

# ASYMPTOTIC BEHAVIOR OF EXTERIOR MINIMAL GRAPHS IN HALF SPACES

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ABSTRACT. In this paper, we studied the asymptotic behavior at infinity of solutions to the minimal surface equation in exterior domains of the half space. We prove that the solution  $u$  tends to a linear function with rate at least  $x_n|x|^{-n}$ . We further obtain a more precise asymptotic expansion. Comparing to the results in exterior domains of the whole space, we do not require any growth condition or dimensional restriction.

## 1. INTRODUCTION

Over the last century, there has been tremendous interest in the Bernstein problem and the asymptotic behavior at infinity of minimal graphs. The Bernstein theorem states that any entire minimal graph over  $\mathbb{R}^n$  must be a hyperplane. It was first proved for  $n = 2$  by Bernstein [Ber27],  $n = 3$  by De Giorgi [DG65],  $n = 4$  by Almgren [Alm66] and  $n \leq 7$  by Simons [Sim68]. For  $n \geq 8$ , there are nonlinear entire minimal graphs constructed by Bombieri, De Giorgi and Giusti [BDGG69]. Under some additional hypotheses, there are many Liouville-type results. If the solution  $u$  has bounded gradient, Moser [Mos61] proved that  $u$  is linear by Harnack inequality; In [CNS88, EH90],  $u$  must be linear if we only require  $|\nabla u(x)| = o(|x|)$ .

For the minimal graphs over half spaces and general convex domains, Edelen and Wang [EW22] proved the following Bernstein-type result: any solution to the minimal surface equation over a convex domain with linear boundary condition must be linear. Unlike the entire case, their result does not need any growth hypothesis of  $u$  or any dimensional restriction, this is due to the linear boundary condition.

For general anisotropic minimal graphs or stable minimal hypersurfaces, see [Sim77, Moo22, MY24][dCP79, FCS80, Pog81, CL23, CLMS24] for the corresponding Bernstein results.

The asymptotic behavior of exterior minimal graphs was studied by many authors. Let  $u$  be a solution of the minimal surface equation outside a bounded subset in  $\mathbb{R}^n$ . For  $n = 2$ , Bers [Ber51] proved that  $\nabla u$  is bounded. Simon [Sim87] extended this result for  $n \leq 7$ . Concerning the asymptotic expansion near infinity, Bers [Ber51] proved that for  $n = 2$ ,  $u$  tends to a linear function plus  $c \log |x|$  at infinity with rate at least  $|x|^{-1}$ . For  $n \geq 3$ , under the additional hypothesis  $\nabla u$  is bounded, using the asymptotic behavior of solutions of uniformly elliptic linear equations, see [GS56, FG57, Gil61, Ser64, Ser65a, Ser65b, SW66],  $u$  tends to a linear function with rate at least  $|x|^{2-n}$ . Furthermore, if  $\nabla u \rightarrow 0$  as  $|x| \rightarrow \infty$ ,

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Schoen [Sch83] proved a more precise expansion, as  $|x| \rightarrow \infty$ ,

$$u(x) = a + b|x|^{2-n} + \sum_{j=1}^n \frac{c_j x_j}{|x|^n} + O(|x|^{-n}),$$

for some constants  $a, b, c_j \in \mathbb{R}$ . Using Kelvin transforms, Han and Wang [HW] improved this expansion to any order. Similar asymptotic expansions for solutions of other non-linear elliptic equations, one can see, [CL03, JLL20] for the Monge-Ampère equation, [HY21] for the maximal surface equation, [LLY20] for the special Lagrangian equation and [LLY20, LL23] for general fully nonlinear equations.

The main purpose of this paper is to extend the above asymptotic expansion for solutions of the minimal surface equation to half spaces. We prove the following result:

**Theorem 1.1.** *Let  $u \in C^2(\overline{\mathbb{R}_+^n} \setminus B_1^+)$  be a solution of minimal surface equation*

$$(1.1) \quad \begin{cases} \mathcal{M}u = \Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = 0 & \text{in } \mathbb{R}_+^n \setminus \overline{B_1^+}, \\ u(x', 0) = l(x') = a + b' \cdot x' & \text{on } \partial \mathbb{R}_+^n \setminus \partial B_1^+, \end{cases}$$

where  $n \geq 2$ ,  $\mathbb{R}_+^n = \{(x', x_n) : x_n > 0\}$  and  $l : \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is a linear function.

Then, as  $|x| \rightarrow \infty$ ,

$$(1.2) \quad u(x) = a + b' \cdot x' + b_n x_n + \frac{c_n x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} + \sum_{j=1}^n \frac{d_j x_j x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^{n+2}} + O(|x|^{-n-\alpha}),$$

for some constants  $b_n, c_n, d_1, \dots, d_n \in \mathbb{R}$  and any  $\alpha \in (0, 1)$ .

Moreover, for any  $m \geq 2$ ,

$$|\nabla u(x) - (b', b_n)| = O(|x|^{-n}), \quad |D^m u(x)| = O(|x|^{1-n-m}).$$

Here and hereafter, the notation  $f = O(g)$  is used to indicate that  $|f|$  is bounded by a constant multiple of  $|g|$  for large  $|x|$ . We now make some remarks about our main theorem.

*Remark 1.2.* We emphasize that in our case of half spaces, we have no requirement on the dimension or the growth of solutions. This is due to the linear boundary condition and the rigidity of minimal graphs over half spaces.

*Remark 1.3.* One can not reduce our result to the asymptotic theorem in exterior domains of the whole space by reflection. Since the rate of solutions tending to a linear function at infinity in exterior domains of the half space is faster than in exterior domains of the whole space, this is due to the linear boundary condition. In particular, when  $n = 2$ , the logarithmic term vanishes.

*Remark 1.4.* We only focus on the asymptotic behavior of  $u$  at infinity, thus the half ball  $B_1^+$  which we excluded in Theorem 1.1 is unimportant, it can be replaced by any compact subset of  $\mathbb{R}_+^n$ . Moreover, the regularity of  $u$  is also not critical. By the global regularity of the minimal surface equation and the standard bootstrap argument, the solution  $u \in C^2(\overline{\mathbb{R}_+^n} \setminus B_1^+)$  in fact implies that  $u$  is smooth up to the boundary in a smaller exterior domain, namely  $u \in C^\infty(\overline{\mathbb{R}_+^n} \setminus B_2^+)$ .

Similar to Han-Wang [HW], using the Kelvin-type transform, we can improve the asymptotic expansion (1.2) to higher order.

**Theorem 1.5.** *Let  $u \in C^2(\overline{\mathbb{R}}_+^n \setminus B_1^+)$  be a solution of (1.1). Then there exists a constant  $b_n \in \mathbb{R}$  and a function  $w$  such that*

$$(1.3) \quad u(x) = a + b' \cdot x' + b_n x_n + |\sqrt{A^{-1}}x|^{2-n} w \left( \frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2} \right) \quad \text{for } x \in \overline{\mathbb{R}}_+^n \setminus B_1^+.$$

and for some  $r > 0$ ,

$$w = 0 \quad \text{on } \sqrt{A^{-1}}\partial\mathbb{R}_+^n \cap B_r.$$

where  $b = (b', b_n)$  and  $A = I - \frac{b \otimes b}{1+|b|^2}$ ,  $A^{-1} = I + b \otimes b$ .

In addition,  $w$  is  $C^{2,\alpha}$  near the origin for any  $\alpha \in (0, 1)$ . Moreover, for  $n \geq 4$ , if  $n$  is even, then  $w \in C^\infty$ ; if  $n$  is odd, then  $w \in C^{n-3,\alpha}$  for any  $\alpha \in (0, 1)$ .

*Remark 1.6.* From the Taylor expansion of  $w$  near the origin, we can improve the asymptotic expansion (1.2) to higher order.

