Examples Sheet

Exercises on Gravitation

The following questions are not arranged in the order of difficulty, although you are suggested to attempt them in order. Some may not be directly related to any part of the lectures.

1. **Mass of Sun** It takes around $500s$ for light to travel from the Sin to the Earth. Using your knowledge about the Earth's revolution about the Sun, show that the solar mass is of 2.0×10^{30} kg.

2. **Period of revolution** A planet of mass M moves along a circle around the Sun with velocity $V = 34.9$ km/s (relative to the heliocentric reference frame). Find the period of revolution of this planet around the Sun.

3. **Orbital speed** The earth orbits around the sun for a period of one year.

   (a) Estimate the average orbital speed of the earth going around the sun. The average Earth-sun distance is 1.5×10^{11} m.

   (b) An asteroid going around the sun has an average orbital speed of 15 km/s. Is the asteroid farther from the sun or closer to the sun as compared to the Earth? Explain your answer.

4. **Maximum Speed** Assume that the earth is not rotating about its axis and that Scientists have developed an engine which can propel vehicles to very high speed on the surface of the earth. What is the maximum possible speed for any such vehicle running on surface of the earth. Earth is a sphere of radius $R = 6400$ km and acceleration due to gravity on the surface is $g = 10$ m/s^2.

5. **Black hole** When the escape speed of a star equals the speed of light, the light cannot escape from the star's surface, and the star will become a "black hole". The theory proves that the escape velocity formula is still correct for this case. What will the radius of the sun be if it becomes a black hole (the current radius is $R = 7 \times 10^8$m)? What is the density of it? How many times is the density larger than the average density of the nucleus (2.3×10^8kgm^{-1})?

6. **Jumping height** A man can jump up to a height of $h_0 = 1$ m on the surface of the earth. What should be the radius of a spherical planet so that the man makes a jump on its surface and escapes out of its gravity? Assume that the man jumps with same speed as on earth and the density of planet is same as that of earth. Take escape speed on the surface of the earth to be 11.2 km/s and radius of earth to be 6400 km.

7. **Falling speed** A planet of uniform density distribution is of radius R and mass M. A rock falls from a height of $3R$ above the surface of the planet. Assume the planet has no atmosphere, show that the speed of the rock when it hits the ground is $v = \sqrt{\frac{3GM}{2R}}$.

8. **Homogeneous planet** If a planet rotates too fast, rocks from its surface will start flying off its surface. If density of a homogeneous planet is ρ and material is not flying off its surface then show that its time period of rotation must be greater then $\sqrt{\frac{3\pi}{G\rho}}$.

9. **New gravitational attraction** Suppose that the gravitational attraction between a star of mass M and a planet of mass m is given by the expression $F = K\frac{Mm}{r^n}$ where K and n are constants. If the orbital speed of the planets were found to be independent of their distance (r) from the star, calculate the time period (T_0) of a planet going around the star in a circular orbit of radius r_0.

10. **Weight difference** Angular speed of rotation of the earth is ω_0. A train is running along the equator at a speed v from west to east. A very sensitive balance inside the train shows the weight of an object as W_1. During the return journey when the train is running at same speed from east to west the balance shows the weight of the object to be W_2. Weight of the object when the train is at rest was shown to be W_0 by the balance. Calculate $W_2 - W_1$.

11. **Projectile time** A body is projected vertically upward from the surface of the earth with escape velocity. Calculate the time in which it will be at a height (measured from the surface of the earth) 8 times the radius of the earth (R). Acceleration due to gravity on the surface of the earth is g.

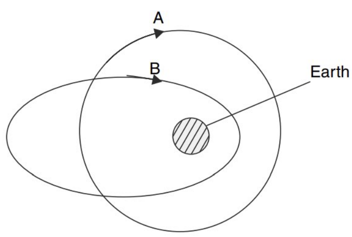
12. **Minimum speed** An astronaut on the surface of the moon throws a piece of lunar rock (mass m) directly towards the earth at a great speed such that the rock reaches the earth.

    Mass of the earth $= M$, Mass of the moon $= \frac{M}{81}$. Radius of the earth $= R$, Distance between the centre of the earth and the moon $= 60R$.

