Special Relativity Solutions

1. Some Lorentz factors

Estimate the Lorentz factor γ for each of the following objects.

i) An electron with a kinetic energy of 1 keV.

ii) The Earth in its orbit around the Sun (The Earth-Sun distance is $\sim$ 500 light seconds).

iii) An Olympic sprinter at their top speed.

In each case you should aim to obtain $\gamma - 1$ correct to at least one significant figure.

The quickest way to answer part *i* is to know that an electron's rest energy is $m_e c^2 = 511\,\text{keV}$ and use the relationship that $\gamma = E/mc^2$ to deduce that

$$\gamma_e = \frac{512}{511} \approx 1 + \frac{1}{500} = 1 + 2 \times 10^{-3} \ .$$

Someone who doesn't know this relationship between E and γ, and doesn't happen to have the electron's rest energy in keV memorised, is much more likely to take the equally valid approach of setting $m_e v^2/2 = 1\,\text{keV}$, which yields a velocity of $v = 1.88 \times 10^7\,\text{m}\,\text{s}^{-1}$, or about $0.06\,c$. Inserting this into $\gamma_v = 1/\sqrt{1 - v^2/c^2}$ gives the answer above. We can note that we were justified in using the Newtonian expression for kinetic energy because $\gamma - 1 \ll 1$,.

For part *ii*, we can use the handy approximation that there are $\sim \pi \times 10^7$ seconds in a year to deduce that the Earth's angular velocity is $2 \times 10^{-7}\,\text{rad}\,\text{s}^{-1}$. Combining this with its orbital radius of roughly 500 light seconds, we obtain an orbital velocity of 10^{-4} times the speed of light, and hence

$$\gamma_{\text{Earth}} \approx 1 + 5 \times 10^{-9} \ .$$

Part *iii* requires a little bit more care than the previous examples. It turns out that Olympic sprinters are so slow that most calculators can't tell the difference between γ and 1. To get around this issue, we can use the Taylor series expansion

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \ldots$$

to explicitly calculate $\gamma - 1$ instead. Only the v^2/c^2 term is really necessary since $v \ll c$ (an approximation which would apply equally well to the previous parts as well). Assuming an Olympic sprinter runs at $\sim 10\,\text{m}\,\text{s}^{-1}$, we find that

$$\gamma_{\text{Usain}} \approx 1 + 6 \times 10^{-16} \ .$$

2. Timelike and spacelike separations

Two events are referred to as timelike separated in a given inertial frame, if the distance between where they occur is smaller than the distance light could travel in the time between when they occur. Similarly, they are referred to as spacelike separated if the distance between them is larger than the distance light could have travelled in the time between them.

i) Show that, if events A and B are timelike separated in a frame $\mathcal{K}$, there exists a frame $\mathcal{K}'$ where they occur in the same place.

ii) Show that, if events A and B are spacelike separated in a frame $\mathcal{K}$, there exists a frame $\mathcal{K}'$ where they occur at the same time.

iii) Show that, if two events are timelike separated in one frame, they will be timelike separated in all inertial frames. Do the same for spacelike separation.

Hint: Since you can always choose your axes such that both events occur along the x axis, you only ever need to deal with Lorentz transformations in the standard configuration.

Taking the suggestion given in the hint we shall always choose our axis such that both events lie on the x axis. In this case we find that A and B are timelike (resp. spacelike) separated if and only if $c|t_A - t_B| > |x_A - x_B|$ (resp. $c|t_A - t_B| < |x_A - x_B|$). Looking at the Lorentz transformations

$$t' = \gamma_v \Big[t - vx/c^2\Big] \qquad \text{and} \qquad x' = \gamma_v \Big[x - vt\Big] \,,$$

we can see that A and B will occur at the same place in $\mathcal{K}'$ if

$$v = \frac{x_A - x_B}{t_A - t_B} \,,$$

and at the same time if

$$v = \frac{c^2(t_A - t_B)}{x_A - x_B} \,.$$

The restrictions regarding timelike and spacelike separation can be seen as essentially equivalent to the restriction that $|v| < c$. The easiest way to show that timelike/spacelike separations are preserved by the Lorentz transformations is to note that they leave $c^2t^2 - x^2$ invariant (full workings in the solution to question 3).

3. The invariant interval

Show explicitly that the interval, defined as

$$s^2 = c^2t^2 - \left(x^2 + y^2 + z^2\right) \,, \tag{1}$$

is invariant under a Lorentz transformation in the standard configuration. That is to say, show that, if we calculated s'^2, which is given by the same expression but with the coordinates from some other inertial frame, then we would find that $s'^2 = s^2$. Qualitatively explain why the interval is also invariant under rotations (there is no need to show this explicitly).

We use the inverse Lorentz transformations

$$t = \gamma_v \Big[t' + vx'/c^2\Big]\,, \quad x = \gamma_v \Big[x' + vt'\Big]\,, \quad y = y'\,, \quad \text{and} \quad z = z'\,,$$

and substitute them into (1). This yields

$$s^2 = \gamma_v^2 \Big[c^2t'^2 + 2vt'x' + v^2x^2/c^2\Big] - \Big(\gamma_v^2 \Big[x'^2 + 2vx't' + v^2t'^2\Big] + y'^2 + z'^2\Big)\,.$$

Collecting together like terms, we can rewrite this in the nicer form

$$s^2 = \gamma_v^2\big(1 - v^2/c^2\big)\big(c^2t'^2 - x'^2\big) - \big(y'^2 + z'^2\big)\,.$$

Recalling the definition that $\gamma_v = 1/\sqrt{1 - v^2/c^2}$, we can see that this expression will then reduce down to

$$s^2 = c^2t'^2 - \big(x'^2 + y'^2 + z'^2\big)\,,$$

thus proving that the interval is invariant under this transformation. Since rotations of our coordinate axes do not change the time coordinate and preserve the Euclidean distance (i.e leave $x^2 + y^2 + z^2$ invariant), they must also preserve the interval. Since any Lorentz transformation can be expressed as a combinations of rotations and a boost in the standard configuration, this is sufficient to show that the interval is invariant under any Lorentz transformation.

