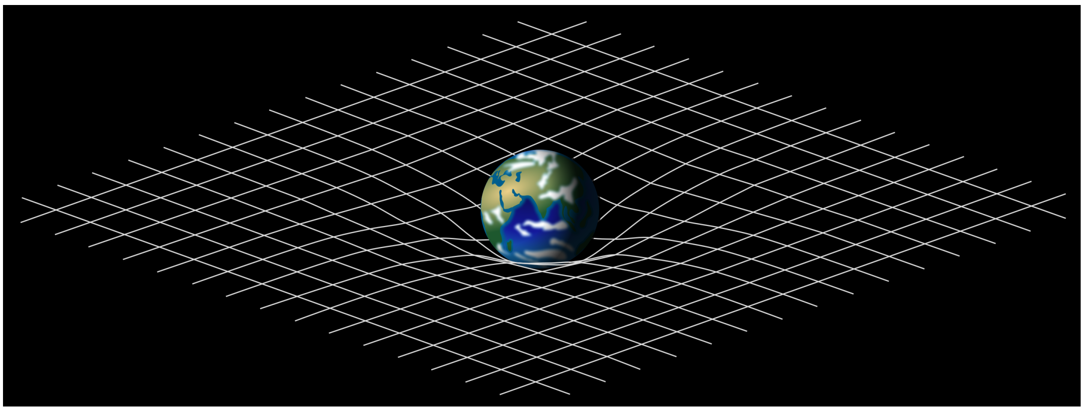
The Parametrised Post Newtonian Expansion of a Central Gravitating Body



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1 Introduction

Arguably the most fundamental principle in all of science is that theories must be tested against experiment in order to confirm their validity. When there exist two or more competing theories this is a fairly straightforward endeavour; simply devise a scenario in which the theories predictions differ from one another by a measurable amount and then carry out the experiment. However, when there is only one serious candidate theory, we would like some way of quantifying how well the theory agrees with experiment. This is the idea behind a test theory, a generalised theory with many free parameters that can be obtained through experiment. In the context of gravitation, a convenient test theory is given by the parametrised post Newtonian approach, which allows us to quantify every possible deviation from Newton's theory of gravity up to a specified given order of expansion in the gravitational potential.

2 Newton's Theory of Gravitation

In order to be complete, a theory of gravitation must be capable of answering three questions:

- What are the times and distances measured by local observers with their clocks and rulers?
- How do massive bodies move under the influence of gravity?
- How do light rays propagate under the influence of gravity?

To answer the first of these questions we need to relate the differential proper time $d\tau$ and length $d\boldsymbol{\ell}$, which are measured by local observers, to the differentials of coordinate time dt and position $d\boldsymbol{r}$, which are used by physicists to chart the motion of bodies. In Newton's theory this relationship takes the trivial form

$$d\tau = dt \qquad \text{and} \qquad d\boldsymbol{\ell} = d\boldsymbol{r} \,, \tag{1}$$

and as such is often left implicit.

In general, describing the motion of massive bodies requires writing down a set of coupled differential equations, which can, in principle, be solved to obtain the trajectories of the bodies. For simplicity, we shall not consider this general case, and instead restrict ourselves to the case of negligible test masses moving in the field of a central gravitating body. This simplification a reasonably good model for the solar system due to the overwhelming gravitational dominance of the Sun. In this case, we simply need to specify the acceleration $\boldsymbol{a}$ of a test mass located at a position $\boldsymbol{r}$ relative to the central body. Newton's law of gravitation is that

$$\boldsymbol{a} = -\frac{m\boldsymbol{r}}{r^3} \,, \tag{2}$$

where m is the mass of the gravitating body.

Finally, there is the question of light rays. To describe the trajectories they take we actually only need to specify the speed of light at each location in space. The ray paths will be determined by Fermat's principle of least time as a consequence of the wave nature of light. Newton never expressed an opinion on whether light was deflected by a gravitational field, leaving it an open question in his works on optics. However, since he never made any attempt to correct astronomical observations for any deflections, he was implicitly assuming that the rays were all straight lines. This is most simply achieved by requiring that the velocity of a light ray $\boldsymbol{c}$ satisfies

$$c^2 = 1\,. \tag{3}$$

3 Kepler's Laws of Planetary Motion

The great success of Newton's theory was its ability to explain Kepler's observational laws of planetary motion

