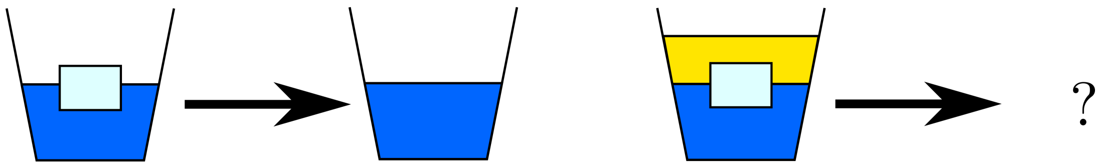
1 Melting Mysteries



An ice cube is floating in a glass of water. Explain why the water level in the glass remains constant as the ice cube melts.

Once again, consider an ice cube floating in a glass of water. However, this time we will pour oil into the glass until the ice cube is completely submerged. Since ice is less dense than water but denser than oil, it will float at the boundary between the two liquid layers. When the cube melts, what happens to the water level and the oil level?

Solution: For qualitative questions like this, it is often not necessary to perform any detailed calculations, although a writing down a few equations can help structure your thinking. Instead, it is most important to identify the key physical principles at play. In this case it is Archimedes' principle, that the upthust on an object is equal to the weight of the fluid it is displacing.

When an ice cube floats in water, it must rest at a height such that the force of upthrust, equal to the weight of the water it is displacing, is equal in magnitude to the cube's own weight. This means that the ice cube displaces a volume of water with the same weight as the cube. When the ice melts, its weight stays the same, so it turns into a volume of water with the same weight as the cube. In other words, the ice cube turns into the same volume of water it was displacing, leading to no change in water level.

When the oil is poured over the ice cube it too exerts a force of upthrust on the cube. As such, the ice cube will have to move up relative to the water level so that it is displacing less than its weight worth of water, with the remainder of its weight being balanced by the upthrust from the oil. The ice cube is displacing less water than before; however upon melting it will still turn into the same volume of water, so the water level will rise. The oil level is determined entirely by the total volume of everything in the glass. When the ice melts its volume is reduced, so the oil level will fall.

2 An Interesting Integral

Evaluate the value of the definite integral

$$\int_{-\infty}^{\infty} \sin(x)e^{-x^2}\, dx\,.$$

Solution: When given a definite integral, a good first step is to look for any special properties of the limits, and if those properties relate to the integrand in some way. In this case the limits are symmetric: one is the negative of the other, and the integrand is an odd function, meaning that

$$\sin(-x)e^{-(-x)^2} = -\sin(x)e^{-x^2}\,.$$

In general, the integral of any odd function between symmetric limits will always be zero. There are two ways of seeing this. The first is algebraic by making the substitution $x = -u$ to the integral, yielding

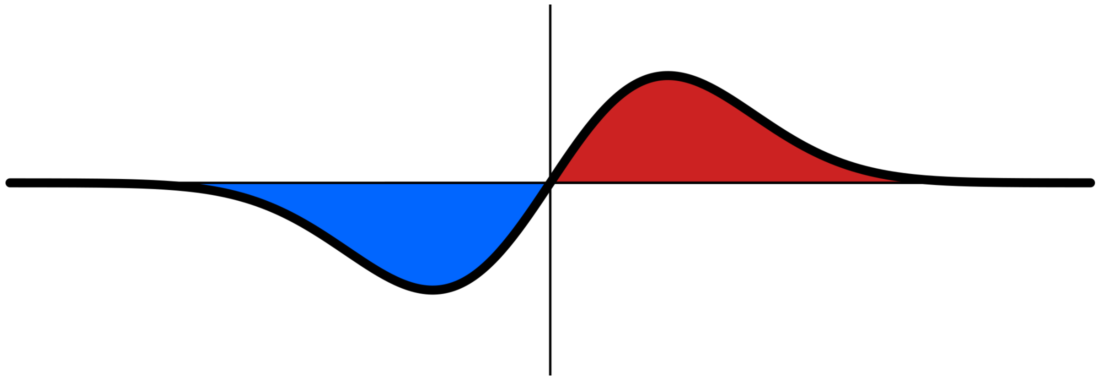
$$\int_{-\infty}^{\infty} \sin(x)e^{-x^2}\,dx = \int_{-(-\infty)}^{-\infty} \sin(-u)e^{-(-u)^2}\,d(-u) = \int_{-\infty}^{\infty} \sin(u)e^{-u^2}\,du\,,$$

where we can see that the minus sign from inverting the argument of the integrand has cancelled with the minus sign from changing the integration variable. However, the limits of the integral have been reversed, and putting them back in the correct order will induce another minus sign leading to

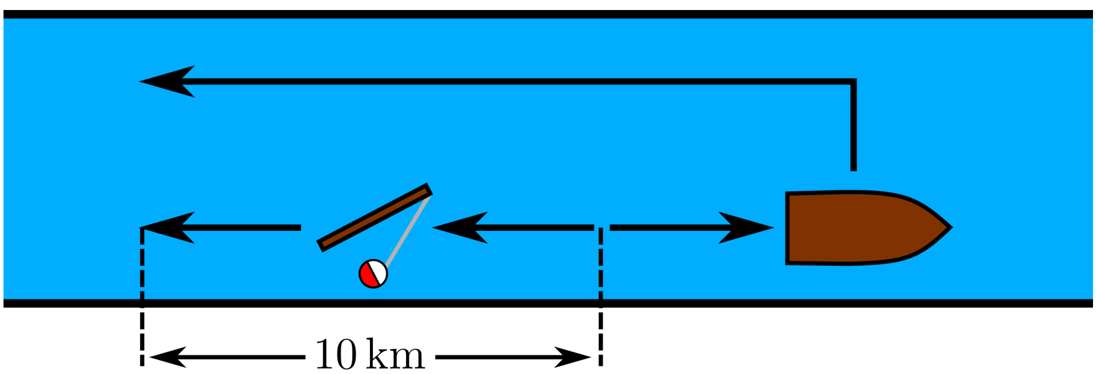
$$\int_{-\infty}^{\infty} \sin(x)e^{-x^2}\,dx = -\int_{-\infty}^{\infty} \sin(u)e^{-u^2}\,du\,.$$

Now, the specific letter we use to represent the variable of integration is not important, so both integrals above must be equal to the same thing. However, this implies that the integral is equal to minus itself, which can only be true if it is equal to zero.

The second approach is graphical. If we look at the graph of any odd function, then we will see that for every positive area to the right of they y axis (red), there is a corresponding negative area to the left (blue). Thus, once we integrate over both sides of the axis, the positive and negative areas cancel, and the total is zero.



3 The Fumbling Fisherman



A fisherman is rowing upstream along a river to reach a good fishing spot. At some point in time, his fishing rod slips out of the boat and is carried away by the flowing river. When the fisherman arrives at his spot 20 minutes later and realises the mistake, he turns around and rows back the way he came. When he eventually reaches his fishing rod and scoops it out of the water, he is 10 km further downriver than when he first dropped the fishing rod. What is the speed of the river?

Solution: One interesting thing to note about this question is that we are actually given very little information. In fact, we are given so little information that we can basically guess the answer without doing any actual physics. We are only given one distance and one time, so the only way we can generate anything with units of speed is to divide the first be the second. Thus, the answer must take the form

$$v = k\frac{10\,\mathrm{km}}{20\,\mathrm{min}}\,,$$

where k is some numerical constant, which will later turn out to be a half.