4. Simultaneous explosions

A train travels past a platform at speed $v = c/2$. As it does so, two small explosions occur at the front and back of the train. From the perspective of an observer inside the train, the explosions occur simultaneously and 10 m apart.

i) When and where do the explosions occur relative to one another from the perspective of an observer on the platform?

ii) Draw spacetime diagrams of these events, in the reference frames of the platform and train respectively.

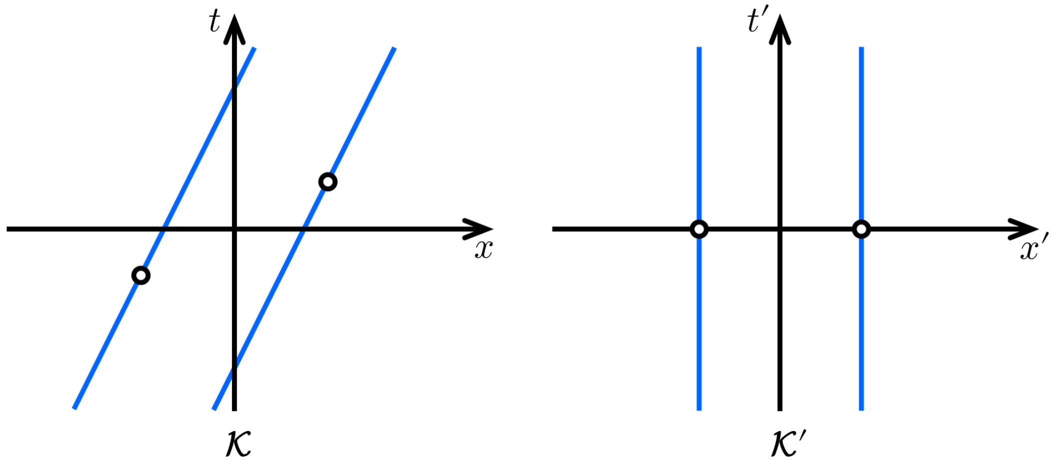
Let $\mathcal{K}$ be the rest frame of an observer on the platform, and $\mathcal{K}'$ the rest frame of an observer inside the train. Using the inverse Lorentz transformations, we can express the coordinate differences between the two events in $\mathcal{K}$ as

$$\Delta t = \gamma_v \Big[\Delta t' + v\Delta x'/c^2\Big] \qquad \text{and} \qquad \Delta x = \gamma_v \Big[\Delta x' + v\Delta t'\Big]\,.$$

Substituting in $v = c/2$, $\Delta x' = 10\,\text{m}$, and $\Delta t' = 0$, we conclude that

$$\Delta t = 19.3\,\text{ns} \qquad \text{and} \qquad \Delta x = 11.5\,\text{m}\,.$$

In words, the explosion at the front of the train occurs 19.3 ns later than the explosion at the back, and at a location 11.5 m farther down the tracks. A nice way of understanding this result is to imagine that the explosions were both triggered by a single radio pulse which originated at the centre of the train. It is pretty clear that in the train's frame this will cause both explosions to happen simultaneously. In the platform frame, the train is length contracted and so its ends are only 8.66 m (or 28.9 light nanoseconds) apart. The radio pulse is still emitted half way along the train; however, in this frame the back of the train is racing towards the wave front, whereas the front of the train is racing away. So if the radio pulse still travels at c, we have no choice but to accept that the pulse reaches the back of the train before it reaches the front. From here some relatively straightforward speed=distance/time type calculations will reproduce the results of the Lorentz transformations.



Spacetime diagrams for part *ii*. The blue lines are the world lines of the front and back of the train respectively. The circles are the events corresponding to each of the explosions.

5. A moving mirror

A stationary light source at the origin of an inertial frame $\mathcal{K}$ emits light of frequency ν_0. A mirror lying in the y, z plane moves towards the light source with velocity v along the x axis. Find an expression for the frequency ν_r of the light reflected by the mirror, as measured by a stationary observer somewhere along the x axis.

Funnily enough, this problem does not really require any application of special relativity to solve. This is because both the light source and the receiver are at rest relative to one another, and so there is no time dilation between them that we need to account for. All we actually need to solve the problem is knowledge that in the rest frame of the source/observer light rays will travel at speed c.

Let us suppose that the light source emits a wavefront at some time t_0. If the distance between the source and the mirror at that point in time is l_0, then the wavefront will reach the mirror at a time t_1 given by

$$c(t_1 - t_0) = l_0 - v(t_1 - t_0) \implies t_1 = t_0 + \frac{l_0}{c+v}\,.$$

Since the mirror is travelling at speed v, the distance between the mirror at the moment of reflection will be $r + l_0 - v(t_1 - t_0)$, where r is the distance between the observer and the source. Thus, the observer will receive the wavefront at a time t_2 given by

$$c(t_2 - t_1) = r + l_0 - v(t_1 - t_0) \implies t_2 = t_0 + \frac{r}{c} + \frac{2l_0}{c+v}\,.$$

