CP3 Differential Equations

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Chapter 1

Introduction

§ 1.1

Definition and examples

1.1.1 Basic Definition and classifications

Definition 1.1 *Differential equation*

A differential equation is an equation that contains a derivative or a differential of one or more functions. They have the general form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

where x is the independent variable, y is the function (or "dependent variable"1) and $y', y'', \ldots, y^{(n)}$ are its first, second, $\ldots$, nth derivatives.

Definition 1.2 *Solution of differential equation*

A solution of an n^{th} order differential equation on an interval $a \leq x \leq b$ is any function possessing all the necessary derivatives, which when substituted for $y, y', y'', \ldots, y^{(n)}$, reduces the differential equation to an identity.

Definition 1.3 *Explicit solution*

An explicit solution to a differential equation is a function $y = f(x)$ that can be substituted directly into the differential equation to satisfy it, i.e.

$$F\left(x, f(x), f'(x), \ldots, f^{(n)}(x)\right) = 0 \text{ on } [a, b]$$

Definition 1.4 *Implicit solution*

An implicit solution to a differential equation is a relation $g(x, y) = 0$ that defines y implicitly in terms of x. This relation may not explicitly solve for y, but it still satisfies the differential equation when substituted into it, i.e.

$$F\left(x, f(x), f'(x), \ldots, f^{(n)}(x)\right) = 0 \text{ on } [a, b] \text{ holds when } g(x, y) = 0 \text{ on } [a, b]$$

But $g(x, y) = 0$ represents at least one real function f on $[a, b]$ such that $y = f(x)$ is an explicit solution on this interval. In other words, the implicit relation $g(x, y) = 0$ can be converted to at least one explicit solution $y = f(x)$ within the given interval.

Definition 1.5 *Linear differential equation*

Generally, an ODE is linear if it has the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x),$$

or, to use Einstein's convention of implied summation over repeated indices, here meant to run from 0 to n,

$$a_i(x)y^{(i)} = f(x)$$

When $f(x) = 0$, the linear ODE is called homogeneous, otherwise it is inhomogeneous.

Homogeneous linear ODEs have the important property that they do not change under an arbitrary rescaling $y \to \lambda y \forall \lambda \neq 0$.

1.1.2 Examples of ordinary differential equations in Physics

1. $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t)$, Newton's second law of motion for a point mass's position in time $\mathbf{r}(t) = (x(t), y(t), z(t))$ in a given force field $\mathbf{F}(x, y, z, t)$
2. $\frac{1}{2}m\dot{r}^2 + \frac{L^2}{mr^2} + U(r) = E$, orbit of a body in a central potential in classical mechanics where E and L are the energy and angular momentum and $U(r)$ is the potential energy, a given function of radius, r
3. $m\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} - k\mathbf{r} = F(t)$, driven harmonic oscillator specifying the motion around an equilibrium where m, γ, and k are constants, $F(t)$ is a given external forcing function in time
4. $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t) + q\mathbf{E}(\mathbf{r}, t)$, Lorentz force in electromagnetism for the motion of a point charge in a given electromagnetic field given by $\mathbf{E}$ and $\mathbf{B}$ everywhere in space and time
5. $\dot{N} = -\lambda N$, radioactive decay in atomic physics; the rate of change is proportional to the number of particles N where λ is the decay constant
6. $C\dot{T} = -\sigma AT^4$, black body radiation cooling in thermodynamics expressing the change of temperature T of a body of area A and heat capacity C, where σ is the Stephan-Boltzmann constant
7. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x) = E\psi$, Schrödinger equation in quantum mechanics which describes the wave function $\psi(x)$ that determines the probability of finding a particle at some location having energy E in a given one dimensional potential well $V(x)$
8. $\ddot{X}^\mu + \Gamma^\mu_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta = 0$, geodesic equation in general relativity - motion of a body in spacetime $X^\alpha(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$, where τ is the time as experienced by the body, the derivatives are with respect to τ, and $\Gamma^\mu_{\alpha\beta}(t, x, y, z)$ are given functions of spacetime, μ, α, β are 4-dimensional (i.e spacetime) indices and summation is assumed over α, and β,

§ 1.2 Existence and uniqueness of solutions

1.2.1 Integral curves and the Cauchy problem

Definition 1.6 *Initial condition*

An initial condition (IC) is the statement that

$$y(t_0) = y_0 \quad \text{for some } (t_0, y_0) \in \mathscr{D}.$$

Definition 1.7 *Boundary condition*

If we were speaking in terms of a spatial variable x, rather than time t, we would call a boundary condition (BC).

This is a very important type of differential equation: it is resolved with respect to derivative, i.e., it is of the general form

$$\dot{y} = f(t, y).$$

This type of ODE allows for a vivid way of thinking of the multiplicity of an ODE's solutions, which we shall now discuss.

The problem of finding the solution of the equation above that satisfies the initial condition is called the initial-value problem, or Cauchy problem.

Does the Cauchy problem always have a solution? If we can find one, is it the only one or are there others? In other words, is there an integral curve that passes through every point $(t_0, y_0) \in \mathscr{D}$ and can these curves ever intersect?

1.2.2 The existence and uniqueness theorem

Theorem. *Existence and uniqueness*
Let $f(t, y)$ and $\partial f / \partial y$ exist and be continuous functions on some open domain $\mathscr{D} \subset \mathbb{R}^2$. Then
(a) $\forall\, (t_0, y_0) \in \mathscr{D}, \exists \Delta t$ such that the Cauchy problem

$$\dot{y} = f(t, y), \quad y\,(t_0) = y_0$$

has a solution in the interval $\mathscr{I} = [t_0 - \Delta t, t_0 + \Delta t]$

(b) This solution is unique, i.e., if $y_1(t)$ and $y_2(t)$ are solutions of equation on the intervals $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, then $y_1(t) = y_2(t) \forall t \in \mathscr{I}_1 \cap \mathscr{I}_2$ (they are the same in the intersection of the intervals where they are solutions).

Essentially the result says that if the function $f(x, t)$ is 'sufficiently nice' then the equation will have a unique solution, at least close to $t = t_0$. However, the result tells us nothing about how large the interval is on which the solution can be defined.

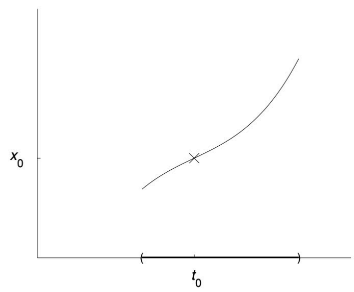

Figure 1.1: Given an initial condition $x\,(t_0) = x_0$, the existence and uniqueness theorem only guarantees the existence of a solution defined on some open interval (marked by the bold line on the horizontal axis) containing the initial time t_0.

Example 1.1 *A not unique solution*
Consider the following Cauchy problem:

$$\dot{y} = y^{2/3}, \quad y(0) = 0.$$

Clearly, $y(t) = 0$ is a solution. But $y(t) = (t/3)^3$ is also a solution, as can be verified by direct substitution. The two integral curves intersect at $t = 0$. What has gone wrong? This is easy to see:

$$f(t, y) = y^{2/3} \quad \Rightarrow \quad \frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \infty \quad \text{at} \quad y = 0.$$

Thus, $\partial f / \partial y$ does not exist at $y = 0$, the conditions of Theorem are violated, and so the Cauchy problem is under no obligation to have a unique solution.

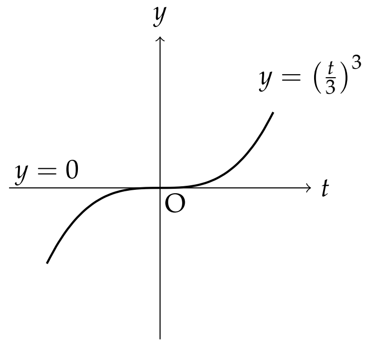

Figure 1.2: Non-uniqueness: two solutions of the example, shown as bold lines, intersect at the origin.

Chapter 2

First order Differential Equations

§ 2.1 Separable differential equations

Suppose our equation is

$$P_1(x)Q_1(y)\mathrm{d}x + P_2(x)Q_2(y)\mathrm{d}y = 0.$$

Let us divide it through by $Q_1(y)P_2(x)$:

$$\frac{P_1(x)}{P_2(x)}\mathrm{d}x + \frac{Q_2(y)}{Q_1(y)}\mathrm{d}y = 0.$$

We must be careful, however, about the possibility that $Q_1(y) = 0$ at some $y = y_0$ or/and $P_2(x) = 0$ at some $x = x_0$, because the reduction is only allowed when $y \neq y_0, x \neq x_0$. We should consider these possibilities.

Example 2.1 Consider

$$xy' + y = y^2 \quad \Rightarrow \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y(y-1) \quad \Rightarrow \quad x\,\mathrm{d}y = y(y-1)\mathrm{d}x$$

✍ This equation is easily separable and, therefore, integrable:

$$\int \frac{\mathrm{d}y}{y(y-1)} = \int \frac{\mathrm{d}x}{x} \Rightarrow \ln|x| + C = \int \frac{\mathrm{d}y}{y-1} - \int \frac{\mathrm{d}y}{y} = \ln\left|\frac{y-1}{y}\right|$$
$$\Rightarrow \quad \frac{y-1}{y} = Cx \quad \Rightarrow \quad y = \frac{1}{1+Cx}$$

Now we should worry about dividing by 0. First consider $x = 0$, it is not a legitimate solution of our original equation.

Now consider $y = 0$ and $y = 1$. Both of these lines are solutions of our equation. One of them, $y = 1$, is, in fact, already covered by $C = 0$, whereas $y = 0$ is new. Thus, the full set of solutions is

$$y = \frac{1}{1+Cx}, \quad y = 0$$

§ 2.2 Exact differentials

A first order differential equation is defined to be an exact differential or full differential if there exists a function $f(x,y)$ whose full differential vanishes as x and y change:

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

If this is zero, the function output does not change, it is a constant.

$$df(x,y) = 0 \quad \Rightarrow \quad f(x,y) = \int df = C$$

By Clairaut's theorem

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Thus, the equation is an exact differential if and only if it may be written as

$$\underbrace{P(x,y)}_{\frac{\partial f}{\partial x}} dx + \underbrace{Q(x,y)}_{\frac{\partial f}{\partial y}} dy = 0 \text{ where } P(x,y) \text{ and } Q(x,y) \text{ must satisfy } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example 2.2 Solve the exact differential equation

$$2xydx + \left(x^2 - y^2\right) dy = 0$$

✍ Let us check if the equation is an exact differential by taking the relevant partial derivatives

$$\frac{\partial}{\partial y} 2xy = 2x, \quad \frac{\partial}{\partial x}\left(x^2 - y^2\right) = 2x.$$

Since these are equal, this is an exact differential. Therefore, there must exist a function $f(x,y)$ which satisfies

$$df = \underbrace{2xy}_{\frac{\partial f}{\partial x}} dx + \underbrace{\left(x^2 - y^2\right)}_{\frac{\partial f}{\partial y}} dy = 0$$

To find this function let us integrate the first condition: $\partial f/\partial x = 2xy$ with respect to x to get

$$f(x,y) = \int \frac{\partial f}{\partial x} dx + C(y) = \int 2xydx + C(y) = x^2 y + C(y)$$

Note that the integration constant for the integral over x is written as $C(y)$ since it be any function of y. To convince ourselves, let us substitute back

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial \left(x^2 y\right)}{\partial x} + \frac{\partial C(y)}{\partial x} = 2xy + 0$$

as required. To find $C(y)$, plug back the result into the definition of $\partial f/\partial y$, and match this to the second term in

$$\frac{\partial f}{\partial y} = x^2 - y^2 \quad \Rightarrow \quad x^2 - y^2 = x^2 + \frac{dC(y)}{dy}.$$

Thus we are left with a separable ODE for $C(y)$

$$\frac{dC(y)}{dy} = -y^2 \quad \Rightarrow \quad C(y) = -\int y^2 dy = -\frac{1}{3} y^3 + C_1$$

where C_1 is any constant real number. Plugging back

$$f(x,y) = x^2 y - \frac{1}{3} y^3 + C_1.$$

The full differential of this expression is the one posed in the problem indeed. Thus $f(x,y)$ must be constant. The implicit general solution is

$$x^2 y - \frac{1}{3} y^3 = C_2.$$

where we have redefined the integration constant as C_2.

