K-Theory and The Index Theorem

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Lecture 1-3 are trivial. So starts from Lecture 4.

1 Lecture 1

Linear algebra.

Topology.

Differential geometry.

2 Lecture 2

Transition function.

Theorem 2.1 (Reconstruction Theorem.)**.**

Examples.

3 Lecture 3

Operations on vector bundles.

1. Dual bundle.
2. Direct sum.
3. Tensor product.
4. Pullback.
5. Subbundles & quotient bundles.

4 Lecture 4

Classifying vector bundles.

Lemma 4.1. *Given a vector bundle E over a compact Hausdorff space X with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ for a finite open covering $\{U_\alpha\}_{\alpha=1}^N$, then there is a continuous map (called* ***classifying map****)*

$$f_E : X \to \mathrm{Gr}_k(\mathbb{R}^{k\times N})$$

such that $f^*\xi_k \cong E$.

Proof. Choose a PoU $\{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$. Define

$$F : E \to \prod_{\alpha=1}^{N} \varphi_\alpha(\pi^{-1}(U_\alpha)) \xrightarrow{\prod \pi^\alpha} \mathbb{R}^{k\times N}$$

by $(x, v) \mapsto \prod_\alpha \rho_\alpha(\pi^\alpha \circ \varphi_\alpha(x, v))_\alpha$, where $\pi^\alpha : \varphi_\alpha(\pi^{-1}(U_\alpha)) \cong U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection. Note that if $(x, v) \notin \pi^{-1}(U_\alpha)$, then $\rho_\alpha(x) = 0$, hence $\rho_\alpha(\pi \circ \varphi_\alpha(x, v)) = 0$.

Obviously, for all x,

$$E_x \xrightarrow{F} \mathbb{R}^{k\times N}$$

is injective and linear.

Define $f_E(x) := F(E_x) \in \mathrm{Gr}_k(\mathbb{R}^{k\times N})$. We need to check $E \cong f_E^*\xi_k$, i.e.

$$\begin{tikzcd} E \arrow[r, "\tilde{F}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi_k \arrow[d] \\ X \arrow[r, "f_E"] & \mathrm{Gr}_k(\mathbb{R}^{k\times N}) \end{tikzcd}$$

is a pullback. Note that $\tilde{F}(x, v) = (f_E(x), F(v))$, so the commutivity is obvious. □

Lemma 4.2. *If $f_{E_0} \simeq f_{E_1}$ are classifying maps, then $E_0 \cong E_1$.*

More generally we have

Lemma 4.3. *If ξ is a vector bundle over Y and $f_0 \simeq f_1 : X \to Y$, then $f_0^*\xi \cong f_1^*\xi$.*

Proof. Let $F : [0, 1] \times X \to Y$ be the homotopy, $E := F^*\xi$ the pullback bundle. We want to show $E|_{0\times X} \cong E|_{1\times X}$.

Note that for all t,

$$\begin{tikzcd} \mathrm{Hom}(E_t, E_t) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}(E, [0, 1] \times E_t) \arrow[d] \\ t \times X \arrow[r, hook] & [0, 1] \times X \end{tikzcd}$$

where the horizon arrows are closed embeddings. Recall that

Lemma 4.4 (Tietze extension theorem for sections)**.**

$$\begin{tikzcd} \xi|_A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi \arrow[d] \\ A \arrow[r, hook] \arrow[u, bend left, "s_A"] & X \arrow[u, dashed, bend right, "\exists s"'] \end{tikzcd}$$

such that $s|_A = s_A$.

For a section $(\mathrm{id}_{E_t} : E_x \cong E_x) \in \mathrm{Hom}(E_t, E_t)$, we have a section s which extends id_{E_t}. Since isomorphism of vector bundle is an open condition, there is a neighborhood Δ_t of t such that $s|_{\Delta_t \times X} : E|_{\Delta_t \times X} \cong \Delta_t \times X$. By compactness of $[0, 1]$, we derived $E|_{0\times X} \cong E|_{1\times X}$, completing the proof. $\square$

Proof of the Tietze extension theorem for section. Recall the original version of Tietze extension theorem:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & X \arrow[d, dashed, "\exists \tilde{f}"] \\ & \mathbb{R} \end{tikzcd}$$

such that $\tilde{f}|_A = f$.