    (a) In its journey, calculate the maximum gravitational potential energy of the rock.

    (b) Find the minimum possible speed of the rock when it enters the atmosphere of the earth.

13. **Intersection point** Satellite A is following a circular path of radius a around the earth another satellite B follows an elliptical path around the earth. The two satellites have same mechanical energy and their orbits intersect. Find the speed of satellite B at the point where its path intersects with the circular orbit of A. Take mass of earth to be M.



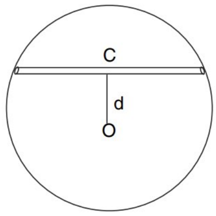
14. **Impulse and work done** A satellite of mass m is going around the earth in a circular orbital at a height $\frac{R}{2}$ from the surface of the earth. The satellite has lived its life and a rocket, on board, is fired to make it leave the gravity of the earth. The rocket remains active for a very small interval of time and imparts an impulse in the direction of motion of the satellite. Neglect any change in mass due to firing of the rocket.

    Mass of the earth $= M$, Radius of the earth $= R$.

    (a) Find the minimum impulse imparted by the rocket to the satellite.

    (b) Find the minimum work done by the rocket engine.

15. **Tunnel along the planet** A tunnel is dug along a chord of non rotating planet at a distance $d = \frac{R}{2}[R =$ radius of the planet $]$ from its centre. A small block is released in the tunnel from the surface of the planet. The block comes to rest at the centre (C) of the tunnel. Assume that the friction coefficient between the block and the tunnel wall remains constant at μ.



    (a) Calculate work done by the friction on the block.

    (b) Calculate μ.

16. **Equilateral triangle** Three material points of same masses m are at the vertices of an equilateral triangle of side d. The system is rotating in free space in such a way that under the mutual gravitational interaction of the three particles, the system is neither expanding nor contracting.

    (a) Find angular velocity ω of this rotation. Universal gravitational constant is G.

    (b) Now the three material points of masses are m_1, m_2 and m_3 respectively, find angular velocity ω of this rotation. Universal gravitational constant is G.

Solution of 1

Proof.

Solution of 2

We have

$$\frac{Mv^2}{r} = \frac{GMm_s}{r^2} \text{ or } r = \frac{Gm_s}{v^2}$$
$$\omega = \frac{v}{r} = \frac{v}{Gm_s/v^2} = \frac{v^3}{Gm_s}$$

(Here m_s is the mass of the Sun.)
So $T = \frac{2\pi Gm_s}{v^3} = \frac{2\pi \times 6.67 \times 10^{-11} \times 1.97 \times 10^{30}}{(34.9 \times 10^3)^3} = 1.94 \times 10^7 \mathrm{sec} = 225$ days.

Solution of 3

$$\frac{mV_0^2}{R} = \frac{GMm}{R^2}$$

If the speed is increased beyond V_0, the vehicle will leave the surface of the earth.

$$\therefore V_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} \simeq 7.9 \mathrm{~km/s}$$

Solution of 4

(a) Average speed of the earth is

$$v = \frac{2\pi r}{T} = \frac{2 \times 3.14 \times 1.5 \times 10^{11}}{365 \times 24 \times 60 \times 60} = 2.98 \times 10^4 \mathrm{~m/s} \simeq 30 \mathrm{~km/s}$$

(b) Speed of bodies orbiting the sun decreases with distance from the sun. Orbital speed for circular orbit is given by

$$v = \sqrt{\frac{GM_s}{r}}$$

Since speed of the asteroid is less than that of the earth, it is farther from the sun.