§ 2.3

First order linear differential equation

2.3.1 Integrating factor for first order linear differential equations

We will begin with a special example:

$$\cos x \frac{dy}{dx} - \sin xy = x$$

By inspection, the LHS results from applying the product rule to the derivative of the product of $\cos xy$, therefore

$$\frac{d}{dx}[\cos xy] = x$$

which is solvable. This motivates the method of integrating factor.

Derivation

Consider standard form of first-order linear differential equations:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Assume $\Lambda(x)$ (which is called *integrating factor*) is a function that can make left-hand side to a total derivative (product rule) by multiplying it:

$$\Lambda(x)\frac{dy}{dx} + \Lambda(x)p(x)y = \Lambda(x)q(x)$$

Remember our aim is to make left-hand side to a complete derivative, then this condition must be satisfied:

$$\Lambda(x)\frac{dy}{dx} + \Lambda(x)p(x)y = \Lambda(x)\frac{dy}{dx} + f'(x)y$$
$$\Lambda'(x) = p(x)\Lambda(x)$$

Then we can use $p(x)$ to express $\Lambda(x)$ by integrating:

$$p(x) = \frac{\Lambda'(x)}{\Lambda(x)}$$
$$\text{In}[\Lambda(x)] = \int p(x)dx$$
$$\Lambda(x) = e^{\int p(x)dx}$$

That $\Lambda(x)$ is called *integrating factor*, we can find it for any given $p(x)$ of a standard linear first order differential equation.

Then we can solve the differential equation, after multiplying the integrating factor, the differential equation has this form:

$$\frac{d[\Lambda(x)y]}{dx} = \Lambda(x)q(x)$$

Then we only need to integrate both sides, then we can get the solution:

$$y = \frac{\int \Lambda(x)q(x)dx}{\Lambda(x)}$$

2.3.2 Method of variation of constants

Method of variation of constants, which is also called method of variation of parameters, can be used to solve first order linear differential equation.

For a first-order equation (in standard form)

$$\frac{dy}{dx} + p(x)y = q(x)$$

The complementary solution to our original (inhomogeneous) equation is the general solution of the corresponding homogeneous equation:

$$\frac{dy}{dx} + p(x)y = 0$$

This homogeneous differential equation can be solved by different methods, for example separation of variables:

$$\begin{aligned} \int \frac{1}{y} dy &= -\int p(x)dx \\ \ln|y| &= -\int p(x)dx + C \\ y &= C_0 e^{-\int p(x)dx} \end{aligned}$$

The complementary solution to our original equation is therefore:

$$y_c = C_0 e^{-\int p(x)dx} = \frac{C_0}{\Lambda(x)}$$

Now we return to solving the non-homogeneous equation:

$$y' + p(x)y = q(x)$$

Using the method variation of parameters, the particular solution is formed by multiplying the complementary solution by an unknown function $C(x)$:

$$y_p = C(x)e^{-\int p(x)dx}$$

By substituting the particular solution into the non-homogeneous equation, we can find $C(x)$:

$$C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx} + p(x)C(x)e^{-\int p(x)dx} = q(x)$$

which give

$$C(x) = \int q(x)e^{\int p(x)dx}dx + C_1 = \int \Lambda(x)q(x)dx + C_1$$

We only need a single particular solution, so we arbitrarily select $C_1 = 0$ for simplicity. Therefore the particular solution is:

$$y_p = e^{-\int p(x)dx} \int q(x)e^{\int p(x)dx}dx = \frac{\int \Lambda(x)q(x)dx}{\Lambda(x)}$$

The final solution of the differential equation is:

$$\begin{aligned} y &= y_c + y_p \\ &= \frac{C_0}{\Lambda(x)} + \frac{\int \Lambda(x)q(x)dx}{\Lambda(x)} \\ &= \frac{\int \Lambda(x)q(x)dx}{\Lambda(x)} \end{aligned}$$

where the constant C_0 can be added into the constant of the last integral.

This gives the same result as the method of integrating factors.

☞ For first order, this method is not very important since it is just integrating factor, but for second order, it is very useful to find the particular solution.

Example 2.3 Consider

$$xy' + (x+1)y = 3x^2e^{-x}$$

The associated homogeneous equation is (assuming $x \neq 0$)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x+1}{x}y \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{y} = -\int \mathrm{d}x\left(1+\frac{1}{x}\right) \quad \Rightarrow \quad y = C\frac{e^{-x}}{x}.$$

To find the solution of the inhomogeneous equation, weaponise the constant: $C \to \psi(x)$. Then

$$y' = \psi'\frac{e^{-x}}{x} - \psi\frac{e^{-x}}{x} - \psi\frac{e^{-x}}{x^2} = -\frac{x+1}{x}y + 3xe^{-x} = -\psi\frac{e^{-x}}{x} - \psi\frac{e^{-x}}{x^2} + 3xe^{-x}$$

which gives that

$$\psi' = 3x^2 \quad \Rightarrow \quad \psi = x^3 + C$$

Therefore, the solution is

$$y = \left(x^3 + C\right)\frac{e^{-x}}{x} = x^3\frac{e^{-x}}{x} + C\frac{e^{-x}}{x}$$

where the first term is the particular integral and the second term is the complementary function.

2.3.3 Integrating factor for inexact differential

For inexact differential

$$\underbrace{P(x,y)}_{\frac{\partial f}{\partial x}}dx + \underbrace{Q(x,y)}_{\frac{\partial f}{\partial y}}dy = 0 \text{ where } \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

We want to turn it into an exact differential by multiplying the integrating factor $\lambda(x,y)$. Now

$$\frac{\partial f}{\partial x} = \lambda(x,y)P(x,y) \qquad \frac{\partial f}{\partial y} = \lambda(x,y)Q(x,y)$$

Since

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

This condition must be satisfied

$$\frac{\partial \lambda P}{\partial y} = \frac{\partial \lambda Q}{\partial x}$$

Thus

$$\frac{\partial \lambda}{\partial y}P + \frac{\partial P}{\partial y}\lambda = \frac{\partial \lambda}{\partial x}Q + \frac{\partial Q}{\partial x}\lambda$$

Let's assume λ, which only depends on x. Hence $\frac{\partial \lambda}{\partial y} = 0$.

$$\Rightarrow \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{d\lambda}{dx}Q$$
$$\Rightarrow \frac{1}{\lambda}d\lambda = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx$$

Then

$$\ln\lambda = \int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx$$

So if also depends on x only, then

$$\lambda = e^{\int f(x)dx}, \text{ where } f(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

Also, we can assume $\lambda = \lambda(y)$:

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\lambda = \frac{d\lambda}{dy}P \quad \Rightarrow \quad \lambda = e^{\int g(y)dy}, \text{ where } g(y) = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Example 2.4 Solve the differential equation $\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}$

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{4x+3y^2}{2xy} \\
\Rightarrow \left(4x+3y^2\right)dx + 2xydy &= 0 \\
P = 4x+3y^2, Q &= 2xy \\
\frac{\partial P}{\partial y} = 6y, \frac{\partial Q}{\partial x} &= 2y \Rightarrow \text{ inexact}
\end{aligned}$$

But notice that $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{2xy}(4y) = \frac{2}{x}$, only depends on x, so we have $\lambda(x) = e^{\int \frac{2}{x}dx} = e^{\ln x^2} = x^2$,

$$\begin{aligned}
&\Rightarrow 2x^3ydy + \left(4x^3 + 3x^2y^2\right)dx = 0 \\
&\frac{\partial f}{\partial x} = 4x^3 + 3x^2y^2 \Rightarrow f(x,y) = x^4 + x^3y^2 \\
&\frac{\partial f}{\partial y} = 2x^3y \Rightarrow f(x,y) = 2x^3y^2 + c \\
&\Rightarrow f(x,y) = x^4 + x^3y^2 + c = 0
\end{aligned}$$

Sometimes to memorise the derived result

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is not easy, so we will assume $\lambda(x)$ or $\lambda(y)$ just depends on x or y and substitute it into the equation to get the solution. Let's see an example.

Example 2.5 Find the solution $y(x)$ that satisfies the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y^2 - x^2 - 2x}{2y} = 0$$

subject to the boundary condition $y(0) = 1$.

✍ We can turn the equation to be

$$2ydy + (y^2 - x^2 - 2x)dx = 0$$

which is obviously an inexact differential.

Now we will just try $\lambda = \lambda(x)$:

$$\frac{\partial(\lambda 2y)}{\partial x} = \frac{\partial(\lambda(y^2 - x^2 - 2x))}{\partial x} \quad \Rightarrow \quad \frac{d\lambda}{dx} = \lambda \quad \Rightarrow \quad \lambda = e^x$$

Then substitute it to original equation:

$$\frac{\partial f}{\partial y} = 2e^x y \quad f = y^2e^x + g(x)$$

Then

$$\frac{\partial g}{\partial x} = y^2e^x + \frac{\partial g}{\partial x} = e^x(y^2 - x^2 - 2x) \quad \Rightarrow \quad g(x) = -\int (x^2+2x)e^x dx = -x^2e^x + c$$

Therefore

$$f = y^2e^x - x^2e^x + c$$

Using the boundary condition, we get $f = 1 + c$, hence the final solution is

$$y = \sqrt{x^2 + e^{-x}}$$

§ 2.4

Substitution Methods

2.4.1 Almost separable differential equations

Consider the first-order differential equation with the form:

$$\frac{dy}{dx} = f(ax + by)$$

For this type of differential equation, we will make the substitution

$$z = ax + by \quad \Rightarrow \quad \frac{dz}{dx} = a + b\frac{dy}{dx} = a + bf(z)$$

which turns the equation to be a linear differential equation.

Example 2.6 Solve the differential equation

$$(x + 2y)\frac{dy}{dx} = 1$$

✍ By using the substitution $z = x + 2y$, we will get

$$\int \frac{z}{z+2} dz = \int dx$$

which gives the solution.

However, notice that $x + 2y \neq 0$, so there is possibility that $z + 2 = 0$, $z + 2$ is a valid solution, which gives that $y = -1 - \frac{1}{2}x$. We can check that it actually satisfies the equation.

Example 2.7 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x - y + 1}$$

✍ Use the substitution

$$u \equiv x - y + 1 \quad \Rightarrow \quad 1 - \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u-1}{u} \quad \Rightarrow \quad u\frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

which is trivially soluble.

2.4.2 Homogeneous differential equations

A first order differential equation is homogeneous if it may be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

abc

We can solve it by making the substitution: $z = \frac{y}{x}$.

Example 2.8 Consider the equation

$$\frac{dy}{dx} = -\frac{2y}{x - \frac{y^2}{x}}$$

✍ We firstly turn the equation to be

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2} = \frac{2\frac{y}{x}}{\frac{y^2}{x^2} - 1}$$

which is a equation that in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Therefore we make the substitution $z = \frac{y}{x}$.
Then we can simplify to get

$$\int \frac{(z^2-1)}{z(z^2-3)} dz = \int -\frac{1}{x} dx \quad \Rightarrow \quad \int \frac{1}{3z} dz + \int \frac{\frac{2}{3}z}{z^2-3} dz = \int -\frac{1}{x} dx$$

which gives

$$\frac{1}{3} \ln|z| + \frac{1}{3} \ln|z^2-3| = -\ln|x| + c \quad \Rightarrow \quad \left(z(z^2-3)\right)^{\frac{1}{3}} = \frac{C}{x}$$

Finally we get further simplify to the implicit solution:

$$y^3 - 3yx^2 = D$$

2.4.3 Almost homogeneous differential equations

The following equation is not a homogeneous equation but it can be turned into a homogeneous equation by a change of variables

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

where f is a given function and $(a_1, a_2, b_1, b_2, c_1, c_2)$ are given constants. Change of variables $X = x + u$ and $Y = y + v$ with appropriately chosen u and v eliminates the constants c_1 and c_2, if they satisfy

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Substituting leads to a homogeneous equation

$$\frac{dY}{dX} = f\left(\frac{a_1X + b_1Y}{a_2X + b_2Y}\right) = f\left(\frac{a_1 + b_1(Y/X)}{a_2 + b_2(Y/X)}\right)$$

Example 2.9 Consider the equation

$$\frac{dy}{dx} = \frac{y+2}{2x+y-4}$$

✍ Following the above procedure:

$$x = 3 + X \quad y = -2 + Y$$

which gives

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{Y}{2X+Y}$$

Then we make the same substitution $z = \frac{Y}{X}$ and continue to calculate, we can find the final solution.