I The shape of a planet's orbit is an ellipse with the Sun located at one focus

II The line joining a planet to the Sun sweeps out area at a constant rate

III The square of a planet's orbital period is proportional to the cube of its major axis

The easiest way of obtaining these laws from Newton's theory is to introduce the specific angular momentum $\boldsymbol{L}$ and eccentricity $\boldsymbol{e}$, defined by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{e} = \frac{\boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{v})}{m} - \frac{\boldsymbol{r}}{r}\,, \tag{4}$$

and use the equation of motion (2) to show that they are both conserved quantities for each planet. Kepler's first law can then be obtained by taking the dot product of the eccentricity with position to yield

$$re\cos\theta = \frac{L^2}{m} - r \implies \frac{L^2}{mr} = 1 + e\cos\theta\,, \tag{5}$$

which is the equation for an ellipse centred on the Sun with semi-latus rectum L^2/m and eccentricity e.

The second law can be obtained by simply noting that the rate at which area is swept out by $\boldsymbol{r}$ is given by

$$\frac{dA}{dt} = \frac{|\boldsymbol{r} \times \boldsymbol{v}|}{2} = \frac{L}{2}\,, \tag{6}$$

which is constant.

Finally, to obtain the third law we can simply evaluate the time period as the area of the ellipse divided by the rate of area being swept out, yielding

$$T = \frac{\pi L^4/m^2(1-e^2)^{3/2}}{L/2} = \frac{2\pi L^3}{m^2(1-e^2)^{3/2}} \implies T^2 = \frac{4\pi^2 a^3}{m}\,, \tag{7}$$

where $a = L^2/m(1-e^2)$ is the semi-major axis of the orbit.

4 The Parametrised Post Newtonian Framework

Newtonian gravity is undeniably an incredibly successful theory demonstrating both explanatory and predictive power. One notable highlight was the discovery of the planet Neptune based on perturbations to the orbit of Uranus. However, its incompatibility with Einstein's special theory of relativity raises the possibility that Newtonian gravity is incomplete. Nonetheless, in light of its historical successes, Newton's theory must emerge as an appropriate low energy limit of any valid relativistic theory of gravitation. As such, a sensible way of analysing candidate theories, especially in the context of the solar system where relativistic effects are small, is in terms of an expansion around the Newtonian theory.

This idea of parametrised post Newtonian framework is to take equations (1–3) and add to them every term allowed by dimensional and symmetry considerations, which is one order higher in the expansion parameters $m/r \sim v^2 \ll 1$. Starting with the expression for the proper time measured by an observer's clock, we obtain

$$d\tau = \left(1 + \alpha_1 \frac{m}{r} + \alpha_2 v^2\right) dt\,. \tag{8}$$

Terms linear in the speed v or involving fractional powers of the mass m are ruled out on the basis of analyticity of the expansion, which seems a reasonable physical requirement. Doing the same for the proper length yields

$$d\boldsymbol{\ell} = \left(1 + \alpha_3 \frac{m}{r} + \alpha_4 \frac{m\boldsymbol{r} \otimes \boldsymbol{r}}{r^3} + \alpha_5 \boldsymbol{v} \otimes \boldsymbol{v} + \alpha_6 v^2\right) d\boldsymbol{r}\,. \tag{9}$$

Assuming the uniqueness of free-fall, the acceleration of a test mass must be independent of its own properties, and the most general extension of (2) is

$$\boldsymbol{a} = -\frac{m\boldsymbol{r}}{r^3} + \beta_1 \frac{m^2\boldsymbol{r}}{r^4} + \beta_2 \frac{m(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{v}}{r^3} + \beta_3 \frac{mv^2\boldsymbol{r}}{r^3}\,. \tag{10}$$

Here, terms linear in $\boldsymbol{v}$ are neglected because they would not conserve energy, leading to runaway instabilities in the solar system which would have prevented it from reaching its current, relatively stable, configuration. Finally, the speed of light rays can be given as

$$\boldsymbol{c} \cdot \left(1 + \gamma_1 \frac{m}{r} + \gamma_2 \frac{m\boldsymbol{r} \otimes \boldsymbol{r}}{r^3}\right) \cdot \boldsymbol{c} + \gamma_3 \frac{m\boldsymbol{r}\cdot\boldsymbol{c}}{r^2} = 1\,. \tag{11}$$