Our idea of proof is inspired by [Sim87] and [HY21]. We first prove the asymptotic expansion (1.2) under the additional linear growth condition, see Theorem 2.1. In this case, we only use tools from the theory of elliptic equations. We estimate the derivatives of  $u$  at infinity, then use the Bernstein-type result in half spaces to prove that any blow-down limit of  $u$  must be linear. Next, using the maximum principle and the Kelvin-type transforms, we prove the asymptotic expansion (1.2).

Without the linear growth condition, we now consider the blow-down of graph  $u$  in the sense of currents. Combining the compactness and Bernstein-type results for mass-minimizing currents, we can also prove that any blow-down limit of graph  $u$  must be a half plane. Then by the uniqueness theorem of tangent cones, see Allard-Almgren [AA81] or Simon [Sim83a, Sim85, AS88], the limit half plane is unique, this half plane is so-called the tangent plane of graph  $u$  at infinity. Moreover, graph  $u$  can be expressed by the graph of some new function with sublinear growth over the tangent plane at infinity. By virtue of the asymptotic expansion for this new function, the asymptotic expansion (1.2) of the original solution  $u$  follows.

This paper is organized as follows. In Section 2, we derive (1.2) for solutions with linear growth, and then prove Theorem 1.1 in Section 3.

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## 2. ASYMPTOTIC BEHAVIOR FOR SOLUTIONS WITH LINEAR GROWTH

Throughout this paper, we adopt the following notations.

- (i)  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ .

- (ii)  $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$  and  $B_r^+(x) = B_r(x) \cap \mathbb{R}_+^n$ .  $B_r = B_r(0)$  and  $B_r^+ = B_r^+(0)$ .
- (iii)  $\Sigma_r(x) = B_r(x) \cap \partial\mathbb{R}_+^n$  and  $\Sigma_r = B_r \cap \partial\mathbb{R}_+^n$ .
- (iv)  $\mathcal{C}_r = \{(x', x_n) : |x'| < r, 0 < x_n < r\}$  be a cylinder, and  $\Gamma_r = \partial\mathcal{C}_r \setminus \{x_n = 0\}$ .
- (v) For  $i, j \in 1, 2, \dots, n$ ,  $\delta_{ij}$  is the Kronecker symbol,  $\delta_{ij} = 1$  for  $i = j$ , and  $\delta_{ij} = 0$  for  $i \neq j$ .

Without loss of generality, up to subtracting some constant from  $u$ , we may assume that  $a = 0$  and  $l(x') = b' \cdot x'$ .

The main result of this section is stated as follows.

**Theorem 2.1.** *Let  $u \in C^2(\overline{\mathbb{R}_+^n} \setminus B_1^+)$  be a solution of (1.1). Assume that  $u$  has at most linear growth, that is*

$$(2.1) \quad |u(x)| \leq K|x| \quad \text{in } \overline{\mathbb{R}_+^n} \setminus B_1^+,$$

for some constant  $K > 0$ . Then the asymptotic expansion (1.2) and (1.3) holds.

Throughout this section,  $C$  denotes a positive constant depending only on  $n$  and  $K$ , whose meaning may be different from line to line. We divide the proof of Theorem 2.1 into three subsections.

**2.1. Global  $C^1$  estimates.** In this subsection, we derive a global gradient bound in a smaller exterior domain. Our approach is modified from [JWZ21]. First, we use the Bernstein technique to reduce the global estimate to the boundary estimate.

**Lemma 2.2.** *Let  $u \in C^2(\overline{\mathbb{R}_+^n} \setminus B_1^+)$  be the solution of the minimal surface equation  $\mathcal{M}u = 0$  in  $\mathbb{R}_+^n \setminus \overline{B}_1^+$ . Assume that  $u$  satisfies the growth condition (2.1). Then, for any  $\rho > 1$ , we have*

$$(2.2) \quad \|\nabla u\|_{L^\infty(\overline{\mathbb{R}_+^n} \setminus B_{4\rho}^+)} \leq C \left( 1 + \|\nabla u\|_{L^\infty(\partial\mathbb{R}_+^n \setminus \partial B_\rho^+)} \right),$$

where the universal constant  $C > 0$  is independent on  $\rho$ .

*Proof.* For any  $x_0 \in \overline{\mathbb{R}_+^n} \setminus B_{4\rho}^+$ , set  $r = \frac{|x_0| - 1}{2}$ . It is clear that  $B_r^+(x_0) \subset \mathbb{R}_+^n \setminus \overline{B}_\rho^+$ . Depending on whether  $x_0$  is closed to  $\partial\mathbb{R}_+^n$ , we split the proof into two cases.

*Case I: Boundary case.* If  $\Sigma_r(x_0) := B_r(x_0) \cap \partial\mathbb{R}_+^n \neq \emptyset$ , then  $\Sigma_r(x_0) \subset \partial\mathbb{R}_+^n \setminus \partial B_\rho^+$ . Set

$$v(x) = u(x) - \inf_{B_r^+(x_0)} u, \quad M := \sup_{B_r^+(x_0)} v$$

$$\gamma(t) = 1 + \frac{t}{M}, \quad \eta(x) = \left( 1 - \frac{|x - x_0|^2}{r^2} \right)^2.$$

Consider the function

$$w(x) := \gamma(v(x))\eta(x) \log |\nabla v(x)|^2, \quad x \in \Omega_r,$$

where  $\Omega_r = \{x \in \overline{B}_r^+(x_0) : |\nabla u(x)| \geq 100\}$ . Clearly,  $w$  is nonnegative in  $\Omega_r$  and positive in the interior of  $\Omega_r$ . Assume that  $w$  attains its positive maximum at  $y_0 \in \Omega_r$ . Since  $\eta = 0$  on  $\partial B_r(x_0)$ , then  $y_0$  cannot belong to  $\partial\Omega_r \cap \partial B_r(x_0)$ .

If  $y_0 \in \partial\Omega_r$ , then either  $|\nabla u(y_0)| = |\nabla v(y_0)| = 100$  or  $y_0 \in \Sigma_r(x_0)$ . For the first case, we have

$$\log |\nabla u(x_0)|^2 \leq w(x_0) \leq w(y_0) \leq 2 \log 100,$$

which implies  $|\nabla u(x_0)| \leq C$ . For the latter case, we have

$$\log |\nabla u(x_0)|^2 \leq w(x_0) \leq w(y_0) \leq 2 \log \|\nabla u\|_{L^\infty(\Sigma_r(x_0))}^2,$$

we deduce that  $|\nabla u(x_0)| \leq C \|\nabla u\|_{L^\infty(\partial\mathbb{R}_+^n \setminus \partial B_\rho^+)}$ .

If  $y_0 \in \text{Int}(\Omega_r)$ , from the calculation in [Wan98] or [Han16, Theorem 3.3.1], we get

$$|\nabla u(x_0)| \leq \exp \left\{ C \left( 1 + \frac{M^2}{r^2} \right) \right\}.$$

Recalling that  $M = \sup_{B_r^+(x_0)} u - \inf_{B_r^+(x_0)} u$  and  $|x_0| = 2r + 1$ . From the growth condition (2.1), we have  $M \leq 2K(3r + 1)$ , therefore,  $|\nabla u(x_0)| \leq C$ .

*Case II: Interior case.* If  $B_r(x_0) \subset \mathbb{R}_+^n \setminus \overline{B}_\rho^+$ . Similar as above, using the interior gradient estimate of minimal surface equation, see [Wan98] or [Han16, Theorem 3.3.1], we can also get

$$|\nabla u(x_0)| \leq \exp \left\{ C \left( 1 + \frac{\omega^2}{r^2} \right) \right\},$$

where  $\omega = \text{osc}_{B_r^+(x_0)} u \leq 2K(3r + 1)$ , hence  $|\nabla u(x_0)| \leq C$ .

In summary, combining all cases, since  $x_0 \in \overline{\mathbb{R}}_+^n \setminus B_{4\rho}^+$  is arbitrary, we obtain (2.2).  $\square$

In the next lemma, we construct upper and lower barriers to bound the gradient of  $u$  on the boundary.