To solve this problem properly, the key insight is that, if the river is flowing at steady speed v, the fisherman will row at a constant speed u relative to the river. This has the consequence that it will take the same 20 minutes the fisherman spent rowing after dropping the rod before he reunites with it after turning around. We can see this algebraically as follows. Let v represent the speed of the river and u the speed of the fisherman relative to the river. Then let t_1 be the time after dropping the rod that the fisherman turns around. The positions of the rod and boat at t_1 are given by

$$x_r = -vt_1 \qquad\qquad x_b = (u-v)t_1\,.$$

At a further time t_2 after the fisherman turns around, the positions will be

$$x_r = -v(t_1+t_2) \qquad\qquad x_b = (u-v)t_1 - (u+v)t_2\,.$$

When the fisherman reunites with his rod we must have

$$x_r = x_b \implies -v(t_1+t_2) = (u-v)t_1 - (u+v)t_2 \implies u(t_1-t_2) = 0 \implies t_1 = t_2\,.$$

Therefore the total time between the fisherman dropping the rod and picking it back up again is $20 + 20 = 40$ minutes, during which time the river carried the rod a distance of 10 km. Thus, the speed of the river is

$$v = \frac{10\,\mathrm{km}}{40\,\mathrm{min}} = 250\,\mathrm{m\,min^{-1}} = 4.17\,\mathrm{m\,s^{-1}}\,.$$

4 Laser Levitation

A person of mass m aims a laser pointer with a wavelength λ straight down and turns it on. Find an expression for the power P that required for the person to be able to hover above the ground?

Solution: Before we dive into the algebra on a question like this, it is worthwhile to make sure we understand what physics it is which is actually responsible for the effect in question. In other words, we should first ask ourselves why the laser would lift the person up in the first place, before we worry about how powerful it needs to be to do so.

The answer is that, since light carries momentum, when it is emitted it must exert a recoil force on the laser to ensure that momentum is conserved. The person will hover when the force from this radiation pressure is equal in magnitude to their weight. Since the momentum of light is crucial to making this work, a good starting point is to write down an equation we know for the momentum of light, such as the De Broglie relation for the momentum of a photon

$$p = \frac{h}{\lambda} \,.$$

Next we need to link this to the recoil force on the laser. To do this we appeal to Newton's second law that force is the rate of transfer of momentum. The total momentum transferred in one unit of time is equal to the number of photons emitted in that time multiplied by the momentum of each photon, so

$$F = \frac{\Delta p}{\Delta t} = \dot{N} p = \frac{\dot{N} h}{\lambda} \,,$$

where $\dot{N}$ is the rate of photon emission, in other words the number of photons that are emitted per second. We have introduced the photon rate $\dot{N}$; however, we are asked to express our answer as a power, i.e the energy transferred per second. This power will just be the photon rate times the energy of one photon

$$E = hf = \frac{hc}{\lambda} \,,$$

which gives

$$P = \dot{N} E = \frac{\dot{N} hc}{\lambda} = Fc \,.$$

Finally, we equate the force to the person's weight mg to find that the required power is

$$P = Fc = mgc \,.$$

For a typical human on Earth $m \sim 10^2\,\mathrm{kg}$, $g \sim 10\,\mathrm{m\,s^{-2}}$, and $c \sim 3 \times 10^8\,\mathrm{m\,s^{-1}}$, giving a power on the order of 300 GW, which would certainly be an impressive laser pointer (for reference, you could get in trouble for bringing a laser stronger than 1 mW into a school without a good reason).

5 Sinusoidal Sketching

Sketch the curve

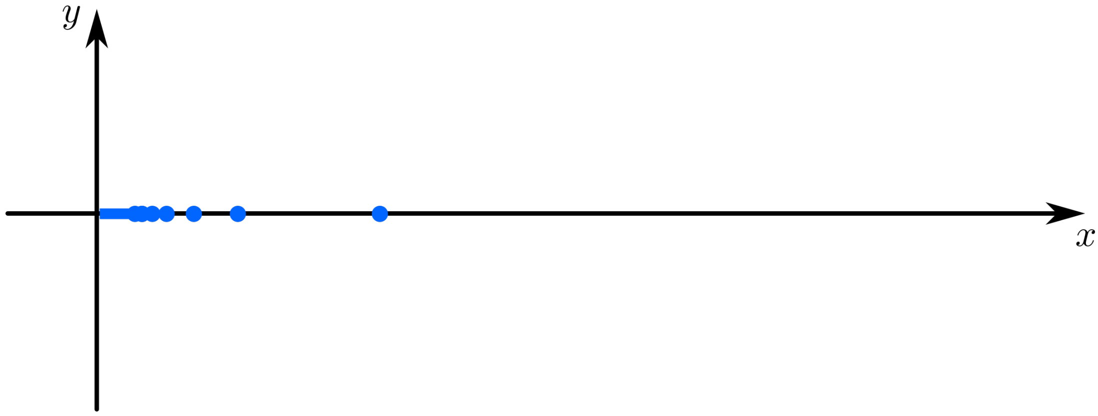
$$y = x \sin\left(\frac{1}{x}\right) \,.$$

Solution: When sketching any graph there are a number of things we can look for which provide helpful starting points. Firstly, it's always worth checking for symmetry. In this case $y(x)$ is an even function, meaning that $y(-x) = y(x)$ so the graph will be symmetric about $x = 0$. As such, we only need to sketch the curve along the positive x axis, since the negative x axis will just be a mirror reflection.

Next we can consider the zeros of the curve where it intersects the x axis. In this case we will have zeros whenever

$$\sin\left(\frac{1}{x}\right) = 0 \implies \frac{1}{x} = \pi, 2\pi, 3\pi, \cdots \implies x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots \,.$$

The important thing to notice is that these zeros get more and more closely packed as they approach the origin, eventually becoming so close together that they are indistinguishable. These zeros a plotted below. The next steps are to consider what happens to the curve at the origin and at infinity.



We can already tell from the zeros that something interesting is happening as x approaches zero, so we shall start there. The issue is that since the $1/x$ in the argument of the sin diverges at the origin, the function begins to oscillate wildly eventually becoming undefined at $x = 0$. However, we can still discuss the limiting behaviour of the curve in a meaningful sense. The key point is that, no matter how wildly it oscillates, sin will always be contained between -1 and 1, which means that y must always be contained between and envelope of $-x$ and x. Thus, as $x \to 0$ $y \to 0$ since the size of the envelope shrinks to zero.

To discuss the behaviour as x becomes large, we can use a small angle approximation to note that

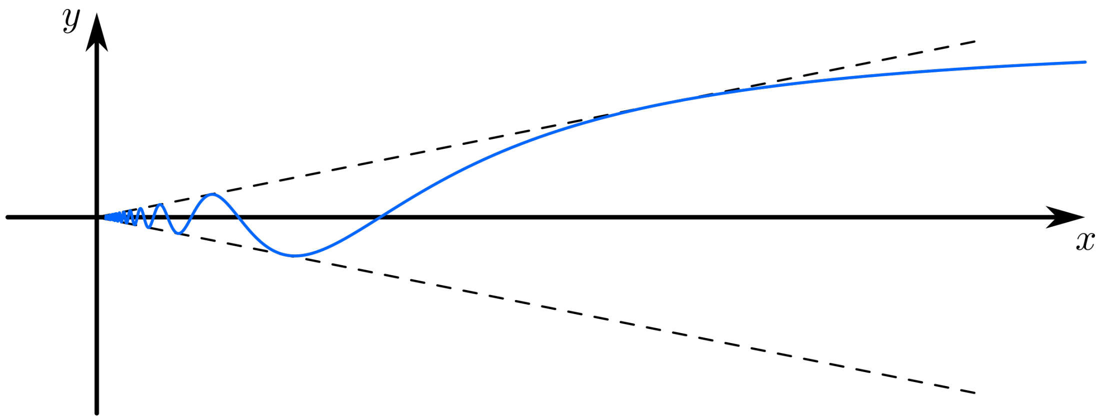
$$\sin\left(\frac{1}{x}\right) \approx \frac{1}{x} \qquad \text{when} \qquad x \gg 1\,,$$

and thus that

$$y = x \sin\left(\frac{1}{x}\right) \approx \frac{x}{x} = 1 \qquad \text{when} \qquad x \gg 1\,.$$

