Different Methods of Calculating Redshift

Consider a two dimensional surface in Lorentzian spacetime $x^\mu(\lambda,\sigma)$, which represents a family of null geodesics between two observers. That is to say, the lines of constant σ should be future directed null geodesics with affine parameter λ, and the curves $\lambda = 0$ and $\lambda = 1$ should be the world lines of the two observers $\mathscr{E}$ (the emitter) and $\mathscr{R}$ (the receiver) respectively.

There are two ways we could define the redshift between $\mathscr{R}$ and $\mathscr{E}$. The first would be

$$1 + z_{cl}(\sigma) = \left(\frac{g_{\mu\nu}(\partial x^\mu/\partial\sigma)(\partial x^\nu/\partial\sigma)|_{\lambda=1}}{g_{\mu\nu}(\partial x^\mu/\partial\sigma)(\partial x^\nu/\partial\sigma)|_{\lambda=0}} \right)^{1/2} .$$

This definition reflects the notion that the redshift should be the ratio between the proper time measured by the receiver and emitter between two closely spaced geodesics. If we assume that wave fronts propagate along null geodesics, then this can be thought of as a classical redshift formula. Alternatively, we could recognise that the 4-momentum of a photon is proportional to the tangent vector of its affinely parametrised geodesic, and thus the redshift should be given by

$$1 + z_{qm}(\sigma) = \frac{g_{\mu\nu}(\partial x^\mu/\partial\tau)(\partial x^\nu/\partial\lambda)|_{\lambda=0}}{g_{\mu\nu}(\partial x^\mu/\partial\tau)(\partial x^\nu/\partial\lambda)|_{\lambda=1}} ,$$

where τ is the proper time measured along a curve of constant λ, which is to say that

$$\frac{\partial\tau}{\partial\sigma} = \left(-g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\sigma} \right)^{1/2} \qquad \frac{\partial\tau}{\partial\lambda} = 0 .$$

This definition reflects the idea that the frequency of a photon is proportional to the scalar product of its 4-momentum with an observers 4-velocity, and so can be taken as a quantum redshift formula. We can see that these two definitions are related by

$$\frac{1 + z_{qm}(\sigma)}{1 + z_{cl}(\sigma)} = \frac{g_{\mu\nu}(\partial x^\mu/\partial\sigma)(\partial x^\nu/\partial\lambda)|_{\lambda=0}}{g_{\mu\nu}(\partial x^\mu/\partial\sigma)(\partial x^\nu/\partial\lambda)|_{\lambda=1}} ,$$

and we would hope that this factor turns out to be equal to unity. We can see that this is the case as follows. Since the curves σ =const. are geodesics, it follows that the vector $\partial x^\mu/\partial\lambda$ must obey the geodesic equation

$$\frac{\mathcal{D}}{\mathcal{D}\lambda}\frac{\partial x^\nu}{\partial\lambda} = 0 .$$

This, together with the fact that the covariant derivatives (with repsect to the Levi-Civita connection) of the metric are zero, allow us to apply the Leibniz rule and conclude that

$$\frac{\partial}{\partial\lambda}\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\lambda} \right) = \frac{\mathcal{D}}{\mathcal{D}\lambda}\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\lambda} \right) = g_{\mu\nu}\frac{\partial x^\nu}{\partial\lambda}\frac{\mathcal{D}}{\mathcal{D}\lambda}\frac{\partial x^\mu}{\partial\sigma} .$$

Since the torsion tensor $T^\mu{}_{\nu\rho}$ is identically zero in the Levi-Civita connection, we must have

$$T^\mu{}_{\nu\rho}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\rho}{\partial\sigma} = \frac{\mathcal{D}}{\mathcal{D}\lambda}\frac{\partial x^\mu}{\partial\sigma} - \frac{\mathcal{D}}{\mathcal{D}\sigma}\frac{\partial x^\mu}{\partial\lambda} - \left[\frac{\partial x}{\partial\lambda},\frac{\partial x}{\partial\sigma}\right]^\mu = 0 .$$

Since, by construction, λ and σ are coordinates, the Lie bracket must vanish, and so we can conclude that

$$\frac{\mathcal{D}}{\mathcal{D}\lambda}\frac{\partial x^\mu}{\partial\sigma} = \frac{\mathcal{D}}{\mathcal{D}\sigma}\frac{\partial x^\mu}{\partial\lambda} .$$

Thus, we have

$$\frac{\partial}{\partial\lambda}\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\lambda}\right) = g_{\mu\nu}\frac{\partial x^\nu}{\partial\lambda}\frac{\mathcal{D}}{\mathcal{D}\sigma}\frac{\partial x^\mu}{\partial\lambda} = \frac{1}{2}\frac{\partial}{\partial\sigma}\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}\right)$$

However, since the curves of constant σ are specifically null geodesics, we know that

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\nu}{\partial\lambda}$$

everywhere. Thus, the partial derivative of this quantity must vanish, allowing us to deduce that

$$\frac{\partial}{\partial\lambda}\left(g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\lambda}\right) = 0 \implies g_{\mu\nu}\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\lambda} = f(\sigma)\,,$$

for some function f. The independence of this quantity on λ means that, for all σ,

$$\frac{1+z_{qm}(\sigma)}{1+z_{cl}(\sigma)} = \frac{f(\sigma)}{f(\sigma)} = 1\,.$$