A special case

Let's consider what happens if there is no solutions for u, v?

Example 2.10 Consider the differential equation

$$\frac{dy}{dx} = \frac{y-3x-2}{2y-6x-5}$$

✍ Substituting $Y = y + a, X = x + v$ gives

$$\frac{dY}{dX} = \frac{Y - a - 3X - 3b - 2}{2Y - 2a - 6X - 6b - 5}$$

To get rid of the constant terms choose $a + 3b = -2, 2a + 6b = -5$. However these are parallel lines in the (a, b) plane and there is therefore no solution.

But looking carefully at the equation, notice that the RHS is a function of just one variable, $z = y - 3x$. This means that it is 'almost separable' and method before works. Therefore we make the substitution

$$u = 2(y - 3x) - 5 \quad \Rightarrow \quad \frac{du}{dx} = \frac{-5u + 1}{u}$$

Finally the solution is

$$-\frac{1}{5} \ln|-5(y - 3x) + 13| - 2(y - 3x) + \frac{26}{5} = x + C$$

2.4.4 Equations solved by interchange of variables

Consider

$$y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$$

This differential equation is non-linear, which is apparently hard to solve. But if we interchange the roles of the dependent and independent variables, it becomes linear: on multiplication by $(\mathrm{d}x/\mathrm{d}y)$ get

$$y^2 + x - 2y \frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

which can be solved using integrating factor.

2.4.5 Method of introduction of parameter

In some cases the equation is not resolved by y' but it may be resolved by y and written in the form

$$y = f(x, y')$$

where $f(x, y')$ is a given function of x and y'. In case where this equation is difficult to solve for y' we can proceed as follows. Introduce $z = y'$, taking the differential of both sides

$$z dx = dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial z} dz$$

Thus,

$$\left(\frac{\partial f}{\partial x} - z \right) dx + \frac{\partial f}{\partial z} dz = 0 \quad \Rightarrow \quad \frac{dz}{dx} = \frac{\frac{\partial f}{\partial z}}{\left(z - \frac{\partial f}{\partial x} \right)}$$

We have turned the equation into the form

$$z' = F(x, z)$$

which is an equation resolved by z'. If $F(x, z)$ is of the types discussed above in this chapter we may solve this for $z(x)$ or $x(z)$. Given that $y = f(x, z)$, this already represents a parametric solution to plot the integral curves in the (x, y) plane as $(x, f(x, z(x)))$ or $(x(z), f(x(z), z))$. If needed, we may proceed by solving for $y(x)$

$$y = \int y' dx + C = \int z(x) dx + C.$$

Example 2.11 Solve

$$y = xy' + y' + x$$

✍ Method 1. Integrating Factor

We firstly rewrite it in standard form:

$$y' - \frac{1}{1+x}y = -\frac{x}{1+x}$$

Then multiplying integrating factor

$$\lambda = e^{-\int \frac{1}{1+x}dx} = \frac{1}{1+x}$$

Thus

$$\begin{aligned}
\frac{d}{dx}\left[y\frac{1}{1+x}\right] &= -\frac{x}{(1+x)^2} \\
y\frac{1}{1+x} &= \int -\frac{x+1}{(1+x)^2} + \frac{1}{(1+x)^2}dx \\
y\frac{1}{1+x} &= -\ln(1+x) - \frac{1}{1+x} + c \\
y &= -(1+x)\ln(1+x) - 1 + c(1+x)
\end{aligned}$$

Method 2. Method of introduction of parameter (Substitution)

Let $z = y' \Rightarrow zdx = dy$, therefore

$$\begin{aligned}
zdx = dy &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial z}dz \\
&= (z+1)dx + (x+1)dz
\end{aligned}$$

which gives

$$\begin{aligned}
-dx &= (x+1)dz \\
\frac{dz}{dx} &= -\frac{1}{1+x} \Rightarrow z = -\ln(1+x) + k \\
y &= \int -\ln(1+x) + kdx \\
&= kx - \left[x\ln(1+x) - \int \frac{x}{1+x}dx\right] \\
&= kx - x\ln(1+x) + \int \frac{1+x}{1+x} - \frac{1}{1+x}dx \\
&= kx - x\ln(1+x) + x - \ln(1+x) + d. \\
&= kx - (1+x)\ln(1+x) + x + d. \\
&= -(1+x)\ln(1+x) + kx + x + d.
\end{aligned}$$

Notice that there is two integration constants so we need to substitute it into the differential equation:

$$\begin{gathered}
-(1+x)\ln(1+x) + kx + x + d = x(-\ln(1+x) + k) - \ln(1+x) + k + x \\
\therefore d = k \\
y = -(1+x)\ln(1+x) + kx + x + k
\end{gathered}$$

Let $k = c - 1$

$$y = -(1+x)\ln(1+x) - 1 + c(1+x)$$

which gives the some result as Method 1.

§ 2.5

Other equations

2.5.1 Bernoulli equation

$$y' + a(x)y = b(x)y^n \quad (n \neq 0, 1)$$

Here $y(t) = 0$ is a solution. Otherwise if $y \neq 0$, then divide by y^n

$$\frac{y'}{y^n} = b(x) - a(x)y^{1-n} \quad \Rightarrow \quad \frac{1}{1-n}\frac{d}{dt}y^{1-n} = b(x) - a(x)y^{1-n}$$

Let $z = y^{1-n}$

$$\frac{1}{1-n}z' = b(x) - a(x)z \quad \Rightarrow \quad z' + (n-1)a(x)z = (n-1)b(x)$$

This is a linear 1st order ODE, which may be solved as discussed above.

Example 2.12 Solve

$$y' + y = xy^{\frac{2}{3}}$$

✍ This is a Bernoulli equation, so

$$y^{-\frac{2}{3}}y' + y^{\frac{1}{3}} = x$$

$$\text{let } z = y^{\frac{1}{3}} \Rightarrow \frac{dz}{dx} = \frac{1}{3}y^{-\frac{1}{3}}y'$$

$$\therefore 3z' + z = x$$

The key of this substitution is when we make it, we can let the first $y^k y'$ term becomes linear.

2.5.2 Ricatti equation

Now consider Riccati's equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

There is no general solution to this equation for arbitrary $a(x), b(x)$, and $c(x)$, but if one particular solution may be guessed $y_P(x)$ then the general solution may be obtained in the form

$$y(x) = z(x) + y_0(x)$$

It turns your Riccati equation into a Bernoulli equation for $z(x)$. Watch:

$$y' = z' + y_0' = az^2 + 2azy_0 + a{y_0}^2 + bz + by_0 + c \quad \Rightarrow \quad z' = [2a(x)y_0(x) + b(x)]\, z + a(x)z^2$$

which is a Bernoulli equation with $n = 2$.

Example 2.13 Solve $y' = y^2 - 2e^x y + e^{2x} + e^x$.

Firstly, we make the guess $y_0 = e^x$

$$e^x = e^{2x} - 2e^{2x} + e^{2x} + e^x$$

So $y_0 = e^x$ is a particular solution. Therefore general solution is $y = z + y_0$, then we substitute it into the differential equation

$$z' + y_0' = z^2 + 2zy_0 + {y_0}^2 - 2e^x z - 2e^x y_0 + e^{2x} + e^x$$

which gives

$$z' = z^2 \quad \Rightarrow \quad \int \frac{1}{z^2} dz = x + c$$
$$z = -\frac{1}{-x + c} \quad \Rightarrow \quad y = -\frac{1}{x + c} + e^x$$

We do not need to memorize the formula, we only need to remember we need firstly find a particular solution, then substitute everything into the equation.

Chapter 3

Second Order Differential Equations

§ 3.1 Autonomous Systems

Definition 3.1 *Autonomous systems*

Autonomous systems are systems which are described by a differential equation that does not explicitly depend on the independent variable, and may be written as

$$\ddot{r} = f(r, \dot{r}), \quad r(t) = ?$$

where $f(r, \dot{r})$ is a given function which does not depend explicitly on t.

Example 3.1 *Newton's second law*

Newton's second law can be turned into a first order equation as follows. Divide both sides by $v = \dot{r}$

$$\frac{\ddot{r}}{\dot{r}} = \frac{dv/dt}{dr/dt} = \frac{dv}{dr} = \frac{f(r,v)}{v}$$

This is now a first order differential equation which specifies how v changes as a function of r, regardless of time. The (r, v) space is referred to as the phase space. Since it is governed by a first order differential equation, the phase space evolution is characterised by a one-parameter family of curves, which gives an understanding of the behaviour of the system, as we will see through examples below. For a given initial condition, the evolution is restricted to any instance of this family, a single curve in phase space.

3.1.1 Conservative systems

Conservative systems are common in physics, where $f(r, v)$ depends explicitly only on r but not on v. In this case the differential equation is separable, since after multiplying by v we get

$$v\frac{dv}{dr} = f(r) \Rightarrow \frac{1}{2}v^2 + V(r) = E \quad \text{where } V(r) = -\int f(r)dr \text{ and } E \text{ is an integration constant}$$

Once this integral is evaluated, we have derived the implicit general solution for $r(t)$, where C and E are the two integration constants, which needs 2 initial conditions to be determined.

The harmonic oscillator and the mathematical pendulum are important examples which turn out to describe a large variety of problems in nature.

Example 3.2 For the pendulum

$$\ddot{\theta} = f(\theta) = -k\sin\theta \quad \Rightarrow \quad \frac{\ddot{\theta}}{\dot{\theta}} = \frac{d\omega/dt}{d\theta/dt} = \frac{d\omega}{d\theta} = -\frac{k\sin\theta}{\omega}$$

where we defined the phase velocity $\omega = \dot{\theta}$. Thus $V(\theta) = -\int f(\theta)d\theta = \int k\sin\theta d\theta = -k\cos\theta$ and

$$\frac{1}{2}\omega^2 - k\cos\theta = E \quad \Rightarrow \quad \omega(\theta) = \pm\sqrt{2E + 2k\cos\theta}$$

Here solutions exist for $E > -k$.
The phases space separates into distinct regions:
The potential $V(\theta)$ has a minimum at $\theta = 2n\pi$ and a maximum at $\theta = (2n+1)\pi$, where n is an integer.
The phase space separates into distinct regions with qualitatively different behaviour:

- At $E = -k$ the pendulum is at rest at the potential minimum $\theta = 0$, a stable fixed point.
- For slightly larger $E \gtrsim -k$, we have $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ and we get harmonic oscillations around the minimum.
- If $-k < E < k$, the pendulum goes back and forth both in θ and in ω around $(\theta, \omega) = (0, 0)$, but now the shape of the curve in phase space is not an ellipse but the curve given by the equation above.
- If $E = k$, the pendulum is fixed at $(\theta, \omega) = (\pi, 0)$, this is an unstable fixed point.
- If $E > k$, the pendulum goes around perpetually respectively either with $\omega > 0$ or with $\omega < 0$ throughout the evolution, such that ω does not change sign.

§ 3.2 Linear differential equations

Mathematically, a higher-order ODE can always be rewritten as a system of first-order ODEs by introducing new variables to represent each derivative up to the $(n-1)$-th order, where n is the order of the original ODE.

For instance, consider a second-order ODE:

$$y'' = f\left(x, y, y'\right)$$

To convert this into a system of first-order ODEs, you introduce a new variable, say u, to represent the first derivative of y :

$$u = y'$$

Now you have two first-order equations:

$$\begin{aligned} y' &= u \\ u' &= f(x, y, u) \end{aligned}$$

This new system contains only first-order derivatives and is equivalent to the original second-order ODE.

More generally,

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

can be recast as this system of n first-order ODEs:

$$\begin{cases} F\left(x, y, p_1, \ldots, p_{n-1}, \dot{p}_{n-1}\right) = 0 \\ y' = p_1 \\ p_1' = p_2 \\ \cdots \\ p_{n-2}' = p_{n-1} \end{cases}$$

Thus, we can talk of n as the order of the original ODE or as the order of the corresponding system of first-order ODEs ($n =$ the number of these ODEs).