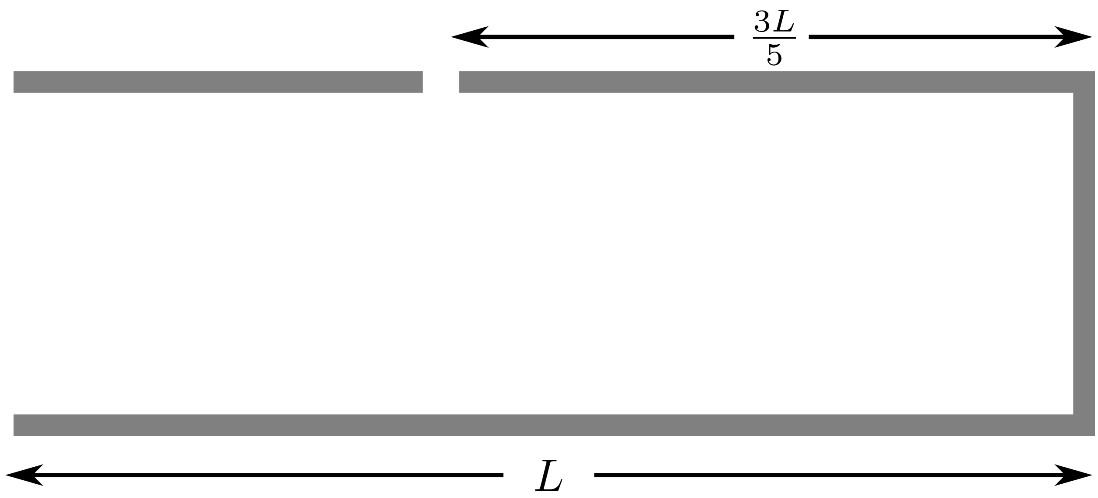
In other words, $y = 1$ is a horizontal asymptote for the curve as x tends towards infinity.

Putting this all together, we need a curve which oscillates inside an envelope $-x < y < x$, going through each of the zeros plotted above, before asymptotically approaching $y = 1$. This will look something like the plot below.

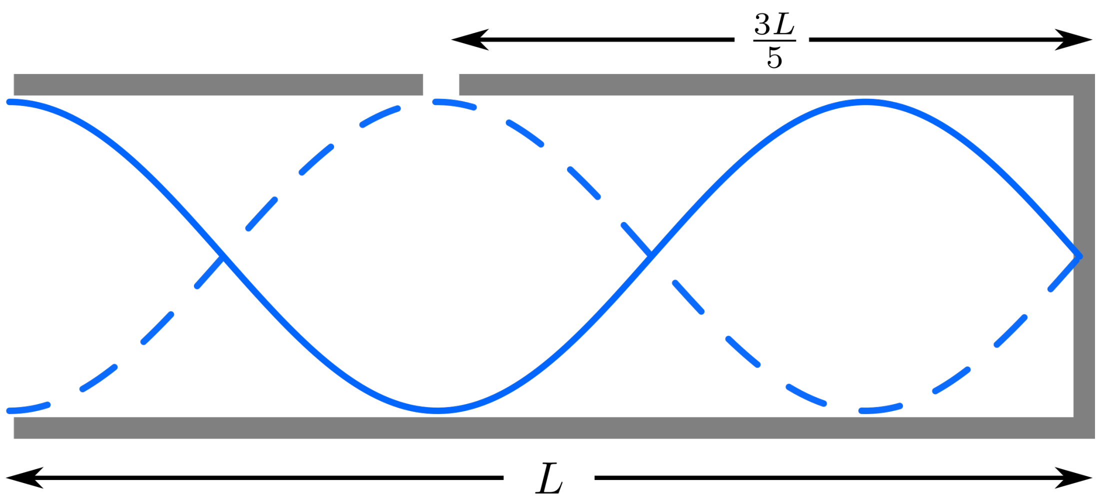


6 Strange Stationary Waves

The diagram below shows a cylindrical pipe of length L that is closed on one end, with a small hole three fifths of the way along its length. Sketch the fundamental stationary wave inside the pipe, and write down an expression for the frequency of the nth harmonic.



Solution: The key knowledge that is needed here is that, for a sound wave in a pipe, the closed end must be a node as there is no space for the air molecules to oscillate, and that the open end must be an antinode because it is always at atmospheric pressure. With this in mind, the small opening in the pipe will also open the pipe to atmospheric pressure, similarly forcing an antinode at that location. The simplest stationary wave with a node at the fixed end, and antinodes at the two openings is shown below.



Between the open end and the hole the wave goes from antinode to node and back to antinode again, which is one half wavelength. Therefore

$$\frac{\lambda}{2} = \frac{2L}{5} \implies \lambda = \frac{4L}{5}\,.$$

More generally, for any stationary wave to exist in this pipe it must be able to fit an odd number of half wavelengths between the opening and the hole. It must be a whole number so that both points can be antinodes, and odd so that when the wave reaches the closed end it is a node. Thus, the wavelength of the nth harmonic can be written as

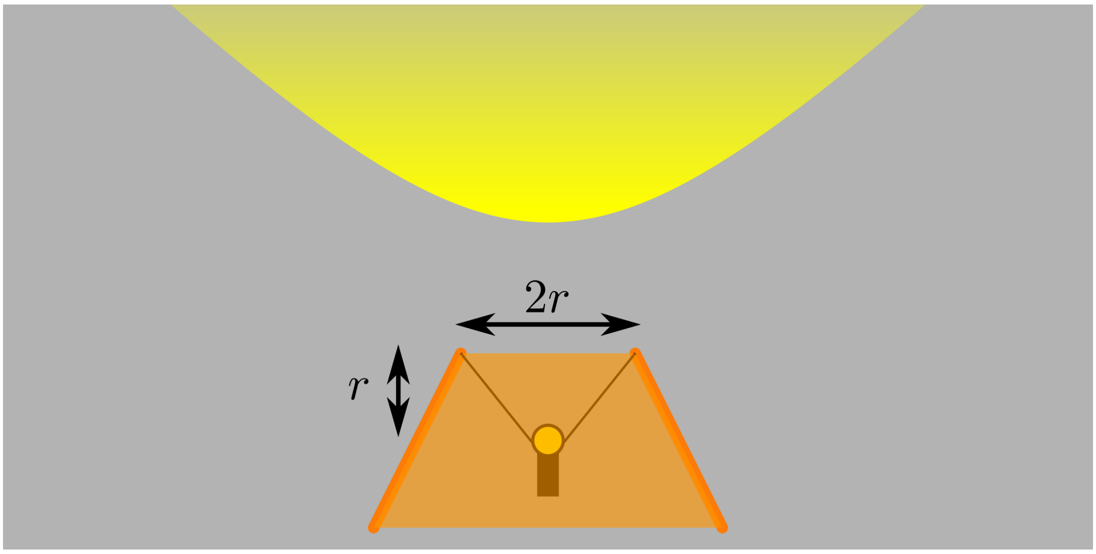
$$\frac{(2n-1)\lambda}{2} = \frac{2L}{5} \implies \lambda = \frac{4L}{5(2n-1)}.$$

To find the frequency, we simply substitute this result into the wave speed equation

$$f = \frac{c_s}{\lambda} = \frac{5(2n-1)}{4}\frac{c_s}{L},$$

where c_s is the speed of sound in air. Interestingly, these frequencies are identical to those that would be obtained from a closed open pipe one fifth the length of the one in this question. This makes sense as every stationary wave inside this pipe has a node at the closed end and an antinode one fifth of the way along its length.

7 Light from a Lampshade



A light fixture is constructed by placing a small bulb inside a lampshade and mounting the assembly on a vertical wall. The lampshade has a circular hole of radius r at its top, a height r above the bulb. The fixture is mounted so that the bulb is a distance $2r$ from the wall. Find an equation for the shape that the light makes on the wall above the lamp.

Solution: This question is mostly about geometry and algebra: the only physics we actually need to know is that light travels in straight lines. With this in mind, the light leaving the top of lampshade is going to be composed entirely of rays, which can go straight from the bulb, which is small enough to be considered a point source of light, through the hole in the top. In other words, these light rays will form a 3D cone, with its tip located at the bulb and its edges inclined at 45° to the vertical.

