AS & A-Level Physics

Lecture Notes

(Draft)

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Contents

Chapter 1

Circular Motion

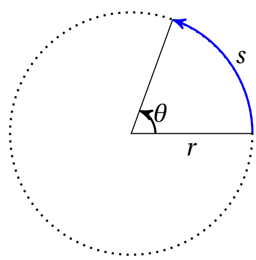
1.1 angular quantities

movement or rotation of an object along a circular path is called **circular motion**

to describe a circular motion, we can use *angular quantities*, which turn out to be more useful than linear displacement , linear velocity , etc.

1.1.1 angular displacement

angular displacement is angle swiped out by object moving along circular



- unit: $[\theta] =$ rad (natural unit of measurement for angles)

  conversion rule: 2π rad $= 360°$
- if two radii form an angle of θ, then length of arc: $s = r\theta$

  two radii subtending an arc of same length as radius form an angle of one **radian**

1.1.2 angular velocity

angular velocity describes how fast an object moves along a circular path

angular velocity is defined as angular displacement swiped out per unit time: $\boxed{\omega = \frac{\Delta\theta}{\Delta t}}$

- unit of: $[\omega] =$ rad s^{-1}, also in radian measures
- angular velocity is a *vector* quantity

  this vector points in a direction normal to the plane of circular motion but in A-level course, we treat angular velocity as if it is a scalar

angular velocity and angular speed may be considered to be the same idea

➢ relation with linear velocity

in interval Δt, distance moved along arc $\Delta s = v\Delta t = r\Delta\theta \quad \Rightarrow \quad \omega = \frac{\Delta\theta}{\Delta t} = \frac{v}{r} \quad \Rightarrow \quad \boxed{v = \omega r}$

this relation between linear speed and angular speed holds at any instant

1.1.3 uniform circular motion

when studying linear motion, we started from motion with constant velocity v

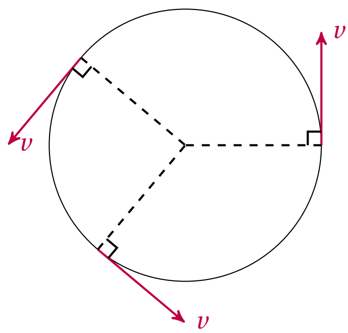
consider the simplest possible circular motion $\longrightarrow$ circular motion with constant ω

analogy with linear motion with constant v

uniform linear motion: $s = vt$

displacement $s \leftrightarrow \theta$, velocity $v \leftrightarrow \omega$

for uniform circular motion, one has: $\boxed{\theta = \omega t}$



➢ time taken for one complete revolution is called **period** T

in one T, angle swiped is 2π, so $\boxed{\omega = \frac{2\pi}{T}}$

➢ uniform circular motion is still *accelerated* motion

speed is unchanged, but *velocity* is changing

direction of velocity always *tangential* to its path, so direction of velocity keeps changing

in general, any object moving along circular path is accelerating

Example 1.1 An object undergoes a uniform motion around a circular track of radius 2.5 m in 40 s, what is its angular speed and linear speed?

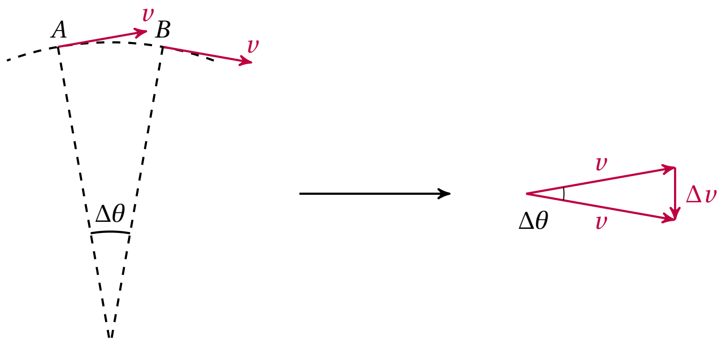
✍ $$\omega = \frac{2\pi}{T} = \frac{2\pi}{40} \approx 0.157 \text{ rad s}^{-1} \qquad v = \omega r = 0.157 \times 2.5 \approx 0.39 \text{ m s}^{-1}$$ □

1.1.4 centripetal acceleration

> **centripetal acceleration** is the acceleration due to the change in direction of velocity vector, it points toward the centre of circular path

consider motion along a circular path from A to B with constant speed v

under small (infinitesimal) duration of time Δt[1]



change in velocity: $\Delta v = 2v\sin\dfrac{\Delta\theta}{2} \approx v\Delta\theta$ (as $\Delta\theta \to 0$, $\sin\Delta\theta \approx \Delta\theta$)

acceleration: $a = \dfrac{\Delta v}{\Delta t} \approx v\dfrac{\Delta\theta}{\Delta t} = v\omega$ (as $\omega = \dfrac{\Delta\theta}{\Delta t}$)

recall relation $v = \omega r$, we find centripetal acceleration: $\boxed{a_c = \dfrac{v^2}{r} = \omega^2 r}$

- direction of centripetal acceleration: always towards centre of circular path
- centripetal acceleration is only responsible for the change in *direction* of velocity

  change in *magnitude* of velocity will give rise to *tangential acceleration*

  this is related to *angular acceleration*[2] , which is beyond the syllabus

1.2 centripetal force

circular motion must involve change in velocity, so object is not in equilibrium

there must be a *net force* on an object performing circular motion

> **centripetal force** (F_c) is the resultant force acting on an object moving along a circular path, and it is always directed towards centre of the circle

- centripetal force causes centripetal acceleration

  using Newton's 2nd law: $\boxed{F_c = m\dfrac{v^2}{r} = m\omega^2 r}$
- F_c is not a new force by nature

[1] A more rigorous derivation can be given by using differentiation techniques

[2] Angular acceleration is analogous to linear acceleration α, defined as rate of change of angular velocity: $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ $(\star)$. Similar to $v = \omega r = \dfrac{ds}{dt}$, the relation $a = \alpha r = \dfrac{dv}{dt}$ also holds.

F_c is a resultant of forces you learned before (weight, tension, contact force, friction, etc.)

➢ F_c acts at right angle to direction of velocity

or equivalently, if $F_{\text{net}} \perp v$, then this net force provides centripetal force for circular motion

➢ effect of F_c: change *direction* of motion, or maintain circular orbits

to change *magnitude* of velocity, there requires a *tangential* component for the net force

again the idea of tangential force is beyond the syllabus

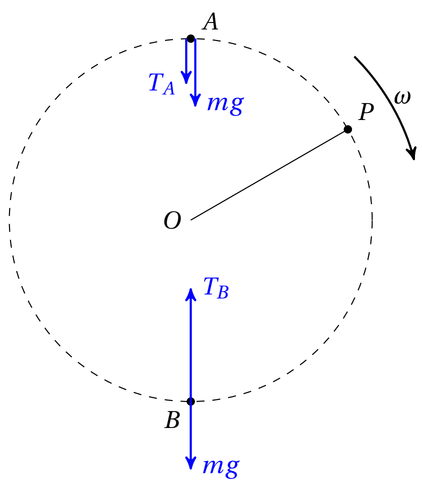
Example 1.2 A rock is able to orbit around the earth near the surface of the earth. Given that radius of the earth $R = 6400$ km and air resistance is ignored, (a) what is its orbital speed? (b) What is the orbital period?

✍ weight of object provides centripetal force: $mg = \dfrac{mv^2}{R}$

orbital speed: $v = \sqrt{gR} = \sqrt{9.81 \times 6.4 \times 10^6} \approx 7.9 \times 10^3 \text{ m s}^{-1}$

period: $T = \dfrac{2\pi R}{v} = \dfrac{2\pi \times 6.4 \times 10^6}{7.9 \times 10^3} \approx 5.1 \times 10^3 \text{ s} \approx 85 \text{ min}$ □

Example 1.3 Particle P of mass $m = 0.40$ kg is attached to one end of a light inextensible string of length $r = 0.80$ m. The particle is whirled at a constant angular speed ω in a vertical plane. (a) Given that the string never becomes slack, find the minimum value of ω. (b) Given instead that the string will break if the tension is greater than 20 N, find the maximum value of ω.



✍ at top of circle (point A): $F_c = T_A + mg = m\omega^2 r \quad \Rightarrow \quad T_A = m\omega^2 r - mg$

at bottom of circle (point B): $F_c = T_B - mg = m\omega^2 r \quad \Rightarrow \quad T_B = m\omega^2 r + mg$

tension is minimum at A, but string being taut requires $T \geq 0$ at any point, so $T_A \geq 0$

$$m\omega^2 r - mg \geq 0 \quad \Rightarrow \quad \omega^2 \geq \frac{g}{r}$$
$$\omega_{\min} = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.81}{0.80}} \approx 3.5 \text{ rad s}^{-1}$$

tension is maximum at B, but string does not break requires $T \leq T_{\max}$, so $T_B \leq T_{\max}$

$$m\omega^2 r + mg \leq T_{\max} \quad \Rightarrow \quad \omega^2 \leq \frac{T_{\max}}{m} - \frac{g}{r}$$
$$\omega_{\max} = \sqrt{\frac{T_{\max}}{m} - \frac{g}{r}} = \sqrt{\frac{20}{0.40} - \frac{9.81}{0.80}} \approx 6.1 \text{ rad s}^{-1}$$ □

Example 1.4 A turntable can rotate freely about a vertical axis through its centre. A small object is placed on the turntable at distance $d = 40$ cm from the centre. The turntable is then set to rotate, and the angular speed of rotation is slowly increased. The coefficient of friction between the object and the turntable is $\mu = 0.30$. If the object does not slide off the turntable, find the maximum number of revolutions per minute.

✍ if object stays on turntable, friction provides the centripetal force required: $f = m\omega^2 d$

increasing ω requires greater friction to provide centripetal force

but maximum limiting friction possible is: $f_{\lim} = \mu N = \mu mg$, therefore

$$f \leq f_{\lim} \quad \Rightarrow \quad m\omega^2 d \leq \mu mg \quad \Rightarrow \quad \omega^2 \leq \frac{\mu g}{d} \quad \Rightarrow \quad \omega_{\max} = \sqrt{\frac{0.30 \times 9.81}{0.40}} \approx 2.71 \text{ rad s}^{-1}$$

period of revolution: $T_{\min} = \dfrac{2\pi}{\omega_{\max}} = \dfrac{2\pi}{2.71} \approx 2.32$ s

number of revolutions in one minute: $n_{\max} = \dfrac{t}{T_{\min}} = \dfrac{60}{2.32} \approx 25.9$ □

Example 1.5 A pendulum bob of mass 120 g moves at constant speed and traces out a circle or radius $r = 10$ cm in a horizontal plane. The string makes an angle $\theta = 25^\circ$ to the vertical. (a) What is the tension in the string? (b) At what speed is the bob moving?

✍ vertical component of tension T_y equals weight

$$T_y = mg \quad \Rightarrow \quad T\cos\theta = mg$$
$$T = \frac{mg}{\cos\theta} = \frac{0.12 \times 9.81}{\cos 25^\circ} \approx 1.3 \text{ N}$$

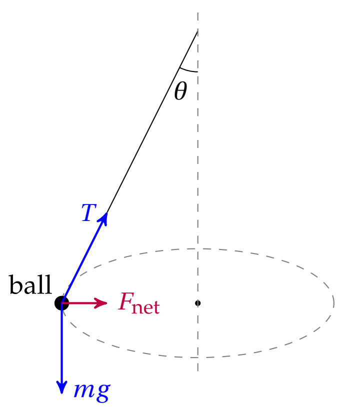
net force equals horizontal component of tension T_x

so component T_x provides centripetal force

$$F_c = T_x \quad \Rightarrow \quad T\sin\theta = \frac{mv^2}{r}$$

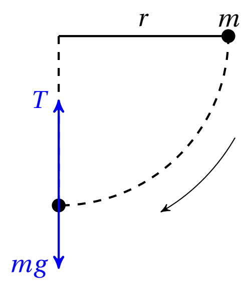
by eliminating T and m, one can find

$$v^2 = \frac{r\tan\theta}{g} = \frac{0.10 \times \tan 25^\circ}{9.81} \quad \Rightarrow \quad v \approx 0.069 \text{ m s}^{-1}$$ □



Example 1.6 A small ball of mass m is attached to an inextensible string of length l. The ball is held with the string taut and horizontal and is then released from rest.

When the ball reaches lowest point, find its speed and the tension in the string in terms of m and l.



✍ energy conservation: G.P.E. loss = K.E. gain

$$mgr = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gr}$$

at lowest point: $F_c = T - mg = m\frac{v^2}{r}$

$$T = mg + m\frac{v^2}{r} = mg + m\frac{2gr}{r} = 3mg$$

□

Question 1.1 A turntable that can rotate freely in a horizontal plane is covered by dry mud. When the angular speed of rotation is gradually increased, state and explain whether the mud near edge of the plate or near the mud will first leave the plate?

Question 1.2 A bucket of water is swung at a constant speed and the motion describes a circle of radius $r = 1.0$m in the vertical plane. If the water does not pour down from the bucket even when it is at the highest position, how fast do you need to swing the bucket?

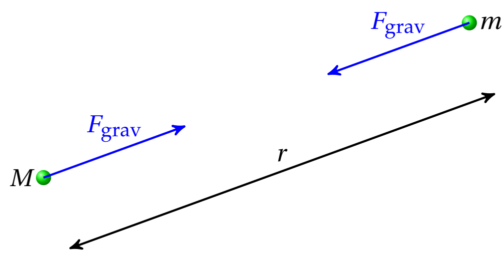
CHAPTER 2

Gravitational Fields

2.1 gravitational forces

2.1.1 Newton's law of gravitation

any object attracts any other object through the gravitational force



gravitational attraction between M and m

Newton's law of gravitation states that gravitational force between two *point* masses is proportional to the product of their masses and inversely proportional to the square of their distance $\left(F_{\text{grav}} \propto \dfrac{Mm}{r^2}\right)$

this law was formulated in *Issac Newton*'s work 'The Principia', or 'Mathematical Principles of Natural Philosophy', first published in 1687

mathematically, gravitational force takes the form: $\boxed{F_{\text{grav}} = \dfrac{GMm}{r^2}}$

$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the *gravitational constant*

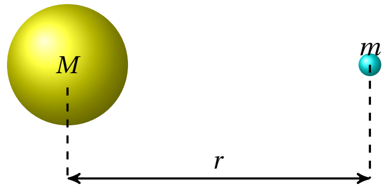
- gravitational force is always *attractive*
- gravity is *universal*, i.e., gravitational attraction acts between *any* two masses
- Newton's law of gravitation refers to *point masses*

  i.e., particles with no size, therefore distance r can be easily defined

➢ a sphere with uniform mass distribution (e.g., stars, planets) can be treated as a *point model*

distance r is taken between centres of the spheres [3]

(see Example 2.8, field lines around a planet *seem* to point towards centre of planet)

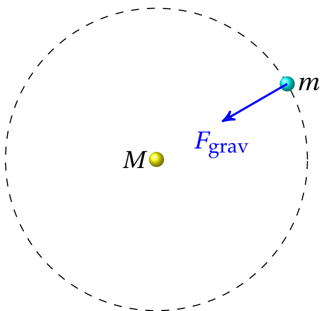


Example 2.1 The Earth can be thought as a uniform sphere of radius $R = 6.4 \times 10^6$ m and mass $M = 6.0 \times 10^{24}$ kg. Estimate the gravitational force on a man of 60 kg at sea level.

✍ $$F = \frac{GMm}{R^2} = \frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 60}{(6.4 \times 10^6)^2} \approx 586 \text{ N}$$ □

Question 2.1 Estimate the gravitational force between you and your deskmate.

2.1.2 planetary motion



a planet/satellite orbiting around a star/earth

a planet/satellite can move around a star/earth in circular orbit

circular motion requires centripetal force

for these objects, gravitational force provides centripetal force

$$F_{\text{grav}} = F_c \quad \Rightarrow \quad \boxed{\frac{GMm}{r^2} = \frac{mv^2}{r}} \quad \text{or} \quad \boxed{\frac{GMm}{r^2} = m\omega^2 r}$$

Example 2.2 GPS (Global Positioning System) satellites move in a circular orbits at about 20000 km above the earth's surface. The Earth has a radius $R = 6.4 \times 10^6$ m and mass $M = 6.0 \times 10^{24}$ kg. (a) Find the speed of GPS satellites. (b) Find its orbital period.

[3]This is known as *shell theorem*: a spherically symmetric shell (i.e., a hollow ball) affects external objects gravitationally as though all of its mass were concentrated at its centre, and it exerts no net gravitational force on any object inside, regardless of the object's location within the shell. (⋆)

✍ $$\frac{GMm}{r^2}=\frac{mv^2}{r} \quad\Rightarrow\quad v=\sqrt{\frac{GM}{r}}=\sqrt{\frac{6.67\times10^{-11}\times6.0\times10^{24}}{6.4\times10^6+2.0\times10^7}}\approx3.9\times10^3\ \mathrm{m\ s^{-1}}$$
$$v=\frac{2\pi r}{T} \quad\Rightarrow\quad T=\frac{2\pi r}{v}=\frac{2\pi\times(6.4\times10^6+2.0\times10^7)}{3.9\times10^3}\approx4.3\times10^4\ \mathrm{s}\approx11.8\ \text{hours}$$ □

Example 2.3 A **geostationary satellite** moves in a circular orbit that appears motionless to ground observers. The satellite follows the Earth's rotation, so the satellite rotates from west to east above equator with an orbital period of 24 hours. Find the radius of this orbit.