**Lemma 2.3.** *Let  $u \in C^2(\overline{\mathbb{R}}_+^n \setminus B_1^+)$  be a solution of (1.1) with linear growth (2.1). Then,*

$$(2.3) \quad \|\nabla u\|_{L^\infty(\partial\mathbb{R}_+^n \setminus \partial B_{10}^+)} \leq C,$$

where  $C > 0$  is a universal constant.

*Proof.* By a rescaling argument, it suffices to prove that

$$(2.4) \quad \|\nabla u\|_{L^\infty(\partial\mathbb{R}_+^n \cap \{10 \leq |x| \leq 20\})} \leq C$$

In fact, if (2.4) holds, for any  $\lambda < 1$ , define  $u_\lambda(x) = \lambda u(\lambda^{-1}x)$ , then  $u_\lambda$  also satisfies (1.1) and (2.1). Applying (2.4), we get

$$\|\nabla u\|_{L^\infty(\partial\mathbb{R}_+^n \cap \{10\lambda^{-1} \leq |x| \leq 20\lambda^{-1}\})} = \|\nabla u_\lambda\|_{L^\infty(\partial\mathbb{R}_+^n \cap \{10 \leq |x| \leq 20\})} \leq C$$

Notice that  $C$  is independent on  $\lambda$ , and  $\lambda < 1$  is arbitrary, thus (2.3) holds.

Since  $u$  equals a linear function  $l$  on the flat boundary, then we only need to bound the normal derivative  $\partial_n u$ . Denote  $\Sigma = \partial\mathbb{R}_+^n \cap \{10 \leq |x| \leq 20\}$ . For any  $x_0 \in \Sigma$ , same as [JWZ21], We construct a  $C^3$  convex domain  $\Omega$  around  $x_0$ . Precisely, let

$$\psi(t) = \frac{64}{35}\sqrt{t} - 2t^2 + \frac{8}{5}t^3 - \frac{3}{7}t^4, \quad \tilde{\psi}(t) = \begin{cases} \psi(t), & t \in [0, 1], \\ \psi(2-t), & t \in [1, 2], \end{cases}$$

and let  $\Omega = \{(x', x_n) : |x' - x_0| \leq 2 + \tilde{\psi}(x_n), 0 < x_n < 2\}$ . Such  $\Omega$  is a bounded  $C^3$  convex domain around  $x_0$ .

Let  $\eta \in C^\infty([0, \infty))$  satisfy  $0 \leq \eta \leq 1$ ,  $\eta \equiv 0$  in  $[0, 1]$  and  $\eta \equiv 1$  in  $[2, \infty)$ . Define

$$\phi(x) = 100K\eta(|x - x_0|) + l(x)(1 - \eta(|x - x_0|)).$$

It is easy to check that  $\phi \geq u$  on  $\partial\Omega$ . Since  $\Omega$  is a  $C^3$  convex domain and  $\phi \in C^3$ , refer to [Han16, Theorem 2.6.1] or [GT83, Theorem 16.10], the Dirichlet problem

$$\begin{cases} \mathcal{M}v = 0 & \text{in } \Omega, \\ v = \phi & \text{on } \partial\Omega \end{cases}$$

has a unique  $C^2(\overline{\Omega})$  solution  $v$ . By the comparison principle,  $u \leq v$  in  $\overline{\Omega}$ . Note that,  $u = l = v$  on  $\Sigma \cap \partial\Omega$ , therefore

$$\partial_n u(x) \leq \partial_n v(x) \leq \|\nabla v\|_{L^\infty(\overline{\Omega})} \leq C, \quad \forall x \in \Sigma \cap \partial\Omega,$$

where  $C > 0$  is universal. Similarly, we can construct a lower barrier to get the lower bound. Finally, we cover  $\Sigma$  by finitely many such domains  $\Sigma \cap \partial\Omega$ , hence we get  $\|\partial_n u\|_{L^\infty(\Sigma)} \leq C$ . □

Combining Lemma 2.2 and Lemma 2.3, we know that

$$\|\nabla u\|_{L^\infty(\mathbb{R}_+^n \setminus B_{40}^+)} \leq C.$$

Therefore, the minimal surface equation becomes uniformly elliptic in an exterior domain.

**2.2. Blow-down argument.** It is well known that the blow-down limits reflect the behavior of  $u$  at infinity. In this subsection, we first prove a decay estimate of  $D^2u$ , then we use this estimate and the Bernstein-type result in half space to show that the blow-down limits of  $u$  must be linear.

**Lemma 2.4.** *Let  $u \in C^2(\mathbb{R}_+^n \setminus B_1^+)$  be a solution of*

$$\begin{cases} \mathcal{M}u = 0 & \text{in } \mathbb{R}_+^n \setminus \overline{B}_1^+, \\ u(x', 0) = l(x') & \text{on } \partial\mathbb{R}_+^n \setminus \partial B_1^+, \\ |u(x)| \leq A|x|^\gamma & \text{in } \mathbb{R}_+^n \setminus B_1^+, \end{cases}$$

for some constant  $A > 0$  and  $\gamma \leq 1$ . Then for  $m = 1, 2$ , we have

$$(2.5) \quad |D^m u(x)| \leq C|x|^{\gamma-m}, \quad \text{in } \mathbb{R}_+^n \setminus B_{100}^+,$$

where  $C > 0$  depending only on  $n, A, \gamma$ .

*Proof.* By the result in subsection 2.1, we have  $\|\nabla u\|_{L^\infty(\mathbb{R}_+^n \setminus B_{40}^+)} \leq C = C(n, A)$ , thus the minimal surface equation is uniformly elliptic in  $\mathbb{R}_+^n \setminus B_{40}^+$ .

For any fixed  $R > 100$ , we will show that  $|D^m u(x)| \leq CR^{\gamma-m}$  on  $\partial B_R \cap \{x_n \geq 0\}$ .

*Case I: Near the boundary.* For any  $x_0 \in \partial B_R \cap \{x_n = 0\}$ . Set

$$\eta(y) = \frac{4}{R}u\left(x_0 + \frac{R}{4}y\right), \quad y \in \overline{B}_2^+,$$

then  $\eta \in C^2(\overline{B}_2^+)$  satisfy

$$\begin{cases} \mathcal{M}\eta = 0 & \text{in } B_2^+, \\ \eta(y', 0) = \tilde{l}(y') & \text{on } \partial B_2^+ \cap \{x_n = 0\}, \end{cases}$$

where  $\tilde{l}$  equals  $l$  plus some constant. By the boundary  $C^{1,\alpha}$  estimate for quasilinear equation, see [GT83, Chapter 13], there exist constants  $\alpha \in (0, 1), C > 0$  depend only on  $n, A$ , such that

$$\|\eta\|_{C^{1,\alpha}(\overline{B}_1^+)} \leq C$$

Let  $a^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$ , then  $\|a^{ij}(\nabla \eta)\|_{C^{0,\alpha}(\overline{B}_1^+)} \leq C$  and  $\lambda I \leq a_{ij} \leq \Lambda I$  in  $\overline{B}_1^+$  for some universal constants  $0 < \lambda \leq \Lambda < \infty$ . By Schauder estimate, for  $m = 1, 2$ , we have

$$4^{1-m} R^{m-1} \|D^m u\|_{L^\infty(B_{R/8}^+(x_0))} = \|D^m \eta\|_{L^\infty(B_{1/2}^+)} \leq C \|\eta\|_{L^\infty(B_2^+)} \leq C R^{\gamma-1}.$$

Hence,  $|D^m u(x)| \leq C R^{\gamma-m} \leq C |x|^{\gamma-m}$  in  $B_{R/8}^+(x_0)$ .