If we restrict our attention to ODEs that are resolved with respect to the highest derivative,

$$y^{(n)} = f\left(x, y, \dot{y}, \ldots, y^{(n-1)}\right),$$

Use matrix notation,

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y})$$

Since all ODEs are (systems of) first-order ODEs. Thus, in the case of linear ODEs, it is formally sufficient to consider systems of the form

$$\boldsymbol{y}' = \mathrm{A}(x) \cdot \boldsymbol{y} + \boldsymbol{f}(x)$$

where $\boldsymbol{y}$ and $\boldsymbol{f}$ are vectors that live in $\mathbb{R}^n$ (which can be extended to $\mathbb{C}^n$ without any complications) and A is an $n \times n$ matrix, in general x-dependent and with elements in $\mathbb{R}$ (or $\mathbb{C}$).

3.2.1 Existence and uniqueness theorem

The 1D existence and uniqueness theorem can be generalised to the n-dimensional case.

Theorem. *Existence and uniqueness for higher order*

Let all $f_i(x, \boldsymbol{y})$ and $\partial f_i / \partial y_j (i, j = 1, \ldots, n)$ exist and be continuous on some open domain $\mathscr{D} \subset \mathbb{R}^{n+1}$. Then

(a) $\forall (t_0, \boldsymbol{y}_0) \in \mathscr{D}, \exists \Delta x$ such that the Cauchy problem has a solution in the interval $\mathscr{I} = [t_0 - \Delta x, t_0 + \Delta x]$.

(b) This solution is unique, i.e., if $\boldsymbol{y}_1(x)$ and $\boldsymbol{y}_2(x)$ are solutions of on the intervals $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, then $\boldsymbol{y}_1(x) = \boldsymbol{y}_2(x) \forall x \in \mathscr{I}_1 \cap \mathscr{I}_2$.

3.2.2 Superposition principle

Consider first the homogeneous version:

$$\boldsymbol{y}' = \mathrm{A}(x) \cdot \boldsymbol{y}.$$

It is very easy to prove, but momentously significant, that if $\boldsymbol{y}_1(x)$ and $\boldsymbol{y}_2(x)$ are solutions, then $\forall$ constants C_1 and C_2,

$$\boldsymbol{y}(t) = C_1 \boldsymbol{y}_1(x) + C_2 \boldsymbol{y}_2(x)$$

is also a solution. This result is called the superposition principle - the fundamental property that makes linear ODEs special.

The superposition theorem clearly indicates the type of solutions homogeneous linear differential equations may have

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

where $C_1, C_2, \ldots, C_n$ are arbitrary generally complex constants.

3.2.3 General solution of homogeneous equations

Due to the linearity of superposition, in the language of linear algebra, this means that solutions can form a linear vector subspace of the space of $\mathbb{R}^n$-valued functions of a single variable.

Since solutions form a linear vector space, we can capture them all if we are able to find a basis in their space. This basis will turn out to be a set of n linearly independent solutions

$$\{\boldsymbol{y}_1(x), \ldots, \boldsymbol{y}_n(t)\},$$

where n is the order of the ODE. Such a set is called the fundamental system of solutions. We can prove that a fundamental system can always be constructed[1].

[1]Reference: Lectures on Ordinary Differential Equations by *Alexander A. Schekochihin*

Therefore for n-th order differential equation, we will have n parameter family of curves. The n-parameter family of curves represents the general solution with n independent integration constants as an arbitrary linear combination of n linearly independent functions, $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. And we need n initial conditions to determine all the integration constants.

3.2.4 Linearly independent functions and the Wronskian

Suppose we found n functions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ which solve an n^{th} order differential equation. To verify that these span the complete set of solutions, we must make sure that they are indeed linearly independent, meaning that

$$C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) = 0 \text{ for all } x \text{ only if } C_1 = C_2 = \cdots = C_n = 0$$

Taking the derivatives of both sides, recall theory in linear algebra, linearly independent functions also satisfy all of the following n conditions:

$$\left.\begin{array}{c} C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) = 0 \\ C_1y_1'(x) + C_2y_2'(x) + \cdots + C_ny_n'(x) = 0 \\ C_1y_1''(x) + C_2y_2''(x) + \cdots + C_ny_n''(x) = 0 \\ \vdots \\ C_1y_1^{(n-1)}(x) + C_2y_2^{(n-1)}(x) + \cdots + C_ny_n^{(n-1)}(x) = 0 \end{array}\right\} \text{for all } x \text{ only if } C_1 = C_2 = \cdots = C_n = 0$$

That is:

$$\begin{pmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ y_1''(x) & y_2''(x) & \ldots & y_n''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for all } x \text{ only if } C_1 = \cdots = C_n = 0$$

The condition for linearly independence is the determinant of matrix does not equal to 0:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ y_1''(x) & y_2''(x) & \ldots & y_n''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0$$

Thus, the determinant formed from the functions and their derivatives in the above way must be nonzero for the functions to be linearly independent. This determinant is called the Wronskian, labelled $W(x)$.

Example 3.3 Show that the functions $y_1(x) = x$ and $y_2(x) = x^2$ are linearly independent.
✍ Here $y_1'(x) = 1, y_2'(x) = 2x$, so

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

This is not zero except at $x = 0$, hence the functions are linearly independent.

§ 3.3

Linear homogeneous ODE with constant coefficient

3.3.1 Homogeneous case

An important special case of linear differential equations, where the solution may be obtained analytically is when the coefficients are constant numbers

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

In order to find fundamental solutions, we 'guess' that they are of the form

$$y(x) = e^{\lambda x}$$

and substitute this into the differential equation. To see that this is reasonable, we know that taking derivatives of $y(x) = e^{\lambda x}$ only multiplies $y(x)$ by λ. So y'', y', and y will all be just some constant times $e^{\lambda x}$; we should be able to cancel the $e^{\lambda x}$ s and all being well end up with an equation that we can solve.

Let us substitute in the differential equation to find λ. In this case $y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}$, and so on, so

$$a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0 \quad \Rightarrow \quad \left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + \lambda + a_0 \right) e^{\lambda x} = 0$$

Therefore $e^{\lambda x}$ is a solution if and only if λ satisfies

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + \lambda + a_0 = 0$$

This is called the characteristic (auxiliary) equation of the differential equation, the left hand side being the characteristic polynomial. The fundamental theorem of algebra states that every n^{th} order polynomial has exactly n generally complex roots (but not necessarily distinct) so that the characteristic polynomial may be written as

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + \lambda + a_0 = a_n (\lambda - \lambda_1)(\lambda - \lambda_2) \times \cdots \times (\lambda - \lambda_n) = 0$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are generally complex numbers, the roots of the characteristic polynomial. If the $(a_1, \ldots, a_n)$ coefficients are real then complex solutions come in pairs. But if $(a_1, \ldots, a_n)$ coefficients are complex, there is no requirement for complex roots to occur in conjugate pairs. The roots can be any complex numbers.

Now we get that the solution of the differential equation is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x} \quad (\lambda_1, \lambda_2, \ldots, \lambda_n \text{ all distinct })$$

If $(\lambda_1, \ldots, \lambda_n)$ are all distinct, this is an n-parameter family of curves, i.e. the complete general solution. However if not all λ_i are distinct but some are equal then their C_i coefficients can be combined into a single coefficient, and the general solution must have additional functions not of this form, we will discuss it later.

Now consider the second order case:

$$\ddot{y} + a\dot{y} + by = 0$$

Solving the auxiliary equation gives

$$\lambda^2 + a\lambda + b = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Unless $a^2 = 4b$, we have two different values of λ and therefore two linearly independent solutions, so we have our general solution:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

- $a^2 > 4b$: then λ_1 and λ_2 are both real and the solution is
$$y(t) = e^{-at/2}\left(C_1 e^{t\sqrt{a^2/4-b}} + C_2 e^{-t\sqrt{a^2/4-b}}\right)$$

- $a^2 < 4b$: in this case, λ_1 and λ_2 are complex:
$$\lambda_{1,2} = -\frac{a}{2} \pm i \underbrace{\sqrt{b - \frac{a^2}{4}}}_{\equiv\Omega}.$$
The solution is
$$y = e^{-at/2}\left(C_1 e^{i\Omega t} + C_2 e^{-i\Omega t}\right),$$
where the constants C_1 and C_2 are, in general, complex. However, we often wants a real solution, representing some real physical quantity. The solution can be recast in two equivalent forms in which the constants can, if we wish, be purely real:
$$y = e^{-at/2}(A\cos\Omega t + B\sin\Omega t) = e^{-at/2}\alpha\cos(\Omega t - \phi),$$
where $A = C_1 + C_2 = \alpha\cos\phi$ and $B = i\,(C_1 - C_2) = \alpha\sin\phi$ are both real if the initial conditions are real.

3.3.2 Solutions for repeated roots

Now consider the special case of $a^2 = 4b$. In this case, the two solutions of the auxiliary equation are degenerate (meaning they are the same):
$$\lambda_1 = \lambda_2 = -\frac{a}{2}$$
This means that we only have one solution available to us:
$$y_1(t) = Ce^{-at/2}$$
To find the second one, we must use method of variation of constant. Namely, we seek the solution in the form
$$y(t) = \psi(t)e^{-at/2}$$
Substituting this back into and recalling that $b = a^2/4$, we find
$$\ddot{\psi} = 0 \quad \Rightarrow \quad \psi(t) = C_1 t + C_2 \quad \Rightarrow \quad y(t) = (C_1 t + C_2)\, e^{-at/2}$$

Example 3.4 Find the general solution to the following differential equation
$$y''' = 0, \quad y(x) = ?$$

✍ We may simply integrate three times to get immediately $C_1 + C_2 x + C_3 x^2$. Alternatively, in this case the characteristic polynomial is $\lambda^3 = (\lambda - 0)^3$, so $\lambda = 0$ is a repeated root with multiplicity 3 . So we recover the solution as $y(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{0x} = C_1 + C_2 x + C_3 x^2$.

This example can be generalised as follows for repeated roots with arbitrary multiplicity.

Example 3.5 Consider the characteristic polynomial of some 7th order constant coefficient linear ODE is
$$(\lambda - a)^3(\lambda - b)^2(\lambda - c)(\lambda - d)$$
where $\{a, b, c, d\}$ are 4 different real numbers, these are the roots of the characteristic equation with multiplicity $3, 2, 1, 1$, respectively, and the general solution of the ODE is
$$y(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{ax} + (C_4 + C_5 x)\, e^{bx} + C_6 e^{cx} + C_7 e^{dx}$$

3.3.3 Cauchy-Euler Equations

Another class of solvable linear differential equations that is of interest are the Cauchy-Euler type of equations, also referred to in some books as Euler's equation. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = f(x)$$

where a, b, c are constants.

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}$$

into original equation, we have

$$[ar(r-1) + br + c]x^r = f(x)$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1) + br + c = f(x)$$

The solutions of Cauchy-Euler equations can be found using this characteristic equation. Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form

$$y_p(x) = c_1x^{r_1} + c_2x^{r_2}$$

It is instructive to know that for Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$$

can be solved using the substitution $x = e^t$:

$$\frac{dx}{dt} = e^t = x, \frac{dt}{dx} = \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$
$$\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

Substituting the derivatives into the equation gives

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = f(t),$$

which is a second order, linear differential equation with constant coefficients that we know how to solve.

Example 3.6 Solve the differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

✍ We could recognise this equation as a Cauchy-Euler equation, then we rearrange it to standard form:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

Then the result is to use the trial solution $y = x^r$.

§3.4

Inhomogenous ODE with constant coefficient

3.4.1 General solution of inhomogeneous equations

We can consider the differential equation as a linear differential operator

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=f(x) \quad \Rightarrow \quad \left(\frac{d^2}{dx^2}+b\frac{d}{dx}+c\right)y=f(x)$$

Therefore $\left(\frac{d^2}{dx^2}+b\frac{d}{dx}+c\right)$ can be considered as a linear map $f:V\to W$. The inhomogeneous solution is to find solutions $\mathbf{x}\in V$ of the equation

$$y=f(\mathbf{x})=\mathbf{b}$$

where $\mathbf{b}\in W$ is a fixed vector. For $\mathbf{b}\neq\mathbf{0}$ this is called an inhomogeneous linear equation and

$$f(\mathbf{x})=\mathbf{0}$$

is the associated homogeneous equation. Its general solution is $\operatorname{Ker}(f)$. The solutions of the inhomogeneous and associated homogeneous equations are related in an interesting way.

