Singular Value Decomposition

An application to Big Data

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Outline

1. Singular Value Decomposition
   - What is it?
   - How can singular values be computed?

2. References

Singular Value Decomposition

Definition of SVD

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$, it can always be found a decomposition such that

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ and $\Sigma \in \mathbb{R}^{m \times n}$.

U and V are two orthogonal matrices and Σ is a diagonal matrix, namely:

$$(\Sigma)_{ij} = \begin{cases} 0, & i \neq j \\ \sigma_i, & i = j \end{cases}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$, $p = \min\{m, n\}$.

Definition of SVD

The non-zero entries of Σ, denoted by σ_i, are called *singular values*.

They are arrenged in a nonincreasing order by convention.

The column vectors $\boldsymbol{u_i}$ of U are called *left singular vectors* and those $\boldsymbol{v_i}$ of V are called *right singular vectors*.

Since in general $m \neq n$, we have:

$$A = \sum_{i=1}^{p} \boldsymbol{u_i} \sigma_i \boldsymbol{v_i}^T$$

Definition of SVD

Theorem

If for some r such that $1 \leq r < p$ we have

$$\sigma_1 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_p = 0$$

then

- $rank(A) = r$
- $A = \sum_{i=1}^{r} \boldsymbol{u_i} \sigma_i \boldsymbol{v_i}^T$

This means that all other $p - r$ dimensions of matrix A are linear combinations of the first r.

Definition of SVD

Lower rank approximation

Let $A \in \mathbb{R}^{m \times n}$ be a matrix whose rank is $rank(A) = r$.
If for a fixed integer value $k < r$ we define

$$A_k = \sum_{i=1}^{k} \sigma_i \boldsymbol{u_i} \boldsymbol{v_i}^T \tag{1}$$

and

$$\mathcal{B} = \left\{ B \in \mathbb{R}^{m \times n} : rank(B) = k \right\}$$

then

$$\min_{B \in \mathcal{B}} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

This result tell us that A_k represents the best approximation (considering the *spectral norm*) of rank k of matrix A.

Singular values computation

To compute the singular values, consider the transponse of A given its decomposition:

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$

The symmetric matrix $A^T A$ is equal to:

$$A^T A = (V\Sigma^T U^T)(U\Sigma V^T) = V\Sigma^T \Sigma V^T$$

Furthermore, this equation can be written as:

$$A^T AV = V\Sigma^T \Sigma$$

This means that the diagonal entries of the square matrix $\Sigma^T \Sigma$, which are the square of the singular values, are the eigenvalues of matrix $A^T A$ and V is the matrix of eigenvectors.

Singular values computation

Similarly, consider the product of AA^T. It is equal to:

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$$

Which means that:

$$AA^T U = U\Sigma\Sigma^T$$

Hence U is the matrix of eigenvectors of AA^T.

Since $rank(A) = r$, only the first r eigenvalues of AA^T and $A^T A$ are non-zero.

References

References

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