✍ $$\frac{GMm}{r^2}=m\omega^2 r \quad\Rightarrow\quad \frac{GMm}{r^2}=m\left(\frac{2\pi}{T}\right)^2 r \quad\Rightarrow\quad r^3=\frac{GMT^2}{4\pi^2}$$
$$r=\left(\frac{GMT^2}{4\pi^2}\right)^{1/3}=\left(\frac{6.67\times10^{-11}\times6.0\times10^{24}\times(24\times3600)^2}{4\pi^2}\right)^{1/3}\approx4.23\times10^7\ \mathrm{m}$$ □

Example 2.4 Assuming the planets in the solar system all move around the sun in circular orbits, show that the square of orbital period is proportional to the cube of orbital radius. [4]

✍ $$\frac{GMm}{r^2}=m\omega^2 r \quad\Rightarrow\quad \frac{GMm}{r^2}=m\left(\frac{2\pi}{T}\right)^2 r \quad\Rightarrow\quad T^2=\frac{4\pi^2}{GM}\cdot r^3$$

G is gravitational constant, M is mass of the sun, so $\frac{4\pi^2}{GM}$ is a constant, so $T^2\propto r^3$ □

Question 2.2 Given that it takes about 8.0 minutes for light to travel from the sun to the earth. (a) What is the mass of the sun? (b) At what speed does the earth move around the sun?

2.1.3 apparent weight

an object's *actual weight* is the gravitational attraction exerted by the earth's gravity

an object's *apparent weight* is the upward force (e.g., normal contact force exerted by ground, tension in a spring balance, etc.) that opposes gravity and prevents the object from falling

apparent weight can be different from actual weight due to vertical acceleration or buoyancy

but if we consider rotation of the earth, this also causes apparent weight to be lessened

object resting on ground is actually rotating together with earth

resultant of gravitational force and contact force should provide centripetal force

for object on equator: $F_{c,\mathrm{eq}}=m\omega^2 R \quad\Rightarrow\quad F_{\mathrm{grav}}-N_{\mathrm{eq}}=m\omega^2 R \quad\Rightarrow\quad N_{\mathrm{eq}}=\frac{GMm}{R^2}-m\omega^2 R$

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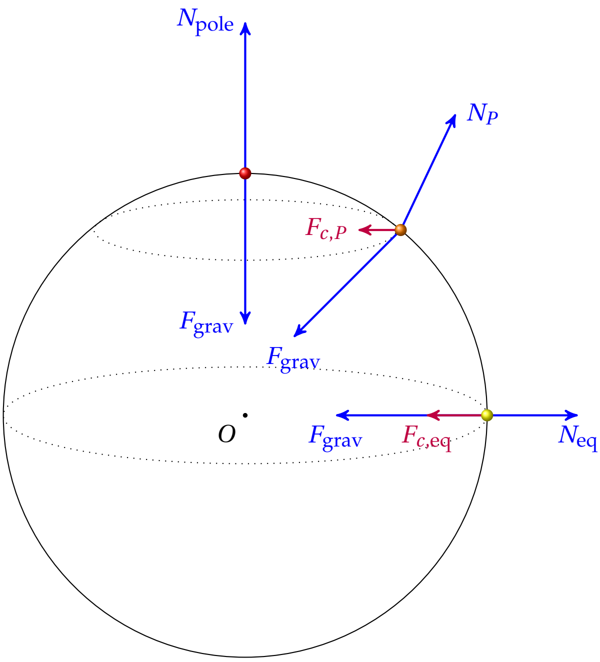
[4] This is known as *Kepler's 3rd law* for planetary motions. In the early 17th century, German astronomer Johannes Kepler discovered three scientific laws which describes how planets move around the sun. This $T^2\propto r^3$ relation not only holds for circular orbits but are also correct for elliptical orbits.

Isaac Newton proved that Kepler's laws are consequences of his own law of universal gravitation, and therefore explained why the planets move in this way. (⋆)



apparent weight at various positions near earth's surface (not to scale)

for object at poles: $F_{c,\text{pole}} = 0 \quad\Rightarrow\quad F_{\text{grav}} - N_{\text{pole}} = 0 \quad\Rightarrow\quad N_{\text{pole}} = \dfrac{GMm}{R^2}$

at lower latitudes, object describe larger circles, hence requires greater centripetal force

this offsets the balancing normal force, so apparent weight decreases near the equator

Example 2.5 A stone of mass 5.0 kg is hung from a newton-meter near the equator. The Earth can be considered to be a uniform sphere of radius $R = 6370$ km and mass $M = 5.97 \times 10^{24}$ kg. (a) What is the gravitational force on the stone? (b) What is the reading on the meter?

✍ gravitational force: $F_{\text{grav}} = \dfrac{GMm}{R^2} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 5.0}{(6.37 \times 10^6)^2} \approx 49.07$ N

centripetal force required: $F_c = m\omega^2 R = m\left(\dfrac{2\pi}{T}\right)^2 R = 5.0 \times \dfrac{4\pi^2}{(24 \times 3600)^2} \times 6.37 \times 10^6 \approx 0.17$ N

apparent weight, or reading on meter: $N = F_{\text{grav}} - F_c = 49.07 - 0.17 \approx 48.90$ N □

Question 2.3 Why astronauts in space stations are said to be *weightless*?

Question 2.4 How do you find the apparent weight of an object at an arbitrary latitude P? Does the apparent weight act vertically downwards? Give your reasons.

2.2 gravitational fields

to explain how objects exert gravitational attraction upon one another at a distance, we introduce the concept of *force fields*

> **gravitational field** is a region of space where a mass is acted by a force

any mass M (or several masses) can produce a gravitational field around it

a test mass m within this field will experience a gravitational force

to describe the effect on a small mass m in the field, we will further introduce

- *gravitational field strength*, to help us compute gravitational force on objects
- *gravitational potential*, to help us compute gravitational potential energy between objects

2.3 gravitational field strength

2.3.1 gravitational field strength

> **gravitational field strength** is defined as gravitational force per unit mass: $\boxed{g = \frac{F_{\text{grav}}}{m}}$

➢ unit of g: $[g] = \text{N kg}^{-1} = \text{ m s}^{-2}$, same unit as acceleration

➢ field strength due to an isolated source of mass M

at distance r from the source, a test mass m is acted by a force: $F_{\text{grav}} = \frac{GMm}{r^2}$

field strength at this position: $g = \frac{F_{\text{grav}}}{m} = \quad \Rightarrow \quad \boxed{g = \frac{GM}{r^2}}$

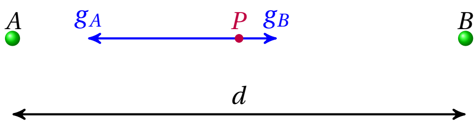
note that the field is produced by the source M, so field strength g depends on M, not m

➢ field strength g is a *vector* quantity, it has a direction

gravitation is *attractive*, so g points towards source mass

to compute combined field strength due to several sources, should perform *vector sum* of contributions from each individual

Example 2.6 Star A of mass 6.0×10^{30} kg and star B of mass 1.5×10^{30} kg are separated by a distance of 2.0×10^{12} m. (a) What is the field strength at the mid-point P of the two stars? (b) If a comet of mass 4.0×10^{6} kg is at the mid-point, what force does it experience?



✍ g_A acts towards A, g_B acts towards B, they are in opposite directions

$$g_P = g_A - g_B = \frac{GM_A}{r_A^2} - \frac{GM_B}{r_B^2} = 6.67 \times 10^{-11} \times \left[\frac{6.0 \times 10^{30}}{(1.0 \times 10^{12})^2} - \frac{1.5 \times 10^{30}}{(1.0 \times 10^{12})^2} \right] \approx 3.0 \times 10^{-4} \text{ N kg}^{-1}$$

force on comet: $F = mg = 4.0 \times 10^{6} \times 3.0 \times 10^{-4} \approx 1.2 \times 10^{3}$ N □

2.3.2 acceleration of free fall

if field strength g is known, gravitational force on an object of mass m is: $F_{\text{grav}} = mg$

if the object is acted by gravity only, then $F_{\text{net}} = F_{\text{grav}} \quad \Rightarrow \quad ma = mg \quad \Rightarrow \quad \boxed{a = g}$ [5]

this shows gravitational field strength gives the acceleration of free fall!

Example 2.7 The earth has a radius of 6370 km. (a) Find the mass of the earth. [6] (b) Find the acceleration of free fall at the top of Mount Everest. (height of Mount Everest $H \approx 8.8$ km)

✍ consider acceleration of free fall near surface of earth:

$$g_s = \frac{GM}{R^2} \quad \Rightarrow \quad 9.81 = \frac{6.67 \times 10^{-11} \times M}{(6.37 \times 10^{6})^2} \quad \Rightarrow \quad M \approx 5.97 \times 10^{24} \text{ kg}$$

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[5] Rigorously speaking, the two m's are different concepts. There is the *inertia* mass, decribing how much an object resists the change of state of motion. There is also the *gravitational* mass, describing the effect produced and experienced by the object in gravitational fields. Yet no experiment has ever demonstrated any significant difference between the two. The reason why the two masses are identical is very profound. We have shown here acceleration of free fall equals gravitational field strength, but Albert Einstein's *equivalence principle* suggests that it is actually impossible to distinguish between a uniform acceleration and a uniform gravitational field. This idea lies at the heart of the *general theory of relativity*, where I should probably stop going further.

[6] British scientist Henry Cavendish devised an experiment in 1798 to measure the gravitational force between masses in his laboratory. He was the first man to yield accurate values for the gravitational constant G. Then he was able to carry out this calculation, referred by himself as 'weighing the world'.

at top of Mount Everest:

$$g_{ME} = \frac{GM}{(R+H)^2} = \frac{6.67\times10^{-11}\times5.97\times10^{24}}{(6.37\times10^6+8.8\times10^3)^2} \approx 9.78\ \mathrm{N\ kg^{-1}} \quad\Rightarrow\quad a_{ME}\approx 9.78\ \mathrm{m\ s^{-2}} \qquad \square$$

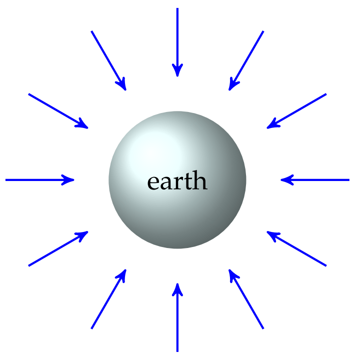
2.3.3 gravitational field lines

gravitational field lines are drawn to graphically represent the pattern of field strength

- *direction* of field lines show the *direction* of field strength in the field
- *spacing* between field lines indicates the *strength* of the gravitational field
- gravitational field lines always end up at a mass

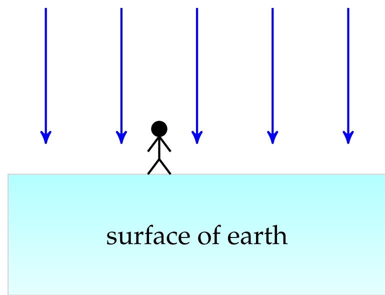
this arises from the attractive nature of gravitation

Example 2.8 field around the earth



radial field (field lines normal to surface)

Example 2.9 field near earth's surface



almost a *uniform* field

(field lines are parallel and equally spaced)

2.4 gravitational potential & potential energy

2.4.1 potential energy

potential energy is the energy possessed by an object due to its position in a force field

work done *by* force field decreases P.E., and work done *against* a force field increases P.E.

let W be work by the force field, then we have: $\boxed{W = -\Delta E_p}$

to define potential energy of an object at a specific point X, we can

(1) choose a position where potential energy is defined to be zero

(2) find work done by force field to bring the object from zero P.E. point to X

(3) consider change in P.E.: $\Delta E_p = E_{p,X} - E_{p,\text{initial}} = E_{p,X} - 0 = E_{p,X}$

but $\Delta E_p = -W$, so P.E. at point X is found: $E_{p,X} = -W$

so potential energy is equal to (negative) work done to move the object to a specific position

gravitational potential energy near earth's surface

we may choose a zero G.P.E. point, for example, $E_p(0) = 0$ at sea level

if mass m is moved up for a height h, work done by gravity is $W = -mgh$[7]

this causes a change in gravitational potential energy $\Delta E_p = -W = mgh$

then at altitude h, G.P.E. can be given by $E_p(h) = mgh$

2.4.2 gravitational potential energy

we are now ready to derive an expression for G.P.E. between two masses M and m

we define $E_p = 0$ at $r = \infty$ (choice of zero potential energy, no force so no G.P.E.), then

> **gravitation potential energy** is equal to the work done by gravitational force to bring a mass to a specific position from *infinity*

consider a mass m at infinity with zero energy and a source mass M at origin

let's find out how much work is done by gravitational force to pull m towards the origin

but F_{grav} varies as inverse square of separation x

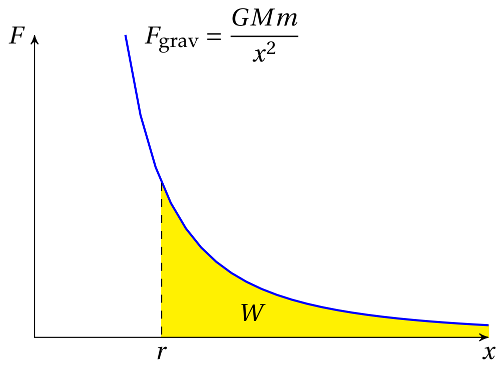
so here we need to evaluate work done by a non-constant force

we can plot a F-x graph, then magnitude of work done equals area under the graph

integrate[8] to evaluate the area: $W = \int_r^\infty \frac{GMm}{x^2}\mathrm{d}x = -\frac{GMm}{x}\Big|_r^\infty = \frac{GMm}{r}$

[7]Negative sign because this is actually work against gravity.

[8]In general, work done by a non-constant force over large distance is $W = \int_{\text{initial}}^{\text{final}} F\mathrm{d}x$.



$$\Delta E_p = -W \quad \Rightarrow \quad E_p(r) - E_p(\infty) = -\frac{GMm}{r}$$

but we have defined $E_p(\infty) = 0$, therefore: $\boxed{E_p(r) = -\frac{GMm}{r}}$

$E_p(r)$ gives G.P.E between masses M and m when they are at distance r from each other

- as $r \to \infty$, $E_p \to 0$, this agrees with our choice of zero G.P.E. point
- potential energy is a *scalar* quantity (negative sign cannot be dropped)
- G.P.E. is always *negative*, this is due to *attractive* nature of gravity

  to separate masses, work must be done to overcome attraction

  so G.P.E. increases with separation r, i.e., G.P.E. is maximum at infinity, which is zero

  G.P.E. between masses at finite separation must be less than zero
- $E_p = mgh$ is only applicable near earth's surface where field is almost *uniform*

  $E_p = -\frac{GMm}{r}$ is a more *general* formula for gravitational potential energy [9]

Example 2.10 A meteor is travelling towards a planet of mass M. When it is at a distance of r_1 from centre of M, it moves at speed v_1. When it is r_2 from M, it moves at speed v_2. Assume only gravitational force applies, establish a relationship between these quantities.

✍ energy conservation: K.E. + G.P.E. = const $\Rightarrow \frac{1}{2}mv_1^2 + \left(-\frac{GMm}{r_1}\right) = \frac{1}{2}mv_2^2 + \left(-\frac{GMm}{r_2}\right)$ □

For our case, x is the displacement away from the source, but gravitational force tends to pull the mass towards the source. F and x are in opposite directions, a negative sign is needed for F. Therefore the work done by gravity to bring mass m from infinity is: $W = \int_\infty^r F\mathrm{d}x = \int_\infty^r \left(-\frac{GMm}{x^2}\right)\mathrm{d}x = +\frac{GMm}{x}\Big|_\infty^r = \frac{GMm}{r}$.

[9]One can recover $\Delta E_p = mg\Delta h$ from $E_p = -\frac{GMm}{r}$. Near the earth's surface, if $r_1 \approx r_2 \approx R$, and $r_1 > r_2$, then we have: $\Delta E_p = E_p(r_1) - E_p(r_2) = -GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = GMm\frac{r_1 - r_2}{r_1 r_2} \approx m\frac{GM}{R^2}\Delta r \xlongequal{g=GM/R^2} mg\Delta h$.

Example 2.11 If an object is thrown from the surface of a planet at sufficiently high speed, it might escape from the influence of the planet's gravitational field. The minimum speed required is called the *escape velocity*. Using the data from previous examples, find the escape velocity from the surface of earth.

✍ assuming no energy loss to air resistance, then total energy is conserved

$$\text{K.E.} + \text{G.P.E. at surface of planet} = \text{K.E.} + \text{G.P.E. at infinity}$$

$$\frac{1}{2}mu^2 + \left(-\frac{GMm}{R}\right) = \frac{1}{2}mv^2 + 0 \quad \xRightarrow{v \geq 0} \quad u^2 \geq \frac{2GM}{R} \quad \Rightarrow \quad u_{\text{min}} = \sqrt{\frac{2GM}{R}}$$

for earth, escape velocity $u_{\text{min}} = \sqrt{\dfrac{2 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6}} \approx 1.12 \times 10^4 \text{ m s}^{-1}$ □

Question 2.5 A planet of uniform density distribution is of radius R and mass M. A rock falls from a height of $3R$ above the surface of the planet. Assume the planet has no atmosphere, show that the speed of the rock when it hits the ground is $v = \sqrt{\dfrac{3GM}{4R}}$.

Question 2.6 A space probe is travelling around a planet of mass M in a circular orbit of radius r. (a) Show that the total mechanical energy (sum of kinetic energy and gravitational energy) of the space probe is $E_{\text{total}} = -\dfrac{2GMm}{r}$. (b) If the space probe is subject to small resistive forces, state the change to its orbital radius and its orbiting speed.

