Special Relativity Problems

1. Some Lorentz factors

Estimate the Lorentz factor γ for each of the following objects.

i) An electron with a kinetic energy of 1 keV.

ii) The Earth in its orbit around the Sun (The Earth-Sun distance is $\sim$ 500 light seconds).

iii) An Olympic sprinter at their top speed.

In each case you should aim to obtain $\gamma - 1$ correct to at least one significant figure.

2. Timelike and spacelike separations

Two events are referred to as timelike separated in a given inertial frame, if the distance between where they occur is smaller than the distance light could travel in the time between when they occur. Similarly, they are referred to as spacelike separated if the distance between them is larger than the distance light could have travelled in the time between them.

i) Show that, if events A and B are timelike separated in a frame $\mathcal{K}$, there exists a frame $\mathcal{K}'$ where they occur in the same place.

ii) Show that, if events A and B are spacelike separated in a frame $\mathcal{K}$, there exists a frame $\mathcal{K}'$ where they occur at the same time.

iii) Show that, if two events are timelike separated in one frame, they will be timelike separated in all inertial frames. Do the same for spacelike separation.

Hint: Since you can always choose your axes such that both events occur along the x axis, you only ever need to deal with Lorentz transformations in the standard configuration.

3. The invariant interval

Show explicitly that the interval, defined as

$$s^2 = c^2t^2 - \left(x^2 + y^2 + z^2\right), \tag{1}$$

is invariant under a Lorentz transformation in the standard configuration. That is to say, show that, if we calculated s'^2, which is given by the same expression but with the coordinates from some other inertial frame, then we would find that $s'^2 = s^2$. Qualitatively explain why the interval is also invariant under rotations (there is no need to show this explicitly).

4. Simultaneous explosions

A train travels past a platform at speed $v = c/2$. As it does so, two small explosions occur at the front and back of the train. From the perspective of an observer inside the train, the explosions occur simultaneously and 10 m apart.

i) When and where do the explosions occur relative to one another from the perspective of an observer on the platform?

ii) Draw spacetime diagrams of these events, in the reference frames of the platform and train respectively.

5. A moving mirror

A stationary light source at the origin of an inertial frame $\mathcal{K}$ emits light of frequency ν_0. A mirror lying in the y, z plane moves towards the light source with velocity v along the x axis. Find an expression for the frequency ν_r of the light reflected by the mirror, as measured by a stationary observer somewhere along the x axis.

6. Twins and spaceships

Two twins, Albert and Emmy, have never travelled at relativistic velocities relative to one another and are thus the same age. One day, Emmy boards a long haul space flight, which will travel at $0.6\,c$ to a star 3 light years away, before turning around and returning to their home planet at the same speed. After one year of waiting at home, Albert decides that he misses his sister, boards a ship of his own, and sets out after her.

i) Albert travels at a speed $0.7\,c$. Once they reunite, which twin will be older, and by how much?

ii) What speed should Albert travel at, if he wants to be the same age as Emmy when they reunite?

7. Measuring distances with clocks

An astronaut in space wants to measure how far away he is from a nearby asteroid. To do this he starts a stopwatch and throws it as hard as he can, directly at the asteroid. It then bounces off, hitting the lap button in the process, and returns to the astronaut, who timed the whole process on his spacesuit's built in clock. He now has access to three time measurements τ_1, τ_2, τ_3 which are: the total time measured by his space suit, the time measured by the stopwatch on the way to the asteroid, and the time measured by the stopwatch during the return journey respectively. Show that the distance to the asteroid r can be given by

$$\frac{r}{c} = f(\tau_1, \tau_2, \tau_3)\,, \tag{2}$$

where f is a function you should determine. Be aware that the collision between the stopwatch and the asteroid may not be perfectly elastic, so you cannot assume that its velocity is the same for both the forward and return journeys.

8. The great spaceship debate

A question on a special relativity exam reads as follows:

"An alien spaceship flies over the Earth at a relativistic speed v. As it does so, the ship fires two probes which embed themselves into the Earth. On the ship, the probes are separated by a distance l_0 along its direction of motion, and are deployed simultaneously. When humans on Earth come to investigate the probes, what will they measure as the distance between them?"

Three students who took the exam discuss this question afterwards and realise that they all gave a different answer.