Locally, the diagram of the section version turns into

$$\begin{tikzcd} (A \cap U_\alpha) \times \mathbb{R}^k \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & U_\alpha \times \mathbb{R}^k \arrow[d] \\ A \cap U_\alpha \arrow[r, hook] \arrow[u, bend left, "s_A^\alpha"] & U_\alpha \arrow[u, dashed, bend right, "\exists s^\alpha"'] \end{tikzcd}$$

the existence of dashed arrow is by the original version of Tietze extension theorem. Since $s_A^\alpha = g_{\alpha\beta} s_A^\beta$ leads to $s^\alpha = g_{\alpha\beta} s^\beta$ on $U_{\alpha\beta}$, via PoU, it *is* a global section. $\square$

Then we can come to the proof that homotopical transition functions give rise to isomorphic vector bundle.

Theorem 4.5. *If* $\{g^0_{\alpha\beta}\}$ *and* $\{g^1_{\alpha\beta}\}$ *are two homotopic transition functions of two vector bundles* E_0 *and* E_1*, then* $E_0 \cong E_1$.

Proof. Let $g^t_{\alpha\beta}(x) : [0, 1] \times U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})$ be the homotopy,

$$\tilde{E} = \frac{\bigsqcup_\alpha [0, 1] \times U_\alpha \times \mathbb{R}^k}{(t, x, v)_\beta \sim (t, x, g^t_{\alpha\beta}(x)v)_\alpha}$$

the bundle over $[0, 1] \times X$. Then the classifying map $f_{\tilde{E}}$ gives the homotopy between f_{E_0} and f_{E_1}, therefore $E_0 \cong f^*_{E_0}\xi_k \cong f^*_{E_1}\xi_k \cong E_1$ as bundle isomorphism from above *lemmata*. $\square$

Some applications.

Example 4.1 (Real vector bundle over $\mathbb{S}^1$.)**.** Let $\mathbb{S}^1 = U_0 \cap U_1$ be the canonical decomposition. $g_{01} : U_{01} \to \mathrm{GL}(k, \mathbb{R})$ is homotopic to $\tilde{g}_{01} : \{\pm 1\} \to \mathrm{GL}(k, \mathbb{R})$. Since $\mathrm{GL}(k, \mathbb{R})$ has two connected component, we write $\mathrm{GL}_{\pm}(k, \mathbb{R}) := \{g | \det g > 0 \text{ or } < 0\}$.

Case I. If $g_{01}(-1)$ and $g_{01}(1)$ are in the same component, say $\mathrm{GL}_+(k, \mathbb{R})$, then $\tilde{g}_{01}$ is homotopic to a constant map to I_k. Hence $E \cong \mathbb{S}^1 \times \mathbb{R}^k$.

Case II. If $g_{01}(1) \in \mathrm{GL}_+(k, \mathbb{R}), g_{01}(-1) \in \mathrm{GL}_-(k, \mathbb{R})$, then g_{01} is homotopic to $\tilde{g}_{01} : 1 \mapsto I_k, -1 \mapsto \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$. Hence

$$E \cong (\mathbb{S}^1 \times \mathbb{R}^{k-1}) \oplus \frac{[0,1] \times \mathbb{R}}{(0, v) \sim (1, -v)} \cong \underline{\mathbb{R}}^{k-1} \oplus \underline{\mathbb{R}}_{-1}.$$

Since $\underline{\mathbb{R}}_{-1} \oplus \underline{\mathbb{R}}_{-1} \cong \underline{\mathbb{R}}^2$, above are all possible cases. The reason for the isomorphism is that $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ can be connected by a path.

Example 4.2 (Complex vector bundle over $\mathbb{S}^1$.)**.** Since $g_{01} : \{\pm 1\} \to \mathrm{GL}(k, \mathbb{C})$ and $\mathrm{GL}(k, \mathbb{C})$ is path-connected, g_{01} is homotopic to a constant map to I_k, therefore $E \cong \mathbb{S}^1 \times \mathbb{C}^k \cong \underline{\mathbb{C}}^k$.

Example 4.3 (Complex vector bundle over $\mathbb{S}^2$.)**.** Let $S^2 = U_0 \cup U_1$ s.t. $g_{01} : U_0 \cap U_1 \sim \mathbb{S}^1 \to \mathrm{GL}(k, \mathbb{C})$. Since $U(k-1) \hookrightarrow U(k) \to \mathbb{S}^{2k-1}$ is a fibration, we have $\pi_1 U(k) \cong \pi_1 U(1) \cong \mathbb{Z}$, hence

$$[\mathbb{S}^1, \mathrm{GL}(k, \mathbb{C})] = [\mathbb{S}^1, \mathrm{U}(k)] \cong \mathbb{Z}(k \geq 1).$$

When $k = 1$, $[\mathbb{S}^1, \mathrm{GL}(1, \mathbb{C})] = [\mathbb{S}^1, \mathbb{S}^1 \times (0, \infty)] = [\mathbb{S}^1, \mathbb{S}^1] \cong \mathbb{Z}$, therefore the isomorphic class L_d of line bundles is classified by a integer d, which equals to the Chern number $d = \langle c_1(L_d), [\mathbb{S}^2] \rangle$.

