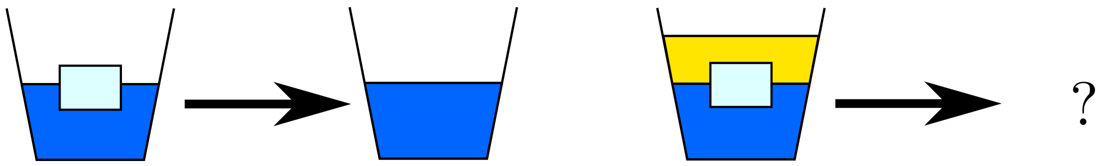
1 Melting Mysteries



An ice cube is floating in a glass of water. Explain why the water level in the glass remains constant as the ice cube melts.

Once again, consider an ice cube floating in a glass of water. However, this time we will pour oil into the glass until the ice cube is completely submerged. Since ice is less dense than water but denser than oil, it will float at the boundary between the two liquid layers. When the cube melts, what happens to the water level and the oil level?

Hint: You will need to consider that both the oil and the water exert an upthrust force on the ice cube based on the volume of each liquid it is displacing. Does the ice cube float higher or lower compared to the original example?

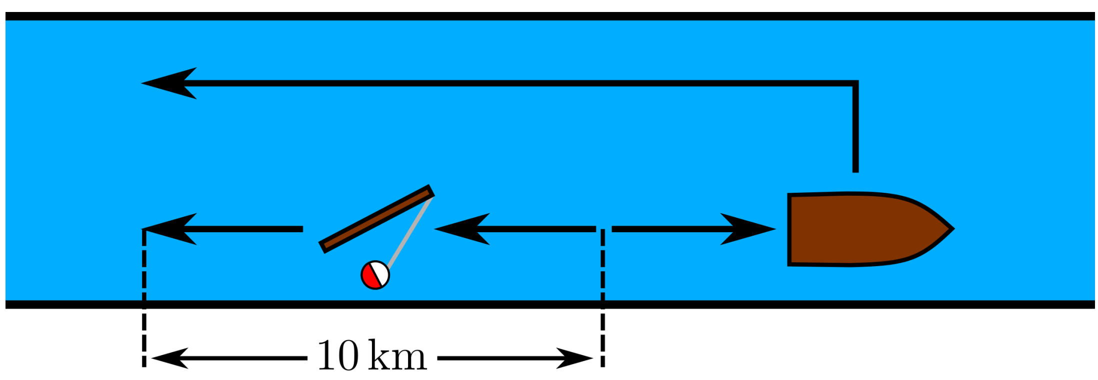
2 An Interesting Integral

Evaluate the value of the definite integral

$$\int_{-\infty}^{\infty} \sin(x)e^{-x^2}\, dx\,.$$

Hint: Sketch the graph of $y = \sin(x)e^{-x^2}$, what do you notice? How is this related to the limits of the integral.

3 The Fumbling Fisherman



A fisherman is rowing upstream along a river to reach a good fishing spot. At some point in time, his fishing rod slips out of the boat and is carried away by the flowing river. When the fisherman arrives at his spot 20 minutes later and realises the mistake, he turns around and rows back the way he came. When he eventually reaches his fishing rod and scoops it out of the water, he is 10 km further downriver than when he first dropped the fishing rod. What is the speed of the river?

Hint: Assuming the fisherman rows equally hard in both directions, what can you conclude about his speed relative to the water?

4 Laser Levitation

A person of mass m aims a laser pointer with a wavelength λ straight down and turns it on. Find an expression for the power P that required for the person to be able to hover above the ground?

Hint: A laser is essentially just a hose which sends out a stream of photons rather than water. How would you calculate the flow rate from a hose required to make the person hover and could you apply a similar idea to the laser?