The shape of the light on the wall is then the section of a vertical plane through this cone. Conic sections come in four different varieties depending on the angle of the plane that is slicing the cone. If the plane is exactly perpendicular to the cone it forms a circle, whereas shallow angles away from this form ellipses. A plane parallel to one edge of the cone will form a parabola and a plane inclined any more steeply will form a hyperbola. Since our plane is parallel to the axis of the cone, we are in this latter case. We can now analyse this problem algebraically.

If we are going to derive an equation to describe the shape on the wall, then we will need to define our coordinate system. A sensible choice is to set the origin to the point on the wall directly behind the light bulb, and to orient our coordinate axes so that x is horizontal, y is vertical, and z is directly out of the wall. In this case the bulb has coordinates $(0, 0, 2r)$.

Next, we can write down the equation for the surface of the cone of light leaving the top of the lampshade. Since the edges of this cone are inclined at 45° to the vertical, at any height this cone traces out a circle centred on the bulb with radius equal to the height above the bulb y. So the equation is

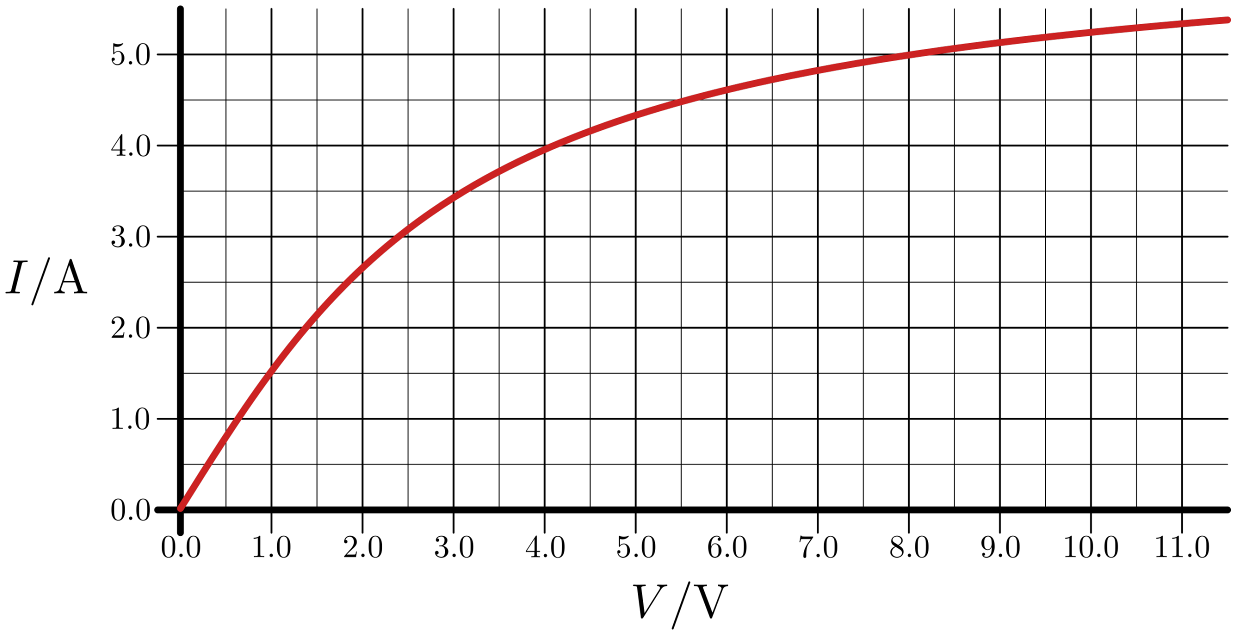
$$x^2 + (z - 2r)^2 = y^2 \,.$$

The wall corresponds to the plane $z = 0$ so we can substitute this into the above equation to obtain

$$x^2 + 4r^2 = y^2 \implies y^2 - x^2 = 4r^2 \,,$$

which is indeed the equation for a hyperbola.

8 Internal and Incandescent



The graph shows the $I - V$ characteristic for an incandescent bulb. This bulb is powered by a cell with an emf of 10 V and an internal resistance of 2.0 Ω. What is the power of the bulb?

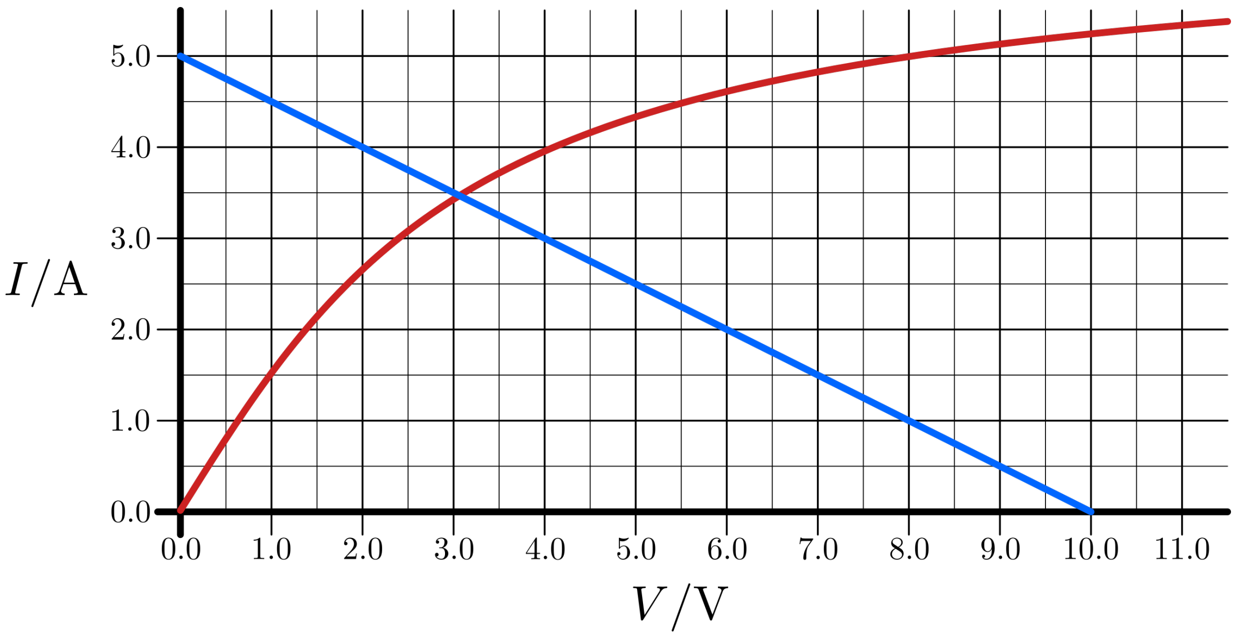
Solution: In a simple series circuit with a cell and one component we have two unknown quantities: the current I and the terminal p.d V. As always, if we have two unknowns, then we will need two simultaneous constraints to solve for them. These constraints will be the relationship between current and p.d in the component. In other words, we can say that

$$I = f(V) \,,$$

where f is some function which depends on the component in question. We don't need to worry about the specific identity of this function because the $I - V$ characteristic we are provided is just a graph of precisely this function. The second constraint comes from the properties of the cell. If it has emf ξ and internal resistance r, then the terminal p.d V must satisfy

$$\xi = V + Ir \, .$$

Both of these equations must apply to the true values of I and V in the circuit, so the problem of finding these values is reduced to solving these two equations simultaneously. Noting that solving simultaneous equations is equivalent to finding the intersection between two curves, we can solve the equations graphically by plotting both equations on the same set of axes. The first is done for us on the $I - V$ characteristic, and the second is just a straight line with a gradient of $-1/r = -1/2\,\Omega^{-1}$ and an x-intercept of $\xi = 10\,\mathrm{V}$, as shown below.



