mathbb{C})$ and $\mathrm{GL}(k, \mathbb{C})$ is path-connected, g_{01} is homotopic to a constant map to I_k, therefore $E \cong \mathbb{S}^1 \times \mathbb{C}^k \cong \underline{\mathbb{C}}^k$.

Example 4.3 (Complex vector bundle over $\mathbb{S}^2$.)**.** Let $S^2 = U_0 \cup U_1$ s.t. $g_{01} : U_0 \cap U_1 \sim \mathbb{S}^1 \to \mathrm{GL}(k, \mathbb{C})$. We have

$$[\mathbb{S}^1, \mathrm{GL}(k, \mathbb{C})] = [\mathbb{S}^1, \mathrm{U}(k)] \cong \mathbb{Z} (k \geq 1).$$

When $k = 1$, $[\mathbb{S}^1, \mathrm{GL}(1, \mathbb{C})] = [\mathbb{S}^1, \mathbb{S}^1 \times (0, \infty)] = [\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$, therefore the isomorphic class L_d of line bundles is classified by a integer d, which equals to the Chern number $d = \left\langle c_1(L_d), [\mathbb{S}^2] \right\rangle$.

It is easy to see that $L_{d_1} \oplus L_{d_2} \cong L_{d_1 + d_2} \oplus \underline{\mathbb{C}}$.

$$g(t) = \begin{bmatrix} z^{\mathrm{i}\, d_1} & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{2} t) & -\sin(\frac{\pi}{2} t) \\ \sin(\frac{\pi}{2} t) & \cos(\frac{\pi}{2} t) \end{bmatrix} \begin{bmatrix} z^{\mathrm{i}\, d_2} & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{2} t) & -\sin(\frac{\pi}{2} t) \\ \sin(\frac{\pi}{2} t) & \cos(\frac{\pi}{2} t) \end{bmatrix}^{-1}$$

is a homotopy between $e^{\mathrm{i}\,\theta} \mapsto \begin{bmatrix} e^{\mathrm{i}\, d_1 \theta} & \\ & e^{\mathrm{i}\, d_2 \theta} \end{bmatrix}$ to $e^{\mathrm{i}\,\theta} \mapsto \begin{bmatrix} e^{\mathrm{i}(d_1 + d_2)\theta} & \\ & 1 \end{bmatrix}$, which are the transition functions.