Theorem. *GS=CF+PI*
If $\mathbf{x}_0\in V$ solves the inhomogeneous equation, that is $f(\mathbf{x}_0)=\mathbf{b}$, then the affine space

$$\operatorname{Ker}(f)+\mathbf{x}_0$$

is the general solution of the inhomogeneous equation.

Proof. If $\mathbf{x}$ is a solution to $f(\mathbf{x})=\mathbf{b}$ then $f(\mathbf{x}-\mathbf{x}_0)=f(\mathbf{x})-f(\mathbf{x}_0)=\mathbf{b}-\mathbf{b}=\mathbf{0}$, so $\mathbf{x}-\mathbf{x}_0\in\operatorname{Ker}f$. Conversely, if $\mathbf{x}\in\mathbf{x}_0+\operatorname{Ker}(f)$, then we can write $\mathbf{x}=\mathbf{x}_0+\mathbf{v}$ for some vector $\mathbf{v}\in\operatorname{Ker}(f)$. Then, $f(\mathbf{x})=f(\mathbf{x}_0+\mathbf{v})=f(\mathbf{x}_0)+f(\mathbf{v})=\mathbf{b}+\mathbf{0}=\mathbf{b}$.

□

In short, the theorem says that the general solution of the inhomogeneous equation is obtained by the sum of all solutions to the homogeneous equation (called complementary function) and a special solution (called particular integral) to the inhomogeneous equation.

There are two methods for finding $y_{\mathrm{PI}}(x)$: for certain simple forms of $f(x)$, it is easy to guess. This is the summary table for simple $f(x)$.

Function $f(x)$	Particular Integral
Constant term	c
Linear function	$ax+b$
Quadratic function	ax^2+bx+c
Exponential function involving e^{px}	ke^{px}
Function involving $\sin px$ or $\cos px$	$a\cos px+b\sin px$

3.4.2 Method of variation of constants

For general $f(x)$, the $y_{\mathrm{P}}(x)$ particular solution may be found from the method of variation of constants just as for the first order equations. Let us see how this works for second order equations.

Assume that we have solved the homogeneous equation

$$y_{\mathrm{H}}(x)=C_1y_{\mathrm{H}1}(x)+C_2y_{\mathrm{H}2}(x)$$

By method of variation: we use two unknown functions instead of constants, $\alpha(x)$ and $\beta(x)$

$$y_{\mathrm{P}}(x) = \alpha(x)y_{\mathrm{H1}}(x) + \beta(x)y_{\mathrm{H2}}(x)$$

and without loss of generality we require this linear combination also satisfies the relation

$$\alpha' y_{\mathrm{H1}} + \beta' y_{\mathrm{H2}} = 0$$

We will obtain these functions by substituting back in the differential equation. First calculate the derivatives

$$y'_{\mathrm{P}} = \alpha y'_{\mathrm{H1}} + \beta y'_{\mathrm{H2}} + \underbrace{\alpha' y_{\mathrm{H1}} + \beta' y_{\mathrm{H2}}}_{0} = \alpha y'_{\mathrm{H1}} + \beta y'_{\mathrm{H2}}, \quad y''_{\mathrm{P}} = \alpha y''_{\mathrm{H1}} + \beta y''_{\mathrm{H1}} + \alpha' y'_{\mathrm{H1}} + \beta' y'_{\mathrm{H2}}$$

In the first equation two terms drop out because of our relation. Substitute back in the equation

$$\begin{aligned}
&a_2 y''_{\mathrm{P}} + a_1 y'_{\mathrm{P}} + a_0 y_{\mathrm{P}} = f(x) \\
&\quad \Rightarrow \quad a_2\left(\alpha y''_{\mathrm{H1}} + \beta y''_{\mathrm{H1}} + \alpha' y'_{\mathrm{H1}} + \beta' y'_{\mathrm{H2}}\right) + a_1\left(\alpha y'_{\mathrm{H1}} + \beta y'_{\mathrm{H2}}\right) + a_0\left(\alpha y_{\mathrm{H1}} + \beta y_{\mathrm{H2}}\right) = f(x) \\
&\Rightarrow a_2\left(\alpha' y'_{\mathrm{H1}} + \beta' y'_{\mathrm{H2}}\right) + \underbrace{\left(a_2 y''_{\mathrm{H1}} + a_1 y'_{\mathrm{H1}} + a_0 y_{\mathrm{H1}}\right)}_{0}\alpha + \underbrace{\left(a_2 y''_{\mathrm{H2}} + a_1 y'_{\mathrm{H2}} + a_0 y_{\mathrm{H2}}\right)}_{0}\beta = f(x)
\end{aligned}$$

Finally we get:

$$a_2\left(\alpha' y'_{\mathrm{H1}} + \beta' y'_{\mathrm{H2}}\right) = f(x)$$

Together with our restriction:

$$\alpha' y_{\mathrm{H1}} + \beta' y_{\mathrm{H2}} = 0$$

We could write these two equations in matrix form:

$$\begin{pmatrix} y_{\mathrm{H1}} & y_{\mathrm{H2}} \\ y'_{\mathrm{H1}} & y'_{\mathrm{H2}} \end{pmatrix}\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x)/a_2 \end{pmatrix}$$

One way to solve this is using Cramer's rule

$$\alpha' = \frac{\begin{vmatrix} 0 & y_{\mathrm{H2}} \\ f(x)/a_2(x) & y'_{\mathrm{H2}} \end{vmatrix}}{\begin{vmatrix} y_{\mathrm{H1}} & y_{\mathrm{H2}} \\ y'_{\mathrm{H1}} & y'_{\mathrm{H2}} \end{vmatrix}} = \frac{-y_2 f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}, \quad \beta' = \frac{\begin{vmatrix} y_{\mathrm{H1}} & 0 \\ y'_{\mathrm{H1}} & f(x)/a_2(x) \end{vmatrix}}{\begin{vmatrix} y_{\mathrm{H1}} & y_{\mathrm{H2}} \\ y'_{\mathrm{H1}} & y'_{\mathrm{H2}} \end{vmatrix}} = \frac{y_{\mathrm{H1}} f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}$$

$$\alpha = \int \frac{-y_{\mathrm{H2}} f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}dx, \quad \beta = \int \frac{y_{\mathrm{H1}} f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}dx$$

Note that the denominator is the Wronskian. Thus

$$\begin{aligned}
\alpha'(x) &= -\frac{1}{W}y_{\mathrm{H2}}f(x), & \beta'(x) &= \frac{1}{W}y_{\mathrm{H1}}f(x) \\
\alpha(x) &= -\int \frac{1}{W}y_{\mathrm{H2}}f(x)\mathrm{d}x, & \beta(x) &= \int \frac{1}{W}y_{\mathrm{H1}}f(x)\mathrm{d}x
\end{aligned}$$

Thus

$$y_{\mathrm{P}} = \alpha(x)e^{\lambda_1 x} + \beta(x)e^{\lambda_2 x} = \frac{\int^x e^{\lambda_2(x-u)}f(u)du - \int^x e^{\lambda_1(x-u)}f(u)du}{a_2\left(\lambda_2 - \lambda_1\right)}.$$

Now Let's discuss why we can make the assumption:

$$\alpha' y_{\mathrm{H1}} + \beta' y_{\mathrm{H2}} = 0$$

It's because, without this restriction, after a cumbersome calculation, we can get the same solution:

$$\alpha = \int \frac{-y_{\mathrm{H2}} f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}dx, \quad \beta = \int \frac{y_{\mathrm{H1}} f(x)/a_2(x)}{y_{\mathrm{H1}}y'_{\mathrm{H2}} - y_{\mathrm{H2}}y'_{\mathrm{H1}}}dx$$

That means the assumption will not change the final solution, but is a trick to simplify the calculation.

Example 3.7 Find a particular integral for the equation

$$\ddot{x} + x = \tan t.$$

Two linearly independent solutions of the homogeneous equation

$$\ddot{x} + x = 0$$

are $x_1(t) = \sin t$ and $x_2(t) = \cos t$, so for a particular integral we try

$$x(t) = u(t)\sin t + v(t)\cos t.$$

The first derivative of $x(t)$ is given by

$$\dot{x} = \dot{u}\sin t + u\cos t + \dot{v}\cos t - v\sin t,$$

and here we impose an additional condition to make sure that there are no second derivatives of u or v in $\ddot{x}$

$$\dot{u}\sin t + \dot{v}\cos t = 0.$$

This means that $\dot{x}$ is given by

$$\dot{x} = u\cos t - v\sin t,$$

and we can differentiate to find

$$\ddot{x} = \dot{u}\cos t - u\sin t - \dot{v}\sin t - v\cos t.$$

Substituting for x and $\ddot{x}$ gives (after some cancellation)

$$\dot{u}\cos t - \dot{v}\sin t = \tan t.$$

Thus

$$\begin{cases} \dot{u}\sin t + \dot{v}\cos t = 0 \\ \dot{u}\cos t - \dot{v}\sin t = \tan t \end{cases}$$

with solution

$$\dot{u} = \sin t \quad \text{and} \quad \dot{v} = \frac{\sin^2 t}{\cos t} = \cos t - \frac{1}{\cos t}.$$

Integrating these two gives

$$u = -\cos t \quad v = \sin t - \ln|\sec t + \tan t|,$$

and so a particular integral is

$$\begin{aligned} x(t) &= -\cos t \sin t + \sin t \cos t - \ln|\sec t + \tan t|\cos t \\ &= -\ln|\sec t + \tan t|\cos t \end{aligned}$$

Example 3.8 Find the solution of $y'' + 3y' + 2y = xe^{-x}$.

Firstly we get the complementary function:

$$\begin{gathered} y'' + 3y' + 2y = 0 \\ \lambda^2 + 3\lambda + 2 = 0 \\ \lambda = -2 \text{ or } \lambda = -1 \\ \therefore y = Ae^{-2x} + Be^{-x} \end{gathered}$$

Then use the variation of constant $y_p = \alpha(x)e^{-2x} + \beta(x)e^{-x}$, and with the assumption $\alpha' e^{-2x} + \beta' e^{-x} = 0$, we have:

$$\begin{pmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ xe^{-x} \end{pmatrix}$$

$$\alpha' = \frac{\begin{vmatrix} 0 & e^{-x} \\ xe^{-x} & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}} = \frac{-xe^{-2x}}{e^{-3x}} = -xe^{x} \Rightarrow \alpha = \int -xe^{x}dx = -xe^{x} + e^{x}$$

$$\beta' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}} = x \quad \Rightarrow \quad \beta = \frac{1}{2}x^2$$

where we use Cramer's rule to find α' and β'. Here when integrate to get α and β, we do not add the integrating constant because these constants can be added to the complementary functions Ae^{-x} and Be^{-2x}.

Finally we have

$$\begin{aligned} y_p &= \left(-xe^{x} + e^{x}\right)e^{-2x} + \frac{1}{2}x^2 e^{-x} \\ &= -xe^{-x} + e^{-x} + \frac{1}{2}x^2 e^{-x} \\ &= e^{-x}\left(\frac{1}{2}x^2 - x + 1\right) \\ \therefore y &= Ae^{-2x} + Be^{-x} + e^{-x}\left(\frac{1}{2}x^2 - x + 1\right) \\ &= C_1 e^{-2x} + C_2 e^{-x} + e^{-x}\left(\frac{1}{2}x^2 - x\right) \end{aligned}$$

In fact, for the inhomogeneous term $f(x) = x^n e^{\alpha x}$, we can directly try the particular integral in this form:

$$y_p = (ax^n + bx^{n-1} + \cdots + m)e^{\alpha x}$$

However, if $\alpha = \lambda$, then we must multiply an extra x:

$$y_p = x(ax^n + bx^{n-1} + \cdots + m)e^{\alpha x}$$

In the example above, the particular integral is indeed in the form:

$$y_p = x(ax + b)e^{-x}$$

Chapter 4

Oscillation

§ 4.1

Forced oscillation and resonance

4.1.1 The forced oscillator

The equation we will be considering in this section describes the physics of a forced, damped, harmonic oscillator:

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = F\cos\omega t.$$

where ω_0 is natural frequency of the oscillator, F is amplitude of the driving force and ω is the frequency of the driving force.