5 Sinusoidal Sketching

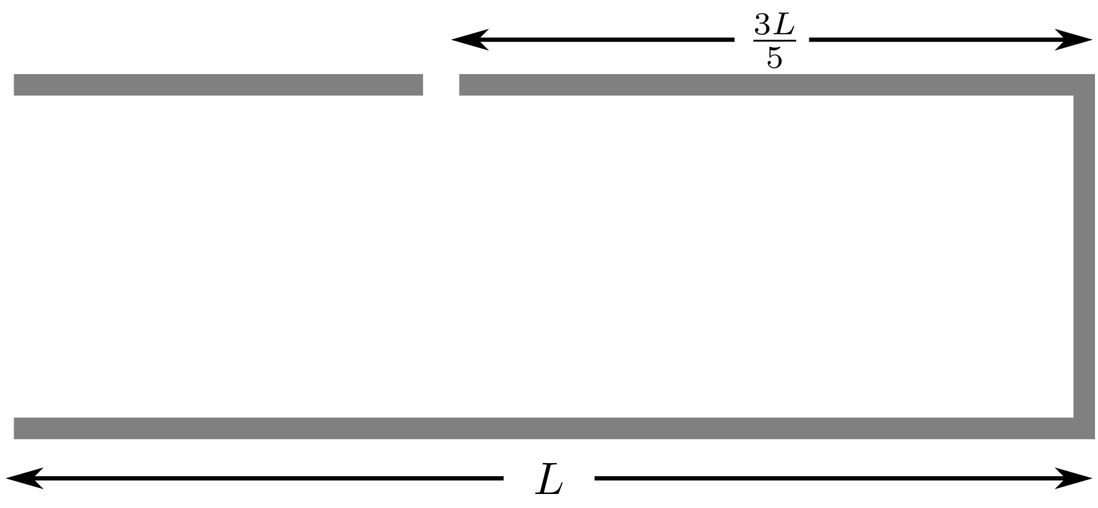
Sketch the curve

$$y = x \sin\left(\frac{1}{x}\right).$$

Hint: What is the range of the sin function and constraints does this place on the values of y? When x is large and therefore $1/x$ is small, what approximations can you use for $\sin(1/x)$?

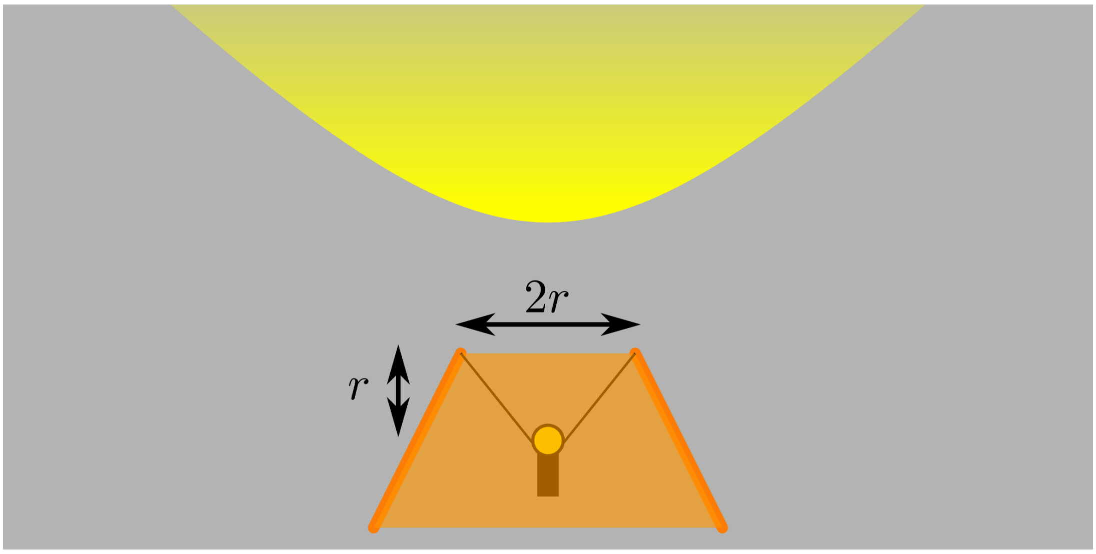
6 Strange Stationary Waves

The diagram below shows a cylindrical pipe of length L that is closed on one end, with a small hole three fifths of the way along its length. Sketch the fundamental stationary wave inside the pipe, and write down an expression for the frequency of the nth harmonic.



Hint: What are the boundary conditions for the stationary wave at the open and closed ends of the pipe? How might this be related to the effect of the hole?