Question 2.7 A *black hole* is a region of spacetime where gravitation is so strong that even light can escape from it. For a star of mass M to collapse and form a black hole, it has to be compressed below a certain radius. (a) Show that this radius is given by $R_S = \dfrac{2GM}{c^2}$, known as the *Schwarzschild radius*[10]. (b) Show that the Schwarzschild radius for the sun is about 3 km.

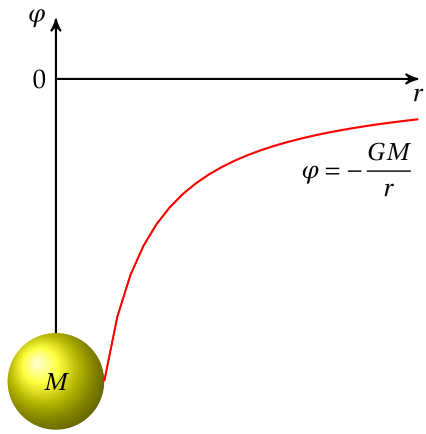
2.4.3 gravitational potential

it is useful to introduce a quantity called *potential* at a specific point in a gravitational field

gravitational potential can be considered as the potential energy per unit mass: $\varphi = \dfrac{E_p}{m}$

[10]When you deal with very strong gravitational fields, Newton's law of gravitation breaks down and effects of Einstein's *general theory of relativity* come into play. The radius of a *Newtonian* black hole being equal to the radius of a Schwarzschild black hole is a mere coincidence.

gravitational potential at a point is defined as the work done to bring *unit* mass from *infinity* to that point



- unit: $[\varphi] = \text{J kg}^{-1}$
- gravitational potential due to an isolated source M
$$\varphi = \frac{E_p}{m} = \frac{-\frac{GMm}{r}}{m} \quad \Rightarrow \quad \boxed{\varphi = -\frac{GM}{r}}$$
- potential at infinity is zero: $\varphi_\infty = 0$

  this is our choice of zero potential point
- gravitational potential is a *scalar*

  combined potential due to several masses equals scalar sum of potential of each individual
- gravitational potential is always *negative*

  again this arises from attractive nature of gravity

  work is done to pull unit mass away from source

  farther from source means higher potential

Example 2.12 A star A of mass $M_A = 1.5 \times 10^{30}$ kg and a planet B of mass $M_B = 6.0 \times 10^{26}$ kg form an isolated astronomical system. Point P is between A and B, and is at distance $r_A = 2.0 \times 10^{12}$ m from A, and distance $r_B = 8.0 \times 10^{10}$ m from B. (a) Find the gravitational potential at P. (b) A meteor is initially at very large distance from the system with negligible speed. It then travels towards point P due to the gravitational attraction. Find its speed when it reaches P.

✍ gravitational potential at P: $\varphi_P = \varphi_A + \varphi_B = \left(-\dfrac{GM_A}{r_A}\right) + \left(-\dfrac{GM_B}{r_B}\right)$
$$\varphi_P = -6.67 \times 10^{-11} \times \left(\frac{1.5 \times 10^{30}}{2.0 \times 10^{12}} + \frac{6.0 \times 10^{26}}{8.0 \times 10^{10}}\right) \approx -5.05 \times 10^7 \text{ J kg}^{-1}$$
gain in K.E. = loss in G.P.E.: $\dfrac{1}{2}mv^2 = m\Delta\varphi \quad \Rightarrow \quad v^2 = 2(\varphi_\infty - \varphi_P) = -2\varphi_P$
$$v = \sqrt{-2 \times (-5.05 \times 10^7)} \approx 1.01 \times 10^4 \text{ m s}^{-1}$$
□

Example 2.13 The Moon may be considered to be an isolated sphere of radius $R = 1.74 \times 10^3$ km. The gravitational potential at the surface of the moon is about -2.82×10^6 J kg^{-1}. (a) Find the mass of the moon. (b) A stone travels towards the moon such that its distance from the centre of the moon changes from $3R$ to $2R$. Determine the change in gravitational potential. (c) If the stone starts from rest, find its final speed.

✍ at surface: $\varphi(R) = -\frac{GM}{R} \quad \Rightarrow \quad -2.82 \times 10^6 = -\frac{6.67 \times 10^{-11} \times M}{1.74 \times 10^6} \quad \Rightarrow \quad M = 7.36 \times 10^{22}\text{ kg}$

from $3R$ to $2R$: $\Delta\varphi = \varphi_{(3R)} - \varphi_{(2R)} = \left(-\frac{GM}{3R}\right) - \left(-\frac{GM}{2R}\right) = \frac{GM}{6R} = \frac{2.82 \times 10^6}{6} \approx 4.70 \times 10^5\text{ J kg}^{-1}$

note this change is a *decrease* in gravitational potential

gain in K.E. = loss in G.P.E.: $\frac{1}{2}mv^2 = m\Delta\varphi \quad \Rightarrow \quad v = \sqrt{2\Delta\varphi} = \sqrt{2 \times 4.70 \times 10^5} \approx 970\text{ m s}^{-1}$ □

Question 2.8 Given that the moon is of radius 1700 km and mass 7.4×10^{22} kg. (a) Find the change in gravitational potential when an object is moved from moon's surface to 800 km above the surface. (b) If a rock is projected vertically upwards with an initial speed of 1800 m s^{-1} from surface, find the rock's speed when it reaches a height of 800 km. (c) Suggest whether the rock can escape from the moon's gravitational field completely.

CHAPTER 3

Oscillation

3.1 oscillatory motion

oscillation refers to a repetitive back and forth motion about it *equilibrium position*

the equilibrium point is a point where all forces on oscillator are balanced

release an object from its equilibrium position from rest, it will stay at rest

examples of oscillation includes pendulum of a clock, vibrating string, swing, etc.

3.1.1 amplitude, period, frequency

to describe motion of an oscillator, we define the following quantities:

- **displacement** (x): distance from the equilibrium position
- **amplitude** (x_0): maximum displacement from the equilibrium position
- **period** (T): time for one complete oscillation
- **frequency** (f): number of oscillations per unit time

frequency is related to period as: $\boxed{f = \frac{1}{T}}$

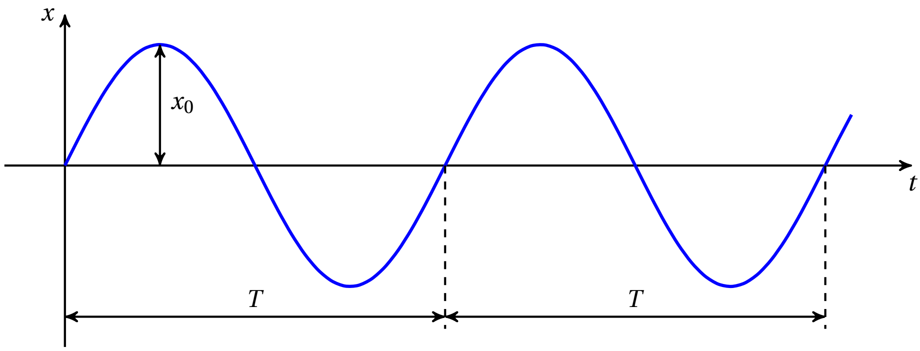
displacement x varies with time t repetitively, so we can plot an x-t graph

amplitude x_0 and period T are labelled on the graph

3.1.2 phase angle

the point that an oscillator has reached within a complete cycle is called **phase angle** (ϕ)

- unit of phase angle: $[\phi] = \text{rad}$



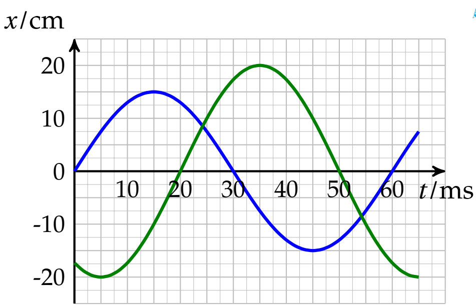
displacement-time graph for a typical oscillator

it looks like an angle, but better think of it as a number telling fraction of a complete cycle

➢ we use **phase difference** $\Delta\phi$ to compare how much one oscillator is ahead of another

$\Delta\phi$ is found in terms of fraction of an oscillation: $\Delta\phi = \frac{\Delta t}{T} \times 2\pi$ (also measured in radians)

Example 3.1 Compare the two oscillations from the x-t graph below.



✍ both have period $T = 60$ ms

frequency $f = \frac{1}{60 \times 10^{-3}} \approx 16.7$ Hz

they are of different amplitudes

one has $x_0 = 15$ cm, the other has $x_0 = 20$ cm

time difference: $\Delta t = 20$ ms

phase difference: $\Delta\phi = \frac{\Delta t}{T} \times 2\pi = \frac{20}{60} \times 2\pi = \frac{2\pi}{3}$ rad

3.1.3 acceleration & restoring force

for any oscillatory motion, consider its velocity and acceleration at various positions

its acceleration must be always pointing towards the equilibrium position

resultant force always acts in the direction to restore the system back to its equilibrium point, this net force is known as the **restoring force**

if at equilibrium position, then no acceleration or restoring force

3.2 simple harmonic oscillation

if an oscillator has an acceleration always proportional to its displacement from the equilibrium position, and acceleration is in opposite direction to displacement, then the oscillator is performing **simple harmonic motion**

many phenomena can be approximated by simple harmonics

examples are motion of a pendulum, molecular vibrations, etc.

complicated motions can be decomposed into a set of simple harmonics

simple harmonic motion provides a basis for the study of many complicated motions [11]

3.2.1 equation of motion

defining equation for simple harmonics can be written as $\boxed{a = -\omega^2 x}$

ω is some constant, so a is proportional to x

the minus sign shows a and x are in opposite directions

general solution to this this equation of motion[12] takes the form: $\boxed{x = x_0 \sin(\omega t + \phi)}$

x_0 represents the amplitude, ω is called the angular frequency, ϕ is the phase angle

angular frequency

- **angular frequency** satisfies the relation: $\boxed{\omega = \frac{2\pi}{T} = 2\pi f}$
- unit of angular frequency: $[\omega] = \text{rad} \cdot \text{s}^{-1}$

[11]This can be done through a mathematical technique known as *Fourier analysis*. For example, a uniform circular motion can be considered as the combination of two simple harmonic motion in x- and y-directions.

[12]You probably know that acceleration can be written as the second derivative of displacement: $a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, so $a = -\omega^2 x$ is equivalent to $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0$, which a *second-order differential equation*. If you do not know how to solve it, you may have the chance to study this in an advanced calculus course.

➢ angular frequency ω is determined by the system's *physical constants* only

if an object is set to oscillate *freely* with no external force, its period will always be the same

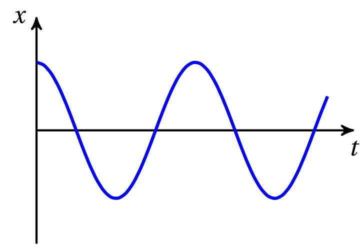
frequency of an free oscillatory system is called the **natural frequency**

phase angle

➢ phase angle ϕ is dependent on *initial conditions* (e.g. initial position and initial speed at $t = 0$?)

➢ in many cases, phase angle term can be avoided if a suitable trigonometric function is chosen

Example 3.2 A simple harmonic oscillator is displaced by 6.0 cm from its rest position and let go at $t = 0$. Given that the period of this system is 0.80 s, state an equation for its displacement-time relation.



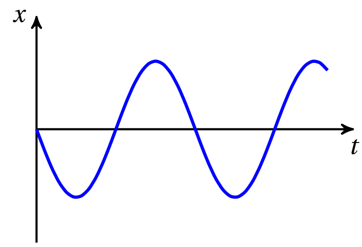
✍ angular frequency: $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80} = \frac{5\pi}{2}\pi \text{ rad s}^{-1}$

initial displacement $x(0) = +x_0 = 6.0$ cm

for displacement-time relation, we use cosine function

$$x(t) = x_0 \cos\omega t \quad \Rightarrow \quad x = 6.0\cos\left(\frac{5\pi}{2}t\right) \qquad \square$$

Example 3.3 A simple harmonic oscillator is initially at rest. At $t = 0$, it is given an initial speed in the negative direction. Given that the frequency is 1.5 Hz and the amplitude is 5.0 cm, state an equation for its displacement-time relation.



✍ angular frequency: $\omega = 2\pi f = 2\pi \times 1.5 = 3\pi \text{ rad s}^{-1}$

initial displacement $x(0) = 0$

for displacement-time relation, we use sine function

$$x(t) = -x_0 \cos\omega t \quad \Rightarrow \quad x = -5.0\sin(3\pi t) \qquad \square$$

3.2.2 examples of simple harmonics

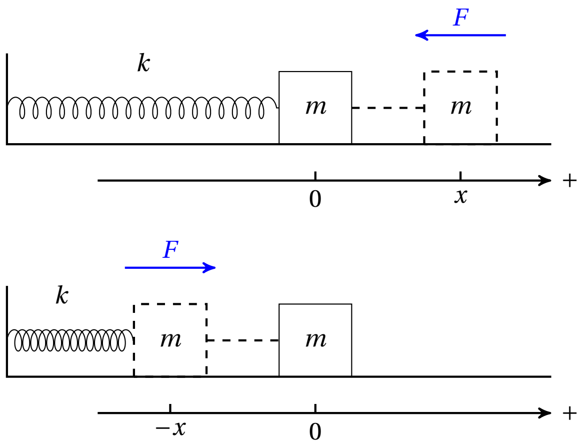
mass-spring oscillator

a mass-spring oscillator system consists of a block of mass m and an ideal spring

when a spring is stretched or compressed by a mass, the spring develops a restoring force

magnitude of this force obeys *Hooke's law*: $F = kx$

direction of this force is in opposite direction to displacement x



restoring force acting on the ideal mass-spring oscillator

take vector nature of force into account, we find

$$F_{\text{net}} = ma \quad \Rightarrow \quad -kx = ma \quad \Rightarrow \quad a = -\frac{k}{m}x$$

spring constant k and mass m are constants, so $a \propto x$

negative sign shows a and x are in opposite directions

so mass-spring oscillator executes simple harmonic motion

compare with $a = -\omega^2 x \quad \Rightarrow \quad \omega^2 = \frac{k}{m} \quad \Rightarrow \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$

period of mass-spring oscillator: $T = \frac{2\pi}{\omega} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$

➢ period and frequency are solely determined by mass of oscillator m and spring constant k

identical mass-spring systems will oscillate at same frequency no matter what amplitude

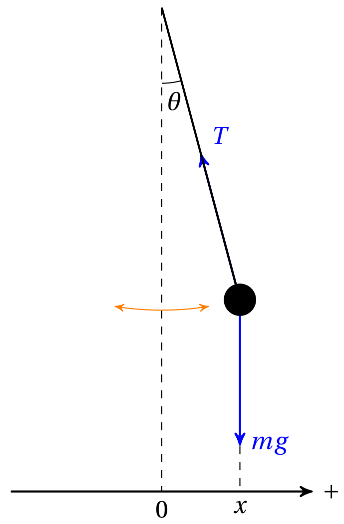
➢ $m \nearrow \Rightarrow T \nearrow$, greater mass means greater inertia, oscillation becomes slower

➢ $k \nearrow \Rightarrow T \searrow$, greater k means stiffer spring, greater restoring force makes oscillation go faster

simple pendulum

a simple pendulum is set up by hanging a bob on a light cord from a fixed point

displace the bob by some angle and release from rest, it can swing freely



one can show this performs simple harmonic motion for *small-angle* oscillation

if angular displacement θ is small, then the pendulum has almost no vertical displacement, the motion can be considered to be purely horizontal

vertically: $T\cos\theta \approx mg \xRightarrow{\cos\theta\approx 1 \text{ as } \theta\to 0} T \approx mg$

horizontally: $-T\sin\theta = ma \xRightarrow{\sin\theta = x/L} a \approx -\dfrac{g}{L}x$

this shows simple pendulum undergoes simple harmonics

compare with defining equation for simple harmonics:

$$a = -\omega^2 x \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{L}}$$

period for a simple pendulum: $\boxed{T = 2\pi\sqrt{\dfrac{L}{g}}}$

- period and frequency of a pendulum are determined by length of the string L only

  as long as angular displacement remains small, frequency does not depend on amplitude

  fix length L, then simple pendulum oscillates at same frequency no matter what amplitude
- $L \nearrow \Rightarrow T \nearrow$, longer pendulums oscillate more slowly
- $g \searrow \Rightarrow T \nearrow$, if there is no gravity ($g = 0$), then the bob will not move at all ($T \to \infty$)

Question 3.1 A cylindrical tube of total mass m and cross sectional area A floats upright in a liquid of density ρ. When the tube is given a small vertical displacement and released, the magnitude of the resultant force acting on the tube is related to its vertical displacement y by the expression: $F_{\text{net}} = \rho g A y$. (a) Show that the tube executes simple harmonic motion. (b) Find an expression for the frequency of the oscillation.

Question 3.2 A small glider moves along a horizontal air track and bounces off the buffers at the ends of the track. Assume the track is frictionless and the buffers are perfectly elastic, state and explain whether the glider describes simple harmonic motion.