If $F = 0$ we recover the unforced damped, harmonic oscillator.

If $F = 0$ and $\gamma = 0$ we are back to simple harmonic motion with natural frequency ω_0.

4.1.2 Transient solution

We are only interested to oscillatory solution so we assume light damping, therefore:

$$x_{CF} = e^{-\frac{\gamma}{2}t}(A\cos\beta t + B\sin\beta t), \quad \beta = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{\frac{1}{2}}$$

This is a transient solution. It depends on the initial conditions (which determine A and B) and it decays to zero. In the steady state it will have died away and it can be ignored.

4.1.3 Steady state solution

On the basis of what we have covered so far, to find the steady state solution (the PI) we would try $x = D\cos\omega t + E\sin\omega t$ or, equivalently, $x = C\cos(\omega t - \varphi)$. However it is much easier to use complex numbers as follows:

$$m\ddot{x}_R + m\gamma\dot{x}_R + m\omega_0^2 x_R = F\cos\omega t = \mathrm{Re}[Fe^{i\omega t}]$$

So for the equation

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = Fe^{i\omega t}$$

To find the PI, try $x = Ce^{i\omega t}$. Substituting in gives

$$-\omega^2 mCe^{i\omega t} + m\gamma i\omega Ce^{i\omega t} + m\omega_0^2 Ce^{i\omega t} = Fe^{i\omega t} \quad \Rightarrow \quad C = \frac{F}{m\left\{\left(\omega_0^2 - \omega^2\right) + i\gamma\omega\right\}}$$

So

$$x = \frac{Fe^{i\omega t}}{m\left\{\left(\omega_0^2 - \omega^2\right) + i\gamma\omega\right\}}$$

Note that

$$m\left\{\left(\omega_0^2-\omega^2\right)+i\gamma\omega\right\}=re^{i\varphi}\quad\text{where}\quad r=m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2},\quad \tan\varphi=\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)}.$$

So the displacement becomes

$$x=\frac{Fe^{i\omega t}}{\left(m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\right)e^{i\varphi}}=\frac{Fe^{(i\omega t-\varphi)}}{m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}.$$

Taking the real part

$$x_R=\frac{F\cos(\omega t-\varphi)}{m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}\equiv A\cos(\omega t-\varphi).$$

The steady state solution is harmonic. It has a different amplitude A, which differs from that of the driving force, and it lags the driving force by a phase φ.

The derivation in this section is tricky at first but easy once you have practised it a few times. You will have to do it often so it is worth getting it straight.

4.1.4 The amplitude response

We have just shown that, for a forcing term with amplitude F, the displacement x (we shall lose the subscript R from now on) has amplitude

$$A=\frac{F}{m\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}.$$

A is a maximum when the denominator is a minimum.

$$\frac{d}{d\omega}\left\{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right\}\Big|_{\omega_R}=0\quad\Rightarrow\quad -4\omega\left(\omega_0^2-\omega_R^2\right)+2\gamma^2\omega=0\quad\Rightarrow\quad \omega_R^2=\omega_0^2-\frac{\gamma^2}{2}$$

The variation of A with ω is plotted in the figure.

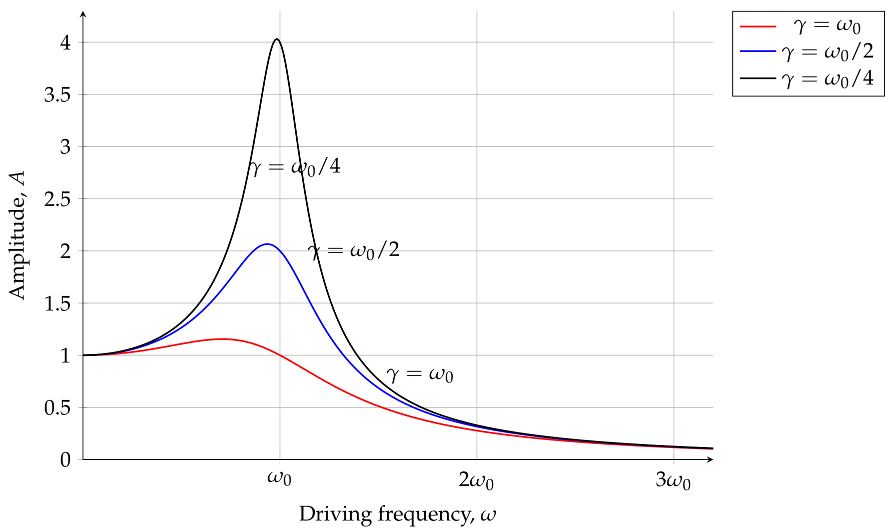

Figure 4.1: Amplitude of a damped, harmonic oscillator for different damping strengths.

- As $\omega\to\infty$, $A\to 0$. This is because the oscillator cannot respond if the driving is too fast.
- For $\omega=0$, $A=\frac{F}{m\omega_0^2}$. The static force is causing a displacement.

- The curve has a maximum at $\omega = \omega_R = \left(\omega_0^2 - \frac{\gamma^2}{2}\right)^{1/2}$.

Differentiating the amplitude response gives the velocity response

$$\dot{x} = \frac{-\omega F \sin(\omega t - \varphi)}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} \equiv -A_V \sin(\omega t - \varphi)$$

with amplitude

$$A_V = \frac{\omega F}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} = \frac{F}{m\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + \gamma^2}}$$

Notice that ω 's only appear in the first bracket in the denominator in the last expression so we can read off that the velocity amplitude has a maximum at $\omega = \omega_0$.

4.1.5 Width of the resonance and the Q-factor

For small damping the amplitude response $A(\omega)$ can be very sharply peaked. This is called resonance, and ω_R is the resonant frequency. It is useful to have a measure of the width of the resonance. A sensible definition is

$$\Delta\omega = \omega_2 - \omega_1 \quad \text{where} \quad A(\omega_1) = A(\omega_2) = \frac{1}{\sqrt{2}}A(\omega_R) \quad (\omega_2 > \omega_R > \omega_1)$$

(The choice of measuring the width of the curve at $\frac{1}{\sqrt{2}}A(\omega_R)$ corresponds to the stored energy $\sim A^2$ being $1/2$ of its maximum value.)

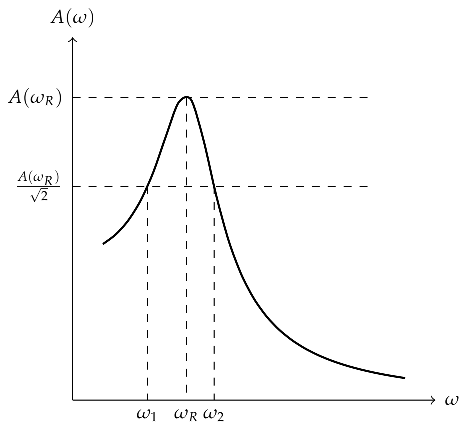

Figure 4.2: Defining the width of the resonance.

To find $\Delta\omega$:

$$A(\omega_1) = \frac{F}{m\sqrt{\left(\omega_0^2 - \omega_1^2\right)^2 + \gamma^2\omega_1^2}} = \frac{1}{\sqrt{2}}A(\omega_R) = \frac{F}{\sqrt{2}m\sqrt{\left(\omega_0^2 - \omega_R^2\right)^2 + \gamma^2\omega_R^2}}$$

$$\Rightarrow \quad \left(\omega_0^2 - \omega_1^2\right)^2 + \gamma^2\omega_1^2 = 2\left\{\left(\omega_0^2 - \omega_R^2\right)^2 + \gamma^2\omega_R^2\right\}$$

Recall that $\omega_R^2 = \omega_0^2 - \frac{\gamma^2}{2}$ so $\omega_0^2 - \omega_R^2 = \frac{\gamma^2}{2}$ and substituting into the equation

$$\left(\omega_0^2 - \omega_1^2\right)^2 + \gamma^2\omega_1^2 = 2\left\{\frac{\gamma^4}{4} + \gamma^2\omega_0^2 - \frac{\gamma^4}{2}\right\} = 2\gamma^2\omega_0^2 - \frac{\gamma^4}{2}$$

This can be solved as a quadratic in ω_1^2 but the answer is messy. It is much neater to identify a small parameter and expand the solution in terms of it (a useful approach for many problems). The relevant small parameter is γ because we expect, on physical grounds, that the damping is small for a sharp resonance. Since γ has the dimensions of frequency so we may write

$$\omega_1 = \omega_0 + a\gamma + O\left(\gamma^2\right)$$

and see if we get a consistent solution. Noting that $O\left(\gamma^2\right)$

$$\left(\omega_0^2 - \omega_1^2\right)^2 = (\omega_0 - \omega_1)^2 (\omega_0 + \omega_1)^2 \approx a^2\gamma^2 4\omega_0^2$$

and substituting the result

$$a^2\gamma^2 4\omega_0^2 + \gamma^2\omega_0^2 = 2\gamma^2\omega_0^2 \quad \Rightarrow \quad 4a^2 = 1 \quad \Rightarrow \quad a = \pm\frac{1}{2}$$

So

$$\omega_1 = \omega_0 - \frac{\gamma}{2}, \quad \omega_2 = \omega_0 + \frac{\gamma}{2}$$

Therefore for small damping, the full width of the amplitude response is

$$\Delta\omega = \omega_2 - \omega_1 = \gamma$$

4.1.6 Power and Energy

We can write down the power supplied

$$P = \text{ driving force } \times \text{ velocity } = F\cos\omega t \times \frac{-\omega F \sin(\omega t - \varphi)}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}.$$

Averaging the time-dependent terms over a cycle, the mean power is

$$\bar{P} = \frac{-\omega F^2}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}\overline{\cos\omega t \sin(\omega t - \varphi)}.$$

where $\overline{\cos\omega t \sin(\omega t - \varphi)} = \overline{\cos\omega t \sin\omega t \cos\varphi} - \overline{\cos^2\omega t \sin\varphi} = -\frac{1}{2}\sin\varphi$. Because $\overline{\cos\omega t \sin\omega t} = 0$ and $\overline{\cos^2\omega t} = \frac{1}{2}$. So

$$\bar{P} = \frac{\omega F^2 \sin\varphi}{2m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}$$

Aslo

$$\tan\varphi = \frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)} \quad \Rightarrow \quad \sin\varphi = \frac{\gamma\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}$$

So

$$\Rightarrow \quad \bar{P} = \frac{\gamma\omega^2 F^2}{2m\left\{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right\}}$$

It has maximum value at $\omega = \omega_0$

$$\bar{P}_{\max} = \frac{F^2}{2m\gamma}$$

We check that this is the same as the energy lost per cycle of a lightly damped, unforced oscillator:

$$F_{\text{damping}} = -m\gamma\dot{x}$$

Recall that

$$\dot{x} = v = \frac{-\omega F \sin(\omega t - \varphi)}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}$$

Energy lost per unit time is

$$\frac{|\Delta E|}{T} = \frac{1}{T}\int_0^T m\gamma\dot{x}^2 dt = m\gamma \cdot \frac{\omega^2 F^2}{m^2\left[\left(\omega_0^2 - \omega^2\right) + \gamma^2\omega^2\right]}\left[\frac{1}{T}\int_0^T \sin^2(\omega t - \varphi)dt\right] = \frac{\gamma\omega^2 F^2}{2m\left[\left(\omega_0^2 - \omega^2\right) + \gamma^2\omega^2\right]}$$

which is same as mean power input.

Quality factor

The energy stored by the oscillator in steady state is

$$\begin{aligned}E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \\ &= \frac{m}{2}\frac{\omega^2 F^2 \sin^2(\omega t-\varphi)}{m^2\left\{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right\}} + \frac{m\omega_0^2}{2}\frac{F^2\cos^2(\omega t-\varphi)}{m^2\left\{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right\}} \\ &= \frac{F^2}{2\gamma^2 m}\end{aligned}$$

at $\omega = \omega_0$.