We now consider a second wavefront emitted by the source one period $T_0 = 1/\nu_0$ after the first. When this second wavefront is emitted the mirror will have moved closer to the source so that it is only a distance $l_0 - vT_0$ away. Thus, the second wavefront will be received at a time given by

$$t_2 + T_r = t_0 + T_0 + \frac{r}{c} + \frac{2(l_0 - vT_0}{c+v}\,,$$

where $T_r = 1/\nu_r$ is the time period of the reflected light received by the observer. Subtracting these equations from one another we find that the period of the reflected radiation is given by

$$T_r = T_0 - \frac{2vT_0}{c+v} = \frac{c-v}{c+v}T_0\,.$$

Therefore, the frequency of the reflected light must be given by

$$\nu_r = \frac{c+v}{c-v}\nu_0\,.$$

We can achieve the same result a little more easily if we actually do make use of special relativity. The mirror is moving towards the light source, and so in the mirror's rest frame $\mathcal{K}'$ the frequency of the incident light will be blueshifted by the standard relativistic Doppler factor

$$\nu_0' = \sqrt{\frac{c+v}{c-v}}\nu_0\,.$$

In its rest frame, an ideal mirror does not change the frequency of the light it reflects, so $\nu_r' = \nu_0'$. However, in this frame, the observer is moving towards the mirror, so when we shift back to $\mathcal{K}$ any light received by the observer must also be Doppler blueshifted

$$\nu_r = \sqrt{\frac{c+v}{c-v}}\nu_r'\,.$$

Combining these results then reproduces our earlier answer. Interestingly, the total blueshift can also be expressed as the result of a single Doppler shift with a velocity equal to the relativistic addition of v with itself

$$\frac{c+v}{c-v} = \sqrt{\frac{c+V}{c-V}} \qquad \text{where} \qquad V = \frac{2v}{1+v^2/c^2}\,.$$

The interpretation of this fact is that, in the mirror's rest frame the image of the light source moves with a velocity v, and to obtain its velocity in the observer's rest frame we have to relativistically add the velocity of the mirror and the image.

6. Twins and spaceships

Two twins, Albert and Emmy, have never travelled at relativistic velocities relative to one another and are thus the same age. One day, Emmy boards a long haul space flight, which will travel at $0.6\,c$ to a star 3 light years away, before turning around and returning to their home planet at the same speed. After one year of waiting at home, Albert decides that he misses his sister, boards a ship of his own, and sets out after her.

i) Albert travels at a speed $0.7\,c$. Once they reunite, which twin will be older, and by how much?

ii) What speed should Albert travel at, if he wants to be the same age as Emmy when they reunite?

Although this problem gives us some specific numbers to make life easier, a general algebraic solution will be given here for completeness. Let T be the time Albert waits on their home planet before setting off, and L be the distance from their home planet to the turn around point. Furthermore let v_E and v_A be the speeds of Emmy and Albert's spaceships respectively. Throughout this question, we shall always work in the rest frame of the planet.

The first question we need to ask is whether or not Albert reaches Emmy before she turns around. Since the turn around happens a distance L from the planet, and a time $L/v_E - T$ after Albert leaves, it follows that he can only catch up before then if

$$v_A > \frac{L}{L/v_E - T} = \frac{Lv_E}{L - v_E T}\,.$$

For the numbers supplied in the question, we would need $v_E > 0.75\,c$, which we do not have, so we will have to worry about the turn around. Once both twins are in their final states of motion (i.e Albert has left and Emmy has turned around) their world lines will be described by

$$x = v_A(t-T) \qquad \text{and} \qquad x = 2L - v_E t \qquad \text{respectively.}$$

The twins reunite when their world lines cross, so to find the coordinates of their reunion, we just need to solve this set of two simultaneous equations. As it happens, we actually only need to the time of the reunion t_R, which is

$$v_A(t_R - T) = 2L - v_E t_R \implies t_R = \frac{2L + v_A T}{v_A + v_E}\,.$$

Inserting the numbers from the question gives us $t_R = 5.15$ years. As a quick sanity check, this is only slightly after Emmy turns around ($t = 5$ years), which makes sense given that Albert's speed was only slightly below the threshold to meet after the turning. We can now calculate the proper time experienced by each twin by noting that, while in motion, they will be time dilated by a standard factor of $\sqrt{1 - v^2/c^2}$. Emmy is in motion at speed v_E, while Albert has a time T of being stationary before moving at v_A. Putting this all together, we obtain

$$\tau_E = \sqrt{1 - \frac{v_E^2}{c^2}}\,t_R = \sqrt{1 - \frac{v_E^2}{c^2}}\left(\frac{2L + v_A T}{v_A + v_E}\right) = 4.12 \text{ years,}$$

for Emmy, and

$$\tau_A = T + \sqrt{1 - \frac{v_A^2}{c^2}}(t_R - T) = T + \sqrt{1 - \frac{v_A^2}{c^2}} \left(\frac{2L - v_E T}{v_A + v_E} \right) = 3.97 \text{ years.}$$

for Albert. So Albert is about 0.15 years (56 days) younger.

The easiest way to answer part *ii* is as follows. First we can immediately rule out any velocity which would allow a reunion before Emmy turns around. This is because inertial world lines always experience more proper time than non inertial ones between the same two events, so we need for Emmy to have accelerated as well as Albert. Using the invariance of the interval between Albert's departure and the reunion, we can deduce that his proper time is given by

$$\tau_A = T + \sqrt{(t_R - T)^2 - \frac{x_R^2}{c^2}}\,.$$

Since Emmy always travels at speed v_E, her proper time is just t_R multiplied by a time dilation factor, so we can express the condition that they are the same age ($\tau_A = \tau_E$) as

$$T + \sqrt{(t_R - T)^2 - \frac{x_R^2}{c^2}} = \sqrt{1 - \frac{v_E^2}{c^2}}\, t_R\,.$$