It is easy to see that $L_{d_1} \oplus L_{d_2} \cong L_{d_1+d_2} \oplus \underline{\mathbb{C}}$.

$$g(t, \theta) = \begin{bmatrix} e^{\mathrm{i}\, d_1 \theta} & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{bmatrix} \begin{bmatrix} e^{\mathrm{i}\, d_2 \theta} & \\ & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{bmatrix}^{-1}$$

is a homotopy between $\theta \mapsto \begin{bmatrix} e^{\mathrm{i}\, d_1 \theta} & \\ & e^{\mathrm{i}\, d_2 \theta} \end{bmatrix}$ and $\theta \mapsto \begin{bmatrix} e^{\mathrm{i}(d_1+d_2)\theta} & \\ & 1 \end{bmatrix}$, which are the transition functions.

5 Lecture 5

Denote $\mathrm{Vect}^k_{\mathbb{R}}(X)$ or $\mathrm{Vect}^k_{\mathbb{C}}(X)$ the isomorphic classes of $\mathbb{R}$ or $\mathbb{C}$ vector bundles of rank k over a topological space X.

The embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} : v \mapsto (v, 0)$ induces $\mathrm{Gr}_k(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_k(\mathbb{R}^{n+1})$. Hence we define $\mathrm{Gr}_k(\mathbb{R}^\infty) = \varinjlim_n \mathrm{Gr}_k(\mathbb{R}^n)$.

Theorem 5.1. *Let X be a compact Hausdorff space. Then there exist bijective correspondences $\mathrm{Vect}^k_{\mathbb{R}}(X) \overset{1:1}{\longleftrightarrow} [X, \mathrm{Gr}_k(\mathbb{R}^\infty)]$ and $\mathrm{Vect}^k_{\mathbb{C}}(X) \overset{1:1}{\longleftrightarrow} [X, \mathrm{Gr}_k(\mathbb{C}^\infty)]$.*

Proof. We only need to prove the real case. Suppose $E \in \mathrm{Vect}^k_{\mathbb{R}}(X)$ with PoUs $\{\rho_\alpha\}_{\alpha=1}^N, \{\rho'_\alpha\}_{\alpha=1}^{N'}$ subordinated, respectively, to $\{U_\alpha\}_{\alpha=1}^N, \{U'_\alpha\}_{\alpha=1}^{N'}$ with trivilizations $\{\varphi_\alpha\}_{\alpha=1}^N, \{\varphi'_\alpha\}_{\alpha=1}^{N'}$, we have

$$\begin{tikzcd} E \arrow[r, "\tilde{F}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi_k \arrow[d] \\ X \arrow[r, "f_E"] & \mathrm{Gr}_k(\mathbb{R}^{k\times N}) \end{tikzcd}$$

where $F(x, v) = \prod_{\alpha=1}^N \rho_\alpha(\pi \circ \varphi_\alpha(x, v))$, $f_E(x) = F(x, E_x) \in \mathrm{Gr}_k(\mathbb{R}^{k\times N})$, and

$$\begin{tikzcd} E \arrow[r, "\widetilde{F'}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \xi_k \arrow[d] \\ X \arrow[r, "f'_E"] & \mathrm{Gr}_k(\mathbb{R}^{k\times N'}) \end{tikzcd}$$

where $F'(x, v) = \prod_{\alpha=1}^{N'} \rho'_\alpha(\pi \circ \varphi'_\alpha(x, v))$, $f'_E(x) = F'(x, E_x) \in \mathrm{Gr}_k(\mathbb{R}^{k\times N'})$. Since

$$\begin{tikzcd} X \arrow[rr, "f_E"] \arrow[d, "f'_E"] & & \mathrm{Gr}_k(\mathbb{R}^{k\times N}) \arrow[d, hook, "j"] \\ \mathrm{Gr}_k(\mathbb{R}^{k\times N'}) \arrow[r, hook, "j'"] & \mathrm{Gr}_k(\mathbb{R}^{k\times(N+N')}) & \subset \mathrm{Gr}_k(\mathbb{R}^\infty) \end{tikzcd}$$

where j is the embedding to the first k components and j' to the last k components. Therefore $t(j \circ f_E) + (1-t)(j' \circ f'_E)$ is the homotopy between $j \circ f_E$ and $j' \circ f'_E$, giving rise to a well-defined homotopy class of $[X, \mathrm{Gr}_k(\mathbb{R}^\infty)]$.