Now, there is a slight subtlety in this expansion. Since the coordinates $t, \boldsymbol{r}$ are treated as labels used to keep track of bodies positions and not observables in their own right, we are free to make a gauge transformation by shifting the coordinates slightly, changing the expansion parameters. In order to make the expansion unique we can impose certain conditions on the coordinates to restrict this freedom. A sensible choice is to choose the coordinates such that the speed of light is the same in all directions at every point in space. This can be done by making the transformation

$$t_{\text{new}} = t + \frac{\gamma_3}{2}\frac{m}{r} \qquad \text{and} \qquad \boldsymbol{r}_{\text{new}} = \boldsymbol{r} - \gamma_2 \frac{m\boldsymbol{r}}{r}\,. \tag{12}$$

When expressed in these new coordinates the form of equations (8–10) is unchanged, since the transformation just mixes together the different terms leading to new values for the αs and βs. However, equation (11) is now considerably simplified by the identity $\gamma_2 = \gamma_3 = 0$. Going forwards we shall always assume that this coordinate condition is imposed.

5 The Classical Tests

Armed with the expansions (8–11), our next task is to determine experimental bounds on the ten parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \gamma_1$, and thus to constrain the landscape of potential theories. The benefit of this approach is that once we have these bounds, all we need to do to test the suitability of any candidate theory is to calculate its predicted values of the parameters, rather than laboriously comparing it to every single experiment individually. Modern experiments are rather complicated and their analysis is fairly long-winded. However, they are usually just more complicated versions of a number of classical tests, which are the experiments we will consider here.

An important class of experiments are those involving interferometers, which measure the phase difference between light beams propagating along two different paths. The fundamental building block of an interferometer is a single arm, which light travels down, before being reflected back to its starting point, where it arrives with some phase difference δ. Let $\boldsymbol{s} = s\boldsymbol{n}$ be the vector pointing along the interferometer arm and let $t_\pm$ be the times taken for light to travel up and down the arm respectively. If the interferometer has a velocity $\boldsymbol{v}$, then to post Newtonian order we have

$$t_\pm = \frac{|\boldsymbol{s} \pm \boldsymbol{v}t_\pm|}{(1+\gamma_1 m/r)^{-1/2}} = s\left(1 \pm \boldsymbol{v}\cdot\boldsymbol{n} + \frac{\gamma_1}{2}\frac{m}{r} + \frac{v^2 + (\boldsymbol{v}\cdot\boldsymbol{n})^2}{2}\right). \tag{13}$$

The total phase difference acquired by the light as it traverses up and down the arm can then be calculated as

$$\delta = f(t_+ + t_-) = 2fs\left(1 + \frac{\gamma_1}{2}\frac{m}{r} + \frac{v^2 + (\boldsymbol{v}\cdot\boldsymbol{n})^2}{2}\right), \tag{14}$$

where f is the frequency of the light source. This expression is not quite complete since both the frequency f and arm length s are being measured in terms of the coordinates t and $\boldsymbol{r}$. In order to make the expression meaningful we must convert to the frequency f_0 and arm length s_0

$$\begin{aligned} f_0 &= \left(1 - \alpha_1\frac{m}{r} - \alpha_2 v^2\right) f\,, \\ s_0 &= \left(1 + \alpha_3\frac{m}{r} + \alpha_4\frac{m(\boldsymbol{r}\cdot\boldsymbol{n})^2}{r^3} + \alpha_5(\boldsymbol{v}\cdot\boldsymbol{n})^2 + \alpha_6 v^2\right) s\,, \end{aligned} \tag{15}$$

which would be measured by a local observer moving with the interferometer. This yields the final expression

$$\begin{aligned} \delta = \delta_0\Bigg(1 &+ (2\alpha_1 - 2\alpha_3 + \gamma_1)\frac{m}{2r} + (1 + 2\alpha_2 - 2\alpha_6)\frac{v^2}{2} \\ &- \alpha_4\frac{m(\boldsymbol{r}\cdot\boldsymbol{n})^2}{r^3} + (1 - 2\alpha_5)\frac{(\boldsymbol{v}\cdot\boldsymbol{n})^2}{2}\Bigg)\,, \end{aligned} \tag{16}$$

where $\delta_0 = 2f_0 s_0$ is the Newtonian phase difference.