*Case II: Interior case.* This case is easier. For any  $x \in \partial B_R \cap \{x_n \geq R/8\}$ . Set

$$\eta(y) = \frac{32}{R} u \left( x + \frac{R}{32} y \right), \quad y \in B_2.$$

Applying the interior  $C^{1,\alpha}$  estimate, see [GT83, Chapter 13], and the Schauder estimate, we can also get

$$32^{1-m} R^{m-1} |D^m u(x)| = |D^m \eta(0)| \leq C \|\eta\|_{L^\infty(B_2)} \leq C R^{\gamma-1}.$$

Hence,  $|D^m u(x)| \leq C R^{\gamma-m} = C |x|^{\gamma-m}$  in  $\partial B_R \cap \{x_n \geq R/8\}$ .

Combining the above two cases, the proof is complete.  $\square$

*Remark 2.5.* By repeating the above argument to improve the regularity of  $\eta$ , we can get the corresponding decay estimates for the higher order derivatives  $D^m u$ ,  $m > 2$ .

Now let  $u \in C^2(\overline{\mathbb{R}}_+^n \setminus B_1^+)$  be a solution of (1.1) with linear growth (2.1). For any  $\lambda > 0$ , consider the rescaled function  $u_\lambda(x) = \lambda u(\lambda^{-1}x)$ ,  $x \in \overline{\mathbb{R}}_+^n \setminus B_\lambda^+$ , then  $u_\lambda$  satisfies

$$(2.6) \quad \begin{cases} \mathcal{M}u_\lambda = 0 & \text{in } \mathbb{R}_+^n \setminus \overline{B}_\lambda^+, \\ u_\lambda(x') = l(x') = b' \cdot x' & \text{on } \partial \mathbb{R}_+^n \setminus \partial B_\lambda^+, \\ |D^m u_\lambda(x)| \leq C |x|^{1-m} & \text{in } \overline{\mathbb{R}}_+^n \setminus B_{100\lambda}^+, \quad m = 0, 1, 2. \end{cases}$$

Using the Arzela-Ascoli theorem and a standard diagonal argument, for any  $\lambda_k \searrow 0$ , there exists a subsequence (still denoted by  $\{\lambda_k\}$ ), such that  $\{u_{\lambda_k}\}$  converges to a function  $u_\infty$  in  $C_{\text{loc}}^1(\overline{\mathbb{R}}_+^n \setminus \{0\})$ . We claim that  $u_\infty$  is a linear function. In fact, from the  $C_{\text{loc}}^1(\overline{\mathbb{R}}_+^n \setminus \{0\})$  convergence,  $u_\infty \in C^1(\mathbb{R}_+^n) \cap C(\overline{\mathbb{R}}_+^n \setminus \{0\})$  solves the minimal surface equation weakly in  $\mathbb{R}_+^n$ . By the interior regularity of the minimal surface equation,  $u_\infty \in C^2(\mathbb{R}_+^n)$ . Note that each  $u_{\lambda_k}$  satisfies the linear growth condition  $|u_{\lambda_k}(x)| \leq C|x|$  in  $\overline{\mathbb{R}}_+^n \setminus B_{100\lambda_k}^+$ ,

passing to limit we get  $|u_\infty(x)| \leq C|x|$  in  $\overline{\mathbb{R}}_+^n \setminus \{0\}$ . Therefore, 0 is a removable singularity by defining  $u_\infty(0) = 0$ . Hence  $u_\infty \in C^2(\mathbb{R}_+^n) \cap C(\overline{\mathbb{R}}_+^n)$  satisfies

$$\begin{cases} \mathcal{M}u_\infty = 0 & \text{in } \mathbb{R}_+^n, \\ u_\infty(x', 0) = l(x') = b' \cdot x' & \text{on } \partial\mathbb{R}_+^n. \end{cases}$$

From the Liouville theorem in half space, see [JWZ21, EW22],  $u_\infty$  is linear. Moreover, there exists  $b_n \in \mathbb{R}$ , such that  $u_\infty(x) = b \cdot x = b' \cdot x' + b_n x_n$ , where  $b = (b', b_n)$ . In the above argument, the blow-down limit obviously depends on the choice of  $\{\lambda_k\}$ . However, in this situation, the blow-down limit is unique.

Let  $v(x) = u(x) - b \cdot x$ , then  $v$  satisfies

$$(2.7) \quad \begin{cases} a^{ij}(\nabla u) \partial_{ij} v = 0 & \text{in } \mathbb{R}_+^n \setminus \overline{B}_{100}^+, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n \setminus \partial B_{100}^+, \\ |v(x)| \leq 2K|x| & \text{in } \overline{\mathbb{R}}_+^n \setminus B_{100}^+, \end{cases}$$

where  $a^{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$ . Since  $\nabla u$  is bounded, therefore  $a_{ij}$  is uniformly elliptic in  $\mathbb{R}_+^n \setminus \overline{B}_{100}^+$ , with some universal ellipticity constants  $0 < \lambda \leq \Lambda < \infty$ .

**Lemma 2.6.**  $\|v\|_{L^\infty(\overline{\mathbb{R}}_+^n \setminus B_{100}^+)} \leq C$  for some universal constant  $C > 0$ .

*Proof.* For any  $r > 0$ , let  $\mathcal{C}_r = \{(x', x_n) : |x'| < r, 0 < x_n < r\}$  be a cylinder, and let  $\Gamma_r := \partial\mathcal{C}_r \setminus \{x_n = 0\}$ .  $\Gamma_r$  is the boundary of  $\mathcal{C}_r$  except the bottom.

Since  $u_{\lambda_k} \rightarrow b \cdot x$  in  $C_{\text{loc}}^1(\overline{\mathbb{R}}_+^n \setminus \{0\})$ , then  $\|\nabla v\|_{L^\infty(\Gamma_{\lambda_k^{-1}})} = \|\nabla u_{\lambda_k} - b\|_{L^\infty(\Gamma_1)} \rightarrow 0$ . Therefore, for any  $\varepsilon > 0$ , there exists  $\lambda_{k_\varepsilon} < 1/100$ , such that

$$|\nabla v| < \varepsilon, \quad \text{on } \Gamma_{\lambda_{k_\varepsilon}^{-1}}.$$

On  $\partial B_{100}^+ \cap \{x_n > 0\}$ , by the linear growth of  $v$ , we have  $|v(x)| \leq 200K$ .

On  $\Gamma_{\lambda_{k_\varepsilon}^{-1}} \cap \{|x'| = \lambda_{k_\varepsilon}^{-1}\}$ , using  $|\nabla v| < \varepsilon$  and  $v|_{\{|x'| \geq 100, x_n = 0\}} = 0$ , we get  $|v(x)| \leq \varepsilon x_n$ .

Similarly, on  $\Gamma_{\lambda_{k_\varepsilon}^{-1}} \cap \{x_n = \lambda_{k_\varepsilon}^{-1}\}$ , we have  $|v(x)| \leq \max_{\Gamma_{\lambda_{k_\varepsilon}^{-1}} \cap \{|x'| = \lambda_{k_\varepsilon}^{-1}\}} |v| + \varepsilon \lambda_{k_\varepsilon}^{-1} \leq 2\varepsilon x_n$ .

By the maximum principle, we obtain,

$$|v(x)| \leq 200K + 2\varepsilon x_n \quad \text{in } \overline{\mathcal{C}}_{\lambda_{k_\varepsilon}^{-1}} \setminus B_{100}^+.$$

Sending  $\varepsilon$  to 0, we conclude  $\|v\|_{L^\infty(\overline{\mathbb{R}}_+^n \setminus B_{100}^+)} \leq C$ . □

*Remark 2.7.* By simply modifying the proof of Lemma 2.4, we deduce that

$$(2.8) \quad |\nabla u(x) - b| = |\nabla v(x)| \leq C|x|^{-1} \text{ and } |D^2 v(x)| \leq C|x|^{-2} \quad \text{in } \overline{\mathbb{R}}_+^n \setminus B_{100}^+.$$

In particular,  $\nabla u$  tends to  $b$  at infinity, which implies the blow-down limit of  $u$  is unique and the tangent plane of graph  $u$  at infinity has the normal vector  $\frac{(-b, 1)}{\sqrt{1+|b|^2}}$ .