Next is to show the 1-1 correspondence. Let $\alpha \in [X, \mathrm{Gr}_k(\mathbb{R}^\infty)]$ and $f \in \alpha$. Suppose $E = f^*\xi_k$. To verify the well-definedness, first notice that, by definition, f is a classifying map. Then since f_E is also a classifying map, there holds $f \sim f_E$ according to the last paragraph, therefore $f_E \in \alpha$. $\square$

Remark. $\mathrm{Vect}^k(X)$ is a semi-ring with operation $\oplus, \otimes$, where $\underline{1} = \underline{\mathbb{R}}$ or $\underline{\mathbb{C}}$, $\underline{0}$ = rank 0 vector bundle.

The clutching(gluing) construction.

Let $X = X_1 \cup X_2, A = X_1 \cap X_2$ closed. Let $E_i \to X_i$ be rank k vector bundles over X_i, respectively, such that there exists a isomorphism $\varphi : E_1|_A \cong E_2|_A$. Then $E_1 \cup_\varphi E_2 \to X$ is a vector bundle, where $E_1 \cup_\varphi E_2 := \frac{E_1 \sqcup E_2}{e \sim \varphi(e), e \in E_1|_A}$.

Check. If $x \in (X_1 - A) \cup (X_2 - A)$, then there exists a neighborhood U of x contained in either $X_1 - A$ or $X_2 - A$, such that

$$(E_1 \cup_\varphi E_2)|_U \cong U \times \mathbb{R}^k.$$

Otherwise $x \in A$. Let V_1 be a neighborhood of x in X_1 such that $\theta_1 : E_1|_{V_1} \cong V_1 \times \mathbb{R}^k$. Let V_2 be a neighborhood of x in X_2 and $\theta_2^A := \varphi^{-1} \circ \theta_1|_{V_1 \cap A} : E_2|_{V_1 \cap A} \cong E_1|_{V_1 \cap A} \longrightarrow (V_1 \cap A) \times \mathbb{R}^k$. Extend it to $\theta_2 : E_2|_{V_2} \cong V_2 \times \mathbb{R}^k$ for a neighborhood V_2 of $V_1 \cap A$ in X_2, such that $\theta_2|_A = \theta_2^A$, which gives rise to

$$\theta_1 \cup_\varphi \theta_2 : (E_1 \cup_\varphi E_2)|_{V_1 \cup V_2} \cong (V_1 \cup V_2) \times \mathbb{R}^k.$$

Proposition 5.2. *1. Let $E \to X = X_1 \cap X_2$ be a vector bundle over X, $E_i = E|_{X_i}$. $I_A : E_1|_A \cong E_2|_A$. Then $E_1 \cup_{I_A} E_2 \cong E$.*

2. *Let $E_i \underset{\cong}{\xrightarrow{\beta_i}} E_i', \varphi : E_1|_A \cong E_2|_A, \varphi' : E_1'|_A \cong E_2'|_A$ such that $\varphi' \circ \beta_1 = \beta_2 \circ \varphi$, then $E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'$*
3. *The clutching construction is compatible with $\oplus, \otimes$.*
4. *If $\varphi_1 \simeq \varphi_2 : E_1|_A \cong E_2|_A$, then $E_1 \cup_{\varphi_1} E_2 \cong E_1 \cup_{\varphi_2} E_2$.*

Proof. Let $I = [1, 2]$ and $(E_1 \times I) \cup_{\{\varphi_t\}} (E_2 \times I) \to X \times I$ be bundles over $X \times I$. Then $E_1 \cup_{\varphi_1} E_2 \cong (E_1 \times I) \cup_{\{\varphi_t\}} (E_2 \times I)|_{X \times 1} \cong (E_1 \times I) \cup_{\{\varphi_t\}} (E_2 \times I)|_{X \times 2} \cong E_1 \cup_{\varphi_2} E_2$. □

Proposition 5.3. *For any topological space X, let $S(X) = C^+(X) \cup_X C^-(X)$ be suspension of X. There is a natural bijective map*

$$\mathrm{Vect}^k_{\mathbb{F}}(S(X)) \overset{1:1}{\longleftrightarrow} [X, \mathrm{GL}(k, \mathbb{F})].$$