We define the quality factor Q for as

$$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost per cycle}} = \frac{\text{stored energy}}{\text{energy lost per radian}}$$

Hence the quality factor is

$$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost per cycle}} = 2\pi\frac{F^2}{2\gamma^2 m}\frac{2m\gamma}{F^2}\frac{\omega_0}{2\pi} = \frac{\omega_0}{\gamma}.$$

Now consider the energy content of the transient motion that the CF describes. Recall that

$$x_{CF} = e^{-\frac{\gamma}{2}t}(A'\cos\beta t + B'\sin\beta t) = \mathrm{e}^{-\gamma t/2}A\cos(\beta t+\psi) = e^{-\frac{\gamma}{2}t}A\cos\eta \quad \text{where } (\eta\equiv\beta t+\psi)$$

So

$$\begin{aligned}x &= e^{-\frac{\gamma}{2}t}A\cos\eta \quad\Rightarrow\quad \dot{x} = -\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}A\cos\eta - \beta e^{-\frac{\gamma}{2}t}A\sin\eta \\ x^2 &= A^2 e^{-\gamma t}\cos^2\eta \quad\Rightarrow\quad \dot{x}^2 = A^2\left[\frac{\gamma^2}{4}e^{-\gamma t}\cos^2\eta + \beta^2 e^{-\gamma t}\sin^2\eta + \gamma\beta e^{-\gamma t}\sin\eta\cos\eta\right]\end{aligned}$$

Therefore

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2x^2 = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2\eta + \gamma\beta\cos\eta\sin\eta + \beta^2\sin^2\eta + \omega_0^2\cos^2\eta\right]$$

For very light damped oscillator, $\gamma/\omega_0 \ll 1$ and recall definition of β, we have

$$\beta = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{\frac{1}{2}} \quad\Rightarrow\quad \left(\frac{\beta}{\omega_0}\right)^2 \approx 1$$

The energy becomes

$$E \approx \frac{1}{2}m\left(\omega_0 A\right)^2\mathrm{e}^{-\gamma t}$$

For very light damped oscillator, the energy decreases in time due to the $e^{-\gamma t}$ factor. The lost energy goes into heat that arises from the damping force.

Since quality factor Q is

$$\begin{aligned}Q &= 2\pi\frac{\text{stored energy}}{\text{energy lost per cycle}} \equiv 2\pi\frac{E(t)}{E\left(t-\pi/\omega_0\right)-E\left(t+\pi/\omega_0\right)} \approx \frac{1}{\mathrm{e}^{\pi\gamma/\omega_0}-\mathrm{e}^{-\pi\gamma/\omega_0}} \\ &= 2\pi\frac{1}{2}\operatorname{csch}\left(\pi\gamma/\omega_0\right) \approx \frac{\omega_0}{\gamma} \quad (\text{ for small } \gamma/\omega_0)\end{aligned}$$

Q is the inverse of the fraction of the oscillator's energy that is dissipated in one period. It is approximately equal to the number of oscillations conducted before the energy decays by a factor of e.

4.1.7 Phase lag

For a force $F \cos \omega t$ the displacement of a damped harmonic oscillator is

$$x_R = \frac{F \cos(\omega t - \varphi)}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}} \equiv A\cos(\omega t - \varphi)$$

The displacement lags the force by a phase φ

$$\tan \varphi = \frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)}$$

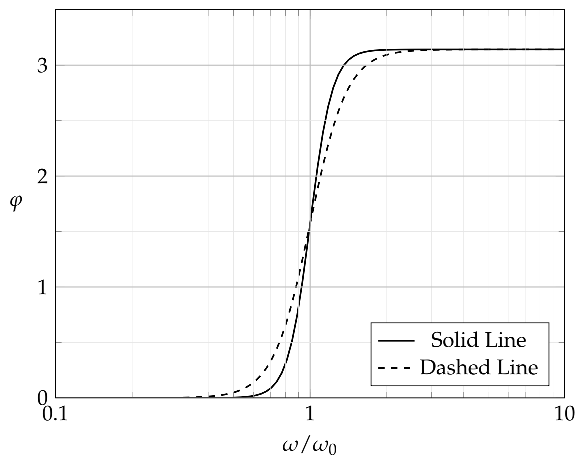

Figure 4.3: Phase of a driven oscillator. Solid lines are for $\gamma = 0.25\omega_0$, dashed lines for $\gamma = 0.5\omega_0$.

The velocity of the oscillator is

$$\begin{aligned} \dot{x} &= -\omega A \sin(\omega t - \varphi) = \omega A \sin(\varphi - \omega t) = \omega A \cos\left(\frac{\pi}{2} - \varphi + \omega t\right) \\ &= \omega A \cos\left(\omega t - \left(\varphi - \frac{\pi}{2}\right)\right) := \omega A \cos\left(\omega t - \varphi_V\right) \end{aligned}$$

so the velocity lags the force by $\varphi_V = \varphi - \frac{\pi}{2}$. Note that the velocity and the driving force are in phase at resonance.

Chapter 5

Systems of Linear Differential Equations

§ 5.1 Introduction

We have explained that for the linear ODEs of arbitrary order n can be written in such a form

$$\dot{\boldsymbol{y}} = \mathrm{A}(t) \cdot \boldsymbol{y} + \boldsymbol{f}(t)$$

In this chapter, we only consider the case that all elements of A is constant coefficients.

We firstly discuss a less general method, which is called decoupling method. Then we will develop the more general method, the eigenvalue method.

5.1.1 Decoupling method

Consider a simple case that A is 2×2 matrix, we could write it explicitly as:

$$\begin{cases} \dot{x} = ax + by + f(t) \\ \dot{y} = cx + dy + g(t) \end{cases}$$

We can find the explicit solutions this pair of coupled linear equations by adding or subtracting equations by proper coefficient to decouple the equations. Let's see an example.

Example 5.1 Solve the coupled differential equations

$$\begin{cases} y' + 2z' + 4y + 10z - 2 = 0 \\ y' + z' + y - z + 3 = 0 \end{cases}$$

where $y = 0$ and $z = -2$ at $x = 0$.

✍ Equation 1 - Equation 2, we get $z' + 3y + 11z - 5 = 0$. Thus y can be represented by z and z', then we differentiate y to get

$$y = \frac{1}{3}\left(5 - 11z - z'\right)$$
$$y' = \frac{1}{3}\left(-11z' - z''\right)$$

Therefore we have

$$\frac{1}{3}\left(-11z' - z''\right) + \frac{1}{3}\left(5 - 11z - z'\right) + z' - z + 3 = 0$$

This is a constant coefficient second order differential equation for z, which can be simplified to

$$z'' + 9z' + 14z = 14 \quad \Rightarrow \quad z = Ae^{-2x} + Be^{-7x} + 1$$

Then y can be solved.

§ 5.2
Eigenvector method

The decoupling method will become extremely complicated when number of equations is more than 2. So we will use a more general method to solve them.

We now reconsider the coupled homogeneous linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

In our matrix notation

$$\dot{\boldsymbol{y}} = \mathrm{A}(t) \cdot \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Notice that since this equation is linear, we have a superposition principle; linear combinations of solutions will still satisfy the equation, i.e. if $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are solutions, then so does

$$\boldsymbol{y}(t) = \alpha \boldsymbol{y}_1(t) + \beta \boldsymbol{y}_2(t)$$

since

$$\begin{aligned} \dot{\boldsymbol{y}} &= \alpha \dot{\boldsymbol{y}}_1 + \beta \dot{\boldsymbol{y}}_2 \\ &= \alpha A \boldsymbol{y}_1 + \beta A \boldsymbol{y}_2 \\ &= A \left[\alpha \boldsymbol{y}_1 + \beta \boldsymbol{y}_2 \right] \\ &= A \boldsymbol{y} \end{aligned}$$

Now the equation looks like the simple linear equation $\dot{x} = ax$, whose solution we know is $x(t) = C\mathrm{e}^{at}$. We will try to find a solution by guessing that it has the same type of exponential dependence on time ($\mathrm{e}^{\lambda t}$ for some λ). Our trial solution is

$$\boldsymbol{y}(t) = \mathrm{e}^{\lambda t} \mathbf{v}$$

In the scalar case, C in $x(t) = Ce^{at}$ is a constant scalar. For the vector case, $\mathbf{v}$ in $y(t) = e^{\lambda t}\mathbf{v}$ plays a similar role but it needs to be a vector because $y(t)$ and its derivative with respect to time are vectors. The matrix A acts on $\mathbf{v}$ to produce another vector, just as a acts on C to produce another scalar in the scalar case.

If we substitute this guess into the equation, then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\mathrm{e}^{\lambda t} \mathbf{v} \right] = A \left[\mathrm{e}^{\lambda t} \mathbf{v} \right]$$

Since $\mathbf{v}$ is a constant vector the $\mathrm{d}/\mathrm{d}t$ on the left-hand side only affects the exponential term:

$$\lambda \mathrm{e}^{\lambda t} \mathbf{v} = \mathrm{e}^{\lambda t} A \mathbf{v}$$

Dividing by the non-zero factor $\mathrm{e}^{\lambda t}$ this becomes

$$A\mathbf{v} = \lambda \mathbf{v}$$

This is known as an eigenvalue equation. There are two possibilities for A in general: the matrix A can be diagonalise or cannot be diagonalised.

For each λ_i

$$\boldsymbol{y}_i(t) = C_i e^{\lambda_i t} \boldsymbol{v}_i,$$

By theorem in linear algebra, $\{\boldsymbol{y}_i(t)\}$ are linearly independent if the matrix is diagonalisable. This gives us a fundamental system, so any solution of the homogeneous equation can be written as

$$\boldsymbol{y}(t) = \sum_{i=1}^{n} C_i \boldsymbol{v}_i e^{\lambda_i t}$$

5.2.1 Diagonalisable system

For eigenvalue equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

Let us suppose that the n roots $\lambda_1, \ldots, \lambda_n$ of the n-th-order polynomial equation

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0$$

are all different, i.e., they are non-degenerate right eigenvalues of A. Then their corresponding eigenvectors $\{v_i\}$ form a basis (not necessarily orthogonal).

Distinct real eigenvalues

For distinct real eigenvalues (no repeated roots), they must can be diagonalised.

For 2×2 case, this means that we have obtained two possible solutions of the differential equation,

$$\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 \quad \text{and} \quad \mathrm{e}^{\lambda_2 t}\mathbf{v}_2.$$

Thus the general solution of $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ can be written as

$$\boldsymbol{y}(t) = A\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + B\mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

Example 5.2 2×2 *system of linear differential equation*

By finding the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the coupled system

$$\dot{x} = x + y$$
$$\dot{y} = 4x - 2y.$$

✍ Rewritten as a matrix equation the problem becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

We will denote the matrix on the right-hand side by A. If we try a solution for the equation $\dot{\mathbf{x}} = A\mathbf{x}$ of the form $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, then as we saw above this leads to this eigenvalue problem

$$\lambda\mathbf{v} = A\mathbf{v}$$

To find the eigenvalues of A we solve the equation $\det(A - \lambda\mathbb{I}) = 0$; this is

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0,$$

which gives the characteristic polynomials:

$$(1-\lambda)(-2-\lambda) - 4 = 0.$$

Multiplying this out and simplifying we obtain

$$\lambda^2 + \lambda - 6 = 0.$$

The solutions of this, $\lambda = 2$ or $\lambda = -3$, are the eigenvalues of A. When $\lambda = 2$, the corresponding eigenvectors v_1 is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $\lambda = -3$, we get

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

The general solution is therefore

$$\mathbf{x}(t) = Ae^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

5.2.2 Inhomogeneous case

Let's go back to inhomogeneous case

$$\dot{\boldsymbol{y}} = \mathrm{A} \cdot \boldsymbol{y} + \boldsymbol{f}(t)$$

where $\boldsymbol{f}(t) \neq 0$.

Then, for the $\boldsymbol{f}(t)$ in the exponential form $e^{\mu t}$, one gets the particular solution as follows:

$$\boldsymbol{y}_{\mathrm{PI}}(t) = \boldsymbol{u}e^{\mu t} \Rightarrow \boldsymbol{u} = \mathrm{A} \cdot \boldsymbol{u} + \boldsymbol{F} \quad \Rightarrow \quad \boldsymbol{u} = -(\mathrm{A} - \mu \mathrm{I})^{-1} \cdot \boldsymbol{F}$$

Note that this only works if μ is not an eigenvalue of A (otherwise the inverse does not exist).