Since the reunion must lie on Emmy's world line after turning, we know that $x_R = 2L - v_E t_R$. Substituting this into the above equation, and simplifying yields

$$\sqrt{\frac{c^2 - v_E^2}{c^2} t_R^2 + \frac{4Lv_E - 2Tc^2}{c^2} t_R + \frac{T^2c^2 - 4L^2}{c^2}} = \sqrt{1 - \frac{v_E^2}{c^2}}\, t_R - T\,.$$

While no one would accuse this expression of being pleasant, it does have the rather nice property that, when we square it the t_R^2 terms cancel leaving behind a linear equation. Thus, we obtain the not altogether too gruesome result

$$t_R = \frac{2L^2}{2Lv_E - \left(1 - \sqrt{1 - v_E^2/c^2}\right) Tc^2}\,.$$

We can once again use $x_R = 2L - v_E t_R$ to find the position of the reunion, and then finally we can use the fact that the reunion must also lie on Albert's world line (i.e $x_R = v_A(t_R - T)$ to deduce that

$$v_A = \frac{2L - v_E t_R}{t_R - T} = \frac{2L^2 v_E - 2\left(1 - \sqrt{1 - v_E^2/c^2}\right) LTc^2}{2L^2 - 2LTv_E + \left(1 - \sqrt{1 - v_E^2/c^2}\right) T^2c^2}\,.$$

Inserting the known values of L, T, and v_E, we obtain the answer

$$v_A = \frac{48c}{73} \approx 0.658\, c\,.$$

While that was a pretty brutal algebraic task, it is much easier if you substitute in the values earlier, as they have been chosen deliberately to make this question more reasonably doable.

7. Measuring distances with clocks

An astronaut in space wants to measure how far away he is from a nearby asteroid. To do this he starts a stopwatch and throws it as hard as he can, directly at the asteroid. It then bounces off, hitting the lap button in the process, and returns to the astronaut, who timed the whole process on his spacesuit's built in clock. He now has access to three time measurements τ_1, τ_2, τ_3 which are: the total time measured by his space suit, the time measured by the stopwatch on the way to the asteroid, and the time measured by the stopwatch during the return journey respectively. Show that the distance to the asteroid r can be given by

$$\frac{r}{c} = f(\tau_1, \tau_2, \tau_3)\,, \tag{2}$$

where f is a function you should determine. Be aware that the collision between the stopwatch and the asteroid may not be perfectly elastic, so you cannot assume that its velocity is the same for both the forward and return journeys.

Let t_f and t_b be the time taken (in the astronaut's frame) for the stopwatch's forwards and backwards journeys respectively. Using the invariance of the interval we can compute the proper times measured along each leg of the stopwatches journey

$$\tau_2^2 = t_f^2 - \frac{r^2}{c^2} \qquad \text{and} \qquad \tau_3^2 = t_b^2 - \frac{r^2}{c^2}\,.$$

Since the astronaut remains at rest, he will just measure the coordinate time, and thus we can say that

$$\tau_1 = t_f + t_b = \sqrt{\tau_2^2 + \frac{r^2}{c^2}} + \sqrt{\tau_3^2 + \frac{r^2}{c^2}}\,.$$

We now have an equation relating the three time measurements to the distance, so all that remains is to rearrange it until we obtain an expression for r. One way of doing this is as follows. First, we isolate and remove one of the square roots

$$\sqrt{\tau_3^2 + \frac{r^2}{c^2}} = \tau_1 - \sqrt{\tau_2^2 + \frac{r^2}{c^2}} \implies \tau_3^2 + \frac{r^2}{c^2} = \tau_1^2 + \tau_2^2 + \frac{r^2}{c^2} - 2\tau_1\sqrt{\tau_2^2 + \frac{r^2}{c^2}}\,.$$

We can now do the same to the second root, which yields

$$2\tau_1\sqrt{\tau_2^2 + \frac{r^2}{c^2}} = \tau_1^2+\tau_2^2-\tau_3^2 \implies 4\tau_1^2\left(\tau_2^2 + \frac{r^2}{c^2}\right) = \tau_1^4+\tau_2^4+\tau_3^4+2\tau_1^2\tau_2^2-2\tau_1^2\tau_3^2-2\tau_2^2\tau_3^2\,.$$

Finally, we can complete the rearrangement to end up with the final result.

$$\frac{r}{c} = \frac{\sqrt{\tau_1^4 + \tau_2^4 + \tau_3^4 - 2\tau_1^2\tau_2^2 - 2\tau_2^2\tau_3^2 - 2\tau_3^2\tau_1^2}}{2\tau_1}\,.$$

The enormous impracticalities in this method aside, it is pretty interesting that it is possible to measure distances using only clocks. In my opinion, this is one of the nicest demonstrations of how fundamentally intertwined space and time are in special relativity.

8. The great spaceship debate

A question on a special relativity exam reads as follows:

"An alien spaceship flies over the Earth at a relativistic speed v. As it does so, the ship fires two probes which embed themselves into the Earth. On the ship, the probes are separated by a distance l_0 along its direction of motion, and are deployed simultaneously. When humans on Earth come to investigate the probes, what will they measure as the distance between them?"

Three students who took the exam discuss this question afterwards and realise that they all gave a different answer.

- Student A argues that, since the spaceship is moving with velocity v, the distance between the two probes should be length contracted to l_0/γ_v.
- Student B argues that nothing in the set-up of the problem should change if instead of dropping two probes, the spaceship dropped a single metal rod of length l_0, and so the distance between the two probes should just be l_0.
- Student C argues that, in the frame of the ship, the Earth is moving at speed v and so it will be length contracted by a factor of $1/\gamma_v$. Thus, they conclude that, to an observer on the Earth, the two probes should be separated by a distance of $\gamma_v l_0$.

Decide which of the students is correct, and explain the flaws in the arguments presented by the other two.

