POISEUILLE'S EQUATION FOR LIQUID-FLOW THROUGH A NARROW TUBE :

Poiseuille made the following assumptions :

(i): The liquid-flow is steady or streamline, ~~with~~ and parallel to the axis of the tube.

ii) The pressure is constant over any given cross-section of the tube.

iii) The liquid in contact with the wall of the tube is stationary.

Let a liquid of coefficient of viscosity η be flowing through a narrow horizontal tube of radius r and length l and when the condition become steady, let the velocity of flow at all points on an imaginary coaxial cylindrical shell of the liquid, of radius x, ~~and~~ be v and the velocity gradient, dv/dx.

In accordance with Newton's law of viscous flow, the backward dragging force on the imaginary liquid shell is given by,

$$F = -\eta A \frac{dv}{dx} = 2\pi x l \eta \frac{dv}{dx}.$$

If the pressure difference across the two ends of the tube be P, the force on the liquid shell, accelaring its forwards $= P \times \pi x^2$.

Where πx^2 is the area of cross-section of the shell.

For the flow to be steady,

$$P \times \pi x^2 = -2 x l \eta \left(\frac{dv}{dx}\right)$$

$$\Rightarrow \quad dv = -\frac{P}{2\eta l} x\,dx.$$

$$\Rightarrow \quad v = \frac{-P}{2\eta l}\int x\,dx = \frac{-Px^2}{4\eta l} + C_1$$

Since, at $x = r$, $v = 0$, we have

$$\frac{-Pr^2}{4\eta l} + C_1 = 0.$$

$$C_1 = \frac{Pr^2}{4\eta l}.$$

Hence,

$$v = \frac{P}{4\eta l}(r^2 - x^2) \quad \text{———} \quad ①.$$

Imagine another coaxial cylindrical shell of the liquid of radius, $x+dx$, enclosing the shell of radius x, the cross sectional area between the two is clearly $2\pi x dx$ and therefore, volume of the liquid flowing through this area is, say, $dQ = 2\pi x v dx$.

$$\Rightarrow \quad Q = \int_0^r 2\pi x v\,dx = \int_0^r 2\pi x \frac{P}{4\eta l}(r^2 - x^2)dx$$

by ①.

$$\Rightarrow \quad Q = \frac{\pi P}{2\eta l}\left[\frac{x^2 r^2}{2} - \frac{x^4}{4}\right]_0^r = \frac{\pi P r^4}{8\eta l}.$$

$$\Rightarrow \quad \boxed{\eta = \frac{\pi P r^4}{8 Q l} = \frac{\pi \rho g r^4}{8 l} \cdot \frac{h}{Q}.}$$