Fundamental Theorem of Calculus

Note, the following proof is borrowed from "Calculus of a Single Variable - AP Edition, 10e" by Larson an Edwards.

Proof

The key to the proof is writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \tag{1}$$

> **Note**
>
> This says that $a = x_0, b = x^n$ and $x_1, x_2, \ldots$ are in between a and b

By pairwise subtraction and addition of like terms you can write:

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned} \tag{2}$$

> **Note**
>
> This says that we are rewriting the original statement. $F(b)$ becomes $F(x_n)$, $F(a)$ becomes $F(x_0)$, and inner terms are added pairwise as $F(x_{n-1}) - F(x_{n-1})$ (which is 0) so we haven't changed the original statement.

By the Mean Value Theorem, you know there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \tag{3}$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and rewrite Equation 2 to obtain:

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] \\ &= \sum_{i=1}^{n}\left[\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \cdot (x_i - x_{i-1})\right] \\ &= \sum_{i=1}^{n} f(c_i)\Delta x_i \end{aligned} \tag{4}$$

Note

This says that we are rewriting Equation 2 using the MVT Equation 3 to get to the familiar representation of a Riemann sum.

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. So by the definition of a definite integral from Riemann sums:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx \tag{5}$$

Thus:

$$\boxed{F(b) - F(a) = \int_a^b f(x)dx} \tag{6}$$