There are two general classes of interferometer experiments which can be employed to probe the parameters which appear in this expression. The first, and most well known, are Michelson–Morley type experiments. These involve interference between two interferometer arms of equal length, such that the phase difference between them depends only on the final two terms in 16. By allowing the rotational and orbital motion of the Earth to change the orientation of the interferometer arms relative to their velocity and looking for fluctuations in the phase difference it is possible to find bounds on the parameters α_4 and α_5. Famously, the Michelson–Morley experiment returned a null result, meaning that no fluctuations in the phase difference were observed, implying

$$\alpha_4 \approx 0 \qquad \text{and} \qquad 1 - 2\alpha_5 \approx 0\,. \tag{17}$$

The second class of interferometric tests are Kennedy–Thorndike type experiments. In these experiments the interferometer arms have a constant length difference, so that the first terms in (16) are relevant, and fluctuations in the phase difference are sought due to changes in speed and gravitational potential as a consequence of the Earth's rotation and the eccentricity of its orbit. Like Michelson–Morley, the Kennedy–Thorndike experiment returned a null result, implying that

$$2\alpha_1 - 2\alpha_3 + \gamma_1 \approx 0 \qquad \text{and} \qquad 1 + 2\alpha_2 - 2\alpha_6 \approx 0\,. \tag{18}$$

A second method of probing the post Newtonian parameters is to investigate shifts of spectral lines. Atomic emission and absorption lines have very narrow bandwidths centred on a definite frequency, as measured by the atom itself. Thus, measuring the frequency of light that reaches us gives us insight into the difference between our proper time and the atoms, and hence into the parameters α_1 and α_2.

The Ives–Stillwell experiment measures the emissions from excited hydrogen atoms flying past a stationary[1] observer at high speed. The measurements are taken in such a way that the classical Newtonian Doppler shift can be averaged out. This leaves behind the transverse Doppler shift

$$1 + z = \frac{d\tau_{\text{obs}}}{d\tau_{\text{atom}}} = \frac{1 + \alpha_1 m/r}{1 + \alpha_1 m/r + \alpha_2 v^2} \implies z = -\alpha_2 v^2\,. \tag{19}$$

A similar experiment involves looking for gravitational redshift in the spectral lines coming from a compact gravitating body such as the Sun. In this case the expected Doppler shift is given by

$$1 + z = \frac{d\tau_{\text{obs}}}{d\tau_{\text{atom}}} = \frac{1}{1 + \alpha_1 m/R} \implies z = -\alpha_1 m/R\,, \tag{20}$$

where we have neglected the terms m/r and v^2 for the observer on Earth, because they will be dominated by the term due to the potential at the surface of the Sun, a consequence of the fact the Sun's radius R is much smaller than the Earth's orbital radius. This redshift was first measured by Brault, and along with the results of Ives and Stillwell suggest that

$$\alpha_1 \approx -1 \qquad \text{and} \qquad \alpha_2 = -\frac{1}{2}\,. \tag{21}$$

[1] Technically speaking, the observer was not stationary as they were moving with the Earth; however, the speed of the hydrogen atoms was much larger than that of the Earth, so this is a reasonable approximation

According to our post Newtonian expansion, the effect of gravitational potential on light rays is to act like a medium with a non-unit refractive index, changing their speed. As such, gravity must also deflect the ray's angle of travel in a manner analogous to refraction. This can be observed by measuring the aberration of distant stars when the Sun is close to the line of sight between the stars and Earth. To calculate the angle of deflection we start with our equation

$$\left(1+\gamma_1\frac{m}{r}\right)c^2 = 1\,, \tag{22}$$

and apply Fermat's principle of least time to obtain the Euler–Lagrange equation of motion

$$\frac{d\boldsymbol{c}}{dt} = -\frac{\gamma_1}{2}\frac{m\boldsymbol{r}}{r^3} + \gamma_1\frac{m(\boldsymbol{r}\cdot\boldsymbol{c})\boldsymbol{c}}{r^3}\,. \tag{23}$$