Student A is correct that the ship is length contracted and thus that, while they are in the spaceship, the two probes will only be separated by a distance of l_0/γ_v. However, they have forgotten that, while the probes may be released simultaneously in the frame of the ship, this is not the case in the frame of the Earth. In the Earth's frame, the probe at the back of the ship will be released earlier than the one at the front by a time delay of $\gamma_v v l_0/c^2$. Since the ship moves with velocity v, this will mean that the total distance between the probes when they land is increased to

$$\frac{l_0}{\gamma_v} + \frac{\gamma_v v^2 l_0}{c^2} = \gamma_v l_0 \left(\frac{1}{\gamma_v^2} + \frac{v^2}{c^2} \right) = \gamma_v l_0 \,.$$

In short, student A's error is essentially the same misunderstanding which underpins the barn-pole paradox.

Student B is correct that replacing the two probes with a singe rod does not change the behaviour of the probes, up until the moment they land on Earth. When they land however, the two probes must be rapidly accelerated to match the velocity of the Earth. As demonstrated by Bell's spaceship paradox, even if these accelerations are identical and simultaneous (as they would be when viewed in the frame of the ship), they still lead to a change in the rest length between the two probes. Thus, if the probes are replaced by a metal rod, the landing will play out differently due to stresses induced in the rod by the extension that occurs when the rod accelerates to the Earth's velocity. As such, the conclusion that the probes will be separated by a distance l_0 is flawed.

Student C's answer is correct. In the frame of the ship the coordinate differences between the two probes impacting the Earth should be $(\Delta t', \Delta x') = (0, l_0)$, so after applying a Lorentz transformation, this becomes $(\Delta t, \Delta x) = (\gamma_v v \ell_0/c^2, \gamma_v l_0)$, so just as predicted the distance between the two probes is $\gamma_v l_0$.

9. Impossible processes

Prove that the following processes are impossible whilst conserving energy and momentum.

i) A photon travelling through free space spontaneously producing an electron-positron pair.

ii) An free electron spontaneously emitting a photon.

iii) A stationary electron scattering a photon and recoiling in a direction perpendicular to the incident photon's momentum.

The quickest way of answering part *i* is to note that we can always find an inertial frame where the energy of a photon is arbitrarily small. Thus, there will always be a frame where the photon's energy is less than $2m_ec^2$. Since the energy of an electron-positron pair must be $2m_ec^2$ plus any kinetic energy, it will be impossible to conserve energy in this frame. If a process is impossible in one frame, then, by the principle of relativity, it must be impossible in all frames.

The easiest way of approaching part *ii* is to note that a necessary condition for energy and momentum conservation is the conservation of invariant mass. The invariant mass of the initial state is just the electron mass m_e. The easiest way of finding the invariant mass of the final state is to evaluate it in the rest frame of the electron, in which case we find that

$$M^2 = m_e^2 = (m_e + E_\gamma/c^2)^2 - E_\gamma^2/c^4 = m_e^2 + 2m_eE_\gamma/c^2 \,.$$

The only way this can be true is if $E_\gamma = 0$, which is essentially the same thing as saying the photon does not exist.

For part *iii* we simply note that, if the electron recoils perpendicularly to the direction of the incident photon, momentum conservation implies that magnitude of the scattered photon's momentum must have increased. This is because the component parallel to the incident photon's momentum must have remained the same, while it gained a perpendicular component to offset the momentum of the recoiling electron. However, the magnitude of the electron's momentum has also increased (from zero), which would imply that both the photon and electron have gained energy in the collision. This is clearly impossible without violating energy conservation.

10. Invariant mass

i) Two massive particles have energies $E^{(1)}$ and $E^{(2)}$ with momenta $p_x^{(1)}, p_y^{(1)}, p_z^{(1)}$ and $p_x^{(2)}, p_y^{(2)}, p_z^{(2)}$ respectively. Using the behaviour of the energy and momentum under Lorentz transformations, or otherwise, show that

$$\frac{E^{(1)}E^{(2)}}{c^4} - \frac{p_x^{(1)}p_x^{(2)} + p_y^{(1)}p_y^{(2)} + p_z^{(1)}p_z^{(2)}}{c^2} = m_1 m_2 \gamma_{1,2} \,, \tag{3}$$

where m_1 and m_2 are the masses and $\gamma_{1,2}$ is Lorentz factor for the relative velocity between the particles.

ii) Hence, show that in a system of n massive particles, the invariant mass is given by

$$M^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \gamma_{i,j} \,, \tag{4}$$

where m_i is the mass of the ith particle, and $\gamma_{i,j} = \gamma_{j,i}$ is the Lorentz factor for the relative velocity between the ith and jth particles. Explain how this formula should be modified if one or more of the particles is massless.

The first step in the solution of part i is to prove that the expression in (3) is Lorentz invariant. This is reasonably straightforward to do by directly substituting in the Lorentz transformations for energy and momentum; however, a slightly quicker method is to note that

$$\frac{E^{(1)}E^{(2)}}{c^4} - \frac{p_x^{(1)}p_x^{(2)} + p_y^{(1)}p_y^{(2)} + p_z^{(1)}p_z^{(2)}}{c^2} = \frac{M_{1,2}^2 - m_1^2 - m_2^2}{2} \,,$$

where $M_{1,2}$ is the invariant mass of the combined two particle system. Since $M_{1,2}$, m_1, and m_2 are all Lorentz invariant, it follows that the expression on the left hand side must be as well. Once we know that the expression is the same in all frames, we simply choose a convenient one to evaluate it in. A suitable choice is the rest frame of particle one. In this frame, the first particle's momentum is zero and so the expression reduces to $E^{(1)}E^{(2)}/c^4$. Furthermore, $E^{(1)} = m_1 c^2$, while $E^{(2)} = \gamma_{1,2} m_2 c^2$, because in this frame the second particle's velocity is equal to the relative velocity between the two particles. Thus,

$$\frac{E^{(1)}E^{(2)}}{c^4} - \frac{p_x^{(1)}p_x^{(2)} + p_y^{(1)}p_y^{(2)} + p_z^{(1)}p_z^{(2)}}{c^2} = m_1 m_2 \gamma_{1,2} \,.$$