Proof. First define the map

$$\Phi : \mathrm{Vect}^k_{\mathbb{F}}(S(X)) \to [X, \mathrm{GL}(k, \mathbb{F})]$$

as follows. For a rank k vector bundle E, since $C^\pm(X)$ are contractible, there exist the triviailzations $\alpha^\pm : E|_{C^\pm(X)} \cong C^\pm(X) \times \mathbb{F}^k$, giving rise to a bundle isomorphism $(\alpha^-|_X) \circ (\alpha^+|_X)^{-1} : X \times \mathbb{F}^k \cong X \times \mathbb{F}^k$. It leads to a continuous map $\varphi_E : X \to \mathrm{GL}(k, \mathbb{F})$.

To show the well-definedness, suppose $E_1 \cong E_2$. Define $\alpha_1^\pm, \alpha_2^\pm$ in similar fashion. Since $\alpha_2^\pm \circ (\alpha_1^\pm)^{-1} : C^\pm(X) \to \mathrm{GL}(k, \mathbb{F})$ are continuous map on contractible space, they are both homotopic to some constant map. Hence $\varphi_{E_1} = (\alpha_1^-|_X) \circ (\alpha_1^+|_X)^{-1} : X \to \mathrm{GL}(k, \mathbb{F})$ and $\varphi_{E_2} = (\alpha_2^-|_X) \circ (\alpha_2^+|_X)^{-1} : X \to \mathrm{GL}(k, \mathbb{F})$ are homotopical.

To show the bijection, we define

$$\Psi : [X, \mathrm{GL}(k, \mathbb{F})] \to \mathrm{Vect}^k_{\mathbb{F}}(S(X)) : \varphi \mapsto (C^+(X) \times \mathbb{F}^k) \cup_\varphi (C^-(X) \times \mathbb{F}^k)$$

by the clutching theorem. It is easy to see that Φ and Ψ are inverse of each other. □

To sum up, we have derived that $\mathrm{Vect}_{\mathbb{C}}^k(X) \xleftrightarrow{1:1} [X, \mathrm{Gr}_k(\mathbb{C}^\infty) \sim B\,\mathrm{U}(k)]$ and $\mathrm{Vect}_{\mathbb{C}}^k(S(X)) \xleftrightarrow{1:1} [X, \mathrm{GL}(\mathbb{C}, k) \sim \mathrm{U}(k)]$.

The collapsing construction.

Proposition 5.4. *Let Y be a closed subset of X, E a vector bundle over X and $p \circ \alpha : E|_Y \cong Y \times \mathbb{R}^k \to \mathbb{R}^k$, then $E/\alpha = E/\sim$, where $e \sim e' \iff p \circ \alpha(e) = p \circ \alpha(e')$, is a vector bundle over X/Y.*

Proof. For $x \in X - Y$, since $X - Y$ is open, there exists a trivialization of E over $X - Y$, hence X/Y.

For the point Y/Y, there exists an open neighborhood U of Y in X such that

$$(E|_U)/\alpha \cong (U/Y) \times \mathbb{R}^k.$$

□

6 Lecture 6

Corollary. *For any vector bundle E over a compact Hausdorff space X. Then there exists a vector bundle F such that $E \oplus F$ is trivial.*

Proof. We only consider the real case. Since X is compact, we can choose a classifying map $f_E : X \to \mathrm{Gr}_k(\mathbb{R}^N)$ for some large N. We have an exact sequence of vector bundles over $\mathrm{Gr}_k(\mathbb{R}^N)$

$$0 \to \gamma_k \to \underline{\mathbb{R}}^N \to \eta_{N-k} \to 0$$

where $\eta_{N-k} = \{(V, v) : V \subset \mathbb{R}^N, \dim V = N - k, v \in V^\perp\}$. Let $F := f_E^* \eta$, we have $E \oplus F = \mathbb{R}^N$. □

Principal bundles and associated bundle.

We first study an important object — frame bundle. Given a vector bundle E over X, the **frame bundle** of E is $\mathrm{Fr}_{\mathrm{GL}}(E) = \bigcup_{x \in X} \{\text{all bases of } E_x\} \to X$, satisfying

1. $\forall \sigma, \sigma' \in \pi^{-1}(x)$, there exists $g \in \mathrm{GL}(k, \mathbb{R})$ such that $\sigma' = \sigma \cdot g$;
2. $\sigma \cdot g = \sigma \iff g = I$.