3.2.3 velocity & acceleration

displacement of simple harmonic oscillator varies with time as: $x = x_0 \sin(\omega t + \phi)$

from this displacement-time relation, we can find velocity and acceleration relations

velocity

to find velocity-time relation, let's recall that velocity v is rate of change of displacement x

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} x_0 \sin(\omega t + \phi) \quad \Rightarrow \quad \boxed{v(t) = \omega x_0 \cos(\omega t + \phi)}$$

by taking $v^2 + \omega^2 x^2$, the sine and cosine terms can be eliminated, we find:

$$v^2 + \omega^2 x^2 = \omega^2 x_0^2 \cos^2(\cdots) + \omega^2 x_0^2 \sin^2(\cdots) = \omega^2 x_0^2$$

this gives velocity-displacement relation: $\boxed{v(x) = \pm\omega\sqrt{x_0^2 - x^2}}$

- at equilibrium position $x = 0$, speed is maximum: $\boxed{v_{\max} = \omega x_0}$
- when $x = \pm x_0$, oscillator is momentarily at rest: $v = 0$

acceleration

acceleration-time relation is found by further taking rate of change of velocity v

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \omega x_0 \cos(\omega t + \phi) \quad \Rightarrow \quad \boxed{a(t) = -\omega^2 x_0 \sin(\omega t + \phi)}$$

this is actually unnecessary, if we compare this with $x(t) = x_0 \sin(\omega t + \phi)$, we have: $a = -\omega^2 x$

we have recovered the definition for simple harmonics

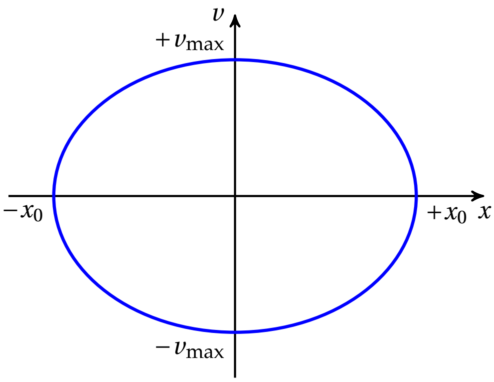
(if $a \propto x$ and in opposite directions to x, then simple harmonic motion)

so acceleration-displacement relation is given by the defining equation explicitly $\boxed{a(x) = -\omega^2 x}$

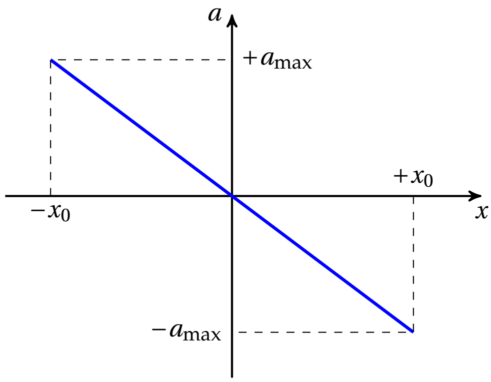
- at equilibrium position $x = 0$, zero acceleration
- when $x = \pm x_0$, acceleration is greatest: $\boxed{a_{\max} = \omega^2 x_0}$

let's take $x = x_0 \sin\omega t$ as example, changes of x, v, a over time are listed below

time t	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
displacement: $x = x_0 \sin\omega t$	0	+max	0	−max	0
velocity: $v = \omega x_0 \cos\omega t$	+max	0	−max	0	+max
acceleration: $a = -\omega^2 x = -\omega^2 x_0 \sin\omega t$	0	−max	0	+max	0



velocity-displacement graph



acceleration-displacement graph

Example 3.4 The motion of a simple pendulum is approximately simple harmonic. As the pendulum swings from one side to the other end, it moves through a distance of 6.0 cm and the time taken is 1.0 s. (a) State the period and amplitude. (b) Find the greatest speed during the oscillation. (c) Find its speed when displacement $x = 1.2$ cm.

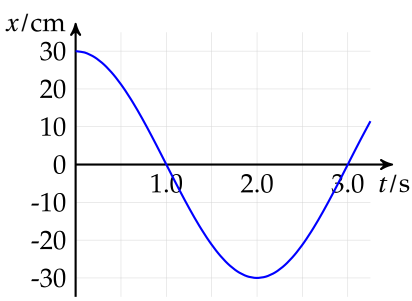
✍ period: $T = 2 \times 1.0 = 2.0$ s, and amplitude: $x_0 = \frac{1}{2} \times 6.0 = 3.0$ cm

angular frequency: $\omega = \frac{2\pi}{T} = \frac{2\pi}{2.0} = \pi$ rad s^{-1}

greatest speed: $v_{max} = \omega x_0 = \pi \times 3.0 \approx 9.4$ cm s^{-1}

speed at 1.2 cm: $v = \omega\sqrt{x_0^2 - x^2} = \pi \times \sqrt{3.0^2 - 1.2^2} \approx 8.6$ cm s^{-1} □

Example 3.5 Given the x-t graph of a simple harmonic oscillator. (a) Find its speed at $t = 0$. (b) Find its greatest speed. (b) Find its acceleration at $t = 1.0$ s.



✍ at $t = 0$, $x = +x_0 \quad \Rightarrow \quad v = 0$ (zero gradient)

from graph: amplitude $x_0 = 30$ cm, period $T = 4.0$ s

angular frequency: $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$ rad s^{-1}

greatest speed: $v_{max} = \omega A = \frac{\pi}{2} \times 30 \approx 47$ cm s^{-1}

at $t = 1.0$ s, $x = 0 \quad \Rightarrow \quad a = 0$

(equilibrium position so no acceleration) □

Question 3.3 Assume the motion of a car engine piston is simple harmonic. The piston completes 3000 oscillations per minute. The amplitude of the oscillation is 4.0 cm. (a) Find the greatest speed. (b) Find the greatest acceleration.

3.2.4 vibrational energy

consider the *ideal* mass-spring oscillator, its vibrational energy consists of two parts:

- kinetic energy of the mass: $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t \xlongequal{v=\pm\omega\sqrt{x_0^2-x^2}} \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- (elastic) potential energy in the spring: $E_p = \frac{1}{2}kx^2 \xlongequal{\omega=\sqrt{\frac{k}{m}}} \frac{1}{2}m\omega^2 x^2$

total energy of the oscillator: $E = E_k + E_p \quad \Rightarrow \quad \boxed{E = \frac{1}{2}m\omega^2 x_0^2}$

➢ although this formula is derived from the mass-spring model

$E = \frac{1}{2}m\omega^2 x_0^2$ can be used to compute vibrational energy of all simple harmonic oscillators

➢ for an ideal system, total energy remains constant

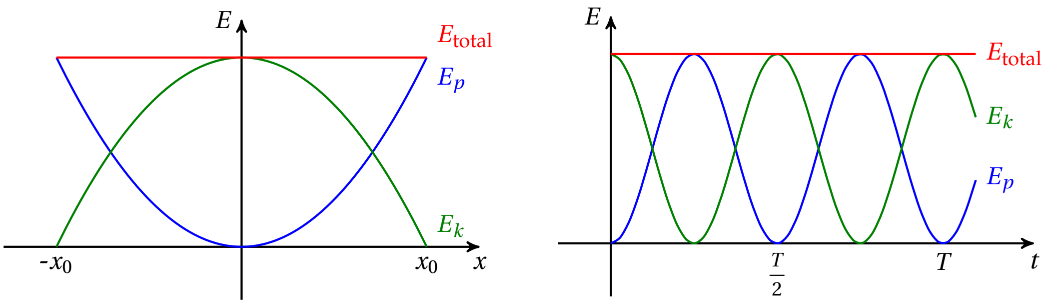
E_k and E_p keep changing, one transfers into another, but total energy is *conserved*

➢ when $x = 0$, $E_k = \max$, $E_p = 0$, vibrational energy is purely kinetic

$$E = E_{k,\max} = \frac{1}{2}mv_{\max}^2 \xlongequal{v_{\max}=\omega x_0} \frac{1}{2}m\omega^2 x_0^2$$

➢ when $x = \pm x_0$, $E_k = 0$, $E_p = \max$, vibrational energy is purely potential

$$E = E_{p,\max} = \frac{1}{2}kx_0^2 \xlongequal{\omega=\sqrt{\frac{k}{m}}} \frac{1}{2}m\omega^2 x_0^2$$



vibrational energy of a mass-spring oscillator

Example 3.6 A block of mass 150 g at the end of a spring oscillates with a period of 0.80 s. The maximum displacement from its rest position is 12 cm. Find the energy of the vibration.

✍ $$E = \frac{1}{2}m\omega^2 x_0^2 = \frac{1}{2}m\left(\frac{2\pi}{T}\right)^2 x_0^2 = \frac{1}{2} \times 0.15 \times \frac{4\pi^2}{0.80^2} \times 0.12^2 \approx 6.7 \times 10^{-2}\ \text{J}$$ □

Question 3.4 An oscillator is given an energy of 20 mJ and starts to oscillate, it reaches an amplitude of 8.0 cm. If we want to double the amplitude, find the vibrational energy required.

3.3 damped oscillations

total vibrational energy stays constant for an ideal system

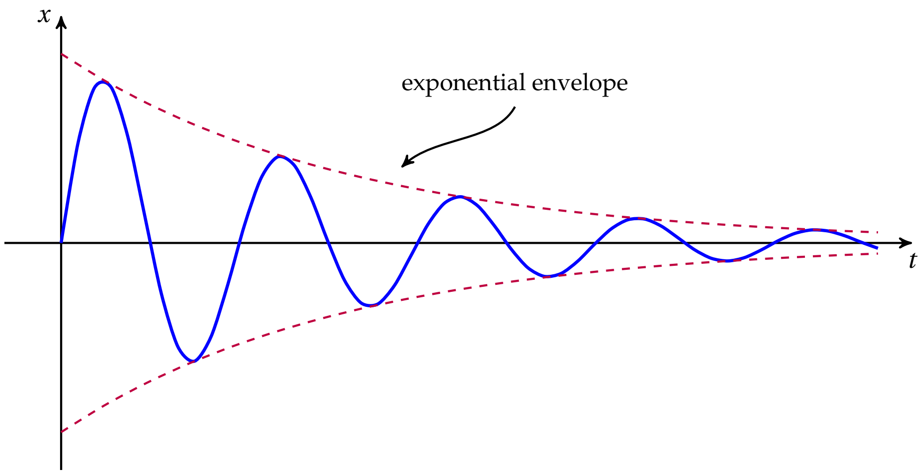
but in reality, there are friction, resistance and viscous forces that oppose motion

amplitude of an oscillator decreases due to energy loss to friction, this is called **damping**

3.3.1 light damping

for a **lightly-damped oscillator**, amplitude decreases *gradually*

oscillator will not stop moving back and forth after quite a few oscillations



- ➢ decrease in amplitude is *non-linear* in time (exponential decay in many cases)
- ➢ frequency and period are (almost) unchanged

Example 3.7 An oscillator is composed of a block of mass $m = 250$ g and a spring of $k = 1.6$ N/cm. It is displaced by 5.0 cm from its rest position and set free. (a) What is its angular frequency? (b) what is the initial vibrational energy? (c) After a few oscillations, 40% of its energy is lost due to damping. What is its new amplitude?

✍ angular frequency: $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{160}{0.25}} \approx 25.3 \text{ rad s}^{-1}$

energy of oscillator: $E = \dfrac{1}{2} m\omega^2 x_0^2 = \dfrac{1}{2} \times 0.25 \times 25.3^2 \times 0.050^2 = 0.20$ J[13]

[13]An easier approach: $E = \dfrac{1}{2} k x_0^2 = \dfrac{1}{2} \times 160 \times 0.050^2 = 0.20$ J.

since $E \propto x_0^2$, so: $\dfrac{E'}{E} = \dfrac{x_0'^2}{x_0^2} \quad \Rightarrow \quad 60\% = \dfrac{x_0'^2}{x_0^2} \quad \Rightarrow \quad x_0' = \sqrt{0.6}x_0 = \sqrt{0.6} \times 5.0 \approx 3.9 \text{ cm}$ □

Question 3.5 A small toy boat of mass 360 g floats on surface of water. It is gently pushed down and then released. During the fist four complete cycles of its oscillation, its amplitude decreased from 5.0 cm to 2.0 cm in a time of 6.0 s. Find the energy loss.

3.3.2 heavy damping

if resistive forces are too strong, there will be no oscillatory motion

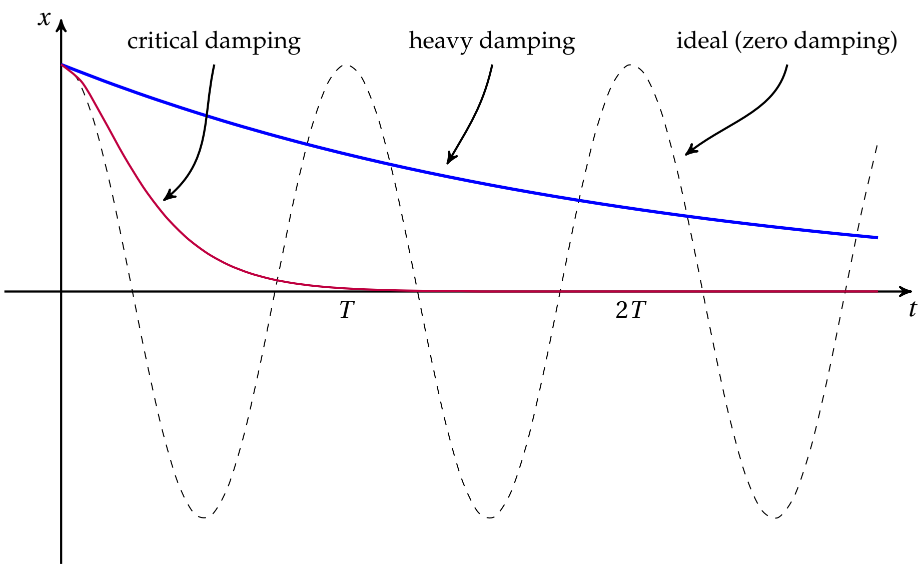
the system will return to the equilibrium position very slowly

this system is said to be **heavily damped**

3.3.3 critical damping

critical damping is the border between light damping and heavy damping

it occurs when system returns to equilibrium in *shortest* time without any oscillation



➢ critical damping is desirable in many engineering designs [14]

[14]When a damped oscillator is required, critically-damped system provides the quickest approach to equilibrium without overshooting, while lightly-damped system reaches the zero position quickly but continues to oscillate, and heavily-damped system reaches zero position in very long time.

examples include door-closing mechanism, shock absorbers in vehicles and artillery, etc.

3.4 forced oscillations

3.4.1 free & forced oscillation

an oscillator moving on its own with no gain or loss of energy is called **free oscillation**

amplitude of the oscillation is constant, its frequency called **natural frequency**

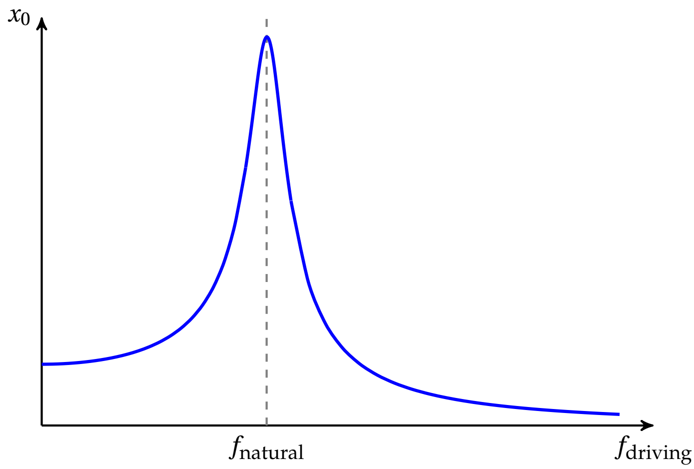
an oscillator may also move under an external driving force, it is **forced oscillation**

frequency of forced oscillator tends to driving frequency after sufficiently long time

3.4.2 resonance

for a forced oscillation system, when frequency of driving force f_{driving} is close to natural frequency f_{natural}, amplitude of oscillator increases rapidly

> when driving frequency of external force equals natural frequency of the system, amplitude of the system becomes maximum, this phenomenon is called **resonance**



resonance is achieved when $f_{\text{driving}} = f_{\text{natural}}$

(amplitude tends to infinity if no damping)

➢ practical application of resonance

- microwave oven – water molecules resonate at microwave frequency and vibrate greatly
- MRI (magnetic resonance imaging) — precession of nuclei resonate at radio frequency, signals are processed to image nuclei of atoms inside a human body in detail
- radio/TV — *RLC* tuning circuits resonate at frequency of signals being received

➢ possible problems caused by resonance

- buildings during earthquake – resonate at frequency of shockwaves and collapse
- car suspension system – going over bumps may give large amplitude vibrations
- bridges and skyscrapers – resonance due to wind conditions

3.4.3 damping & resonance

an oscillation system can be subject to both driving force and resistive force

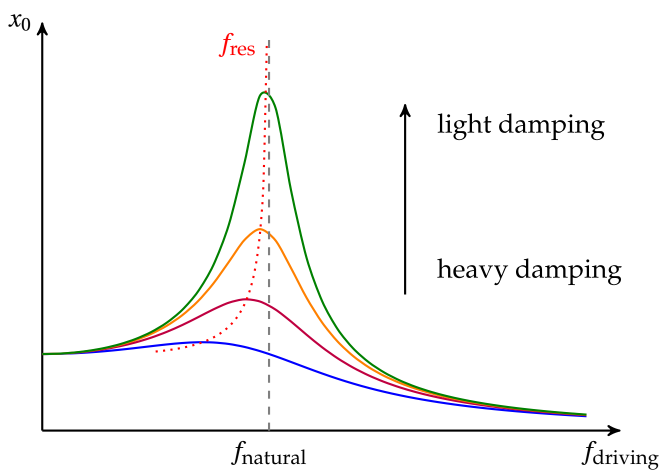
resonance behaviour will be changed by damping effects

➢ damping decreases amplitude of oscillation at all frequencies

greater damping causes resonance peak to become *flatter*

engineering systems are often deliberately damped to minimise resonance effect

➢ damping also shifts resonance frequency (slightly reduced for light damping)



resonance effect for various damping conditions

Chapter 4

Ideal Gases

4.1 gas molecules

4.1.1 motion of gas particles

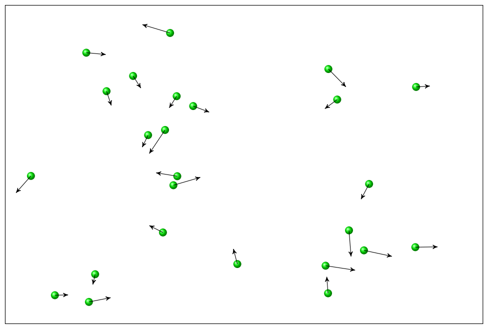
gas consists of a large number of molecules

gas molecules move *randomly* at high speeds

➢ randomness results from *collisions* of fast-moving molecules in the gas

for an individual molecule, its velocity changes constantly as it collides with other molecules

for the gas at any instant, there is a range of velocities for molecules



motion of gas molecules in a container

➢ experimental evidence of random motion: **Brownian motion**

dust or smoke particles in air undergo jerky random motion (viewed through microscope)

this is due to collisions with gas molecules that move randomly

➢ speed of gas molecules depend on temperature

molecules move faster at higher temperature[15]

4.1.2 amount of molecules

there are a huge number of molecules in a gas

we introduce **amount of substance** to measure the size of a collection of particles

➢ unit of amount of substance: $[n] = \text{mol}$

[15]We will prove this statement later in this chapter.

one **mole** is defined as the amount carbon-12 atoms in a sample of 12 grams

➢ 1 mole of substance contains 6.02×10^{23} particles

this number is called **Avogadro constant**: $N_A = 6.02 \times 10^{23}\ \text{mol}^{-1}$ [16]

conversion between number of molecules and amount of substance: $\boxed{N = nN_A}$

➢ it is useful to introduce the notion of molar mass M

molar mass of a substance is defined as the mass of a given sample divided by the amount of substance: $M = \dfrac{m}{n}$

- amount of substance $= \dfrac{\text{mass of sample}}{\text{molar mass}}$, or $n = \dfrac{m}{M}$
- mass of single molecule $= \dfrac{\text{molar mass}}{\text{Avogadro constant}}$, or $m_0 = \dfrac{M}{N_A}$

Example 4.1 Find the number of molecules in 160 grams of argon-40 gas.