**2.3. Asymptotic expansion.** Now we are ready to give the proof of Theorem 2.1. We first construct the upper barrier to prove the 0-order expansion, then improve it to higher order by a Kelvin-type transform.

**Lemma 2.8.** *Let  $v(x) = u(x) - b \cdot x$  as above, then there exists universal constants  $C > 0$ ,  $R_1 \geq 1000$ , such that*

$$|v(x)| = |u(x) - b \cdot x| \leq \frac{Cx_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} \quad \text{in } \overline{\mathbb{R}}_+^n \setminus B_{R_1}^+.$$

*Proof.* From (2.7), (2.8) and Lemma 2.6,  $v$  satisfies the following uniformly elliptic equation

$$\begin{cases} a^{ij}(\nabla u)\partial_{ij}v = 0 & \text{in } \mathbb{R}_+^n \setminus \overline{B}_{1000}^+, \\ v = 0 & \text{on } \partial\mathbb{R}_+^n \setminus \partial B_{1000}^+, \\ |v(x)| \leq C & \text{in } \overline{\mathbb{R}}_+^n \setminus B_{1000}^+, \\ |\nabla v(x)| \leq C|x|^{-1} & \text{in } \overline{\mathbb{R}}_+^n \setminus B_{1000}^+. \end{cases}$$

where  $a^{ij}(\nabla u) = \delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2}$ . Since  $|\nabla u(x) - b| = |\nabla v(x)| \leq C|x|^{-1}$ , then the coefficient matrix tends to a constant matrix at infinity with the same rate, that is,

$$(2.9) \quad |a^{ij}(\nabla u(x)) - a_\infty^{ij}| \leq C|x|^{-1} \quad \text{in } \overline{\mathbb{R}}_+^n \setminus B_{1000}^+,$$

where  $a_\infty^{ij} = \delta_{ij} - \frac{b_i b_j}{1 + |b|^2}$ . It is clear that  $\lambda I \leq (a_\infty^{ij}) \leq \Lambda I$  for some universal constants  $0 < \lambda \leq \Lambda < \infty$ . We now denote the uniformly elliptic operators  $a^{ij}(\nabla u)\partial_{ij}$  and  $a_\infty^{ij}\partial_{ij}$  by  $L$  and  $L_0$  respectively.

We shall construct an upper barrier to bounded  $v$  at infinity, our argument is modified from [GS56, JLL20]. Let

$$h(x) = \frac{x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n}.$$

From a direct computation, one can check that  $L_0 h = 0$  in  $\mathbb{R}_+^n \setminus \{0\}$ , and  $h(x) \sim \frac{x_n}{|x|^n}$ ,  $|\nabla h(x)| \sim |x|^{-n}$ ,  $|D^2 h(x)| \sim |x|^{-n-1}$  in  $\overline{\mathbb{R}}_+^n \setminus B_{1000}^+$ . Here the notation  $f \sim g$  means  $cg \leq f \leq Cg$  for some universal constant  $c, C > 0$ .

Consider  $\bar{v} = h - h^\alpha$  for some  $\alpha > 0$  to be fixed later, then

$$\begin{aligned} \partial_i \bar{v} &= \partial_i h - \alpha h^{\alpha-1} \partial_i h, \\ \partial_{ij} \bar{v} &= (1 - \alpha h^{\alpha-1}) \partial_{ij} h - \alpha(\alpha - 1) h^{\alpha-2} \partial_i h \partial_j h. \end{aligned}$$

Thus  $L_0 \bar{v} = (1 - \alpha h^{\alpha-1}) L_0 h - \alpha(\alpha - 1) h^{\alpha-2} a_\infty^{ij} \partial_i h \partial_j h$ . We choose  $\alpha \in (1, 2)$ , then

$$\begin{aligned} L_0 \bar{v} &= -\alpha(\alpha - 1) h^{\alpha-2} a_\infty^{ij} \partial_i h \partial_j h \leq -\alpha(\alpha - 1) \lambda h^{\alpha-2} |\nabla h|^2 \\ &\leq -C_1 \alpha(\alpha - 1) \left( \frac{x_n}{|x|^n} \right)^{\alpha-2} \frac{1}{|x|^{2n}} \end{aligned}$$

Moreover,

$$\begin{aligned}
|D^2\bar{v}| &\leq C_2 (|D^2h| + h^{\alpha-1}|D^2h| + h^{\alpha-2}|Dh|^2) \\
&\leq C_2 \left( \frac{1}{|x|^{n+1}} + \left( \frac{x_n}{|x|^n} \right)^{\alpha-1} \frac{1}{|x|^{n+1}} + \left( \frac{x_n}{|x|^n} \right)^{\alpha-2} \frac{1}{|x|^{2n}} \right) \\
&\leq C_2 \left( \frac{1}{|x|^{n+1}} + \left( \frac{x_n}{|x|^n} \right)^{\alpha-2} \frac{1}{|x|^{2n}} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
L\bar{v} &= (a^{ij}(\nabla u) - a_{\infty}^{ij})\partial_{ij}\bar{v} + L_0\bar{v} \\
&\leq C_2 \frac{1}{|x|} \left( \frac{1}{|x|^{n+1}} + \left( \frac{x_n}{|x|^n} \right)^{\alpha-2} \frac{1}{|x|^{2n}} \right) - C_1\alpha(\alpha-1) \left( \frac{x_n}{|x|^n} \right)^{\alpha-2} \frac{1}{|x|^{2n}} \\
&= \frac{C_2}{|x|^{n+2}} + \left( \frac{C_2}{|x|} - C_1\alpha(\alpha-1) \right) \frac{x_n^{\alpha-2}}{|x|^{n\alpha}}.
\end{aligned}$$

If  $|x| \geq \frac{2C_2}{C_1\alpha(\alpha-1)}$ , then

$$\begin{aligned}
L\bar{v} &\leq \tilde{C} \left( \frac{1}{|x|^{n+2}} - \frac{\alpha(\alpha-1)}{2} \frac{x_n^{\alpha-2}}{|x|^{n\alpha}} \right) \leq \frac{\tilde{C}}{|x|^{n+2}} \left( 1 - \frac{\alpha(\alpha-1)}{2} \frac{x_n^{\alpha-2}}{|x|^{n\alpha-n-2}} \right) \\
&\leq \frac{\tilde{C}}{|x|^{n+2}} \left( 1 - \frac{\alpha(\alpha-1)}{2} \frac{1}{|x|^{n\alpha-n-\alpha}} \right).
\end{aligned}$$

Therefore, if  $(n-1)\alpha-n < 0$ , i.e. we fix some  $\alpha \in \left(1, \frac{n}{n-1}\right)$ , then there exists a universal constant  $R_0 > 1000$ , such that  $L\bar{v} \leq 0$  in  $\overline{\mathbb{R}}_+^n \setminus B_{R_0}^+$ .

For any  $r > 0$ , same as Lemma 2.6, let  $\mathcal{C}_r, \Gamma_r$  be the cylinder and its boundary without bottom respectively. Define  $\mathcal{E}_r$  be the ellipsoid  $\{|x|^2 + (b \cdot x)^2 < r^2\}$ , and let  $\mathcal{E}_r^+ = \mathcal{E}_r \cap \mathbb{R}_+^n$ . We choose  $R_1 > 1000$  satisfying  $B_{R_0}^+ \subset \mathcal{E}_{R_1}^+$ . It is easy to see that  $\mathcal{E}_{R_1}^+ \subset B_{R_1}^+$ .