Assuming a small deflection angle χ, and approximating the Earth as very far away from the Sun we can say that

$$\chi = \int_{-\infty}^{\infty} dt\,\frac{dc_\perp}{dt} = \int_{-\infty}^{\infty} dt\,\left(-\frac{\gamma}{2}\frac{mr_\perp}{r^3} + \gamma_1\frac{m(\boldsymbol{r}\cdot\boldsymbol{c})c_\perp}{r^3}\right)\,, \tag{24}$$

where the subscript $\perp$ indicates vector components perpendicular to the light ray's incoming direction of travel. In order to evaluate this integral to the desired order of approximation, we can use the functions $\boldsymbol{r}(t)$ and $\boldsymbol{c}(t)$ to Newtonian order

$$\boldsymbol{r}(t) = (t, b, 0) \qquad \text{and} \qquad \boldsymbol{c}(t) = (1,0,0)\,,$$

where b is the impact parameter of the ray, approximately equal to its closest approach to the Sun. This yields

$$\chi = \frac{\gamma_1}{2}\int_{-\infty}^{\infty} dt\,\frac{mb}{(b^2+t^2)^{3/2}} = \gamma_1\frac{m}{b}\,. \tag{25}$$

Eddington famously measured this deflection by observing the shifts relative to their usual positions of stars when the Sun was almost directly in front of them. In order to actually see the stars without being overpowered by sunlight Eddington had to wait until a total solar eclipse. This measurement gave a value of

$$\gamma_1 \approx 4\,. \tag{26}$$

Finally, we come to the task of measuring the three β parameters that control the motion of freely falling bodies like the planets in the solar system. The most readily apparent effect of the β parameters is the perihelion precession of planetary orbits. To calculate the change in argument of perihelion advance per revolution ε, we start by differentiation the eccentricity vector to find that

$$\frac{d\boldsymbol{e}}{dt} = \beta_1\frac{m\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{v})}{r^4} + 2\beta_2\frac{(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{v}\times(\boldsymbol{r}\times\boldsymbol{v})}{r^3} + \beta_3\frac{v^2\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{v})}{r^3}\,. \tag{27}$$

It turns out to be more convenient of we rewrite this equation slightly in the form

$$\frac{d}{dt}\left(\left(1+\frac{2\beta_2 m}{r}\right)\boldsymbol{e}\right) = \beta_1\frac{m\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{v})}{r^4} + 2\beta_2\frac{(\boldsymbol{r}\cdot\boldsymbol{v})m\boldsymbol{r}}{r^3} + \beta_3\frac{v^2\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{v})}{r^3}\,. \tag{28}$$

Now, assuming that the precession of the perihelion in a single orbit is small, we can calculate ε we by evaluating the integral

$$\varepsilon = \frac{1}{e}\int_0^T dt\,\frac{d}{dt}\left(\left(1+\frac{2\beta_2 m}{r}\right)e_\perp\right), \tag{29}$$

where $e_\perp$ is the component of the eccentricity perpendicular to its initial direction. Tidying up this integral a little gives us

$$\varepsilon = \int_0^T dt\,\left(-\beta_1\frac{m}{r}\cos\theta\frac{L}{r^2} - \beta_2\frac{d(m/r)}{dt}\sin\theta - \beta_3 v^2\cos\theta\frac{L}{r^2}\right) \tag{30}$$

We can simplify the equation somewhat by noting that

$$\frac{L}{r^2} = \frac{d\theta}{dt}, \tag{31}$$

and hence that we must have

$$\varepsilon = \int_0^{2\pi} d\theta\,\left(-\beta_1\frac{m}{r}\cos\theta - \beta_2\frac{d(m/r)}{d\theta}\sin\theta - \beta_3 v^2\cos\theta\right). \tag{32}$$

To post Newtonian order, it is sufficient to evaluate this integral using the Newtonian trajectory of the planet, which is described by

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta \qquad \text{and} \qquad v^2 = \frac{2m}{r} - \frac{m}{a}. \tag{33}$$

This leads to the result

$$\varepsilon = \frac{m\pi}{a(1-e^2)}\left(2\beta_2 - \beta_1 - 2\beta_3\right), \tag{34}$$

or alternatively after substituting in Kepler's third law

$$\varepsilon = \frac{4\pi^3 a^2}{T^2(1-e^2)}\left(2\beta_2 - \beta_1 - 2\beta_3\right). \tag{35}$$

The most famous measurements of perihelion precession were on the orbit of Mercury, since both its close proximity to the Sun, and its reasonably eccentric orbit made the effect more pronounced. In fact, the effect was so noticeable that it was measured before there was any theoretical interest in post Newtonian theories of gravitation, and was instead theorised to be a consequence of an additional hidden planet gravitationally perturbing Mercury's orbit. The measurements show that

$$2\beta_2 - \beta_1 - 2\beta_3 \approx 6. \tag{36}$$

The final two β parameters are hard to determine from orbital motions, but their values must satisfy

$$\beta_2 + 2\beta_3 \approx 2 \qquad \text{and} \qquad \beta_2 - \gamma_1 \approx 0, \tag{37}$$

in order to maintain the uniqueness of free fall for objects which have a significant electromagnetic contribution to their masses. These results are confirmed by the Ëotvös experiment.