✍ amount of gas: $n = \dfrac{m}{M} = \dfrac{160\ \text{g}}{40\ \text{g mol}^{-1}} = 4.0\ \text{mol}$

number of gas molecules: $N = nN_A = 4.0\ \text{mol} \times 6.02 \times 10^{23}\ \text{mol}^{-1} \approx 2.41 \times 10^{24}$ □

Question 4.1 Find the mass of a sample of uranium-235 that contains 6.0×10^{20} atoms.

4.1.3 pressure (qualitative view)

when gas molecules collide with walls of container and rebound, they are acted by a force

by Newton's third law, gas molecules must exert a reaction force on container in return

contributions from many molecules give rise to a pressure

Example 4.2 If a gas is heated with its volume fixed, how does the pressure change?

✍ at higher temperature, gas molecules move faster

they will collide *harder* and produce a greater force upon each collision

they will also collide more *frequently* with the container

[16] In 2018, IUPAC suggested a new definition of the mole, which is defined to contain exactly 6.02×10^{23} particles. This new definition fixed numerical value of the Avogadro constant, and emphasized that the quantity 'amount of substance' is concerned with counting number of particles rather than measuring the mass of a sample.

so pressure of the gas will increase □

Question 4.2 If you pump gas into a bicycle tyre, state and explain how the pressure changes.

Question 4.3 A fixed amount of gas is allowed to expand at constant temperature, state and explain how the pressure changes.

4.2 ideal gas

4.2.1 ideal gas equation

a gas that satisfies the equation $\boxed{pV = nRT}$ or $\boxed{pV = NkT}$ at any pressure p, any volume V, and thermodynamic temperature T is called an **ideal gas**

molar gas constant: $R = 8.31\ \mathrm{J\ mol^{-1}\ K^{-1}}$

Boltzmann constant: $k = 1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}}$

values of R and k apply for any ideal gas, i.e., they are *universal* constants

➢ recall conversion between number of molecules and amount of substance: $\boxed{N = nN_A}$

we have relation between the constants: $R = kN_A$, or $k = \dfrac{R}{N_A}$

➢ one must use *thermodynamic temperature* in the equation

thermodynamic temperature is measured in kelvins (K), so it is also called the *Kelvin scale*[17]

conversion between Kelvin temperature and Celsius temperature: $\boxed{T_K(\mathrm{K}) \underset{+273}{\overset{-273}{\rightleftharpoons}} T_C(^\circ\mathrm{C})}$

real gases

real gas behaves ideally at sufficiently high temperature and low pressure

- at very low temperatures, real gas will condense into liquid or solid
- at very high pressures, intermolecular forces become important

however, under normal conditions (room temperature $T \approx 300$ K and standard atmospheric pressure $p \approx 1.0 \times 10^5$ Pa), there is no significant difference between a real gas and an ideal gas

[17]We will discuss in details about Kelvin scale in §5.1.1 and §5.3.2.

so ideal gas approximation can be used with good accuracy for most of our applications

Example 4.3 A sealed cylinder of volume of 0.050 m^3 contains 75 g of air. The molar mass of air is 29 g mol^{-1}. (a) Find the air pressure when its temperature is 30°C. (b) The gas is allowed to expand with its pressure fixed. Find the temperature of the gas when the volume doubles.

✍ amount of gas: $n = \frac{m}{M} = \frac{75}{29} \approx 2.59$ mol

pressure at 30°C: $p = \frac{nRT_1}{V_1} = \frac{2.59 \times 8.31 \times (30+273)}{0.050} \approx 1.30 \times 10^5$ Pa

pressure fixed, so $V \propto T \quad \Rightarrow \quad \frac{T_2}{T_1} = \frac{V_2}{V_1} = 2 \quad \Rightarrow \quad T_2 = 2 \times (30+273) = 606\text{ K} = 333°\text{C}$ □

Example 4.4 A gas cylinder holding 5000 cm^3 of air at a temperature of 27 °C and a pressure of 6.0×10^5 Pa is used to fill balloons. Each balloon contains 1000 cm^3 of air at 27 °C and 1.0×10^5 Pa when filled. (a) Find the initial amount of gas in the cylinder. (b) Find the number of balloons that can be filled.

✍ initial amount of gas in cylinder: $n_0 = \frac{p_0 V}{RT} = \frac{9.0 \times 10^5 \times 5000 \times 10^{-6}}{8.31 \times (27+273)} \approx 1.203$ mol

final amount of gas in cylinder: $n_{\text{remain}} = \frac{pV}{RT} = \frac{1.0 \times 10^5 \times 5000 \times 10^{-6}}{8.31 \times (27+273)} \approx 0.201$ mol[18]

amount of gas in each balloon: $n_b = \frac{pV_b}{RT} = \frac{1.0 \times 10^5 \times 1000 \times 10^{-6}}{8.31 \times (27+273)} \approx 0.040$ mol

number of balloons: $N = \frac{n_0 - n_{\text{remain}}}{n_b} = \frac{1.203 - 0.201}{0.040} \approx 25$ □

Example 4.5 A storage cylinder has a volume of 5.0×10^{-4} m^3. The gas is at a temperature of 300 K and a pressure of 4.0×10^6 Pa. (a) Find the number of molecules in the cylinder. (b) The gas molecules slowly leak from the cylinder at a rate of 1.6×10^{16} s^{-1}. Find the time, in days, after which the pressure will reduce by 5.0%.

✍ initial number of molecules: $N_0 = \frac{p_0 V}{kT} = \frac{4.0 \times 10^6 \times 5.0 \times 10^{-4}}{1.38 \times 10^{-23} \times 300} \approx 4.83 \times 10^{23}$

volume fixed, so $N \propto p \quad \Rightarrow \quad \frac{\Delta N}{N_0} = \frac{\Delta p}{p_0} = 5.0\%$

number of molecules escaped: $\Delta N = 0.05 \times 4.83 \times 10^{23} \approx 2.42 \times 10^{22}$

time needed: $t = \frac{2.42 \times 10^{22}}{1.6 \times 10^{16}} \approx 1.51 \times 10^6\text{ s} \approx 17.4$ days □

[18] Air will leave the cylinder to fill balloons only if pressure inside the cylinder is higher than pressure of the balloon. When the two pressures become equal, no more balloons can be filled, there will be some air remain in cylinder.

Question 4.4 Containers A has a volume of 2.5×10^{-2} m^3 contains a gas at a temperature of 17°C and pressure of 1.3×10^5 Pa and . Another container B of same size holds a gas at same temperature and a pressure of 1.9×10^5 Pa. The two containers are initially isolated from each another. (a) Find the total amount of molecules. (b) The two containers are now connected through a tube of negligible volume. Assume the temperature stays unchanged, find the final pressure of the gas.

Question 4.5 The air in a car tyre can be assumed to have a constant volume of 3.0×10^{-2} m^3 . The pressure of this air is 2.8×10^5 Pa at a temperature of 25°C. The pressure is to be increased using a pump. On each stroke 0.015 mol of air is forced into the tyre. If gas has a final pressure of 3.6×10^5 Pa and final temperature of 28°C. Find the number of strokes of the pump required.

4.2.2 empirical laws

historically, the ideal gas law was first stated by *Émile Clapeyron* in 1834:

for a fixed amount of gas, $\boxed{\dfrac{PV}{T} = \text{const}}$

his work was based on the empirical Boyle's law, Charles's law, and Gay-Lussac's law

we will next recover these laws from the ideal gas equation

Boyle's law

Boyle's law was discovered by *Robert Boyle* in 1662, based on experimental observations

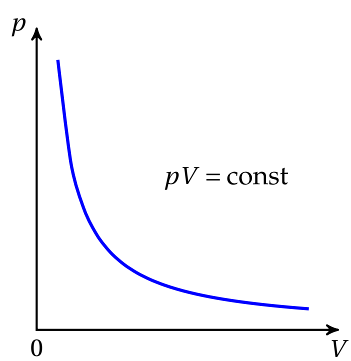
if temperature T remains constant, then

$$\boxed{pV = \text{const}}, \text{ or } \boxed{p \propto \frac{1}{V}}$$

i.e., pressure p of gas is inversely proportional to volume V

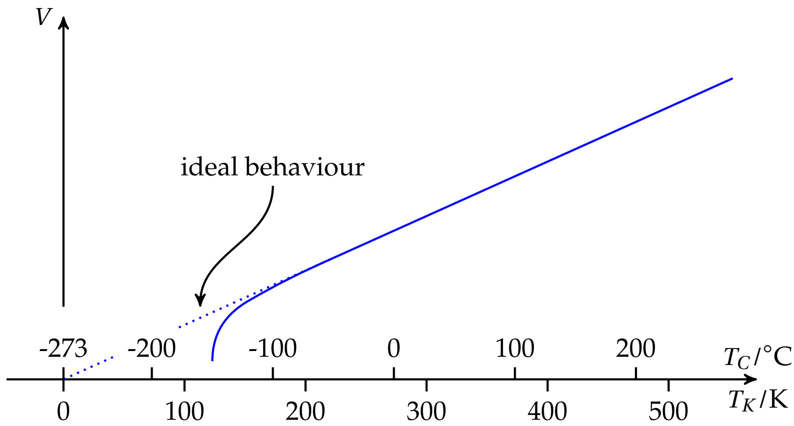
- for a gas with fixed temperature: $p_1V_1 = p_2V_2$
- a thermodynamic process for which temperature is kept constant is called an *isothermal* process

p-V relation for an isothermal process is shown



Charles's law

Charles's law was discovered by *Jacques Charles* in 1787, based on experimental observations



if pressure p remains constant, then: $\boxed{\frac{V}{T} = \text{const}}$, or $\boxed{V \propto T}$

i.e., volume V of gas is directly proportional to its temperature T

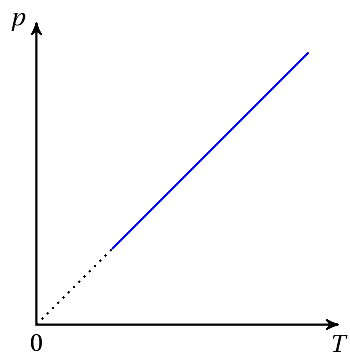
- proportionality relation only applies if Kelvin scale is used
- a thermodynamic process for which pressure is kept constant is called an *isobaric* process

  V-T relation for an isobaric process is shown
- Charles's law implies that volume of gas tends to zero at a certain temperature

  historically this is how the idea of *absolute zero* first arose
- as $T \to 0$, a real gas condenses into solid

  there will be deviation from ideal behaviour (dotted line)

Gay-Lussac's law



Gay-Lussac's law was discovered by *Joseph Louis Gay-Lussac* between 1800 and 1802

if volume V remains constant, then

$$\boxed{\frac{p}{T} = \text{const}}, \text{ or } \boxed{p \propto T}$$

i.e., pressure p is directly proportional to temperature T

➢ a thermodynamic process for which volume is kept constant

is called an *isochoric* process, or *isometric* process

p-T relation for an isochoric process is shown

➢ behaviour of real gas again deviates from ideal behaviour (dotted line) as $T \to 0$

4.3 kinetic theory of ideal gases

kinetic model of gases: a theory based on microscopic motion of molecules of a gas that explains its macroscopic properties

4.3.1 assumptions of ideal gas model

kinetic theory of the ideal gas model is based on the following assumptions:

- gas molecules are in constant *random* motion
- *intermolecular separation* is much greater than size of molecules

  volume of molecules is negligible compared to volume occupied by gas
- *intermolecular forces* are negligible
- collisions between molecules are perfectly *elastic*, i.e., no kinetic energy lost
- molecules travel in straight line between collisions

Example 4.6 A mass of 20 g helium-4 at a temperature of 37°C has a pressure of 1.2×10^5 Pa. Each helium-4 atom has a diameter of 280 pm. (a) Find the volume occupied by the gas. (b) Find the volume of atoms in this gas. (c) Compare the two volumes, suggest whether this gas can be considered as an ideal gas.

✍ number of helium molecules: $N = nN_A = \frac{m}{M} \times N_A = \frac{20}{4.0} \times 6.02 \times 10^{23} \approx 3.01 \times 10^{24}$

volume of gas: $V_{\text{gas}} = \frac{NkT}{p} = \frac{3.01 \times 10^{24} \times 1.38 \times 10^{-23} \times (37+273)}{1.2 \times 10^5} \approx 0.107 \text{ m}^3$

volume of one atom: $V_{\text{atom}} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (140 \times 10^{-12})^3 \approx 1.15 \times 10^{-29} \text{ m}^3$

volume of all atoms: $V_{\text{atoms}} = NV_{\text{atom}} = 3.01 \times 10^{24} \times 1.15 \times 10^{-29} \text{ m}^3 \approx 3.46 \times 10^{-5} \text{ m}^3$

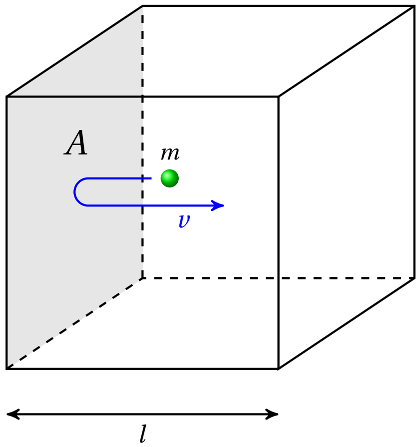
$V_{\text{gas}} \gg V_{\text{atoms}}$, so negligible volume of molecules compared to volume of gas this gas can approximate to an ideal gas □

4.3.2 pressure (quantitative view)

we are ready to derive a formula for pressure due to ideal gas

pressure of gas is due to collision of gas molecules with container

let's first consider the effect of one single molecule moving in one dimension only, and then generalise the result to a gas containing N molecules moving in all three dimensions



one gas molecule moving in 1-D

let's assume this single molecule only moves in x-direction (see figure)

change in momentum when colliding with wall: $\Delta P_x = mv_x - (-mv_x) = 2mv_x$[19]

time interval between collisions: $\Delta t = \frac{2l}{v_x}$

average force acting: $F_x = \frac{\Delta P_x}{\Delta t} = \frac{2mv_x}{\frac{2l}{v_x}} = \frac{mv_x^2}{l}$

average pressure: $p_x = \frac{F}{A} = \frac{mv_x^2}{lA} = \quad \Rightarrow \quad p_x = \frac{mv_x^2}{V}$

generalisation to N molecules moving in 3-D

– N molecules so N times the contributions to pressure

  but there is a *distribution* of speeds for N molecules, so should take average of v^2

[19]In this section we use P for momentum of a particle and p for pressure of a gas to avoid confusion.