For part ii, we start from the definition of the invariant mass

$$M^2 = \frac{\left(\displaystyle\sum_{i=1}^{n} E^{(i)}\right)^2}{c^4} - \frac{\left(\displaystyle\sum_{i=1}^{n} p_x^{(i)}\right)^2 + \left(\displaystyle\sum_{i=1}^{n} p_y^{(i)}\right)^2 + \left(\displaystyle\sum_{i=1}^{n} p_y^{(i)}\right)^2}{c^2} \,.$$

Expanding out the brackets, we find that this can be rewritten as

$$M^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{E^{(i)}E^{(j)}}{c^4} - \frac{p_x^{(i)}p_x^{(j)} + p_y^{(i)}p_y^{(j)} + p_z^{(i)}p_z^{(j)}}{c^2}\right) \,.$$

We now just substitute in the result from part i to arrive at the final answer

$$M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_i m_j \gamma_{i,j} \, .$$

When one or more of the particles is massless we run into a slight complication since the Lorentz factors involving the massless particles are divergent. We can determine the appropriate replacement by going back to our derivation of the key result in part i. If the jth particle is massless, but the ith is not, then the derivation works essentially the same way by going into the rest frame of the massive particle. Thus, we simply replace $m_j \gamma_{i,j}$ with the energy of particle j, in the rest frame of particle i, divided by c^2. If both particles are massless, then we replace $m_i m_j \gamma_{i,j}$ with $M_{i,j}$.

11. Particle accelerator

A linear accelerator uses a potential difference V in order to accelerate charged particles from rest up to relativistic speeds. Show that the speed of a particle after passing through this accelerator is given by

$$\frac{u}{c} = \frac{\sqrt{V^2 + 2XV}}{X + V} \, , \tag{5}$$

where $X = mc^2/q$ is the mass to charge ratio of the particle multiplied by c^2.

If a particle has charge q, then it will gain an energy qV when it passes through a potential difference V. Thus, if it was initially at rest, its energy will be given by

$$E = mc^2 + qV \, .$$

We can calculate the speed of a particle using its energy by applying the formula

$$\frac{u}{c} = \frac{pc}{E} = \frac{\sqrt{E^2 - m^2c^4}}{E} \, .$$

Thus, the speed of the particle leaving the accelerator is given by

$$\frac{u}{c} = \frac{\sqrt{(mc^2 + qV)^2 - m^2c^4}}{mc^2 + qV} = \frac{\sqrt{q^2V^2 + 2mc^2qV}}{mc^2 + qV} \, .$$

Dividing the numerator and denominator by q and defining $X = mc^2/q$ then yields the desired expression

$$\frac{u}{c} = \frac{\sqrt{V^2 + 2mc^2V/q}}{mc^2/q + V} = \frac{\sqrt{V^2 + 2XV}}{X + V} \, .$$

This same approach could be used to obtain an equivalent formula for any process in which a fixed amount of kinetic energy is transferred to a particle.

12. An inelastic collision

A particle of mass m_1 travelling at speed u collides with a stationary particle of mass m_2. Following the collision, the two particles coalesce into a single particle of mass M travelling at speed v. Find expressions for M and v in terms of m_1, m_2, and u. Show that these expressions reduce to the Newtonian results when $u \ll c$.

Applying energy and momentum conservation to this collision tells us that

$$\gamma_u m_1 c^2 + m_2 c^2 = \gamma_v M c^2 \quad \text{and} \quad \gamma_u m_1 u = \gamma_v M v \,.$$

We now have two simultaneous equations for the two unknowns M and v, so from here the solution is simply a matter of algebra. The most efficient way to proceed is probably to first divide the two equations by one another to obtain

$$v = \frac{\gamma_v M v}{\gamma_v M} = \frac{\gamma_u m_1 u}{\gamma_u m_1 + m_2} \,.$$

The easiest way of finding the mass of the new particle is essentially to apply the mass-energy-momentum relationship

$$M = \sqrt{(\gamma_v M)^2 - (\gamma_v M v/c)^2} = \sqrt{(\gamma_u m_1 + m_2)^2 - (\gamma_u m_1 u/c)^2} \,.$$

Simplifying the above expression slightly, we find that the mass is given by

$$M = \sqrt{m_1^2 + m_2^2 + 2\gamma_u m_1 m_2} \,.$$

Importantly, this expression is symmetric under the interchange of the two masses. This makes sense because M should be frame invariant and we can swap the roles of the two initial masses by switching to the rest frame of m_1. We can take Taylor series expansions of these expressions to obtain

$$v = \frac{m_1 u}{m_1 + m_2} + \frac{m_1 m_2 u^3}{2(m_1 + m_2)^2 c^2} + \mathcal{O}(u^5/c^4) \,,$$

and

$$M = m_1 + m_2 + \frac{m_1 m_2 u^2}{2(m_1 + m_2) c^2} + \mathcal{O}(u^4/c^4) \,.$$

When $u \ll c$, we can clearly see that $v \simeq m_1 u/(m_1 + m_2)$ and $M \simeq m_1 + m_2$. These are exactly the results that would be expected in Newtonian mechanics by applying the conservation of mass and momentum.