- Student A argues that, since the spaceship is moving with velocity v, the distance between the two probes should be length contracted to l_0/γ_v.
- Student B argues that nothing in the set-up of the problem should change if instead of dropping two probes, the spaceship dropped a single metal rod of length l_0, and so the distance between the two probes should just be l_0.
- Student C argues that, in the frame of the ship, the Earth is moving at speed v and so it will be length contracted by a factor of $1/\gamma_v$. Thus, they conclude that, to an observer on the Earth, the two probes should be separated by a distance of $\gamma_v l_0$.

Decide which of the students is correct, and explain the flaws in the arguments presented by the other two.

9. Impossible processes

Prove that the following processes are impossible whilst conserving energy and momentum.

i) A photon travelling through free space spontaneously producing an electron-positron pair.

ii) An free electron spontaneously emitting a photon.

iii) A stationary electron scattering a photon and recoiling in a direction perpendicular to the incident photon's momentum.

10. Invariant mass

i) Two massive particles have energies $E^{(1)}$ and $E^{(2)}$ with momenta $p_x^{(1)}, p_y^{(1)}, p_z^{(1)}$ and $p_x^{(2)}, p_y^{(2)}, p_z^{(2)}$ respectively. Using the behaviour of the energy and momentum under Lorentz transformations, or otherwise, show that

$$\frac{E^{(1)}E^{(2)}}{c^4} - \frac{p_x^{(1)}p_x^{(2)} + p_y^{(1)}p_y^{(2)} + p_z^{(1)}p_z^{(2)}}{c^2} = m_1 m_2 \gamma_{1,2} \,, \tag{3}$$

where m_1 and m_2 are the masses and $\gamma_{1,2}$ is Lorentz factor for the relative velocity between the particles.

ii) Hence, show that in a system of n massive particles, the invariant mass is given by

$$M^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j \gamma_{i,j} \,, \tag{4}$$

where m_i is the mass of the ith particle, and $\gamma_{i,j} = \gamma_{j,i}$ is the Lorentz factor for the relative velocity between the ith and jth particles. Explain how this formula should be modified if one or more of the particles is massless.

11. Particle accelerator

A linear accelerator uses a potential difference V in order to accelerate charged particles from rest up to relativistic speeds. Show that the speed of a particle after passing through this accelerator is given by

$$\frac{u}{c} = \frac{\sqrt{V^2 + 2XV}}{X + V} \,, \tag{5}$$

where $X = mc^2/q$ is the mass to charge ratio of the particle multiplied by c^2.

12. An inelastic collision

A particle of mass m_1 travelling at speed u collides with a stationary particle of mass m_2. Following the collision, the two particles coalesce into a single particle of mass M travelling at speed v. Find expressions for M and v in terms of m_1, m_2, and u. Show that these expressions reduce to the Newtonian results when $u \ll c$.

13. Δ^+ baryon decay

A Δ^+ baryon ($m_\Delta c^2 = 1232\,\mathrm{MeV}$) can decay into a proton ($m_p c^2 = 938\,\mathrm{MeV}$) and a neutral pion ($m_\pi c^2 = 135\,\mathrm{MeV}$). Determine the energies of the daughter particles from this decay, as measured in the rest frame of the Δ^+.

14. The Greisen-Zatsepin-Kuzmin limit

High energy cosmic ray protons can interact with photons from the cosmic microwave background via the reaction $p^+\gamma \to p^+\pi^0$. That is to say, the net effect of the collision is to convert a photon into a neutral pion. Taking the typical energy of a CMB photon to be $E_\gamma \approx 400\,\mu\mathrm{eV}$, estimate the minimum energy required by the proton for this process to be possible. You may assume that the proton and photon collide head on.

[$m_p c^2 = 938\,\mathrm{MeV}$ and $m_\pi c^2 = 135\,\mathrm{MeV}$]

15. Annihilation angle

A positron with kinetic energy E_K annihilates with a stationary electron to produce a pair of photons. Find the minimum and maximum values of the angle between the two photons' momenta. What is the minimum kinetic energy required for a possibility of two photons at right angles to one another?

16. Railgun rocket

A particular type of rocket is designed to travel through dust clouds in space. It generates thrust using a built in railgun-like mechanism to take in the dust accelerate it and then eject it out the back of the rocket. With its batteries uncharged the rocket has a mass $2m_0/3$, and when the batteries are charged its mass increases to m_0. In the rocket's rest frame, the railgun transfers a, very small, fixed amount of kinetic energy per unit mass to the dust it accelerates. If the rocket starts from rest with its battery fully charged, estimate its speed once the battery has been fully depleted.