- in three-dimensional space, we have: $v^2 = v_x^2 + v_y^2 + v_z^2$

  but molecules have no preference in any specific direction, so: $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \dfrac{\langle v^2 \rangle}{3}$

  pressure should be shared equally among three dimensions: $p = p_x = p_y = p_z$

therefore we find the pressure of an ideal gas is given by: $\boxed{p = \dfrac{Nm\langle v^2 \rangle}{3V}}$

➢ $\langle v^2 \rangle$ is the *mean square velocity* of gas molecules

we can further define r.m.s. (root mean square) velocity: $v_{\text{rms}} = \sqrt{\langle v^2 \rangle}$

gas molecules in random motion so there exists a range of velocities

we cannot tell exact velocity of a specific molecule, but can only tell mean values

➢ N is number of molecules, m is mass of one molecule

then Nm gives total mass of the gas, and $\dfrac{Nm}{V}$ gives gas density ρ

we can rewrite the pressure formula as: $\boxed{p = \dfrac{1}{3}\rho\langle v^2 \rangle}$

(pressure depends only on density and mean square speed of molecules)

➢ physical interpretation of the formula

- $N \nearrow \;\Rightarrow$ more molecules, more collisions $\Rightarrow p \nearrow$
- $m \nearrow \;\Rightarrow$ greater mass, greater force upon collision $\Rightarrow p \nearrow$
- $v \nearrow \;\Rightarrow$ strike container harder, also more often $\Rightarrow p \nearrow$
- $V \nearrow \;\Rightarrow$ spend more time in gas, less frequent collision with container $\Rightarrow p \searrow$

4.3.3 kinetic energy

we now have two equations for ideal gases:

$$\begin{cases} pV = nRT\,, \text{ or } pV = NkT & \text{ideal gas law} \\ p = \dfrac{Nm\langle v^2 \rangle}{3V} & \text{pressure law} \end{cases}$$

compare the two equations: $pV = \dfrac{1}{3}Nm\langle v^2 \rangle = NkT \quad \Rightarrow \quad m\langle v^2 \rangle = 3kT$

mean kinetic energy of a single molecule in a gas is: $\boxed{\langle E_k \rangle = \dfrac{1}{2}m\langle v^2 \rangle = \dfrac{3}{2}kT}$

mean K.E. of ideal gas molecules is *proportional* to its thermodynamic temperature

➢ useful relation for molecular speeds: $\boxed{v_{\text{rms}}^2 \propto T}$

recall our statement in §4.1.1, higher temperature means higher speed for molecules

➢ we only talk about *translational* K.E. here

molecules have this energy because they are moving through space

total kinetic energy may also include *rotational* K.E. and *vibrational* K.E. [20]

➢ $\langle E_k \rangle = \frac{3}{2}kT$ gives the *mean*, or *average* K.E. per molecule

gas molecules exchange energies with each other upon collisions

for an individual molecule, its K.E. is not a constant

but mean K.E. is constant, which depends on temperature T only

➢ in a mixture of several gases, K.E. is shared *equally* among its components

this is because of repeated collisions between particles

though all molecules have same K.E., heavier molecules will move more slowly

Example 4.7 Air consists of oxygen (O_2, molar mass 32 g mol^{-1}) and nitrogen (N_2, molar mass 28 g mol^{-1}). (a) Calculate the mean translational kinetic energy of these molecules at 300 K. (b) Estimate the typical speed for each type of the molecule.

✍ mean K.E. of single molecule: $\langle E_k \rangle = \frac{3}{2}kT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 \approx 6.21 \times 10^{-21}$ J

$$\langle E_k \rangle = \frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}kT \quad \Rightarrow \quad \frac{1}{2}\frac{M}{N_A}\langle v^2 \rangle = \frac{3}{2}kT \quad \Rightarrow \quad \langle v^2 \rangle = \frac{3kN_AT}{M} = \frac{3RT}{M}$$

for oxygen molecule: $v_{O_2} \approx \sqrt{\frac{3 \times 8.31 \times 300}{0.032}} \approx 483$ m s^{-1}

for nitrogen molecule: $v_{N_2} \approx \sqrt{\frac{3 \times 8.31 \times 300}{0.028}} \approx 517$ m s^{-1} □

Example 4.8 A cylinder container initially holds a gas of helium-4 at a temperature of 54°C. (a) Find the mean square speed of these helium atoms. (b) If the temperature is raised to 540°C, find the r.m.s. speed of the atoms.

✍ mass of one helium-4 atom: $m = 4\text{u} = 4 \times 1.66 \times 10^{-27} \approx 6.64 \times 10^{-27}$ kg

[20] There is an important result in classical thermal physics, known as the *equipartition of energy theorem*. It states that the average energy per molecule is $\frac{1}{2}kT$ for each independent *degree of freedom*. A molecule can move in three directions, corresponding to three translational degrees of freedom, thus its mean translational kinetic energy is $\frac{3}{2}kT$. For a polyatomic gas (each molecule consists of several atoms), apart from translational motion , it has additional rotational degrees of freedom and different vibrational modes, so its average energy can be calculated by counting the total number of degrees of freedom.

$$\text{at } 54°\text{C: } \frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \quad \Rightarrow \quad \langle v^2\rangle = \frac{3kT}{m} = \frac{3\times 1.38\times 10^{-23}\times(54+273)}{6.64\times 10^{-27}} \approx 2.04\times 10^6 \text{ m}^2\text{ s}^{-2}$$

$$\text{note relation between } v \text{ and } T\text{: } \langle v^2\rangle \propto T \quad \Rightarrow \quad \frac{\langle v'^2\rangle}{\langle v^2\rangle} = \frac{T'}{T} \quad \Rightarrow \quad v'_{\text{rms}} = \sqrt{\frac{T'}{T}}\times v_{\text{rms}}$$

$$\text{at } 540°\text{C: } v'_{\text{rms}} = \sqrt{\frac{540+273}{54+273}}\times\sqrt{2.04\times 10^6} \approx 2.25\times 10^3 \text{ m s}^{-1} \qquad \square$$

Question 4.6 A fixed mass of gas expands to twice its volume at constant temperature. (a) How does its pressure change? (b) How does mean kinetic energy change?

Question 4.7 In order for a molecule to escape from the gravitational field of the earth, it must have a speed of 1.1×10^6 m s^{-1} at the top of the atmosphere. (a) Estimate the temperature at which helium-4 atoms could have this speed. (b) Helium atom actually escape from top of the atmosphere at much lower temperatures, explain how this is possible.

Chapter 5
Thermodynamics

5.1 thermal physics basics

5.1.1 temperature scales

- Celsius scale (unit: °C)

  0°C defined as temperature of ice-water mixture

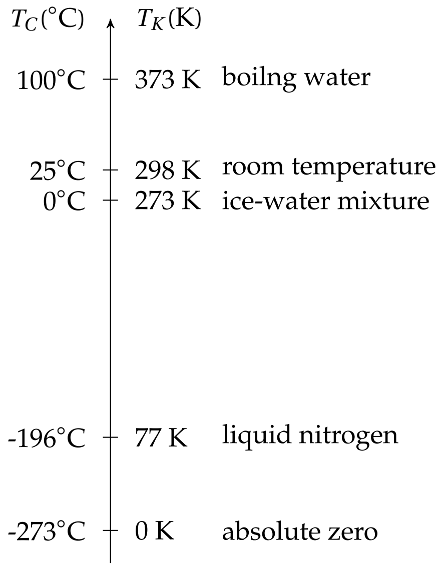
  100°C defined as temperature of boiling water

- Kelvin scale (unit: K)

  0 K (*absolute zero*) is lowest temperature possible

- conversion rule: $T_K(\mathrm{K}) \overset{-273}{\underset{+273}{\rightleftharpoons}} T_C(°\mathrm{C})$

- change of 1°C equals change of 1 K



5.1.2 kinetic theory of matter

there are three common states of matter: solid, liquid and gas

they have very different physical properties (density, compressibility, fluidity, etc.)

but deep down, they are all composed of a large number of small molecules

in the **kinetic theory of matter**, we look at microscopic behaviour at molecular level (arrangement, motion, intermolecular forces, separation, etc.)

microscopic behaviour of molecules cause differences in *macroscopic* properties of matter

- solid: molecules close together, tightly bonded, vibrate about their positions
- liquid: molecules quite close together, vibrate but has some freedom to move about
- gas: molecules widely separated, free from neighbours, move rapidly

5.1.3 specific latent heat

it requires heat energy to *melt* a solid or *boil* a liquid

melting and boiling usually occur at a fixed temperature

thermal energy to cause the change of state at a constant temperature is called *latent heat*

amount of latent heat needed depends on mass of substance: $\boxed{Q = Lm}$

> we define **specific latent heat** (L) as the thermal energy required to change the state of *unit* mass of substance with no change in temperature is called

- unit of specific latent heat: $[L] = \mathrm{J} \cdot \mathrm{kg}^{-1}$
- specific latent heat is an *intensive* property

  i.e., L does not depend on size or shape of sample, L depends on type of substance only
- for melting, L is called *specific latent heat of fusion*

  for boiling, L is called *specific latent heat of vaporisation*
- latent heat is related to breaking bonds and increasing intermolecular separation

  vaporisation requires larger increase in particle separation than fusion

  for a given substance, $L_{\text{vapour}} > L_{\text{fuse}}$

Example 5.1 A 3.0 kW electric kettle contains 0.5 kg of water already at its boiling point. Neglecting heat losses, determine how long it takes to boil dry. ($L_{\text{water}} = 2.26 \times 10^6$ J kg^{-1})

✍ heat required: $Q = mL = 0.50 \times 2.26 \times 10^6 = 1.13 \times 10^6$ J

time needed: $t = \dfrac{Q}{P} = \dfrac{1.13 \times 10^6}{3.0 \times 10^3} \approx 380 \text{ s} \approx 6.3 \text{ min}$ □

Example 5.2 A student measures specific latent heat of fusion for ice. He uses an electric heater to melt ice but the insulation is not perfect. The experiment is carried out twice, with the heater operating at different powers. Use the data table to calculate specific latent heat of fusion.

	Power (W)	time interval (min)	mass of ice melted (g)
test 1	60	3.0	40.4
test 2	90	3.0	56.6

✍ there exists heat gain from surroundings, so effective power $P_{\text{eff}} = P_{\text{heater}} + P_{\text{sur}}$

heat energy to melt ice: $Q = mL = (P_{\text{heater}} + P_{\text{sur}})t$

$$\begin{cases} 40.4 \times L = (60 + P_{\text{sur}}) \times 3.0 \times 60 \\ 56.6 \times L = (90 + P_{\text{sur}}) \times 3.0 \times 60 \end{cases} \Rightarrow \begin{cases} L \approx 333 \text{ J g}^{-1} \\ P_{\text{sur}} \approx 14.8 \text{ W} \end{cases}$$ □

Question 5.1 A student designs an experiment to determine the specific latent heat of fusion L of ice. Some ice at 0°C is heated with an electric heater. The experiment is carried out twice and the following data are obtained.

	energy supply from heater (J)	time interval (min)	mass of ice melted (g)
heater off	0	10.0	14.3
heater on	21000	5.0	70.0

(a) Suggest why two sets of readings are taken. (b) Find specific latent heat of fusion for ice.

5.1.4 specific heat capacity

heating a substance could cause an increase in its temperature

heat required is proportional to its mass m and temperature change ΔT: $\boxed{Q = cm\Delta T}$

> we define **specific heat capacity** (c) as the thermal energy required per unit mass of substance to cause an increase of one unit in its temperature

- unit of specific heat capacity: $[c] = \text{J kg}^{-1}\text{ K}^{-1}$ or $\text{J kg}^{-1}\text{ °C}^{-1}$
- c is also an *intensive* property, i.e., independent of size or shape of the sample

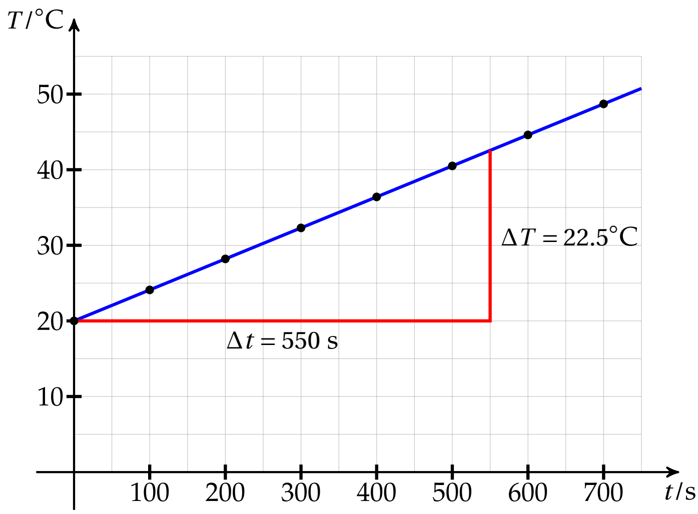
Example 5.3 A 1.00 kg aluminium block is heated using an electrical heater. The current in the heater is 4.2 A and the p.d. across is 12 V. Measurements of the rising temperature are represented by the graph. Determine specific heat capacity of aluminium.

✍ energy supplied: $Q = cm\Delta T \quad \Rightarrow \quad IV\Delta t = cm\Delta T \quad \Rightarrow \quad c = \dfrac{IV}{m\frac{\Delta T}{\Delta t}}$

$\dfrac{\Delta T}{\Delta t}$ is gradient of fitting line: $\dfrac{\Delta T}{\Delta t} = \dfrac{22.5}{550} \approx 4.09 \times 10^{-2}$ °C s^{-1}

specific heat capacity: $c = \dfrac{4.2 \times 12}{1.00 \times 4.09 \times 10^{-2}} \approx 1230 \text{ J kg}^{-1}\text{ °C}^{-1}$ □

Example 5.4 A block of 30 g ice at −20°C is added to a large cup of 270 g water at 80°C. Assume there is no energy lost, what is the final temperature of the mixture? (data: specific



heat capacity of water is 4200 J kg^{-1} K^{-1}, specific heat capacity of ice is 2100 J kg^{-1} K^{-1}, specific latent heat of ice is 3.3×10^5 J kg^{-1}.

✍ energy lost by hot water = energy gain by ice cube

$$\underbrace{4200 \times 0.27 \times (80 - T)}_{95\,°\text{C water} \to T\,°\text{C water}} = \underbrace{2100 \times 0.030 \times [0 - (-20)]}_{-20\,°\text{C ice} \to 0\,°\text{C ice}} + \underbrace{3.3 \times 10^5 \times 0.030}_{0\,°\text{C ice} \to 0\,°\text{C water}} + \underbrace{4200 \times 0.030 \times (T - 0)}_{0\,°\text{C water} \to T\,°\text{C water}}$$

$$90720 - 1134T = 1260 + 9900 + 126T$$

$$T = \frac{90720 - 1260 - 9900}{1134 + 126} \approx 63°\text{C}$$

□

Question 5.2 A mixture contains 5% silver and 95% of gold by weight. Some gold is melted and the correct weight of silver is added. The initial temperature of silver is 20°C. Use the data to calculate the initial temperature of gold so that the final mixture is at melting point of gold.

	silver	gold
melting point (K)	1240	1340
specific heat capacity (solid or liquid) (J kg^{-1} K^{-1})	235	129
specific latent heat of fusion (kJ kg^{-1})	105	628

5.2 internal energy

we now consider the total energy within a thermodynamic system

molecules in a system undergo random motion, so they have kinetic energy there are potential energy between molecules due to intermolecular interaction

internal energy is defined as the sum of random kinetic energy of molecules and potential energy between molecules: $\boxed{U = E_k + E_p}$

- internal energy is a *state function* of the system

  it only depends on current state of system, not on process to arrive at this state

5.2.1 kinetic energy

- internal energy counts K.E. due to random motion at molecular level

  K.E. of macroscopic motion of the system as a whole is not included
- mean K.E. of molecules is directly proportional to temperature: $\boxed{E_k \propto T}$

  K.E. of molecules depends on temperature only

  higher temperature means molecules move faster, vibrate more intensively, etc.

5.2.2 potential energy

- internal energy counts P.E. due to force fields *within* the system

  P.E. of the system as a whole due to *external* force fields is not included
- P.E. between molecules depends on intermolecular separation and chemical bonding

  in general, greater intermolecular separation means greater P.E.[21]: $\boxed{r \nearrow \Leftarrow E_p \nearrow}$

  breaking intermolecular bonds also causes an increase in P.E.

  mean P.E. of gas > mean P.E. of liquid > mean P.E. of liquid solid

5.2.3 internal energy of ideal gas

for ideal gas, mean K.E. of one molecule: $E_k = \dfrac{3}{2}kT$

[21]Intermolecular separation does not necessarily increase during melting processes. A typical counter example is melting of ice into water, for which intermolecular separation actually decreases (density of water > density of ice), but potential energy of the system will still increase because hydrogen bonds between H_2O molecules are broken.

there is no intermolecular force, so P.E. of ideal gas is defined to be zero: $E_p = 0$

internal energy per molecule: $U = E_k + E_p = \frac{3}{2}kT$

hence internal energy of ideal gas is purely kinetic and directly proportional to temperature

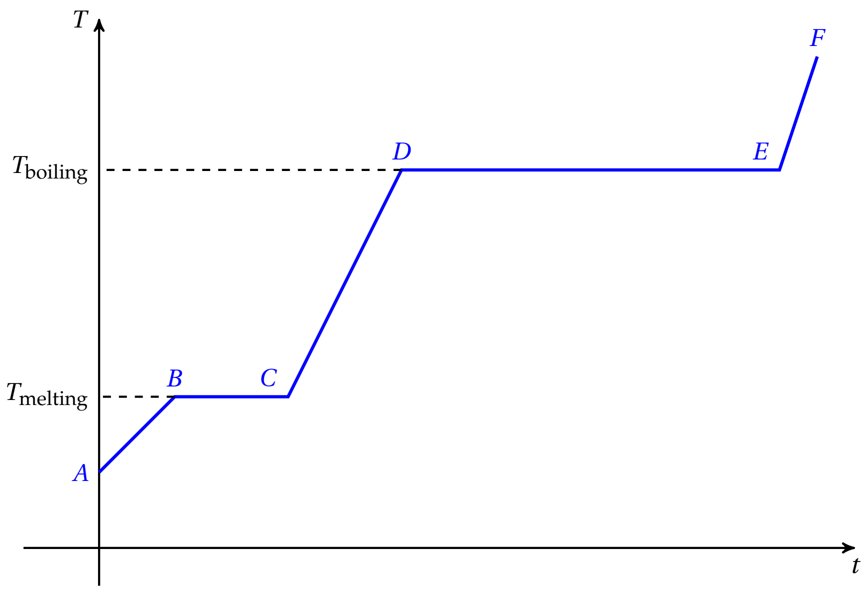
total internal energy of the gas: $U_{\text{gas}} = NU \quad \Rightarrow \quad \boxed{U_{\text{gas}} = \frac{3}{2}NkT}$

5.2.4 change of states

consider a substance being heated from its solid state

it will melt into a liquid and further vaporise into its gaseous state

we now look into the changes of internal energy during each stage



– AB (solid state): $T \nearrow \Rightarrow E_k \nearrow$, greater vibration for solid particles

  but no (significant) change in mean separation[22] $\Rightarrow$ no change in P.E.