There is a right action

$$\mathrm{Fr}_{\mathrm{GL}}(E) \times \mathrm{GL}(k, \mathbb{R}) \to \mathrm{Fr}_{\mathrm{GL}}(E) : (\sigma, g) \mapsto \sigma \cdot g$$

which is free ($\sigma \cdot g = \sigma \iff g = I$) and transitive along each fiber $\pi^{-1}(x)$.

We can define local triviailzation as follows. There exists an open cover $X = \{U_\alpha\}$, such that

$$\begin{tikzcd} \mathrm{Fr}(E)|_{U_\alpha} \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr] \arrow[rr, phantom, shift right=4, "\circlearrowleft"] & & U_\alpha \times \mathrm{GL}(k, \mathbb{R}) \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

Fixed a local frame $\sigma^\alpha : U_\alpha \to \mathrm{Fr}(E)|_{U_\alpha}$. For all $p \in \mathrm{Fr}(E)|_{U_\alpha}$, since σ^α and p are frames, there exists a $g_p \in \mathrm{GL}(k, \mathbb{R})$ such that $p = \sigma^\alpha \cdot g_p$. Therefore we can define $\varphi_\alpha(p) = (\pi(p), g_p) \in U_\alpha \times \mathrm{GL}(k, \mathbb{R})$.

Generally we can define the notions of frame bundle and associated bundle.

Definition 6.1. Given a Lie group G, a principal G-bundle over X is a topological space P with a map $\pi : P \to X$ and a right G-action $P \times G \to P$ satisfying

1. that G-action is free and transitive along each fibre $\pi^{-1}(x)$
2. the local triviality: for a over covering $\{U_\alpha\}$ of X, there is a G-equivariant homeomorphism

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr] & \circlearrowleft & U_\alpha \times G \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

   G is called the structure group of P.

Example 6.1.
1. $\mathrm{Fr}_{\mathrm{GL}}(E)$ is a principal $\mathrm{GL}_k(\mathbb{R})$-bundle for any real bundle E of rank k. Similarily for the complex case.
2. If E is equipped with Euclidean metric, then $\mathrm{Fr}_{\mathrm{O}}(E) = \bigcup_x \{\text{orthonormal frames of } E_x\}$ is a principal $\mathrm{O}(k)$-bundle. In complex case, $\mathrm{Fr}_{\mathrm{U}}(E)$ is a principal $\mathrm{U}(k)$-bundle.

For a principal bundle P over X with transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ w.r.t. each local section of $P|_{U_\alpha}$ over U_α. If $g_{\alpha\beta}(x) \in H, \forall \{U_\alpha\}, \forall x$, for some $H < G$, then the principal G-bundle can be reduced to a principal H-bundle Q, where $Q|_{U_\alpha} \cong U_\alpha \times H$ and $Q = \frac{\bigsqcup_\alpha U_\alpha \times H}{(x,h)_\beta \sim (x, g_{\alpha\beta}(x) h)_\alpha}$.

Example 6.2.
1. The $\mathrm{GL}(k)$-frames bundle $\mathrm{Fr}_{\mathrm{GL}}(E)$ admits an $\mathrm{O}(k)$-reduction for any real vector bundle E, or an $\mathrm{U}(k)$-reduction for any complex vector bundle E.
   There is no obstruction from $\mathrm{GL}(k, \mathbb{R})$ to $\mathrm{O}(k)$ or $\mathrm{GL}(k, \mathbb{C})$ to $\mathrm{U}(k)$, since $\mathrm{GL}(k, \mathbb{R})$ and $\mathrm{O}(k)$ are homotopy equivalent.
2. E is an oriented real vector bundle if $\mathrm{Fr}_{\mathrm{O}}(E)$ has an $\mathrm{SO}(k)$-reduction $\iff$ the transition functions $g_{\alpha\beta}$ can be chosen in $\mathrm{SO}(k)$ $\iff$ $[\mathrm{sgn}(g_{\alpha\beta})] = 0 \in \check{H}^1(\{U_\alpha\}; \mathbb{Z}_2)$. If $\{U_\alpha\}$ is a good covering, then $[\mathrm{sgn}(g_{\alpha\beta})] \in H^1(X; \mathbb{Z}_2)$ is called the 1st Stiefel-Whitney class.
   To show the last equivalence, suppose that $[\mathrm{sgn}(g_{\alpha\beta})] = 0$ in Čech cohomology. Then there exists $\xi_\alpha : U_\alpha \to \mathbb{Z}_2$ such that $\mathrm{sgn}(g_{\alpha\beta}) = \zeta_\alpha \cdot \zeta_\beta^{-1}$ $\iff$ $\mathrm{sgn}(\zeta_\alpha^{-1} g_{\alpha\beta} \zeta_\beta) = 1$ $\iff$ $\det E$ is trivial $\iff$ $\det(g_{\alpha\beta}) = 1$. Therefore the oriented o.n. frame bundle of E exists.
3. Similarly for a complex vector bundle. $\mathrm{Fr}_{\mathrm{SU}}(E)$ exists $\iff$
   - $\det_{\mathbb{C}}(E)$ is trivial $\iff$