A careful reading from the graph reveals that the two lines meet at values

$$I = 3.47\,\mathrm{A} \qquad\qquad V = 3.07\,\mathrm{V} \, .$$

The power can then be calculated straightforwardly as

$$P = IV = 10.7\,\mathrm{W} \, .$$

9 A Difficult Derivative

Find the derivative of the function

$$y = \log_x 5 \, .$$

Solution: Often, when manipulating logarithms it can a good idea to utilise the definition of the logarithm and exponentiate both sides of an equation. In this case we can say, by definition, that

$$y = \log_x 5 \implies x^y = 5\,.$$

One strategy would be to take this new equation and differentiate it implicitly, then solve for dy/dx. This works, provided you know how to differentiate x^y. A slightly more accessible approach is to use the surprisingly useful trick

$$x = e^{\ln x} \implies x^y = e^{y\ln x} = 5\,.$$

We can now take natural logs of both sides of this equation to end up with

$$y \ln x = \ln 5 \implies y = \frac{\ln 5}{\ln x}\,,$$

which is straightforward to differentiate with the chain rule, yielding

$$\frac{dy}{dx} = -\frac{\ln 5}{(\ln x)^2}\frac{d\ln x}{dx} = -\frac{\ln 5}{x(\ln x)^2}\,.$$

10 Spinning Saucer

A uniform circular disk with a mass m and a radius R rotates about an axis through its centre with an angular speed ω. What is its kinetic energy?

Solution: When unsure how to make progress with a question like this, a good strategy can be to first ask yourself why the question is hard. In this case, we know that the kinetic energy of a mass m moving at speed v is given by

$$E = \frac{1}{2}mv^2\,;$$

however, the problem we encounter is that the rotating disk does not have a single uniform speed. Instead, the speed of the material in the disk varies from 0 in the centre to $R\omega$. So, let us get started by approaching a similar version of this problem that does not have this difficulty. If instead of a disk, we consider a ring of negligible thickness, then every point on the ring would move with the same speed $R\omega$, and we could say that

$$E = \frac{1}{2}mR^2\omega^2\,.$$

In order to apply this to our problem, we need to recognise that a disk is actually composed of lots of rings of different radii all joined together. If we consider the ring of material between the radii r and $r + dr$, then all the material will move with a speed $r\omega$, giving it a kinetic energy of

$$dE = \frac{1}{2}dmr^2\omega^2\,,$$

where dm is the mass of the infinitesimal ring. The total kinetic energy of the disk can be obtained by adding together the kinetic energies of all its parts. A sum over an infinite number of infinitesimal energies is just an integral so we have

$$E = \int dE = \int \frac{1}{2}dmr^2\omega^2\,.$$

Now all we need to do is express the infinitesimal mass dm in terms of r and dr so we can actually evaluate the integral. To do this we note that the ring has a circumference $2\pi r$ and a thickness dr, diving it an area

$$dA = 2\pi r\, dr\,.$$

Since the disk is uniform, the mass of any part of it is just the area multiplied by the total mass m and divided by the total area πR^2, giving

$$dm = \frac{m}{\pi R^2} dA = \frac{2mr}{R^2} dr\,.$$

Putting this all together

$$E = \int \frac{1}{2} dm r^2 \omega^2 = \int_0^R \frac{mr^3\omega^2}{R^2} dr = \left[\frac{mr^4\omega^2}{4R^2}\right]_0^R = \frac{1}{4} mR^2\omega^2\,.$$

11 Paternoster Pendulum

A simple pendulum with a length l in a gravitational field g has a period given by

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

How would this expression be modified for the period of a pendulum inside a lift which was accelerating upwards with acceleration a?

Solution: The easiest way to answer this question is to invoke Einstein's equivalence principle that inside a sealed box with acceleration a, all physics from your point of view behaves as if you were not accelerating, but instead located in an additional gravitational field of strength a in the opposite direction. This is why your body feels heavier when a lift accelerates upwards and lighter when accelerating downwards; from your point of view the strength of gravity is actually changing. With this in mind, inside the lift the gravitational field strength will effectively be $g + a$, leading to a period

$$T = 2\pi\sqrt{\frac{l}{g+a}}\,.$$

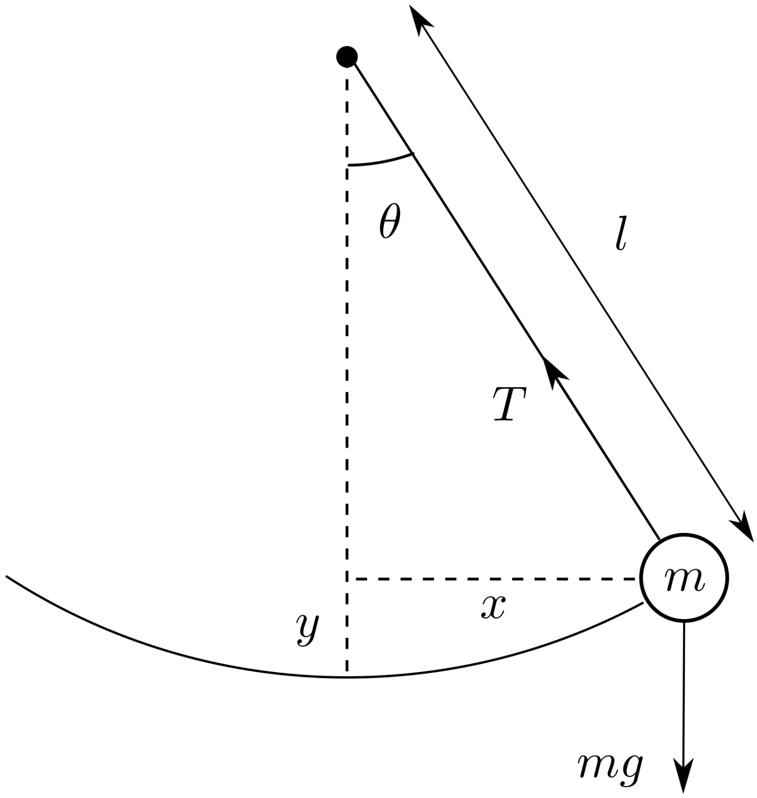
However, supposing we were not familiar with Einstein's equivalence principle, it should still be possible to reach this conclusion using standard Newtonian mechanics. To start, let us review the derivation of the period for a simple pendulum in the absence of acceleration.

If we resolve the forces on the pendulum bob and apply Newton's second law, then we obtain the equations of motion

$$m\frac{d^2x}{dt^2} = -T\sin\theta \qquad \text{and} \qquad m\frac{d^2y}{dt^2} = T\cos\theta - mg\,.$$

Using a bit of trigonometry we can say that

$$\sin\theta = \frac{x}{l} \qquad \text{and} \qquad \cos\theta = \frac{l-y}{l}\,.$$



Under the small angle approximation we can say that $\cos\theta \approx 1$ and therefore that $y \approx 0$, meaning that the second derivative of y is also equal to zero. Thus, we can say that

$$T \approx mg \implies m\frac{d^2x}{dt^2} = -\frac{mg}{l}x\,,$$

which when compared to the standard equation for simple harmonic motion

$$\frac{d^2x}{dt^2} = -\omega^2 x\,,$$

allows us to conclude that

$$\omega^2 = \frac{g}{l} \implies T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}\,.$$