– BC (*melting*): latent heat goes into breaking intermolecular bonds, $r \nearrow \Rightarrow E_p \nearrow$

  but melting occurs at constant temperature[23] $\Rightarrow$ no change in K.E.

[22] A typical solid material expands when it is heated, so intermolecular separation will increase slightly.

[23] Here we talk about *pure substance*, which changes from solid into liquid at a particular temperature, called the *melting point*. But for a *mixture* of substances, melting may occur over a *range* of temperatures. It is also possible for a substance to decompose before they change states.

- *CD* (liquid state): $T \nearrow \Rightarrow E_k \nearrow$, greater vibration and free motion

  but no (significant) change in mean separation $\Rightarrow$ no change in P.E.
- *DE* (*boiling*): molecules break free, $r \nearrow \Rightarrow E_p \nearrow$

  boiling occurs at constant temperature[24] $\Rightarrow$ no change in K.E.
- *EF* (gas state): $T \nearrow \Rightarrow E_k \nearrow$ particles move even faster

  particles completely separated, no intermolecular force, so constant $E_p = 0$

Question 5.3 For a particular substance, why is the specific latent heat of vaporisation much greater than the specific latent heat of fusion?

evaporation

liquid changes into gas without boiling $\longrightarrow$ **evaporation**

particles move randomly, i.e., they move at various speeds

some molecules move fast enough to break free

➢ *cooling effect*: evaporation causes a decrease in temperature of the liquid

most energetic molecules escaped, those remain in the liquid have less energy, $E_k \searrow \Rightarrow T \searrow$

➢ rate of evaporation increases with temperature, surface area of liquid

➢ different between boiling and evaporation

	boiling	evaporation
occurrence	throughout the liquid	at surface only
temperature	occur at boiling point	occur at any temperature
bubble formation	bubbles formed	no bubbles
rate of process	fast	slow

5.2.5 first law of thermodynamics

internal energy of a system changes upon heat transfer or doing work

[24] Again we only concern pure substances. Mixtures that boil over a range of temperatures or substance decompose before phase transition are not considered here.

first law of thermodynamics states that the increase in internal energy equals sum of heat supply to the system and work done on the system: $\boxed{\Delta U = Q + W}$

➢ first law of thermodynamics is an extension of the law of conservation of energy

➢ sign conventions for Q and W

- $Q > 0$ if heat supplied to system
- $Q < 0$ if heat released by system to surroundings
- $W > 0$ if work done *on* system by external

  i.e., if system is compressed and volume decreases, then $W > 0$
- $W < 0$ if system does work *against* surroundings

  i.e., system expands and volume increases, then $W < 0$

➢ amount of heat energy: $Q = \begin{cases} cm\Delta T & \text{(if no change of state)} \\ Lm & \text{(during change of state)} \end{cases}$

➢ amount of work is related to pressure and change of volume

if volume changed at *constant* pressure, then $W = F\Delta s = pA\Delta s \Rightarrow \boxed{W = p\Delta V}$[25]

if no change of volume, then no work is done

Example 5.5 A gas is heated by supplying it with 25 kJ of energy. The gas expands so that the volume increases by 0.10 m^3. Assume the gas has a fixed pressure of 150 kPa during the process. Calculate the change in internal energy.

✍ amount of work done: $W = p\Delta V = 150 \times 0.10 = 15$ kJ

but gas expands means work is done against surroundings, so this is negative work

change in internal energy: $\Delta U = Q + W = (+25) + (-15) = +10$ kJ □

Example 5.6 Use the idea of internal energy and the first law of thermodynamics, explain why boiling water requires heat supply.

✍ boiling occurs at constant temperature, so $\Delta E_k = 0$

but separation between molecules increased, so $\Delta E_p > 0$

[25]If pressure changes with volume during a thermodynamics process, then work done $W = \int p\mathrm{d}V$. Alternatively, we can evaluate the area under a p-V graph to find the work done.

by definition, internal energy $U = E_k + E_p$, so $\Delta U > 0$

during boiling, there is an increase in volume, so work against surroundings, $W < 0$

recall first law of thermodynamics $\Delta U = Q + W$, must have $Q > 0$

this means heat must be supplied for boiling processes □

Example 5.7 When you pump up a bicycle tyre, the temperature of air inside the tyre will go up. Explain why this happens using the first law of thermodynamics.

✍ pumping up tyre involves compressing gas, so positive work is done: $W > 0$

for each stroke, there is little time for heat transfer, so $Q \approx 0$

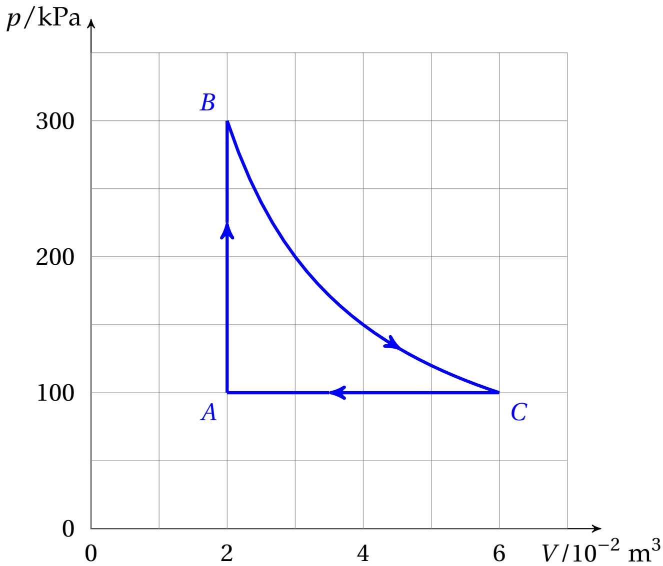
according to first law of thermodynamics $\Delta U = Q + W \quad \Rightarrow \quad \Delta U > 0$

by definition, internal energy $U = E_k + E_p \quad \Rightarrow \quad \Delta U = \Delta E_k + \Delta E_p > 0$

but for gas, there is negligible intermolecular force, so $\Delta E_p = 0$, then must have $\Delta E_k > 0$

K.E. of molecules is proportional to temperature, higher K.E. so higher temperature □

Example 5.8 An ideal gas of 0.080 mol is initially at state A and then undergoes a cycle $ABCA$. The variation of its pressure p with its volume V is shown on the graph.



Temperature of state A is 300 K. The magnitude of work on gas from state B to C is 6570 J.

For each stage $A \to B$, $B \to C$ and $C \to A$ during the cycle, determine work done and heat supply to the gas, and also find the change in internal energy.

✍ work done depends on change in volume

$A \to B$: no change in volume, so $W_{AB} = 0$

$B \to C$: $|W_{BC}| = 6570$ J, but expansion implies $W < 0$, so $W_{BC} = -6570$ J

$C \to A$: $|W_{CA}| = p\Delta V_{CA} = 1 \times 10^5 \times (6-2) \times 10^{-2} \quad \Rightarrow \quad W_{CA} = +4000$ J (compression so $W > 0$)

change in internal energy of ideal gas depends on change in temperature

$A \to B$: same V but $p_B = 3p_A$, so $T_B = 3T_A = 900$ K

$$\Delta U_{AB} = \frac{3}{2}Nk\Delta T_{AB} = \frac{3}{2} \times 0.80 \times 6.02 \times 10^{23} \times 1.38 \times 10^{-23} \times (900-300) \approx +5980 \text{ J}$$

$B \to C$: note that $p_B V_B = p_C V_C$, so $T_B = T_C$, no change in temperature, so $\Delta U_{BC} = 0$

$C \to A$: $\Delta U_{CA} = \frac{3}{2}Nk\Delta T_{CA} = \frac{3}{2} \times 0.80 \times 6.02 \times 10^{23} \times 1.38 \times 10^{-23} \times (300-900) \approx -5980$ J

for cycle $ABCA$, same initial and final state, so total change in internal energy must be zero

one can check that $\Delta U_{\text{cycle}} = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA} = 0$

to find supply of thermal energy, we apply first law of thermodynamics: $\Delta U = Q + W$

$A \to B$: $+5980 = Q_{AB} + 0 \quad \Rightarrow \quad Q_{AB} = +5980$ J

$B \to C$: $0 = Q_{BC} + (-6570) \quad \Rightarrow \quad Q_{BC} = +6570$ J

$C \to A$: $-5980 = Q_{CA} + (+4000) \quad \Rightarrow \quad Q_{CA} = -9980$ J

the table below summarises all energy changes during the cycle $ABCA$

change	W/J	Q/J	ΔU/J
$A \to B$	0	+5980	+5980
$B \to C$	-6570	+6570	0
$C \to A$	+4000	-9980	-5980

□

Question 5.4 Show that when n mol of gas is heated at a fixed volume, thermal energy required to raise the temperature by 1.0 K is nR.

Question 5.5 Two identical balloons A and B hold the same amount of gas at the same initial temperature. They are given the same amount of heat. Suppose volume of A is fixed, while B is allowed to expand, compare the final temperatures of the gases in the two balloons.

5.3 temperature

5.3.1 temperature & thermal energy

➢ temperature can be considered as a *relative* measure of thermal energy

temperature can tell the *direction* of thermal energy flow

heat always (spontaneously) flows from high temperature regions to colder regions[26]

➢ if two objects in contact have the same temperature, then there is no net heat transfer

the two objects are said to be in **thermal equilibrium**

➢ if two systems A and B are each in thermal equilibrium with a third system C, A and B are also in thermal equilibrium, this is called the **zeroth law of thermodynamics** [27]

Question 5.6 A student thinks that temperature measures the amount of heat in an object. Suggest why this statement is incorrect with examples.

5.3.2 absolute zero

mean K.E. of molecules is a microscopic description of temperature T

minimum K.E. occurs if molecules do not move at all (completely frozen)[28]

this corresponds to the lowest possible temperature, called **absolute zero**

➢ **Kelvin scale of thermodynamic temperature** is defined based on absolute zero as 0 K[29]

[26]This is the consequence of the *second law of thermodynamics*, which is concerned with the direction of natural processes. The law states that the total *entropy*, a quantity that counts the number of microstates of a system, of an isolated system can never decrease over time. You will learn more about entropy if you study A-Level chemistry.

[27]This law is important for the formulation of thermal physics. The physical meaning of the law was expressed by Maxwell: "All heat is of the same kind." The zeroth law allows us to give the mathematical definition of temperature.

[28]In the *classical* description, there is no reason not allowing a molecule to cease motion. However, due to *quantum mechanical effects*, kinetic energy of a system cannot be zero even at absolute zero.

[29]More precisely, 0 K for absolute zero and 273.16 K for water triple point.

- conversion rule between Celsius scale and Kelvin scale: $\boxed{T_K(\mathrm{K}) \underset{+273.15}{\overset{-273.15}{\rightleftharpoons}} T_C(^\circ\mathrm{C})}$ [30]
- Kelvin scale is said to be an *absolute scale*

  zero of Kelvin scale does not depend on property of a specific substance

  in contrast, zero of Celsius scale is based on properties of water
- it is impossible to remove any more energy from a system at 0 K (or -273.15 °C)

  but there is no practicable means to bring a physical system to exactly 0 K [31]

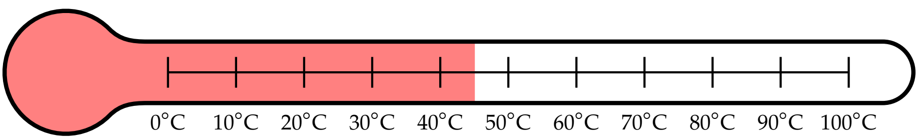
5.3.3 thermometer

a **thermometer** is a device which can be used to measure temperature

liquid-in-glass thermometer

basic principle: liquid expands in volume at higher temperature

examples include alcohol thermometer, mercury-in-glass thermometer, etc.



resistance temperature detectors (RTD)

basic principle: resistance of electronic element changes with temperature

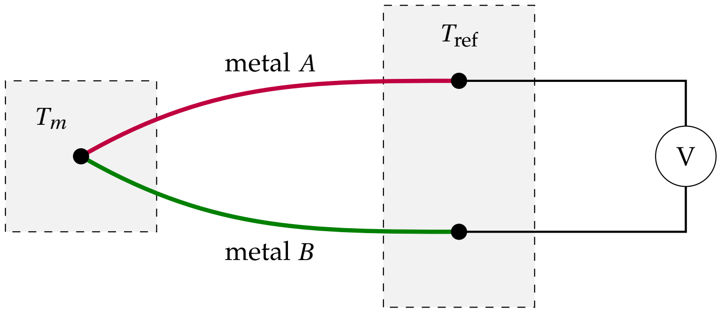
metal wires and thermistor are both used in RTD elements

thermocouple

basic principle: difference in temperature can produce a *thermoelectric voltage* across junctions, thermocouple measures temperatures by means of this voltage

[30]The numerical value of 273.15 will only be quoted in this section. For everywhere else in the notes, we will use the less precise value 273 for simplicity.

[31]This is known as the *third law of thermodynamics*, which states that it is impossible, no matter how idealized, to reduce the temperature of any closed system to absolute zero in a finite number of operations.



typical configuration of a thermocouple unit

two pieces of different metal wires are joined at their ends

if there exists a temperature difference between the ends, a *thermoelectric voltage* is developed

this voltage depends on temperature difference, captured by some characteristic function

in practice, we place the *measurement* junction in an environment of unknown temperature

the other end, or the *reference* junction, is at a known temperature

temperature difference is deduced from the voltage reading

hence the desired temperature can be determined

➢ features of a thermometer:

- *range*: whether the thermometer can measure very low or very high temperatures
- *sensitivity*: whether a small change in temperature can be detected
- *response time*: whether changes in temperature can be immediately measured
- *linearity*: whether changes in temperature are proportional to changes in output

	liquid-in-glass	RTD wire	thermistor	thermocouple
valid range	narrow	wide	narrow	very wide
sensitivity	low	fair	high	high
response time	slow	fast	fast	very fast
linearity	good	good	limited	non-linear

Chapter 6

Electrostatics

6.1 electric forces

6.1.1 Coulomb's law

charged objects will attract or repel one another through the electric force

> **Coulomb's law** states that the electric force between two electrically charged particles is proportional to their charges and inversely proportional to the square of their separation: $\boxed{F = \dfrac{Qq}{4\pi\epsilon_0 r^2}}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$ is the *permittivity of free space*

$k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \mathrm{N\ m^2\ C^{-2}}$ is a useful constant for calculations

this law was first published by French physicist *Charles Augustin de Coulomb* in 1785

- charges Q, q in Coulomb's law are *point charges*
- for uniformly charged spheres, they can be thought as point charges

  separation r is taken to be centre-to-centre distance
- symbolically, sign of F can tell direction of the electric force

  for like charges (both positive or both negative), $Q_1Q_2 > 0 \Rightarrow F > 0 \Rightarrow$ *repulsion*

  for opposite charges (one positive and one negative), $Q_1Q_2 < 0 \Rightarrow F < 0 \Rightarrow$ *attraction*

Example 6.1 The hydrogen atom has a radius of about 53 pm. Estimate the electric force between the proton and the orbiting electron.

✍ $$F = \frac{Q_pQ_e}{4\pi\epsilon_0 r^2} = 8.99 \times 10^9 \times \frac{(1.60 \times 10^{-19})^2}{(53 \times 10^{-12})^2} \approx 8.2 \times 10^{-8}\ \mathrm{N}$$ □

Question 6.1 Two protons are separated by a distance r. Find the ratio of the electric force to the gravitational force between them.

6.1.2 electric fields

to explain how charges affect each other at a distance, we introduce notion of *electric fields*

electric field is a region of space where a charged object is acted by a force

any charge Q (or several charges) can produce an electric field

any test charge q within the field will experience an electric force

Next, we will introduce the concepts of *electric field strength* and *electric potential*, and see how they are related to the force acting on a charged object and the potential energy it possesses.

You might have noticed that Coulomb's law for electrostatic forces and Newton's law of gravitation are both *inverse square laws*, it turns out that electric fields are very similar to gravitational fields in various aspects.

6.2 electric field strength

6.2.1 electric field strength

electric field strength is defined as electric force per unit positive charge: $\boxed{E = \frac{F}{q}}$

- unit of E: $[E] = \mathrm{N\,C^{-1}} = \mathrm{V\,m^{-1}}$[32]
- field strength due to an isolated source of *point* charge Q

  a small test charge q at distance r is acted by a force: $F = \frac{Qq}{4\pi\epsilon_0 r^2}$

  field strength at this point: $E = \frac{F}{q} \quad \Rightarrow \quad \boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$

  the field is produced by Q, so field strength only depends on the source Q
- if the source is a charged *sphere* of radius R with uniform charge distribution

  viewed from *outside* the sphere, it acts like a point charge concentrated at the centre [33]

[32] You will later find in §6.4.3 the deeper reason why $\mathrm{V\,m^{-1}}$ is also a reasonable unit for field strength.