- $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SU}(k) \iff$
- the 1st Chern class $c_1(E) \in H^2(X;\mathbb{Z})$ vanishes.

Definition 6.2. Given a principal G-bundle and a linear representation over real(or complex) space

$$G \xrightarrow{\rho} \mathrm{GL}(V) = \mathrm{GL}(k,\mathbb{R})$$

(respectively, $G \xrightarrow{\rho} \mathrm{GL}(V) = \mathrm{GL}(k,\mathbb{C})$), then

$$P \times_\rho V = \frac{P \times V}{(p,v) \sim (p \cdot g, \rho(g)^{-1}(v))}$$

is a vector bundle, called an **associated bundle** of P.

Example 6.3. 1. $E^{\otimes n}$ is an associated bundle of $\mathrm{Fr}_{\mathrm{GL}}(E)$ w.r.t.

$$\rho_n : \mathrm{GL}(k,\mathbb{R}) \to \mathrm{GL}((\mathbb{R}^k)^{\otimes n}) : g \mapsto g^{\otimes n}.$$

2. $E \otimes E^* = \mathrm{End}(E)$ is an associated bundle of $\mathrm{Fr}_{\mathrm{GL}}(E)$ w.r.t.

$$\rho = \mathrm{Ad} : \mathrm{GL}(k,\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^k \otimes (\mathbb{R}^k)^*) = \mathrm{GL}(M_k(\mathbb{R}))$$

where $\rho(g) \cdot A = gAg^{-1}$.

Definition 6.3. $\Gamma(X, P \times_\rho V) = \{f : P \xrightarrow{C^0} V : f(p \cdot g) = \rho(g)^{-1} f(p)\}$

7 Lecture 7

<u>K^0-group</u>

Let X be compact Hausdorff and $\mathrm{Vect}_{\mathbb{C}}(X)$ the set of isomorphic classes of complex vector bundle over X.

Example 7.1. $\mathrm{Vect}_{\mathbb{C}}(pt) = \{0, 1, 2, \cdots\} \cong \mathbb{N}$.

It is easy to see that $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$ is an abelian semigroup, $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus, \otimes)$ is a semiring.

Definition 7.1. The K-Theory (K^0-group) of a compact Hausdorff space X is the group completion of the semi-group $(\mathrm{Vect}_{\mathbb{C}}, \oplus)$.(=the Grothendieck group of the category of complex vector bundles over X.)

Let $K(S)$ denote the group completion of an abelian semigroup S. It satisfies either of the two equivalent conditions

1. universal property. $K(S)$ is an abelian group s.t. there is a semigroup homomorphism

$$\alpha : S \to K(S)$$

and for any abelian group G and any semigroup homomorphism $\gamma : S \to G$, there exists a unique group homomorphism $\chi : K(S) \to G$ s.t. $\chi \circ \alpha = \gamma$.

$$\begin{tikzcd} S \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\circlearrowleft"] & K(S) \arrow[dl, dashed, "\exists \chi"] \\ & G & \end{tikzcd}$$

2. $K(S) = \frac{S \times S}{\{(s,s) | s \in S\}}$. Here $\alpha : S \to K(S) : a \mapsto [a, 0]$. We also have $-[a, b] = [b, a]$, $[a, b] = [a] - [b]$, where $[a] := [a, 0]$.

Remark. If S is an abelian group, $K(S) = S$, $[a, b] \mapsto a - b$.

We denote $K^0(X) = K(\mathrm{Vect}_{\mathbb{C}}(X), \oplus) =: K(X)$.