Example 5.3 Consider

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{5t}, \\ \dot{y} = x + 2y \end{cases}$$

✍ To solve this, consider the homogeneous equation first. The matrix is

$$\mathrm{A} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \det(\mathrm{A} - \lambda \mathbf{I}) = \lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 1$$

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore, the general solution of the homegeneous equation is

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathrm{CF}} = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}.$$

Now let us look for the particular solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathrm{PI}} = \begin{pmatrix} A \\ B \end{pmatrix}e^{5t} \Rightarrow A = 3, \quad B = 1.$$

The values of the constants are obtained by direct substitution of the trial solution into the equation. Final answer:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{5t}.$$

If such a simple method of finding $\boldsymbol{y}_{\mathrm{PI}}$ does not work, the universal panacea is to vary constants: let $C_1 \to \psi_1(t), C_2 \to \psi_2(t)$, substitute into the equation and solve for $\psi_{1,2}$.

Repeated eigenvalue

Example 5.4 Find the general, real solutions of the following inhomogeneous systems of ODEs

$$\begin{cases} \dot{x} = 4x + 3y - 3z, \\ \dot{y} = -3x - 2y + 3z, \\ \dot{z} = 3x + 3y - 2z + 2e^{-t} \end{cases}$$

✍ By solving the eigenvalue equation, we find when $\lambda = -2$, eigenvector is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

And when $\lambda = 1$, which is a root with multiplicity 2. The eigenvectors are:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} x(t) &= C_3 e^{-2t} - (C_1 - C_2)\, e^t \\ y(t) &= C_1 e^t - C_3 e^{-2t} \\ z(t) &= C_2 e^t + C_3 e^{-2t} \end{aligned}$$

Combine with particular integral, we find the general solution is

$$\begin{aligned} x(t) &= C_3 e^{-2t} - (C_1 - C_2)\, e^t + 3e^{-t} \\ y(t) &= C_1 e^t - C_3 e^{-2t} - 3e^{-t} \\ z(t) &= C_2 e^t + C_3 e^{-2t} + 2e^{-t} \end{aligned}$$

It seems that there are only two linearly independent solutions when you only consider about e^t and e^{-2t}, but in fact the key is the eigenvectors, we have three linearly independent eigenvectors so they span the entire solution space for the homogeneous system.

Complex eigenvalues

Normally, the question will ask for real solutions so we need to take the real part of the solution.

Example 5.5 *Hermitian system with complex eigenvalues*
Find the general, real solutions of the following inhomogeneous system of ODEs

$$\begin{cases} \dot{x} = -5x + y - 2z + \cosh t \\ \dot{y} = -x - y + 2\sinh t + \cosh t \\ \dot{z} = 6x - 2y + 2z - 2\cosh t \end{cases}$$

✍ Firstly solving the characteristic equation to get λ, which is -2, $-1+i$ and $-1-i$.
When $\lambda = 2, v = (-1,-1,1), \lambda = -1+i, v = (-1,-i,2)$ and when $\lambda = -1-i, v = (-1,i,2)$.
Then we only need to take the real part of solution, which is the complementary functions. Finally, we need to find the particular integral, which is easy.

Assume $\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{PI} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cosh t + \begin{pmatrix} D \\ E \\ F \end{pmatrix} \sinh t.$

We have

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} D \\ E \\ F \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

which gives that $A = B = C = 0, D = 1, E = 1, F = -2$.

5.2.3 Changing coordinates

We can change to a coordinate system that uses the eigenvectors as axes.

Recall the change of basis theorem in linear algebra, we have

$$A' = P^{-1}AP$$

In the context of diagonalisation, it is actually

$$D = P^{-1}AP \quad \Rightarrow \quad A = PDP^{-1}$$

where

$$P = (\boldsymbol{v}_1 \dots \boldsymbol{v}_n)\,, \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{pmatrix} \equiv \text{diag}\,\{\lambda_i\}$$

And P^{-1} is the change of coordinate matrix[1] that transforms vector in original basis S to vector in eigenvector basis S':

$$\boldsymbol{y}' = P^{-1}\boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{y} = P\boldsymbol{y}'$$

Now substitute to our differential equation

$$\dot{\boldsymbol{y}} = Ay \quad \Rightarrow \quad P\dot{\boldsymbol{y}}' = AP\boldsymbol{y}'$$

which is

$$\begin{aligned} \dot{\boldsymbol{y}}' &= P^{-1}AP\boldsymbol{y}' \\ &= D\boldsymbol{y}' \end{aligned}$$

Since D is a diagonal matrix, so our system of ODEs breaks into n decoupled equations:

$$\dot{y}'_i = \lambda_i y'_i \quad \Rightarrow \quad y'_i(t) = C_i e^{\lambda_i t}$$

Then the general solution is

$$\boldsymbol{y}(t) = P\boldsymbol{y'}(t) = \sum_i^n C_i \boldsymbol{v}_i e^{\lambda_i t}$$

which is same as the solution before.

Example 5.6 *Change of eigenvector basis to solve differential equations*

By means of an appropriate coordinate transformation decouple the equations

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x - 2y \end{aligned}$$

and hence write down their general solution.

[1]Reference: my CP3 Linear Algebra Notes

✍ Rewriting the equation in matrix form we have

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{A} \mathbf{x}.$$

We found the eigenvalues and eigenvectors of the matrix A for the previous example: they are

$$\lambda_1 = 2 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -3 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

We need

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} x' + y' \\ x' - 4y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}}_{P} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

('x $= P\mathbf{x}'$'). Multiplying the extreme left- and right-hand sides of this equation by P^{-1} will give $\mathbf{x}'$ in terms of $\mathbf{x}$,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

Computing the inverse we arrive at

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e.

$$x' = \frac{1}{5}(4x + y) \quad \text{and} \quad y' = \frac{1}{5}(x - y).$$

Referred to these new coordinate axes the equation becomes

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t} &= P^{-1}AP\mathbf{x}' \\ &= \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \mathbf{x}' \\ &= \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 2 & 12 \end{pmatrix} \mathbf{x}' \\ &= \frac{1}{5} \begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix} \mathbf{x}' \\ &= \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}'. \end{aligned}$$

So in our new variables we obtain the decoupled equations

$$\frac{dx'}{dt} = 2x' \quad \frac{dy'}{dt} = -3y'.$$

The solutions of these can easily be seen to be $x'(t) = A\mathrm{e}^{2t}$ and $y'(t) = B\mathrm{e}^{-3t}$, and so the solution of the original equation can be recovered,

$$\mathbf{x}(t) = A\mathrm{e}^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

5.2.4 Hermitian system

Suppose now that our matrix is Hermitian: $A = A^{\dagger}$. Recall the spectrum theorem in linear algebra, we have

(i) All eigenvalues of A are real.

(ii) Eigenvectors for different eigenvalues are orthogonal.

(iii) There exists an ortho-normal basis of V consisting of eigenvectors of f: $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n\}$.

Therefore, we can always diagonalise the Hermitian matrix even if not all the eigenvalues of A are non-degenerate.

5.2.5 Non-Hermitian system *

For non-Hermitian system, the diagonalisability of non-Hermitian matrices is not guaranteed.

If the characteristic polynomial has n distinct roots, then A is diagonalisable. This occurs when characteristic polynomial has repeated roots and the degeneracy is less than the multiplicity, the matrix is not diagonalisable.

We have to develop some method to deal with non-Hermitian system.

Jordan form and Jordan chain

We use the method of Jordan Chain, the principle of Jordan Chain is very complicated so we only consider the algorithm:

$$\begin{aligned}\boldsymbol{y}(t) =&\xi_1(t)\boldsymbol{v}_1 + \cdots + \xi_k(t)\boldsymbol{v}_k\\ =&e^{\lambda_1 t}\left[C_1\boldsymbol{v}_1 + C_2\left(t\boldsymbol{v}_1 + \boldsymbol{v}_2\right) + C_3\left(\frac{t^2}{2}\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{v}_3\right) + \ldots\right.\\ &\left.+C_k\left(\frac{t^{k-1}}{(k-1)!}\boldsymbol{v}_1 + \cdots + t\boldsymbol{v}_{k-1} + \boldsymbol{v}_k\right)\right]\\ =&\ C_1\boldsymbol{y}_1(x) + \cdots + C_k\boldsymbol{y}_k(t).\end{aligned}$$

Example 5.7 Consider

$$\begin{cases}\dot{x} = -x + y - 2z\\ \dot{y} = 4x + y\\ \dot{z} = 2x + y - z\end{cases}$$

First, find the eigenvalues:

$$A = \begin{pmatrix}-1 & 1 & -2\\ 4 & 1 & 0\\ 2 & 1 & -1\end{pmatrix} \quad\Rightarrow\quad \det(\mathrm{A} - \lambda\mathrm{I}) = (1+\lambda)^2(1-\lambda) = 0 \quad\Rightarrow\quad \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 1.$$

The eigenvector corresponding to the non-degenerate eigenvalue $\lambda_3 = 1$ satisfies

$$\begin{pmatrix}-2 & 1 & -2\\ 4 & 0 & 0\\ 2 & 1 & -2\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0 \quad\Rightarrow\quad x = 0, y = 2z \quad\Rightarrow\quad \boldsymbol{v}_3 = \begin{pmatrix}0\\ 2\\ 1\end{pmatrix}.$$

We now expect to find two eigenvectors or an eigenvector and an associated vector corresponding to $\lambda_1 = -1$:

$$\begin{pmatrix}0 & 1 & -2\\ 4 & 2 & 0\\ 2 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 0 \quad\Rightarrow\quad y - 2z = 0, 2x + y = 0 \quad\Rightarrow\quad \boldsymbol{v}_1 = \begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}.$$

No other eigenvector can be found, so we test our confidence in Jordan's theorem by seeking an associated vector according to Jordan's chain:

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad y - 2z = -1, 2x + y = 1 \quad \Rightarrow \quad v_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

where we let $z = 2$, then we get $y = 3$ and $x = -1$. Note that the generalised eigenvector is not unique, we can add any multiple of v_1 to get the general form of v_2. For example, if we let $z = 4$, then $y = 7$ and $x = -3$, which gives $v_2 + 2v_1$.

Success, we have our Jordan basis! Jordan's theorem tells us that we should not be able to continue this Jordan chain. Let us verify this:

$$\begin{pmatrix} 0 & 1 & -2 \\ 4 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = v_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \Rightarrow \quad y - 2z = -1, 4x + 2y = 3, 2x + y = 2.$$

Indeed, this has no solution. Now with full confidence that the method works, hence general solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left\{ C_1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + C_2 \left[\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right] \right\} e^{-t} + C_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^t.$$

Example 5.8 Find the general solutions of the following homogeneous systems of ODEs:

$$\begin{cases} \dot{x} = 2x - 5y - 8z \\ \dot{y} = 7x - 11y - 17z \\ \dot{z} = -3x + 4y + 6z \end{cases}$$

✍ For the matrix

$$A = \begin{pmatrix} 2 & -5 & -8 \\ 7 & -11 & -17 \\ -3 & 4 & 6 \end{pmatrix}$$

We have $\lambda_1 = \lambda_2 = \lambda_3 = -1$, the multiplicity is 3.

When $\lambda = -1$:

$$\begin{pmatrix} 3 & -5 & -8 \\ 7 & -10 & -17 \\ -3 & 4 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -5 & -8 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -5 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Now let's find generalised eigenvectors v_2:

$$\begin{pmatrix} 3 & -5 & -8 \\ 7 & -10 & -17 \\ -3 & 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad v_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

For v_3:

$$\left(\begin{array}{ccc|c} 3 & -5 & -8 & 3 \\ 7 & -10 & -17 & 2 \\ -3 & 4 & 7 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -5 & -5 & 15 \\ 1 & 0 & -1 & -4 \\ 0 & -1 & -1 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = 1$, we get $y = -4$ and $x = -3$.

Therefore the general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-t} \left\{ C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right] + C_3 \left[\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} \right] \right\}$$