Solution of 5

Solution of 6

For a jump of $h_0 = 1 \mathrm{~m}$ on the earth, speed required is given by

$$\frac{1}{2}mV^2 = mgh_0 \quad \Rightarrow V \simeq \sqrt{20} \mathrm{~m/s}$$

Escape speed on the surface of a planet is

$$V_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi GR^2\rho}{3}}$$
$$\therefore \quad \frac{V_{\text{esc}}^{\text{planet}}}{V_{\text{esc}}^{\text{earth}}} = \frac{R_{\text{planet}}}{R_{\text{earth}}}$$

We want $V_{\text{esc}}^{\text{planet}} = \sqrt{20} \mathrm{~m/s}$ And it is given that $V_{esc}^{\text{earth}} = 11.2 \mathrm{~km/s}$

$$R_{\text{planet}} = \frac{\sqrt{20} \times (6400 \mathrm{~km})}{11200} = 2.5 \mathrm{~km}$$

Solution of 7

Solution of 8

The rocks start flying away from the equator of the planet if

$$\omega^2 R \geq g \quad \Rightarrow \omega \geq \sqrt{\frac{g}{R}}$$

$$\Rightarrow \frac{2\pi}{T} \geq \sqrt{\frac{g}{R}} \quad \Rightarrow \quad \frac{T}{2\pi} \leq \sqrt{\frac{R}{g}}$$

$$T \leq 2\pi\sqrt{\frac{R}{g}}$$

For rocks to not fly away

$$T \geq 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R}{\frac{GM}{R^2}}}$$

$$= 2\pi\sqrt{\frac{R^3}{G \cdot \frac{4}{3}\pi R^3 \cdot \rho}} = \sqrt{\frac{3\pi}{G\rho}}$$

Solution of 9

$$\frac{mV^2}{r} = K\frac{Mm}{r^n}$$

V is independent of r if $n = 1$

$$\therefore \quad V^2 = KM$$

$$V = \sqrt{KM}$$

Time period $T_0 = \frac{2\pi r_0}{V} = \frac{2\pi r_0}{\sqrt{KM}}$

Solution of 10

When the train is at rest

$$W_0 = mg - \frac{mV_0^2}{R} \quad [V_0 = \omega_0 R, \omega_0 = \text{ angular speed of the earth }]$$

When the train is moving from West to East $V_1 = V_0 + v$

$$\therefore W_1 = mg - \frac{m\left(V_0 + v\right)^2}{R}$$

For train running due west

$$V_2 = V_0 - v$$

$$
\begin{aligned}
&W_2 = mg - \frac{m\left(V_0 - v\right)^2}{R} \\
&\therefore W_2 - W_1 = \frac{m}{R}\left[\left(V_0 + v\right)^2 - \left(V_0 - v\right)^2\right] \\
&= \frac{m}{R}\left[4V_0 v\right] = \frac{4m\left(\omega_0 R\right) v}{R} = 4m\omega_0 v \\
&= 4\frac{W_0}{g}\omega_0 v
\end{aligned}
$$

Solution of 11

A body projected with escape speed will have total energy equal to zero. Let the speed of the body be v when it is at a distance x from the centre of the earth. Energy conservation gives- $\frac{1}{2}mv^2 - \frac{GMm}{x} = 0$

$$
\Rightarrow \frac{v^2}{2} - \frac{GM}{x} = 0 \quad \Rightarrow v = \frac{\sqrt{2GM}}{x} \Rightarrow \frac{dx}{dt} = \frac{\sqrt{2GM}}{\sqrt{x}}
$$

$$
\begin{aligned}
&\Rightarrow \int_R^{9R} \sqrt{x} dx = \sqrt{2GM} \int_0^t dt \\
&\Rightarrow \frac{2}{3}\left[x^{3/2}\right]_R^{9R} = \sqrt{2GM} \cdot t \\
&\Rightarrow \frac{2}{3}(26)R^{3/2} = \sqrt{2GM} \cdot t \\
&t = \frac{52}{3}\frac{R^{3/2}}{\sqrt{2GM}} = \frac{52\sqrt{R}}{3\sqrt{\frac{2GM}{R^2}}} \\
&= \frac{52}{3}\sqrt{\frac{R}{2g}}
\end{aligned}
$$

Solution of 12

(a) As the rock moves up from the lunar surface, its KE decreases and PE increases. There is a point in its path where the gravitational field of the earth balances the field due to the moon (say this point is at a distance x from the centre of the earth). Beyond this point the KE of the rock once again begins to increases as gravity of the earth becomes more powerful. Hence, PE is maximum at distance x from the centre of the earth.