7 Light from a Lampshade



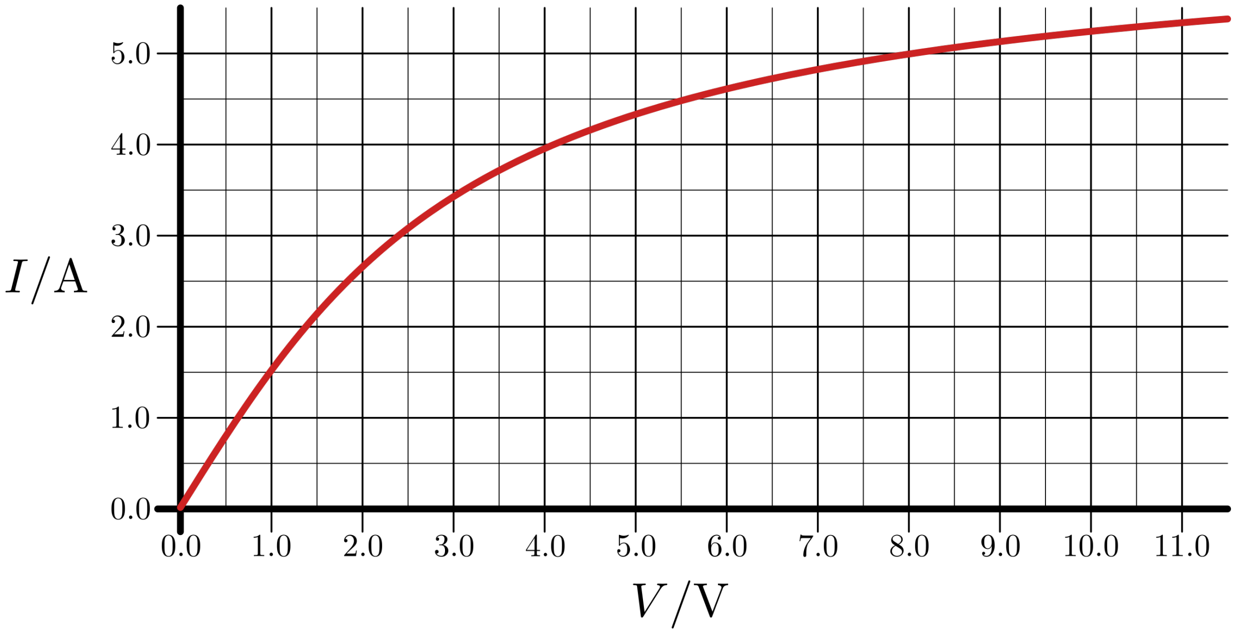
A light fixture is constructed by placing a small bulb inside a lampshade and mounting the assembly on a vertical wall. The lampshade has a circular hole of radius r at its top, a height r above the bulb. The fixture is mounted so that the bulb is a distance $2r$ from the wall. Find an equation for the shape that the light makes on the wall above the lamp.

Hint: At what angle to the vertical do light rays leave the top of the lampshade? What shape does this light make in 3 dimensional space?

8 Internal and Incandescent

The graph shows the $I - V$ characteristic for an incandescent bulb. This bulb is powered by a cell with an emf of $10\,\mathrm{V}$ and an internal resistance of $2.0\,\Omega$. What is the power of the bulb?

Hint: How would you solve this question for a fixed resistor? Can you find a graphical interpretation of that solution?



9 A Difficult Derivative

Find the derivative of the function

$$y = \log_x 5 \,.$$

Hint: Can you relate the logarithm of a number with the base x to the logarithm with a more natural base, such as e.

10 Spinning Saucer

A uniform circular disk with a mass m and a radius R rotates about an axis through its centre with an angular speed ω. What is its kinetic energy?

Hint: What would the kinetic energy of a thin ring be if it was rotating in this way? Can you find a way to break the disk down into rings and add their kinetic energies together?