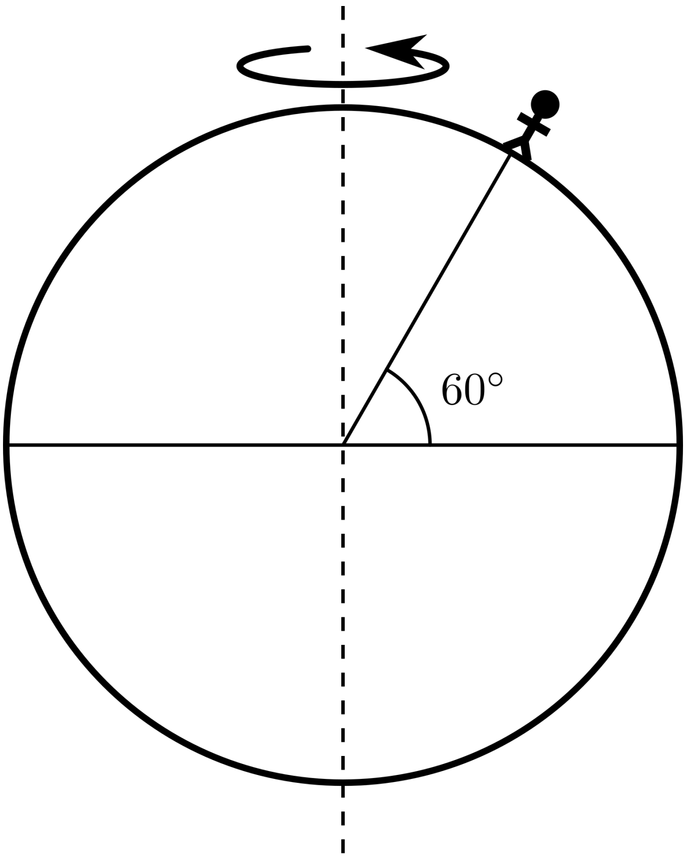
The derivation with acceleration proceeds very similarly, we simply note that y is being measured relative to the accelerating point in the left where the pendulum is secured, so the equation of motion becomes

$$m\frac{d^2y}{dt^2} + ma = T\cos\theta - mg \implies m\frac{d^2y}{dt^2} = T\cos\theta - m(g+a)\,,$$

and then continue the derivation in exactly the same way.

12 The Sky from Skye

At any moment in time, half of the sky is visible to us, while the other half is hidden behind the Earth underneath the horizon. As the Earth rotates, different parts of the sky come into view; however, even over the course of a full day, some regions of the sky will never come into view, for



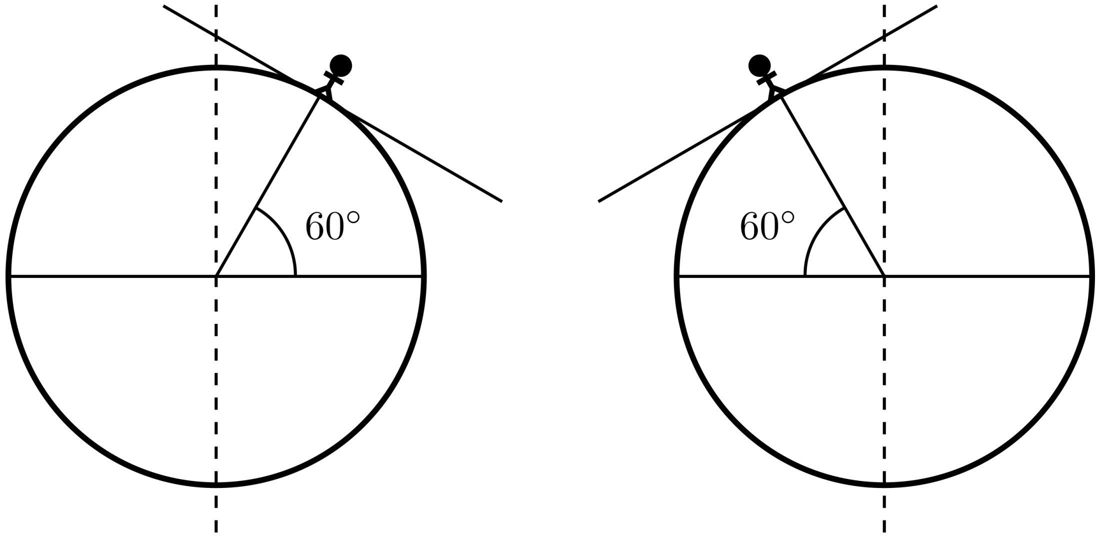
example the star Alpha Centauri is never visible from the UK, while Polaris is never visible from Australia. This is further complicated by the fact that some stars are not visible, even when they are above the horizon. For example, in the Summer the Sun is directly in front of the constellation Cancer, making those stars essentially invisible. However, over the course of a full year the Earth's orientation relative to the Sun changes and this issue can be avoided.

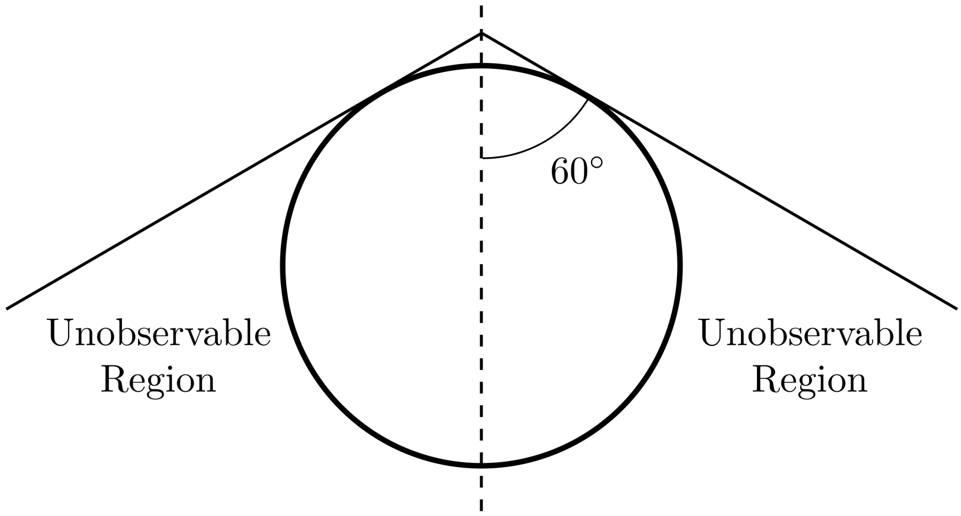
The Isle of Skye of the coast of Scotland is located at a latitude of approximately 60°, meaning that the angle between it and the equator is equal to 60°. What fraction of the stars in the sky would be visible from Skye over the course of a full year?

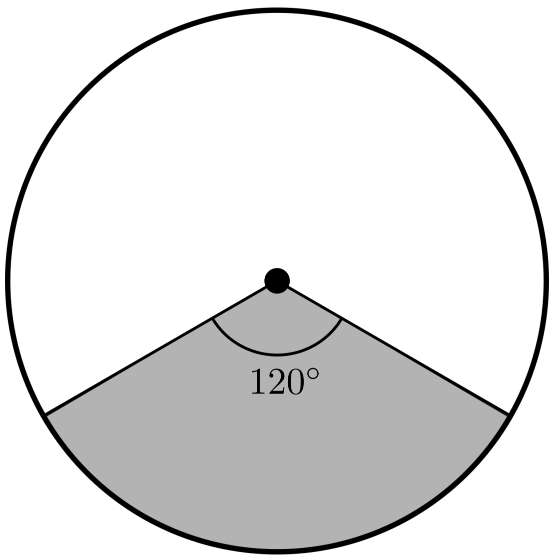
Solution: The first step to solving this problem is to draw onto the diagram the horizon at a particular moment in time. This horizon will be the plane tangential to the surface of the Earth at the point of observation. As the Earth rotates the plane will rotate with it and the region of space that is unobservable will be the area contained by the plane as it sweeps around. This unobservable region will take the shape of a cone making an angle of 60° to its axis at the apex, such that the plane representing the horizon is always tangent to the surface of the cone.

Now technically this cone has its apex slightly above the surface of the Earth. However, compared to the distances to the stars, the Earth is so small that we can treat any point on or slightly above its surface as being in the same place. As such, if we imagine a sphere of observable stars centred on the Earth, the hidden region will be a segment with angle 120°.