Lemma 7.1. *Any element of $K(X)$ is of the form $[E] - [n]$ where $[n] := [\underline{\mathbb{C}}^n]$.*

Proof. Let $[E_0, F_0] \in K(X)$, then there exists F_0' such that $F_0 \oplus F_0' = \underline{\mathbb{C}}^n$ for some n. Then $[E_0, F_0] = [E_0 \oplus F_0', F_0 \oplus F_0'] = [E_0 \oplus F_0', \underline{\mathbb{C}}^n] = [E] - [n]$. □

Proposition 7.2. *Let $\mathscr{C}$ be the category of compact Hausdorff spaces, then the K^0-group is a contravariant, homotopy functor*

$$K^0 : \mathscr{C} \to \mathscr{A}bel.$$

By contravariant we mean that for $f : X \to Y$, we have $K^0(f) = f^* : K^0(Y) \to K^0(X)$. By homotopy functor we mean that if $f_1 \sim f_2 : X \to Y$, then $f_1^* = f_2^* : K^0(Y) \to K^0(X)$, i.e. $K^0(X)$ is a homotopy invariant of X.

K-Theory as a generalised(extraordinary) cohomology theory

Recall that for X a topological space, the singular cohomology $H^i(X; \mathbb{Z})$ is a sequence of contravariant homotopy functor

$$\mathscr{C} \to \mathscr{A}bel.$$

We also have a relative version. Let $\mathscr{C}^2 = \{(X, A) \in \mathscr{X} \times \mathscr{X} : A \subset X\}$. Then for each $i \in \mathbb{Z}$, $H^i(X, A) : \mathscr{C}^2 \to \mathscr{A}bel$ is a contravariant homotopy functor. Moreover, we can define a natural transformation (called **boundary map**) $\delta^i : H^i(A) \to H^{i+1}(X, A)$ such that the following axioms holds:

1. Excision axiom. If $(X, A) \in \mathscr{C}^2, U \subset A$ s.t. $\bar{U} \subset \mathring{A}$, then the natural inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces a natural isomorphism $H^i(X, A) \cong H^i(X \setminus U, A \setminus U)$.
2. Exactness. For any $(X, A) \in \mathscr{C}^2$, the inclusions $i : A \hookrightarrow X$ and $j : (X, \varnothing) \to (X, A)$, then there exists a long exact sequence

$$\cdots \to H^{n-1}(A) \xrightarrow{\delta^{n-1}} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta^n} H^{n+1}(X, A) \to \cdots$$

3. Dimension axiom. $H^i(pt;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i \neq 0 \end{cases}$

8 Lecture 8

If X is compact, we simply set $K^0(X) = K(\mathrm{Vect}_{\mathbb{C}}(X))$. If X is non-compact but locally compact, then we set

$$K^0(X) = \ker\{K^0(X^+) \to K^0(pt)\},$$

where X^+ is the one-point compactification of X.

Eilenberg-Steenod Axioms. The contravariant, homotopy functors

$$K^i : \mathscr{C}^2 \to \mathscr{A}bel$$

satisfy

1. Excision axiom. $\forall (X, A) \in \mathscr{C}^2, U \subset A, \bar{U} \subset \mathring{A}$, then $K^i(X - U, A - U) \xrightarrow{\cong} K^i(X, A)$.
2. Exactness. For i

We let $\mathscr{C}$ denote the category of compact spaces, $\mathscr{C}^+$ the category of based compact spaces, $\mathscr{C}^2$ the category of compact pairs $\{(X, A) : A \subset X\}$.

Definition 8.1. $K^0(X, A) = \tilde{K}^0(X/A) = \ker(K^0(X/A) \to K^0(pt))$. $\tilde{K}^0 : \mathscr{C}^+ \to \mathscr{A}bel$

Remark. $K^0(X) = K^0(X, \varnothing) = \tilde{K}^0(X/\varnothing) = \tilde{K}^0(X) \oplus \mathbb{Z}$.

Lemma 8.1. *Given a compact pair (X, A). If A is contractible, the quotient map $q : X \to X/A$ induces a bijection*

$$q^* : \mathrm{Vect}^k_{\mathbb{C}}(X/A) \to \mathrm{Vect}^k_{\mathbb{C}}(X), \forall k.$$

In particular,

$$K(X/A) \cong K(X),$$

$$\tilde{K}(X/A) \cong \tilde{K}(X), x_0 \in A.$$

Proof. □

Lemma 8.2. *Let $(X, A) \in \mathscr{C}^2$. Suppose $A \in \mathscr{C}^+$ and X, A connected. Then*

$$K^0(X, A) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A)$$

is exact iff

$$\tilde{K}^0(X/A) \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A)$$

is exact.

Consider the suspend construction.

Definition 8.2. For $n \geq 1$, for $X \in \mathscr{C}^+$, $\tilde{K}^{-n}(X) = \tilde{K}^0(\Sigma^n X)$.

9 Lecture 9

Bott's periodicity.