On  $\partial\mathbb{R}_+^n \setminus \partial B_{R_0}^+$ , we have  $v = \bar{v} = 0$ . Then on  $\partial\mathcal{E}_r^+ \cap \mathbb{R}_+^n$ , since  $\nabla v$  is bounded, then  $|v(x)| \leq Cx_n$ . Now

$$\bar{v}(x) = \frac{x_n}{R_1^n} - \frac{x_n^\alpha}{R_1^{n\alpha}} \geq \tilde{C}^{-1}x_n \geq \tilde{C}^{-1}|v(x)| \quad \text{on } \partial\mathcal{E}_r^+ \cap \mathbb{R}_+^n.$$

From the proof in Lemma 2.6, for any  $\varepsilon > 0$ , there exists a  $R_\varepsilon > 1000$ , such that  $|\nabla v| \leq \varepsilon$  on  $\Gamma_{R_\varepsilon}$ , thus  $|v(x)| \leq 2\varepsilon x_n$  on  $\Gamma_{R_\varepsilon}$ . Consequently,

$$\begin{cases} L(\tilde{C}\bar{v} + 2\varepsilon x_n) \leq Lv = 0 \leq L(-\tilde{C}\bar{v} - 2\varepsilon x_n) & \text{in } \mathcal{C}_{R_\varepsilon} \setminus \mathcal{E}_{R_1}^+, \\ -\tilde{C}\bar{v} - 2\varepsilon x_n \leq v \leq \tilde{C}\bar{v} + 2\varepsilon x_n & \text{on } \partial(\mathcal{C}_{R_\varepsilon} \setminus \mathcal{E}_{R_1}^+). \end{cases}$$

By the maximum principle, we get

$$|v(x)| \leq \tilde{C}\bar{v} + 2\varepsilon x_n \quad \text{in } \overline{\mathcal{C}}_{R_\varepsilon} \setminus \mathcal{E}_{R_1}^+.$$

Sending  $\varepsilon \rightarrow 0$ , we finally prove that

$$|v(x)| \leq \tilde{C}\bar{v} \leq \tilde{C} \frac{x_n}{\left(\sqrt{|x|^2 + (b \cdot x)^2}\right)^n} \quad \text{in } \mathbb{R}_+^n \setminus B_{R_1}^+,$$

here we used the fact  $\mathcal{E}_{R_1}^+ \subset B_{R_1}^+$ , the proof is complete.  $\square$

*Remark 2.9.* Same as Remark 2.7, applying the modified Lemma 2.4 with  $\gamma = 1 - n$ , we can improve (2.8) to

$$(2.10) \quad |\nabla u(x) - b| = |\nabla v(x)| \leq C|x|^{-n} \text{ and } |D^2 v(x)| \leq C|x|^{-n-1} \quad \text{in } \mathbb{R}_+^n \setminus B_{R_2}^+,$$

for some  $R_2 > R_1$ . Moreover, the asymptotic behavior of higher order derivatives follows from Remark 2.5.

*Proof of Theorem 2.1. Step 1. Kelvin type transform.* We first introduce some notations. Set

$$\begin{aligned} A(x) &= (a^{ij}(\nabla u)) = I - \frac{\nabla u(x) \otimes \nabla u(x)}{1 + |\nabla u|^2} = \left( \delta_{ij} - \frac{u_i(x)u_j(x)}{1 + |\nabla u(x)|^2} \right), \\ A &= (a_{\infty}^{ij}) = I - \frac{b \otimes b}{1 + |b|^2} = \left( \delta_{ij} - \frac{b_i b_j}{1 + |b|^2} \right). \end{aligned}$$

One can easily check that  $A^{-1} = I + b \otimes b = (\delta_{ij} + b_i b_j)$ . From (2.10), we have  $|A(x) - A| \leq C|x|^{-n}$  in  $\mathbb{R}_+^n \setminus B_{R_2}^+$ . Let  $H = \sqrt{A^{-1}}\mathbb{R}_+^n$  be a new half space, and  $\partial H = \sqrt{A^{-1}}\partial\mathbb{R}_+^n$  be its boundary.

Introduce a new variable  $z = \frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2}$ , and let  $w$  be a function in  $\overline{H} \cap B_{r_0}$  for some  $r_0 > 0$  such that

$$(2.11) \quad u(x) = b \cdot x + |\sqrt{A^{-1}}x|^{2-n}w \left( \frac{\sqrt{A^{-1}}x}{|\sqrt{A^{-1}}x|^2} \right) \quad \text{for } x \in \mathbb{R}_+^n \setminus B_{R_2}^+.$$

Clearly,  $w$  is smooth except the origin. From a direct computation, we have

$$|z|^{n+2}\Delta w(z) = \text{tr}(AD^2u(x))$$

Since  $\text{tr}(A(x)D^2u(x)) = 0$ , then  $f(z) := \Delta w(z)$  satisfies

$$|f(z)| \leq C|z|^{-n-2}|A(x) - A||D^2u(x)| \leq C|z|^{n-1} \quad \text{in } \overline{H} \cap \overline{B}_{r_1} \setminus \{0\},$$

for some  $r_1 > 0$ . Since  $n \geq 2$ , then  $f \in C^{0,1}(\overline{H} \cap \overline{B}_{r_1})$ . By the boundary Schauder estimates, we deduce that  $w \in C^{2,\alpha}(\overline{H} \cap \overline{B}_{r_1/2})$  for any  $\alpha \in (0, 1)$ . Consequently

$$(2.12) \quad \left| w(z) - w(0) - \nabla w(0) \cdot z - \frac{1}{2}z^T D^2 w(0)z \right| \leq C|z|^{2+\alpha} \quad \text{in } \overline{H} \cap \overline{B}_{\frac{r_1}{2}}.$$

Moreover, since  $u(x) = b \cdot x$  on  $\partial\mathbb{R}_+^n \setminus \partial B_1^+$ , then  $w = 0$  on  $\partial H \cap B_{r_0}$ , which implies that all tangential derivatives of  $w$  along  $\partial H$  at 0 must vanish. Denote  $\varepsilon_i = \sqrt{A^{-1}}e_i$ ,  $i = 1, 2, \dots, n$ , here  $\varepsilon_1, \dots, \varepsilon_{n-1}$  are tangential vectors of  $\partial H$ . Hence

$$w(0) = \nabla_{\varepsilon_i} w(0) = D_{\varepsilon_i \varepsilon_j}^2 w(0) = 0, \quad i, j = 1, 2, \dots, n-1.$$

Note that  $|\sqrt{A^{-1}}x|^2 = x^T A^{-1}x = |x|^2 + (b \cdot x)^2$ , then  $z = \sum_{i=1}^n \frac{x_i \varepsilon_i}{|x|^2 + (b \cdot x)^2}$ . Under the  $x$  coordinate, (2.12) becomes

$$\left| \left( \sqrt{|x|^2 + (b \cdot x)^2} \right)^{n-2} (u(x) - b \cdot x) - \frac{\nabla_{\varepsilon_n} w(0) x_n}{|x|^2 + (b \cdot x)^2} + \sum_{j=1}^n \frac{D_{\varepsilon_j \varepsilon_n}^2 w(0) x_j x_n}{(|x|^2 + (b \cdot x)^2)^2} \right| \leq C \left( \sqrt{|x|^2 + (b \cdot x)^2} \right)^{-2-\alpha}.$$

Let  $c_n = \nabla_{\varepsilon_n} w(0)$  and  $d_j = D_{\varepsilon_j \varepsilon_n}^2 w(0)$ ,  $j = 1, 2, \dots, n$ , then we get

$$\left| u(x) - b \cdot x - \frac{c_n x_n}{\left( \sqrt{|x|^2 + (b \cdot x)^2} \right)^n} + \sum_{j=1}^n \frac{d_j x_j x_n}{\left( \sqrt{|x|^2 + (b \cdot x)^2} \right)^{n+2}} \right| \leq C |x|^{-n-\alpha}$$

in  $\mathbb{R}_+^n \setminus B_{R_3}^+$  for some  $R_3 > R_2$  and any  $\alpha \in (0, 1)$ .

*Step 2. Higher regularity of  $w$ .* Using the relation (2.11) and the minimal surface equation for  $u$ , we compute the explicit equation that  $w$  satisfies.

