Singular Value Decomposition

An application to Big Data

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Outline

Definition of SVD

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$, it can always be found a decomposition such that

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and $\Sigma \in \mathbb{R}^{m \times n}$.
U and V are two orthogonal matrices and Σ is a diagonal matrix, namely:

$$(\Sigma)_{ij} = \begin{cases} 0, & i \neq j \\ \sigma_i, & i = j \end{cases}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$, $p = min\{m, n\}$.

Definition of SVD

The non-zero entries of Σ, denoted by σ_i, are called *singular values*. They are arrenged in a nonincreasing order by convention.
The column vectors $\boldsymbol{u_i}$ of U are called *left singular vectors* and those $\boldsymbol{v_i}$ of V are called *right singular vectors*.

Since in general $m \neq n$, we have:

$$A = \sum_{i=1}^{p} \boldsymbol{u_i} \sigma_i \boldsymbol{v_i}^T$$

Definition of SVD

Singular values computation

To compute the singular values, consider the transponse of A given its decomposition:

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$