13. Δ^+ baryon decay

A Δ^+ baryon ($m_\Delta c^2 = 1232\,\mathrm{MeV}$) can decay into a proton ($m_p c^2 = 938\,\mathrm{MeV}$) and a neutral pion ($m_\pi c^2 = 135\,\mathrm{MeV}$). Determine the energies of the daughter particles from this decay, as measured in the rest frame of the Δ^+.

Let E_p and E_π be the energies carried by the proton and pion respectively. Let p be the magnitude of the momentum of each of the daughter particles. We know that p must be the same for each particle since the momenta must be equal and opposite so that the total momentum is zero, as required by momentum conservation. Applying the mass-energy-momentum relation for each particle, we can deduce that

$$E_p^2 - p^2c^2 = m_p^2c^4 \quad \text{and} \quad E_\pi^2 - p^2c^2 = m_\pi^2c^4\,.$$

Combining these two equations allows us to eliminate p, and conclude that the two energies are related by

$$E_p^2 - E_\pi^2 = m_p^2c^4 - m_\pi^2c^4\,.$$

The final constraint needed to close this system of equations comes from energy conservation. All of the Δ^+ baryon's rest energy must be converted into the energy of the two daughter particles, so we must also have

$$E_p + E_\pi = m_\Delta c^2\,.$$

From here we simply need to solve two simultaneous equations in two unknowns. This is relatively straightforward; however, one particularly elegant method is to note that

$$E_p - E_\pi = \frac{E_p^2 - E_\pi^2}{E_p + E_\pi} = \frac{m_p^2c^4 - m_\pi^2c^4}{m_\Delta c^2}\,.$$

We can now add/subtract this equation from the sum of the energies to obtain the final answers

$$E_p = \frac{m_\Delta^2c^4 + m_p^2c^4 - m_\pi^2c^4}{2m_\Delta c^2} \quad \text{and} \quad E_\pi = \frac{m_\Delta^2c^4 + m_\pi^2c^4 - m_p^2c^4}{2m_\Delta c^2}\,.$$

As a quick sanity check, we note that the expression for E_π can be obtained from E_p by simply swapping m_π and m_p, which we would expect to be the case. Plugging in the measured values of the masses give us

$$E_p = 966\,\mathrm{MeV} \quad \text{and} \quad E_\pi = 266\,\mathrm{MeV}\,.$$

If we want to know the value of the momentum p, then it is a simple matter of taking one of these energies and substituting it back into the appropriate mass-energy-momentum relation to find

$$p = \frac{\sqrt{m_\Delta^4 + m_p^4 + m_\pi^4 - 2\left(m_\Delta^2m_p^2 + m_p^2m_\pi^2 + m_\pi^2m_\Delta^2\right)}}{2m_\Delta}c\,.$$

Evaluating this with the given masses tells us that $pc = 230\,\mathrm{MeV}$.

14. The Greisen-Zatsepin-Kuzmin limit

High energy cosmic ray protons can interact with photons from the cosmic microwave background via the reaction $p^+\gamma \to p^+\pi^0$. That is to say, the net effect of the collision is to convert a photon into a neutral pion. Taking the typical energy of a CMB photon to be $E_\gamma \approx 400\,\mathrm{\mu eV}$, estimate the minimum energy required by the proton for this process to be possible. You may assume that the proton and photon collide head on.

$[m_p c^2 = 938\,\mathrm{MeV}$ and $m_\pi c^2 = 135\,\mathrm{MeV}]$

The easiest way of approaching this problem is to make use of the invariant mass M, defined by

$$E_{\mathrm{tot}}^2 - p_{\mathrm{tot}}^2 c^2 = M^2 c^4 \,.$$

The invariant mass M has the properties that it is the same in all inertial frames, and that it is the same before and after the collision. The first following from the behaviour of energy and momentum under Lorentz transformation, and the second from the conservation of energy and momentum.

Let us first evaluate the invariant mass for the final state, consisting of a proton and a neutral pion. If we choose a frame where the total momentum is zero, then the invariant mass will just be the total energy divided by c^2. Since, a particle's energy is always larger than its rest energy, we can deduce that

$$M \geq m_p + m_\pi \,,$$

with equality if and only if the two particles are at rest relative to one another. We can now evaluate the invariant mass of the initial particles. Assuming that the photon and proton collide head on, since this maximises the invariant mass, we will have

$$M^2 c^4 = (E_p + E_\gamma)^2 - (p_p - p_\gamma)^2 c^2 \,.$$

Expanding the brackets and substituting in the mass-energy-momentum relations for the proton and photon, we find that

$$M^2 c^4 = m_p^2 c^4 + 2E_\gamma \left(E_p + \sqrt{E_p^2 - m_p^2 c^4} \right) \,.$$

Combining this with our inequality that M must be greater than or equal to the sum of m_p and m_π, we can deduce that E_p must satisfy

$$E_p + \sqrt{E_p^2 - m_p^2 c^4} \geq \frac{2 m_p m_\pi c^4 + m_\pi^2 c^4}{2E_\gamma} \,.$$

Here our life can be made substantially easier if we make a small approximation. Since $E_\gamma \ll m_\pi c^2$, we can see that we must have $E_p \gg m_p c^2$. Thus, the left hand side of the above inequality is equal to $2E_p$ to a very good approximation. Thus

$$E_{p,\mathrm{min}} \simeq \frac{2 m_p m_\pi c^4 + m_\pi^2 c^4}{4E_\gamma} = 1.7 \times 10^{14}\,\mathrm{MeV} \,.$$

We can see that this minimum energy is eleven orders of magnitude larger than the proton rest energy, and so the approximation that $p_p c \simeq E_p$ will be very good indeed. Nonetheless, if we want to obtain an exact answer, we should proceed as follows. Firstly, we rearrange the inequality into

$$\sqrt{E_p^2 - m_p^2 c^4} \geq \frac{2 m_p m_\pi c^4 + m_\pi^2 c^4}{2E_\gamma} - E_p \,.$$