Differentiating the relation (2.11), we have

$$(2.13) \quad u_i = b_i + |z|^{n-2} P_i, \quad u_{ij} = |z|^{n-2} (Q_{ik} Q_{jl} w_{kl} + R_{ij}),$$

where

$$\begin{aligned} P_i &= |z|^2 \langle \varepsilon_i, \nabla w \rangle - [(n-2)w + 2\langle z, \nabla w \rangle] \langle \varepsilon_i, z \rangle, & Q_{ik} &= \varepsilon_{ik} |z|^2 - 2\langle \varepsilon_i, z \rangle z_k, \\ R_{ij} &= -[(n-2)w + w\langle z, \nabla w \rangle] |z|^2 \langle \varepsilon_i, \varepsilon_j \rangle + n[(n-2)w + 4\langle z, \nabla w \rangle] \langle \varepsilon_i, z \rangle \langle \varepsilon_j, z \rangle \\ &\quad - n|z|^2 [\langle \varepsilon_i, z \rangle \langle \varepsilon_j, \nabla w \rangle + \langle \varepsilon_j, z \rangle \langle \varepsilon_i, \nabla w \rangle]. \end{aligned}$$

Now the minimal surface equation  $(1 + |\nabla u|^2) \Delta u - u_i u_j u_{ij} = 0$  becomes

$$\begin{aligned} 0 &= \left[ \left( 1 + |b + |z|^{n-2} P|^2 \right) \delta_{ij} - (b_i + |z|^{n-2} P_i) (b_j + |z|^{n-2} P_j) \right] u_{ij} \\ &= [(1 + |b|^2) \delta_{ij} - b_i b_j] u_{ij} \\ &\quad + |z|^{2n-4} \{ 2\langle b, P \rangle \delta_{ij} - b_i P_j - b_j P_i \} \{ Q_{ik} Q_{jl} w_{kl} + R_{ij} \} \\ &\quad + |z|^{3n-6} \{ |P|^2 \delta_{ij} - P_i P_j \} \{ Q_{ik} Q_{jl} w_{kl} + R_{ij} \}. \end{aligned}$$

Note that the first term  $= (1 + |b|^2) \text{tr}(AD^2 u) = (1 + |b|^2) |z|^{n+2} \Delta w$ . After a lengthy computation and rearrangement, we finally get

$$(2.14) \quad \left\{ \begin{aligned} &(1 + |b|^2) \delta_{kl} \\ &-4(n-2) \sqrt{1 + |b|^2} |z|^{n-4} \langle b, z \rangle z_k z_l w \\ &+ |z|^{n-2} (\text{Polynomial of } z, w, \nabla w) \\ &+ |z|^{2n-4} (\text{Polynomial of } z, w, \nabla w) \end{aligned} \right\} w_{kl} = \left\{ \begin{aligned} &|z|^{n-4} (\text{Polynomial of } z, w, \nabla w) \\ &+ |z|^{n-2} (\text{Polynomial of } z, w, \nabla w) \\ &+ |z|^{2n-6} (n-2) [(n-2)z + 2\langle z, \nabla w \rangle]^3 (b \cdot z)^2 \\ &+ |z|^{2n-4} (\text{Polynomial of } z, w, \nabla w) \end{aligned} \right\}$$

Here the terms  $|z|^{n-4}$  and  $|z|^{n-2}$  are not smooth for odd  $n$  or small  $n$ .

For  $n \geq 4$ , if  $n$  is even, then the coefficient and the right hand side of (2.14) are smoothly dependent on  $w$ . Recall that  $w \in C^{2,\alpha}(\overline{H} \cap \overline{B}_{r_1/2})$  for any  $\alpha \in (0, 1)$ . By the boundary Schauder estimate and the bootstrap argument,  $u$  is smooth near the origin. If  $n$  is odd, since the term  $|z|^{n-4}$  is  $C^{n-5,1}$ , then we can only improve the regularity of  $w$  to  $C^{n-3,\alpha}(\overline{H} \cap \overline{B}_{r_2})$  for some  $r_2 > 0$ .  $\square$

### 3. PROOF OF THEOREM 1.1 AND THEOREM 1.5

Without the linear growth condition (2.1), we lack the decay estimates, namely Lemma 2.4, hence we cannot ensure the convergence of the blow-down sequence  $\{u_\lambda\}$ . Geometrically, this problem is not caused by a geometric defect in the surface we are studying, but rather by a defect in the function  $u$  that represents this surface. Therefore, we now focus on the surface itself, rather than its representation function  $u$ .

We first introduce some preliminaries about geometric measure theory, one can refer to [Sim83b] for details. Denote the  $k$ -dimensional Hausdorff measure by  $\mathcal{H}^k$ . Given an open subset  $U$  of  $\mathbb{R}^{n+1}$ ,  $\mathcal{D}^k(U)$  denotes the space of all smooth  $k$ -forms with compact support in  $U$ .

A  $k$ -current in  $U$  is a continuous linear functional on  $\mathcal{D}^k(U)$ , the set of all  $k$ -currents in  $U$  is denoted by  $\mathcal{D}_k(U)$ . The boundary of a  $k$ -current  $T$  is defined to be a  $k-1$  current  $\partial T \in \mathcal{D}_{k-1}(U)$  satisfying

$$\partial T(\omega) = T(d\omega), \quad \forall \omega \in \mathcal{D}^{k-1}(U).$$

A  $k$ -current  $T$  is said to be *integer multiplicity rectifiable* in  $U$ , if there is a countably  $k$ -rectifiable subset  $M_T$  of  $U$ , an  $\mathcal{H}^k \llcorner M_T$ -measurable simple unit  $k$ -vector  $\xi_T$  orienting  $T_x M_T$  for  $\mathcal{H}^k$ -a.e.  $x$ , and an  $\mathcal{H}^k \llcorner M_T$ -measurable positive integer valued function  $\theta_T$ , such that

$$T(\omega) = \int_{M_T} \langle \omega, \xi_T \rangle \theta_T d\mathcal{H}^k, \quad \forall \omega \in \mathcal{D}^k(U).$$

Here the integer valued function  $\theta_T$  is called the multiplicity function of  $T$ . In this case, the corresponding mass measure of  $T$  is  $\mu_T = \theta_T d\mathcal{H}^k \llcorner M_T$ , i.e. for any open  $W \subset U$ ,

$$\mathbb{M}_W(T) = \mu_T(W) = \int_{M_T \cap W} \theta_T d\mathcal{H}^k.$$

An integer multiplicity rectifiable  $k$ -current  $T$  is said to be *integral* if both  $\mu_T$  and  $\mu_{\partial T}$  are Radon measures. An integral  $k$ -current  $T$  is said to be mass-minimizing in an open set  $U' \subset U$ , if for any  $W \subset \subset U'$  and any integral  $k$ -current  $S$  in  $U'$  with  $\partial S = 0$ ,  $\text{spt} S \subset W$ , we have

$$\mathbb{M}_W(T) \leq \mathbb{M}_W(T + S).$$

*Remark 3.1.* Let  $M$  be a  $k$ -dimensional oriented  $C^1$  submanifold of an open set  $U \subset \mathbb{R}^{n+1}$ , and let the continuous function  $\xi(x) = \pm \tau_1 \wedge \cdots \wedge \tau_k$  be the orientation of  $M$ , where  $\tau_1, \dots, \tau_k$  is an orthonormal basis for  $T_x M$ . Then there is a corresponding multiplicity 1 rectifiable  $k$ -current  $\llbracket M \rrbracket \in \mathcal{D}_k(U)$  defined by

$$\llbracket M \rrbracket(\omega) = \int_M \langle \omega, \xi \rangle d\mathcal{H}^k, \quad \forall \omega \in \mathcal{D}^k(U).$$

*Remark 3.2.* Let  $E$  be a subdomain of  $\mathbb{R}^{n+1}$ , then  $\llbracket E \rrbracket \in \mathcal{D}_{n+1}(U)$  is integral if and only if the characteristic function  $\chi_E \in BV_{\text{loc}}(U)$ , or equivalently,  $E$  has locally finite perimeter in  $U$ .