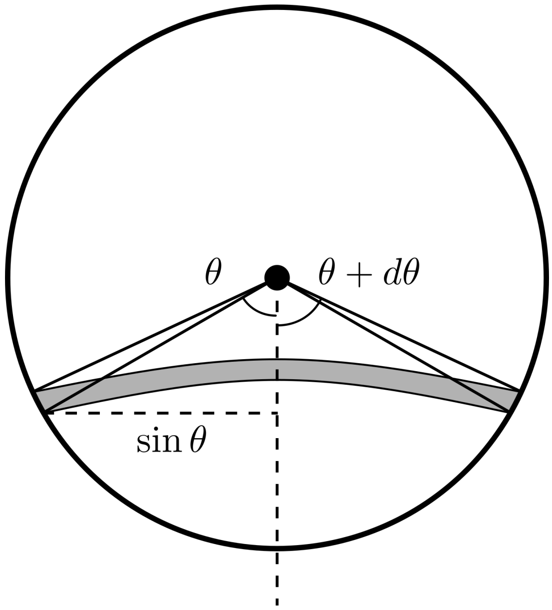
At this point it is incredibly tempting to claim that one third of the sky is permanently hidden from view, since 120° is one third of 360°. However, in doing so we would be neglecting the 3D nature of space and the fact that the full sky takes the shape of a sphere, not a circle.







What we actually need to find is the fraction of the celestial sphere's surface area which is contained within the hidden cone. To work out the surface area of a segment on a sphere, it is helpful to first consider a thin ring on the surface containing points which make angles between θ and $\theta + d\theta$ to some axis.



Setting the radius of the sphere to 1, since it will just cancel out of the final answer anyway, this ring is a circle with radius $\sin\theta$. Moreover, since the thickness of the ring subtends an angle $d\theta$, the area of the ring will be its circumference times the thickness

$$dA = 2\pi \sin\theta \, d\theta \, .$$

To find the total surface area up to some angle, we simply need to add together the areas of each of the individual rings which make it up. Adding together an infinite number of infinitesimal areas is nothing more than an integral, so for example the total surface area of the sphere, which extends from angles of 0 to π rad is

$$A = \int dA = \int_0^{\pi} 2\pi \sin\theta \, d\theta = [-2\pi\cos\theta]_0^{\pi} = 4\pi \, .$$

The unobservable area of the celestial sphere extends from 0 to 60°, i.e from 0 to $\pi/3$ rad, which means that the unobservable fraction is

$$\frac{1}{4\pi}\int_0^{\pi/3} 2\pi \sin\theta \, d\theta = \left[-\frac{\cos\theta}{2}\right]_0^{\pi/3} = \frac{1}{4} \, .$$

In other words, three quarters of the sky is visible from the isle of Skye.

13 Integrating Inverses

Show that

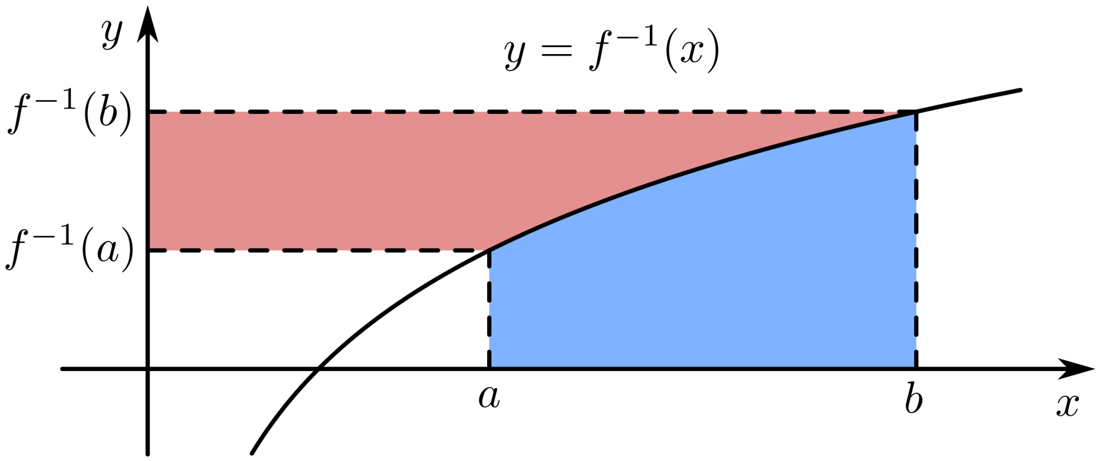
$$\int \ln x \, dx = x \ln x - x + \text{const.} \, .$$

Now obtain a general formula for the integral of an inverse function

$$\int f^{-1}(x) \, dx \, ,$$

expressing your answer in terms of the inverse, $f^{-1}(x)$, the original function $f(x)$, and its antiderivative $F(x)$. Verify that your formula reproduces the result above when $f(x) = e^x$.

Solution: While you can solve this problem using traditional integration methods, it is conceptually clearer if we draw a diagram of what we are doing.



If we look at the graph, then we can see that the integral of $f^{-1}(x)$ is represented by the area under the graph shaded in blue. This area can be calculated as the difference between two rectangles, giving $bf^{-1}(b) - af^{-1}(a)$, minus the red shaded area between the curve and the y axis. Here we can recognise that since the curve of an inverse function is just a reflected version of the original functions curve, this red area can be calculated as an integral of the original function. That is to say

$$\int_a^b f^{-1}(x)\,dx = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(y)\,dy\,.$$

If $F(x)$ is an antiderivative of $f(x)$ we can write this as

$$\int_a^b f^{-1}(x)\,dx = bf^{-1}(b) - af^{-1}(a) - F\left(f^{-1}(b)\right) + F\left(f^{-1}(a)\right)\,,$$

from which we can conclude that

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x)) + \text{const.}\,.$$

Mathematically, what we have actually done here is used integration by parts to conclude that

$$\int y dx = xy - \int x dy\,,$$

and then substituted in the relations $y = f^{-1}(x)$ and $x = f(y)$.

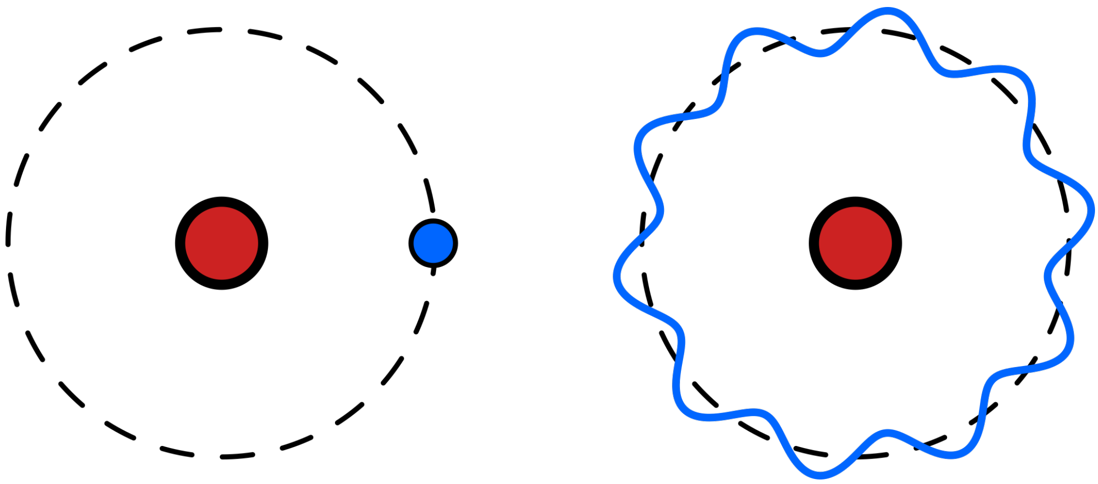
Either way, once we have our result we can substitute in

$$f(x) = e^x \implies f^{-1}(x) = \ln x \qquad F(x) = e^x\,,$$

which allows us to conclude that

$$\int \ln x\,dx = x\ln x - e^{\ln x} + \text{const.} = x\ln x - x + \text{const.}\,.$$

14 Hydrogen Harmonics



The Bohr model of the hydrogen atom has the electron following circular orbits around the stationary proton. However, acknowledging the wave–particle duality of the electron, Bohr's model only allows the electron to exist in stationary wave orbits, where the electrons de Broglie wavelength fits a whole number n times into the circumference of the orbit. Use this model to derive an expression for the energy levels of the hydrogen atom.