$$
\begin{aligned}
&\frac{GM}{x^2} = \frac{GM/81}{(60R - x)^2} \\
&\Rightarrow \left(\frac{60R - x}{x}\right)^2 = \frac{1}{81} \\
&\Rightarrow \frac{60R - x}{x} = \frac{1}{9} \\
&\Rightarrow 54R = x
\end{aligned}
$$

$$
\begin{aligned}
U_{\max} &= -\frac{GMm}{54R} - \frac{G\frac{M}{81}m}{6R} \\
&= -\frac{GMm}{R}\left[\frac{1}{54} + \frac{1}{81 \times 6}\right] = -\frac{GMm}{54R}\left[1 + \frac{1}{9}\right] = -\frac{5GMm}{243R}
\end{aligned}
$$

(b) If stone is projected such that its speed is just zero when it is at a distance of x from the earth, it will reach the surface of the earth with least KE

$$K_{\text{min}} + PE_{\text{near earth}} = PE_{\text{at } x}$$

$$K_{\text{min}} - \frac{GMm}{R} - \frac{G\frac{M}{81}m}{59R} = -\frac{GMm}{54R} - \frac{G\frac{M}{81}m}{6R}$$

$$K_{\text{min}} = \frac{GMm}{R}\left[1 + \frac{1}{81 \times 59} - \frac{1}{54} - \frac{1}{81 \times 6}\right] = \frac{GMm}{R}\left[\frac{81 \times 59 \times 6 + 6 - 9 \times 59 - 59}{81 \times 59 \times 6}\right]$$

$$K_{\text{min}} = \frac{14045}{14337}\frac{GMm}{R}$$

Solution of 13

Hint: At the point of intersection both the satellites have same PE. Since they have same mechanical energy, their KE will be same at the point of intersection.

So speed is $\sqrt{\frac{GM}{a}}$

Solution of 14

(a) Orbital speed $V_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{2GM}{3R}} = \sqrt{\frac{2gR}{3}}$ For escape the satellite must have speed given by $\frac{1}{2}mV^2 - \frac{GMm}{r} = 0$

$$V = \sqrt{\frac{2GM}{r}} = \sqrt{2}V_0$$

$$\therefore \text{ Impulse needed } = m\left(\sqrt{2}V_0\right) - mV_0$$

$$= (\sqrt{2} - 1)mV_0 = (\sqrt{2} - 1)m\sqrt{\frac{2GM}{3R}}$$

(b) $W = \frac{1}{2}m\left(\sqrt{2}V_0\right)^2 - \frac{1}{2}mV_0^2 = \frac{1}{2}mV_0^2 = \frac{GMm}{3R}$

Solution of 15

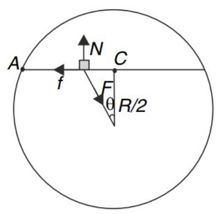
(a) Gravitational potential at $A, V_A = -\frac{GM}{R}$ Potential at C

$$V_c = -\frac{GM}{R^3}\left[\frac{3}{2}R^2 - \frac{1}{2}\left(\frac{R}{2}\right)^2\right] = -\frac{11}{8}\frac{GM}{R}$$

$\therefore$ Loss in gravitational PE of the block

$$= -\frac{GMm}{R} - \left(-\frac{11GMm}{8R}\right) \quad = \frac{3GMm}{8R}$$

$\therefore$ Work done by friction $= -\frac{3GMm}{8R}$



(b) At any intermediate position (θ) shown in the figure

$$F\cos\theta = N$$
$$\frac{GMm}{R^3}\left(\frac{R/2}{\cos\theta}\right)\cdot\cos\theta = N$$

$\therefore N = \frac{GMm}{2R^2}$ = a constant $\therefore$ Work done by friction $= -\mu N(AC)$

$$-\frac{3GMm}{8R} = -\mu\frac{GMm}{2R^2}\left(\frac{\sqrt{3}R}{2}\right)$$
$$\Rightarrow \mu = \frac{\sqrt{3}}{2}$$

Solution of 16

(a)

$$\omega = \sqrt{\frac{3Gm}{d^3}}$$

(b)

$$\omega = \sqrt{G\frac{m_1+m_2+m_3}{d^3}}$$