We now square both sides yielding

$$E_p^2 - m_p^2 c^4 \geq \left(\frac{2m_p m_\pi c^4 + m_\pi^2 c^4}{2E_\gamma}\right)^2 - 2E_p \left(\frac{2m_p m_\pi c^4 + m_\pi^2 c^4}{2E_\gamma}\right) + E_p^2 \,,$$

which can be simplified to give

$$E_p \geq \frac{2m_p m_\pi c^4 + m_\pi^2 c^4}{4E_\gamma} + \frac{E_\gamma m_p^2 c^4}{2m_p m_\pi c^4 + m_\pi^2 c^4} \,.$$

The second term has a numerical value of 1.3 meV, clearly negligible compared to the first term, which was our previous approximate answer.

15. Annihilation angle

A positron with kinetic energy E_K annihilates with a stationary electron to produce a pair of photons. Find the minimum and maximum values of the angle between the two photons' momenta. What is the minimum kinetic energy required for a possibility of two photons at right angles to one another?

The invariant mass of the electron-positron pair can be calculated as

$$M^2 c^4 = (E_K + 2m_e c^2)^2 - \left((E_K + m_e c^2)^2 - m_e^2 c^4\right) = 4m_e^2 c^4 + 2m_e c^2 E_K \,.$$

The invariant mass of the pair of photons can be calculated as

$$M^2 c^4 = (E_1 + E_2)^2 - (E_1^2 + E_2^2 + 2E_1 E_2 \cos\theta) = 2E_1 E_2 \left(1 - \cos\theta\right) \,,$$

where E_1 and E_2 are their energies and θ is the angle between them. Thus, the angle is given by

$$\cos\theta = 1 - \frac{2m_e^2 c^4 + m_e c^2 E_K}{E_1 E_2} \,.$$

Since the angles we are considering are between 0 and π, the minimum angle will be the one which maximises $\cos\theta$, which requires the maximisation of $E_1 E_2$, subject to the constraint imposed on them by energy conservation that $E_1 + E_2 = 2m_e c^2 + E_K$. It follows from the AM-GM inequality (or equivalently elementary calculus) that this maximum will occur when

$$E_1 = E_2 = m_e c^2 + \frac{E_K}{2} \,,$$

leading to the expression

$$\theta_{\min} = \arccos\left(\frac{2E_K^2 + 4m_e c^2 E_K}{E_K^2 + 4m_e c^2 E_K + 4m_e^2 c^4} - 1\right) \,.$$

Since the product $E_1 E_2$ can be made arbitrarily small while still respecting energy conservation (just let $E_1 \to 0$), we can always make $\cos\theta = -1$, and so the maximum angle is always just

$$\theta_{\max} = \pi \,.$$

The values of E_1 and E_2 at which this occurs will be the minimum and maximum possible values of the photon energies.

If we want to have a possibility of photons at right angles then we will need $\theta_{\min} \leq \pi/2$. Since $\cos \pi/2 = 0$, this requirement is equivalent to demanding that

$$\frac{2E_K^2 + 4m_e c^2 E_K}{E_K^2 + 4m_e c^2 E_K + 4m_e^2 c^4} \geq 1\,.$$

Multiplying through by the denominator, which is always positive, and simplifying, this inequality can be rewritten as

$$E_K^2 - 4m_e^2 c^4 \geq 0 \implies E_K \geq 2m_e^2 c^4\,,$$

where we have used the fact that the kinetic energy must be positive to choose one root of the quadratic.

16. Railgun rocket

A particular type of rocket is designed to travel through dust clouds in space. It generates thrust using a built in railgun-like mechanism to take in the dust accelerate it and then eject it out the back of the rocket. With its batteries uncharged the rocket has a mass $2m_0/3$, and when the batteries are charged its mass increases to m_0. In the rocket's rest frame, the railgun transfers a, very small, fixed amount of kinetic energy per unit mass to the dust it accelerates. If the rocket starts from rest with its battery fully charged, estimate its speed once the battery has been fully depleted.

Let us suppose that at some moment the rocket has rapidity ξ. We consider an infinitesimal interval over which the mass of the rocket changes by dm. The only way for the mass of the rocket to change is for energy to be used from the batteries, so an energy $-dmc^2$ must have been transferred to the dust via the railgun. In the rocket's rest frame, the dust has an initial velocity of $\tanh \xi$ towards the rocket. We know that the kinetic energy per unit mass transferred to the dust is very small so we can approximate that the momentum transferred to it is

$$dp = \frac{dp}{dE}(-dmc^2) = \frac{d\sinh\xi}{d\cosh\xi}(-dmc) = -dmc\coth\xi\,.$$

By conservation of momentum the rocket must receive an equal and opposite amount of momentum, which will equal $mcd\xi$, leading to the fundamental equation

$$mcd\xi = -dmc\coth\xi \implies -\frac{dm}{m} = \tanh\xi\, d\xi = d\big(\ln(\cosh\xi)\big)\,.$$

Integrating this equation we deduce that the rapidity of the rocket, when its total mass is m, can be given by

$$\cosh\xi = \frac{m_0}{m}\,,$$

which implies that the rocket's speed is given by

$$u = c\tanh\xi = c\sqrt{1 - 1/\cosh^2\xi} = c\sqrt{1 - m^2/m_0^2}\,.$$

Once the rocket has depleted its battery its mass will be $m = 2m_0/3$, in which case the speed will be

$$u = \frac{\sqrt{5}}{3}c = 2.23 \times 10^8\,\mathrm{m\,s^{-1}}\,.$$