6 The General Theory of Relativity

As a closing note, we shall derive the values of the post Newtonian parameters predicted by Einstein's general theory of relativity and demonstrate that the agree with experiment. In general relativity, the spacetime around a spherically symmetric mass is described by the Schwarzschild metric. This metric can take several different forms depending on the particular choice of coordinates employed, but in Eddington gauge it is

$$ds^2 = -\left(\frac{2r-m}{2r+m}\right)^2 dt^2 + \left(1+\frac{m}{2r}\right)^4 \left(dx^2+dy^2+dz^2\right) . \tag{38}$$

Expanding this metric to post Newtonian order yields

$$ds^2 = -\left(1-\frac{2m}{r}+\frac{2m^2}{r^2}\right) dt^2 + \left(1+\frac{2m}{r}\right) \left(dx^2+dy^2+dz^2\right) . \tag{39}$$

The proper time measured by clocks can be evaluated to the desired order directly from this metric

$$d\tau = \sqrt{-ds^2} = \left(1-\frac{m}{r}-\frac{v^2}{2}\right) dt\,, \tag{40}$$

The proper length measured by a local observer's ruler is a little bit more complicated, since we need to account for the lack of simultaneity between the observer and our coordinate frame. However, careful use of the metric (39) to transform $\boldsymbol{dr}$ into the observer's frame and then projecting onto their spatial hyperplane yields

$$d\boldsymbol{l} = \left(1+\frac{m}{r}+\frac{\boldsymbol{v}\otimes\boldsymbol{v}}{2}\right) d\boldsymbol{r}\,. \tag{41}$$

The speed of light can be obtained from (39) by imposing the condition that $ds^2 = 0$ along a light ray, and thus that

$$-\left(1-\frac{2m}{r}\right)+\left(1+\frac{2m}{r}\right)c^2 = 0 \implies \left(1+\frac{4m}{r}\right)c^2 = 1\,. \tag{42}$$

Finally, we can obtain the equation of motion for free falling bodies by noting that in general relativity they move along geodesics. Thus, the motion of falling bodies can be obtained from the Lagrangian

$$\mathscr{L} = -\sqrt{-ds^2} = -\sqrt{1-\frac{2m}{r}+\frac{2m^2}{r^2}-\left(1+\frac{2m}{r}\right)v^2}\,. \tag{43}$$

Expanding to post Newtonian order we obtain

$$\mathscr{L} = -1+\frac{v^2}{2}+\frac{m}{r}-\frac{m^2}{2r^2}+\frac{v^4}{8}+\frac{3mv^2}{2r}\,. \tag{44}$$

The Euler–Lagrange equation that follows is then

$$\frac{d}{dt}\left(\left(1+\frac{3m}{r}\right)\boldsymbol{v}+\frac{v^2\boldsymbol{v}}{2}\right) = -\frac{m\boldsymbol{r}}{r^3}+\frac{m^2\boldsymbol{r}}{r^4}-\frac{3mv^2\boldsymbol{r}}{2r^3}\,, \tag{45}$$

which to the appropriate order is equivalent to

$$\boldsymbol{a} = -\frac{m\boldsymbol{r}}{r^3}+\frac{4m^2\boldsymbol{r}}{r^4}+\frac{4m(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{v}}{r^3}-\frac{mv^2\boldsymbol{r}}{r^3}\,. \tag{46}$$

The comparison between general relativity and our generalised post Newtonian theory is shown in the table below

Parameter	α_1	α_2	α_3	α_4	α_5	α_6	β_1	β_2	β_3	γ_1
GR value	-1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	4	4	-1	4

These values are fully consistent with all of the classical tests. Modern tests are now becoming so precise that they can probe beyond the post Newtonian order, and are still finding agreement with general relativity.