11 Paternoster Pendulum

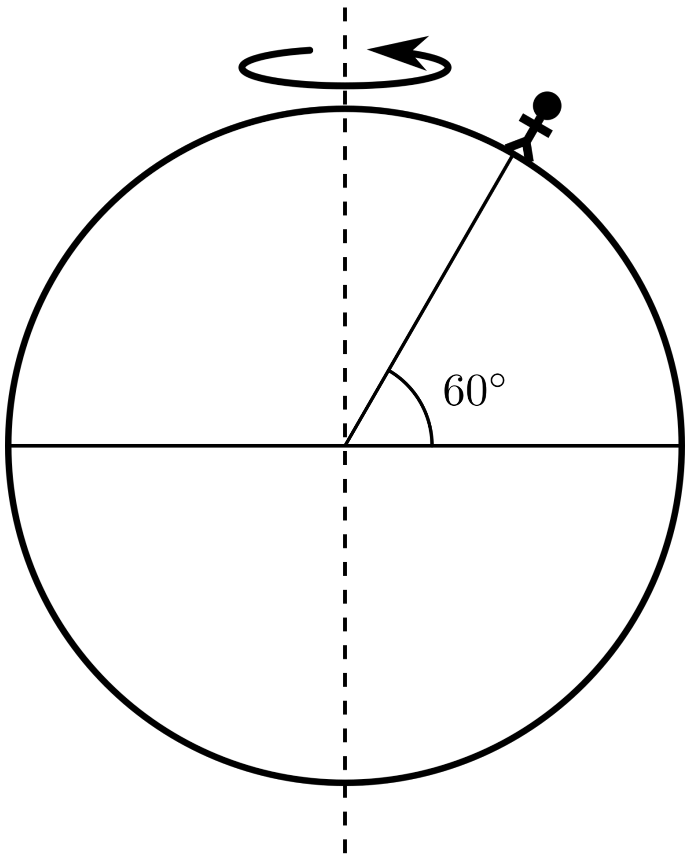
A simple pendulum with a length l in a gravitational field g has a period given by

$$T = 2\pi\sqrt{\frac{l}{g}} \,.$$

How would this expression be modified for the period of a pendulum inside a lift which was accelerating upwards with acceleration a?

Hint: How would you derive the period of a regular simple pendulum? What assumptions do you need to modify to take the derivation into the accelerating lift?

12 The Sky from Skye



At any moment in time, half of the sky is visible to us, while the other half is hidden behind the Earth underneath the horizon. As the Earth rotates, different parts of the sky come into view; however, even over the course of a full day, some regions of the sky will never come into view, for example the star Alpha Centauri is never visible from the UK, while Polaris is never visible from Australia. This is further complicated by the fact that some stars are not visible, even when they are above the horizon. For example, in the Summer the Sun is directly in front of the constellation Cancer, making those stars essentially invisible. However, over the course of a full year the Earth's orientation relative to the Sun changes and this issue can be avoided.

The Isle of Skye of the coast of Scotland is located at a latitude of approximately 60°, meaning that the angle between it and the equator is equal to 60°. What fraction of the stars in the sky would be visible from Skye over the course of a full year?

Hint: The entire sky can be thought of as the surface of a sphere. How are you going to calculate the fraction of the sphere that is visible?

13 Integrating Inverses

Show that

$$\int \ln x \, dx = x \ln x - x + \text{const.}.$$

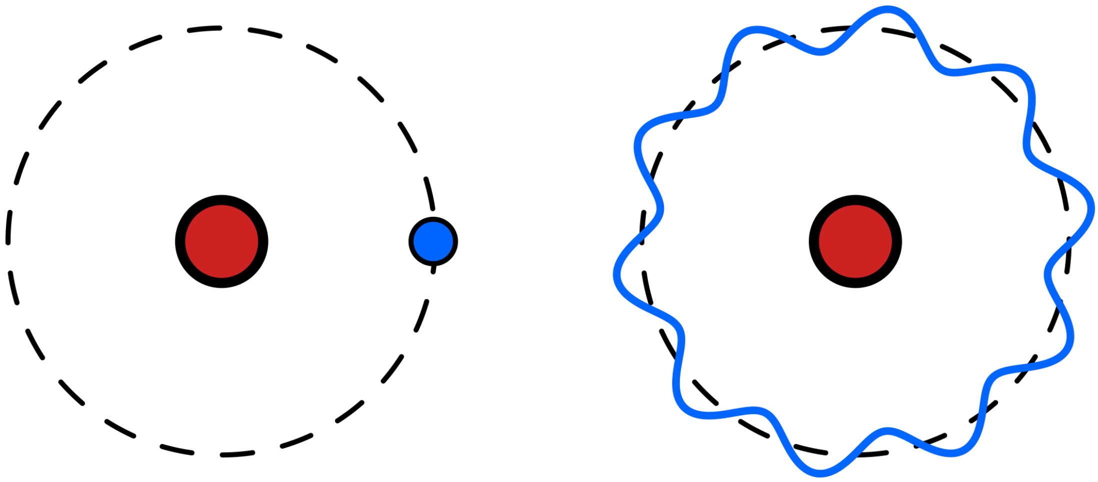
Now obtain a general formula for the integral of an inverse function