Now we prove Theorem 1.1. Our idea is to view graph  $u$  as a multiplicity 1 integral current, then use the compactness theorem to extract a convergent subsequence of the blow-down of graph  $u$ . By the rigidity of mass minimizing currents in half spaces, the limit must be a half hyperplane. Finally applying the uniqueness theorem of tangent cones and Theorem 2.1 to conclude the result.

*Proof of Theorem 1.1 and Theorem 1.5.* Let  $G = \text{graph } u = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}_+^n \setminus \overline{B}_1^+\}$ . Then  $G$  is a  $C^2$  hypersurface in  $\mathbb{R}^{n+1}$ . Equipping  $G$  with upward orientation, then  $G$  can be viewed as a multiplicity 1 integral  $n$ -current in  $\mathbb{R}^{n+1}$ . Since  $u$  solves the minimal surface equation in  $\mathbb{R}_+^n \setminus \overline{B}_1^+$ , then  $G$  is mass-minimizing in  $(\mathbb{R}_+^n \setminus \overline{B}_1^+) \times \mathbb{R}$ . Moreover,  $G$  has the following form

$$G = \partial \llbracket U \rrbracket \llcorner \left( (\mathbb{R}_+^n \setminus \overline{B}_1^+) \times \mathbb{R} \right),$$

where  $U = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}_+^n \setminus \overline{B}_1^+, x_{n+1} < u(x)\}$ .

By simply modifying the proof in [GT83, Chapter 16], we have the following upper volume bound: for any ball  $D_r(y) \subset \mathbb{R}^{n+1}$ , there holds  $\mathcal{H}^n(G \cap D_r(y)) \leq Cr^n$ , for some universal constant  $C > 0$ .

Similar as before, for any  $\lambda > 0$ , let  $u_\lambda(x) = \lambda u(\lambda^{-1}x)$ , and let  $G_\lambda = \text{graph } u_\lambda = \{(x, u_\lambda(x)) \in \mathbb{R}^{n+1} : x \in \mathbb{R}_+^n \setminus \overline{B}_\lambda^+\}$ . Then  $G_\lambda$  can be also viewed as a multiplicity 1 integral  $n$ -current in  $\mathbb{R}^{n+1}$  which is mass-minimizing in  $(\mathbb{R}_+^n \setminus \overline{B}_\lambda^+) \times \mathbb{R}$ . Moreover,  $G_\lambda$  also has the form

$$G_\lambda = \partial \llbracket U_\lambda \rrbracket \llcorner \left( (\mathbb{R}_+^n \setminus \overline{B}_\lambda^+) \times \mathbb{R} \right),$$

where  $U_\lambda = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}_+^n \setminus \overline{B}_\lambda^+, x_{n+1} < u_\lambda(x)\}$ . The upper volume bound also holds for  $G_\lambda$ .

Using the standard compactness theorem, see [Sim83b, Chapter 7], for any  $\lambda_k \searrow 0$ , there is a subsequence (still denoted by  $\{\lambda_k\}$ ), and a multiplicity 1 integral  $n$ -current  $T = \partial \llbracket V \rrbracket \llcorner (\mathbb{R}_+^n \times \mathbb{R})$  such that  $T = \lim_k G_{\lambda_k}$  in sense of weak convergence of currents, and  $\chi_{U_{\lambda_k}} \rightarrow \chi_V$  in  $L_{\text{loc}}^1(\mathbb{R}_+^n \times \mathbb{R})$ ,  $\mathcal{H}^n \llcorner G_{\lambda_k} \rightarrow \mathcal{H}^n \llcorner \text{spt } T$  in the sense of Radon measures. Moreover,  $T$  is mass-minimizing in  $\mathbb{R}_+^n \times \mathbb{R}$ .

Since  $u$  has linear boundary condition, i.e.  $u = l$  on  $\partial \mathbb{R}_+^n \setminus \partial B_1^+$ , by [EW22, Lemma 2.2], each  $G_{\lambda_k}$  has the following form

$$G_{\lambda_k} = (\partial \llbracket E_{\lambda_k} \rrbracket - \llbracket L_{\lambda_k} \rrbracket) \llcorner (\mathbb{R}^{n+1} \setminus (\overline{B}_{\lambda_k} \times \mathbb{R})),$$

where

$$E_{\lambda_k} = U_{\lambda_k} \cup \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus (\overline{B}_{\lambda_k} \times \mathbb{R}) : x_n < 0, x_{n+1} < l(x')\},$$

$$L_{\lambda_k} = \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus (\overline{B}_{\lambda_k} \times \mathbb{R}) : x_n < 0, x_{n+1} = l(x')\}$$

Passing to the limit, we deduce that  $T$  has the form

$$T = \partial \llbracket \tilde{E} \rrbracket - \llbracket \tilde{L} \rrbracket,$$

where

$$\begin{aligned}\tilde{E} \cap (\mathbb{R}_+^n \times \mathbb{R}) &= V \cap (\mathbb{R}_+^n \times \mathbb{R}), \\ \tilde{E} \setminus (\mathbb{R}_+^n \times \mathbb{R}) &= \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n < 0, x_{n+1} < l(x')\}, \\ \tilde{L} &= \{(x', x_n, x_{n+1}) \in \mathbb{R}^{n+1} : x_n < 0, x_{n+1} = l(x')\}\end{aligned}$$

Therefore, by the Bernstein type theorem for mass-minimizing integral  $n$ -currents in half spaces with linear boundary, see [EW22, Theorem 1.2], we deduce that  $T = \llbracket H \rrbracket$  for some half hyperplane  $H$  in  $\mathbb{R}^{n+1}$ . Applying the theorem of the uniqueness of tangent cones, see [AA81] or [Sim83a, Sim85, AS88], the half hyperplane  $H$  is unique, i.e. it is independent on the choice of  $\{\lambda_k\}$ . Moreover, choosing a new coordinate on  $H$ ,  $G$  can be expressed by the graph of a function  $h$  with sublinear growth over the exterior domain in  $H$ . To be specific, There exists  $R > 0$ , and a function  $h \in C^2(H \setminus D_R)$ , such that

$$G \setminus D_R = \{y + h(y)\nu : y \in H \setminus D_R\},$$

where  $\nu$  is the unit normal vector of  $H$ , and  $h$  has sublinear growth

$$h(y) = o(|y|) \text{ or equivalently, } \lim_{|y| \rightarrow \infty} \frac{|h(y)|}{|y|} = 0.$$

*Claim.*  $H$  can not be vertical, that is  $\nu \cdot e_{n+1} \neq 0$ .

Suppose to the contrary,  $H$  is a vertical half hyperplane, without loss of generality, we can assume  $\nu = e_j$  for some  $j \in \{1, 2, \dots, n\}$ . Since  $G$  is a minimal hypersurface in  $(\mathbb{R}_+^n \setminus \overline{B}_1^+) \times \mathbb{R}$ , then  $h$  also satisfies the minimal surface equation under the coordinate on  $H$ . Since  $h$  has sublinear growth, by Theorem 2.1, we deduce that  $h(y) = a + O(|y|^{1-n})$ . In particular,  $h$  is bounded, which means that the height of  $G$  in the  $e_j$  direction must be bounded. It contradicts the definition of  $G$ . Since  $G = \text{graph } u$ , and  $u$  is defined on  $\mathbb{R}_+^n \setminus B_1^+$ , then in the  $e_j$  direction,  $G$  must be extended to infinity. Therefore, the claim holds.

Since  $\nu \cdot e_{n+1} \neq 0$ , then we can assume that  $H = \{(x', x_n, x_{n+1}) : x_n > 0, x_{n+1} = b \cdot x\}$ , where  $b = (b', b_n)$  for some  $b_n \in \mathbb{R}$ . Under the coordinate on  $H$ ,  $h$  has the asymptotic expansion  $h(y) = a + O(|y|^{1-n})$ . In particular,  $h$  is bounded which implies  $u$  has at most linear growth. Applying Theorem 2.1, the proof is complete.  $\square$

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**Conflict of interest.** The authors declare they have no financial interests.

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