Solution: As with any circular motion, if the electron is to follow a circular orbit around the proton, then there must be a force acting on it to provide the required centripetal acceleration. In this case, the force is going to be the electrostatic attraction to the proton, so we can equate the force from Coulomb's law to the centripetal force, yielding

$$F_{\text{coulomb}} = \frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r} = F_{\text{cent}} \, .$$

Furthermore, if we impose the Bohr quantisation condition that there are n de Broglie wavelengths in one circumference of the orbit, we obtain

$$2\pi r = n\lambda = \frac{hn}{mv} \, .$$

We now have two simultaneous equations in the two unknowns r and v, so we can solve them in the standard way by using one equation to eliminate an unknown from the other, which gives us

$$r = \frac{h^2 n^2 \varepsilon_0}{\pi m e^2} \qquad v = \frac{e^2}{2\varepsilon_0 hn} \, .$$

The energy of the electron is given by the sum of its kinetic and potential energies

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} \, ,$$

which after substituting in the above results can be rearranged to give

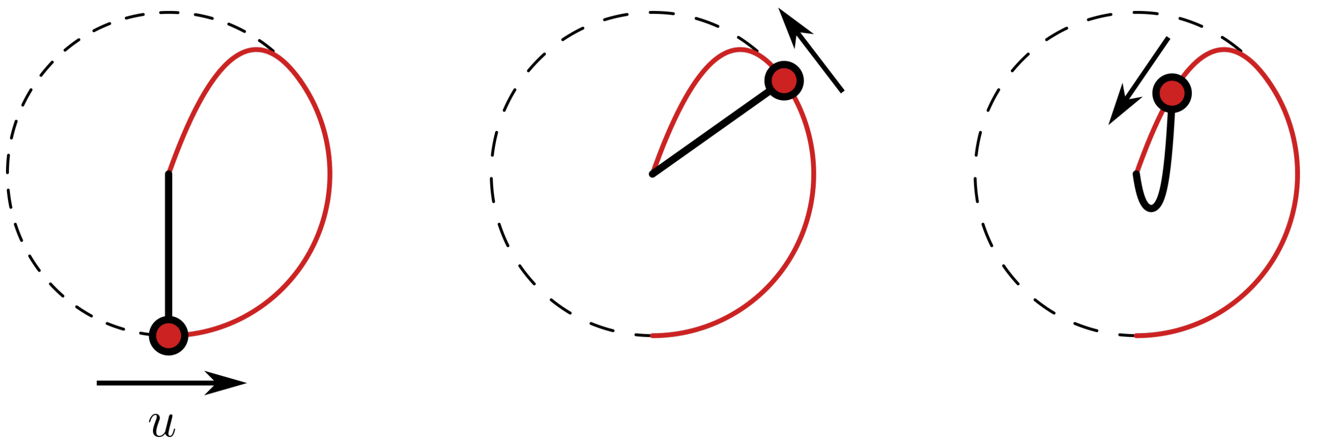
$$E = -\frac{me^4}{8\varepsilon_0^2 h^2 n^2} = -\frac{E_H}{n^2} \, ,$$

where

$$E_H = \frac{me^4}{8\varepsilon_0^2 h^2} \simeq 13.6\,\mathrm{eV}\,,$$

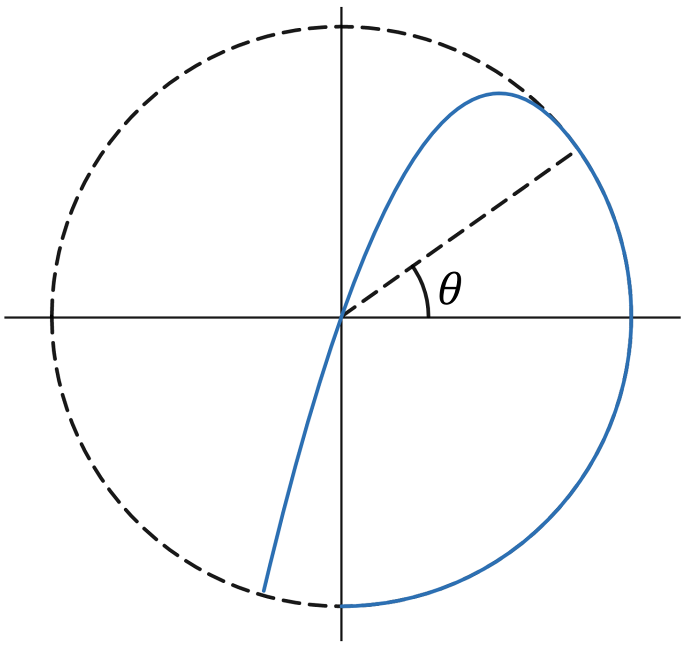
is the ionisation energy of the hydrogen atom.

15 Catching At The Centre



A small ball bearing of mass m is suspended by a light string of length r from a small cup. The ball bearing is given an impulse so that it starts moving with a horizontal velocity u. Initially the ball will move in a circle of radius r centred on the cup. However, at some moment the string goes slack and the ball falls as a projectile. Find an expression for the value of u such that the ball will land in the cup at the centre of the circle.

Solution: A good starting point is to analyse the most interesting part of the ball bearing's



motion: the moment at which the string goes slack and it transitions from circular to projectile motion. Firstly, by resolving forces on the ball and equating the radial component to the centripetal force we can conclude that the tension in the string T is given by

$$T - mg\sin\theta = \frac{mv^2}{r} \implies T = \frac{mv^2}{r} - mg\sin\theta\,,$$

where θ is the angle between the string and the horizontal. The string will go slack when its tension is equal to zero, i.e when

$$T = 0 \implies \frac{mv^2}{r} = mg\sin\theta \implies v^2 = gr\sin\theta\,.$$

After the string goes slack, the ball bearing will move as a projectile with an initial speed v inclined at an angle of θ to the vertical. We can now find the relationship between v and θ such that the ball passes through the centre of the circle and lands in the cup. To do this we will express the motion in two different directions as functions of time, and then eliminate the time to obtain a single constraint on v and θ. Conventionally we would do this with the horizontal and vertical directions; however, there is no rule that the two directions we pick have to be perpendicular. In this case it is actually slightly easier to resolve horizontally (no acceleration) and radially (no initial velocity). The relevant equations are

$$vt\sin\theta = r\cos\theta \quad \text{and} \quad \frac{1}{2}gt^2\sin\theta = r\,.$$

The simplest way to eliminate t from these equations is to square the first and divide by the second. This yields

$$\frac{v^2t^2\sin^2\theta}{\frac{1}{2}gt^2\sin\theta} = \frac{r^2\cos^2\theta}{r} \implies \frac{2v^2\sin\theta}{g} = r\cos^2\theta \implies v^2 = \frac{gr}{2}\frac{1-\sin^2\theta}{\sin\theta}\,,$$

where we have used the identity $\sin^2\theta + \cos^2\theta = 1$ in order to express everything in terms of $\sin\theta$. We now have two simultaneous equations in two unknowns (v and θ) so solving them is simply a matter of algebra. Equating our two expressions for v^2 we obtain

$$v^2 = gr\sin\theta = \frac{gr}{2}\frac{1-\sin^2\theta}{\sin\theta} \implies 3\sin^2\theta = 1$$

Feeding this back into the expression for v we find that

$$\sin\theta = \frac{1}{\sqrt{3}} \quad \text{and} \quad v^2 = \frac{gr}{\sqrt{3}}\,.$$

Finally, we note that in order to reach the point where the string went slack, the ball had to be raised through a height

$$\Delta h = r(1+\sin\theta)\,.$$

As such, we can solve for the initial velocity u by applying energy conservation

$$u^2 = v^2 + 2gr(1+\sin\theta)\,.$$

Putting this all together we find that the final answer is

$$u^2 = gr\left(2+\sqrt{3}\right)\,.$$