$$\int f^{-1}(x) \, dx \,,$$

expressing your answer in terms of the inverse, $f^{-1}(x)$, the original function $f(x)$, and its antiderivative $F(x)$. Verify that your formula reproduces the result above when $f(x) = e^x$.

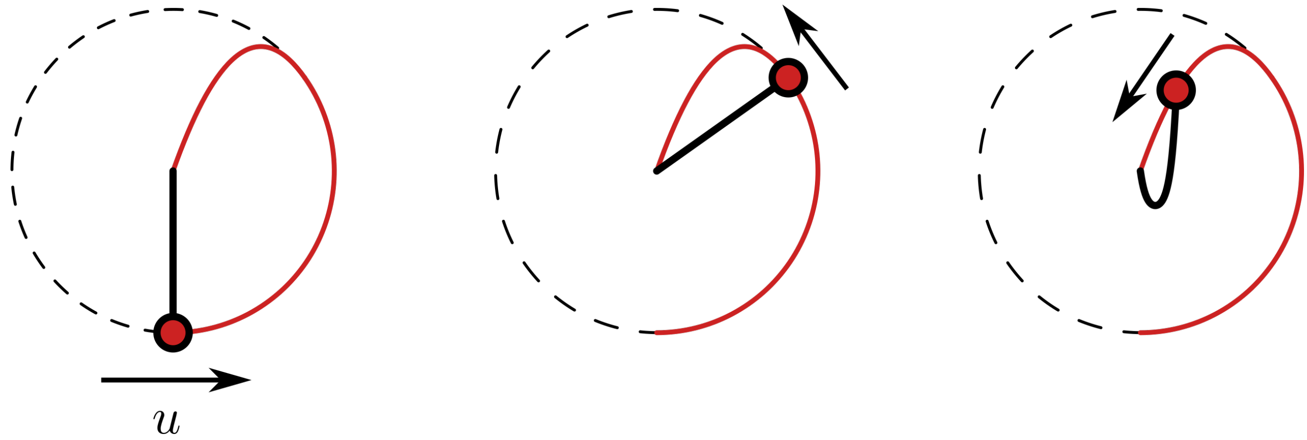
Hint: Consider the definite integral of $f^{-1}(x)$ between limits a and b. Sketch a graph showing this area. What is the graphical relationship between a function and its inverse? Can you use this to work out some other areas on the graph?

14 Hydrogen Harmonics



The Bohr model of the hydrogen atom has the electron following circular orbits around the stationary proton. However, acknowledging the wave–particle duality of the electron, Bohr's model only allows the electron to exist in stationary wave orbits, where the electrons de Broglie wavelength fits a whole number n times into the circumference of the orbit. Use this model to derive an expression for the energy levels of the hydrogen atom.

Hint: A circular orbit has two unknowns, the radius and the speed, so you will need two constraints to get an answer for the energy. The first is the Bohr condition on the de Broglie wavelength. What other equation could you use to relate the radius and speed of a circular orbit?



15 Catching At The Centre

A small ball bearing of mass m is suspended by a light string of length r from a small cup. The ball bearing is given an impulse so that it starts moving with a horizontal velocity u. Initially the ball will move in a circle of radius r centred on the cup. However, at some moment the string goes slack and the ball falls as a projectile. Find an expression for the value of u such that the ball will land in the cup at the centre of the circle.

Hint: The tension in the string will be whatever it needs to be to keep the radial component of the resultant force on the ball equal to the centripetal force. What can we say about the tension at the moment the string goes slack?