[33] A brief explanation is given in Example 6.4.

therefore, $E = \dfrac{Q}{4\pi\epsilon_0 r^2}$ also holds for field strength at $r > R$[34]

where r is the distance from the point of interest to *centre* of the sphere

➢ field strength E is a *vector* quantity, it has a direction

to compute combined field strength due to several sources, should perform *vector sum* of contributions from each individual

➢ direction of field strength depends on the source charge Q

for positive source ($Q > 0$): field points away from the source

for negative source ($Q < 0$): field points towards the source

➢ electric force on a charge q can be found if field strength E is known

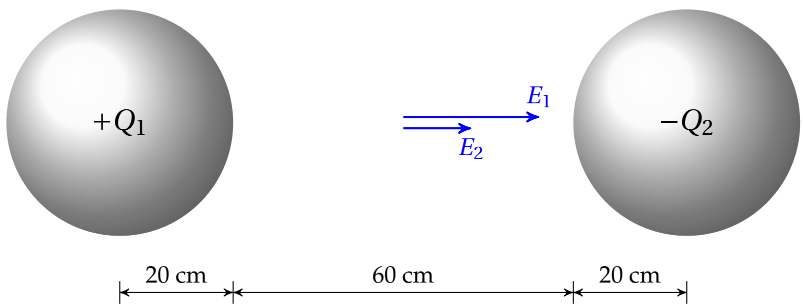
magnitude of electric force: $F = Eq$

direction of force: same direction as E if $q > 0$, but opposite to E if $q < 0$

Example 6.2 A *Van de Graaff generator* produces sparks when its surface electric field strength 4.0×10^4 V cm^{-1}. If the diameter of the sphere is 40 cm, what is the charge on it?

✍ $$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad\Rightarrow\quad Q = 4\pi\epsilon_0 E r^2 = 4\pi\times8.85\times10^{-12}\times4.0\times10^6\times0.20^2 \approx 1.8\times10^{-5}\text{ C}$$ □

Example 6.3 Two identical metal spheres of radius 20 cm carry charges $+2.0$ μC and -1.0 μC respectively. There is a 60 cm gap between them. (a) Find the electric field strength midway along the line joining their centres. (b) A dust particle carrying a charge of -1.3×10^{-8} C is at this position. Find the electric force it experiences.



✍ field strengths due to the two spheres are in same direction

$$E = E_1 + E_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{r_1^2} + \frac{Q_2}{r_2^2}\right) = 8.99\times10^9\times\left(\frac{2.0\times10^{-6}}{0.25^2} + \frac{1.0\times10^{-6}}{0.25^2}\right) \approx 4.32\times10^5\text{ N C}^{-1}$$

[34]For electric field strength *inside* a conducting sphere, detailed discussions are given in §6.4.2.

field strength points to the right

force on dust particle: $F = Eq = 4.32 \times 10^5 \times 1.4 \times 10^{-8} \approx 5.6 \times 10^{-3}$ N

dust particle is negatively-charged means force is opposite to field strength

so force on dust particle acts to the left □

Question 6.2 When the charge on the Van de Graaff generator is 4.0×10^{-7} C, the electric field strength at the sphere's surface is 2.4×10^6 V m^{-1}. Determine the additional charge added to the sphere if the field strength at the surface becomes 3.0×10^6 V m^{-1}.

Question 6.3 Two positively charged particles A and B are situated in a vacuum. Point P lies on the line joining the centres of the two spheres and is a distance x from A. Sketch the variation with x of electric field strength E due to the two particles.

6.2.2 electric field lines

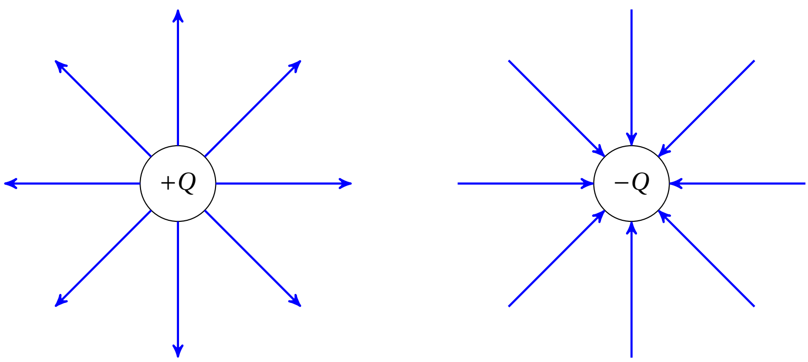
we can use **electric field lines** to visualize an electric field

- *arrows* of field lines show direction of the field

  field lines always tend to leave positive charge, and end up at negative charges

- *density* or spacing of lines show strength of the field

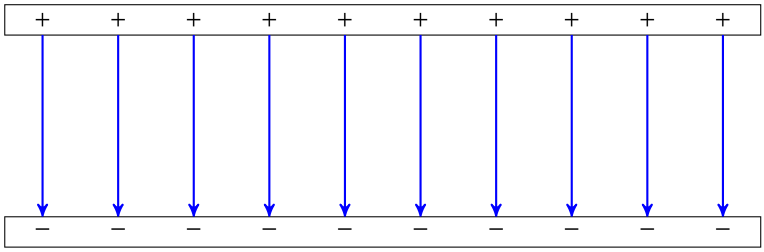
Example 6.4 Sketch the electric field around a positively-charged sphere or a negatively charged sphere, and explain why they can be considered as point charges.



field lines of either case are *radial*, i.e., perpendicular to surface

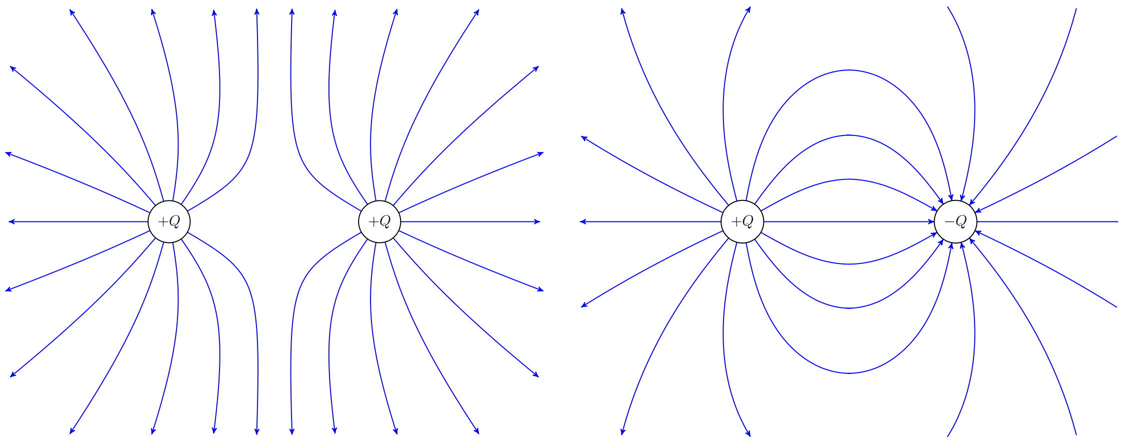
field lines appear to start from or converge towards centre of the sphere so charged spheres act like point charges □

Example 6.5 Field lines between two oppositely-charged large metal plates.



field lines are *parallel* and equally spaced, so this is a *uniform* electric field □

Example 6.6 Field pattern due to two charges of equal magnitude.



two positive charges two opposite charges □

6.3 potential & potential energy

6.3.1 electric potential energy

gain/loss in **electric potential energy** is defined as work done against/by electric force

(compare everything in this section with what you have learned about gravitational P.E.!)

let's start to derive the electrical P.E. between two charges Q and q separated by r

again we define $E_p = 0$ at $r = \infty$ (choice of zero potential energy, no force so no P.E.), then

electric potential energy is equal to the work done by electric force to bring a charge to a specific position from *infinity*

moving a test charge q from $r=\infty$ to a distance of r from Q

work done by electric force: $W=\int_\infty^r F dr=\int_\infty^r \frac{Qq}{4\pi\epsilon_0}\frac{\mathrm{d}x}{x^2}=-\frac{Qq}{4\pi\epsilon_0 x}\Big|_\infty^r=-\frac{Qq}{4\pi\epsilon_0 r}$

since $\Delta E_p=-W$, we find: $E_p(r)-E_p(\infty)=\frac{Qq}{4\pi\epsilon_0 r}$

but $E_p(\infty)=0$, so electric P.E. between two charges Q and q is $\boxed{E_p(r)=\frac{Qq}{4\pi\epsilon_0 r}}$

- as $r\to\infty$, $E_p\to 0$, this agrees with our definition for zero P.E. point
- for like charges, $Qq>0$, so $E_p>0$

  to bring like charges closer, work must be done to overcome their *repulsion*, P.E. increases

  minimum P.E. $E_p(\infty)=0$ at infinity, so positive P.E. at finite r
- for opposite charges, $Qq<0$, so $E_p<0$

  to pull opposite charges apart, work must be done to overcome their *attraction*, P.E. increases

  maximum P.E. $E_p(\infty)=0$ at infinity, so negative P.E. at finite r
- electric P.E. is a *scalar quantity*, sign is important

  repulsion implies positive P.E., and attraction implies negative P.E.

  sign of P.E. is hidden in polarities of charges

Example 6.7 In α-particle scattering experiment, we fire α-particles ($^4_2\alpha$) towards a thin gold ($^{197}_{79}$Au) foil in hope of gaining information about the nucleus. The size of a typical nucleus is about 10^{-14}m, what is the minimum initial speed for α-particles so that radius of gold nucleus can be determined?

✍ as α-particle approaches the nucleus, it slows down due to the repulsive interaction

kinetic energy decreases and electric potential energy increases

if it gets close enough to the nucleus before coming to a stop, nuclear radius can be estimated

$$\text{K.E. loss} = \text{P.E. gain} \quad \Rightarrow \quad \frac{1}{2}mu^2 - \underbrace{\frac{1}{2}mv^2}_{0} = E_p(r) - \underbrace{E_p(\infty)}_{0} \quad \Rightarrow \quad \frac{1}{2}mu^2 = \frac{Qq}{4\pi\epsilon_0 r}$$

$$\frac{1}{2} \times 4 \times 1.66 \times 10^{-27} \times u^2 = \frac{79 \times 1.60 \times 10^{-19} \times 2 \times 1.60 \times 10^{-19}}{4\pi \times 8.85 \times 10^{-12} \times 10^{-14}} \quad \Rightarrow \quad u \approx 3.3 \times 10^7 \text{ m s}^{-1} \qquad \square$$

Question 6.4 A metal sphere of radius 20 cm carries a charge of 5.0×10^{-7} C. A proton is sent towards the sphere at a speed of 1.8×10^6 m s^{-1}. Can the proton reach the surface of the sphere?

6.3.2 electric potential

it is also useful to define the *electric potential* for any specific point in a field

electric potential can be thought as the electric potential energy per unit charge: $V = \dfrac{E_p}{q}$

> **electric potential** is the work needed to bring a unit positive charge from infinity

- unit: $[V] = \text{J C}^{-1} = \text{V}$
- electric potential due to an isolated source Q: $V = \dfrac{E_p}{q} = \dfrac{\frac{Qq}{4\pi\epsilon_0 r}}{q} \quad \Rightarrow \quad \boxed{V = \dfrac{Q}{4\pi\epsilon_0 r}}$
- potential at infinity vanishes: $V_\infty = 0$
- electric potential can take both signs

  the sign depends on whether unit positive charge is repelled or attracted by the source

  for positively-charged sources $V > 0$, while for negatively-charged sources $V < 0$
- electric potential is a *scalar* quantity

  to find combined potential due to multiple charges, add up contributions of each charge

Example 6.8 An electron is accelerated from rest through a potential difference of 600 V. Find the final speed of the electron.

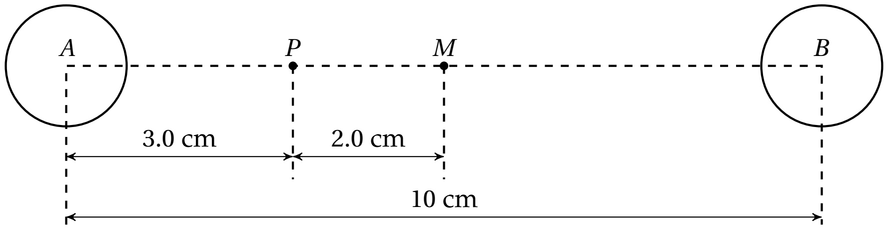
✍ gain in K.E. = change in electric P.E. $\quad \Rightarrow \quad \dfrac{1}{2}mv^2 = q\Delta V$

$$v = \sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2.60 \times 10^{-19} \times 600}{9.11 \times 10^{-31}}} \approx 1.45 \times 10^7 \text{ m s}^{-1} \qquad \square$$

Example 6.9 Two small metal spheres A and B are in a vacuum. Sphere A has charge +20 pC and sphere B has charge +84 pC. The arrangement is shown below.

(a) Find the electric potential at point P and point M respectively.

(b) Find the work done to move an α-particle from P to M.



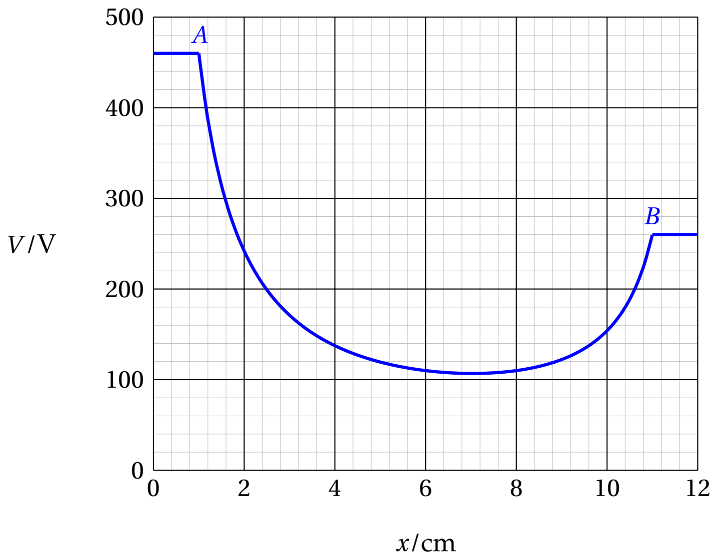
✍ combined electric potential: $V = V_A + V_B = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{Q_A}{r_A} + \dfrac{Q_B}{r_B}\right)$

at P: $V_P = 8.99 \times 10^9 \times \left(\dfrac{+20 \times 10^{-12}}{0.030} + \dfrac{+84 \times 10^{-12}}{0.070}\right) \approx 16.8$ V

at M: $V_M = 8.99 \times 10^9 \times \left(\dfrac{+20 \times 10^{-12}}{0.050} + \dfrac{+84 \times 10^{-12}}{0.050}\right) \approx 18.7$ V

from P to M: $W = \Delta E_p = q\Delta V = q(V_M - V_P) = 2 \times 1.60 \times 10^{-19} \times (18.7 - 16.8) \approx 6.2 \times 10^{-19}$ J □

Example 6.10 A and B are two positively-charged spheres of radius 1.0 cm. A proton P initially at rest on the surface of A moves along the line joining the centres of the two spheres. The variation with distance x from the centre of A of electric potential V at point P is given.



(a) Find the maximum speed as the proton moves from A to B.

(b) Find the speed when the proton reaches surface of B.

✍ increase in K.E. = loss in P.E., so: $\frac{1}{2}mv^2 - 0 = q\Delta V \quad \Rightarrow \quad \frac{1}{2}mv^2 = q(V_A - V_P)$

maximum speed when ΔV is maximum, or $V_P = 107$ V becomes minimum (at $x = 7.2$ cm)

$$\frac{1}{2} \times 1.67 \times 10^{-27} \times v_{\text{max}}^2 = 1.60 \times 10^{-19} \times (460 - 107) \quad \Rightarrow \quad v_{\text{max}} \approx 2.60 \times 10^5 \text{ m s}^{-1}$$

at surface of B, $V_P = 260$ V (at $x = 11.0$ cm)

$$\frac{1}{2} \times 1.67 \times 10^{-27} \times v_B^2 = 1.60 \times 10^{-19} \times (460 - 260) \quad \Rightarrow \quad v_B \approx 1.96 \times 10^5 \text{ m s}^{-1} \qquad \square$$

Question 6.5 Electrical breakdown occurs when electric field strength at surface of a metal sphere exceeds 5.0×10^6 N C^{-1}. Given that the radius of the sphere is 16 cm. What is the electric potential at the surface when electrical breakdown occurs?

Question 6.6 Two charged particles A and B are separated by 20 cm. P is a point on the line AB. Given that particle A carries charge +7.2 μC, and electric potential is zero where $AP = 5.0$ cm. Find the electric charge of B.

Question 6.7 A particle with specific charge (ratio of its electric charge to its mass) $+9.58 \times 10^7$ C kg^{-1} is moving towards a fixed metal sphere. The sphere has a potential of +500 V. The initial speed of the particle is 3.0×10^5 m s^{-1} when it is a large distance from the sphere. Determine whether the particle can reach the surface of the sphere.

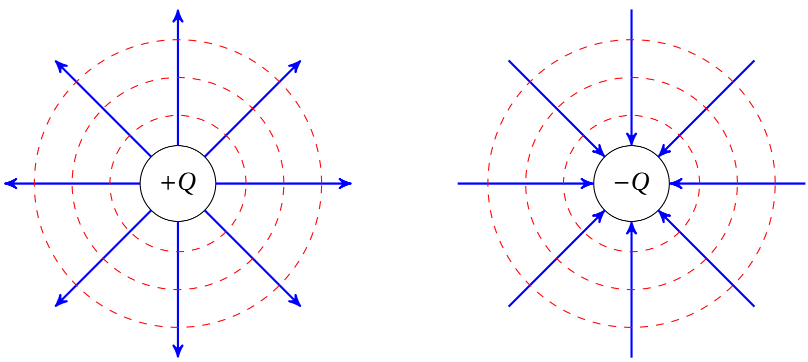
6.3.3 equipotential lines

to show potential distributions, we draw **equipotential lines** [35]

points on same equipotential line have constant electric potential

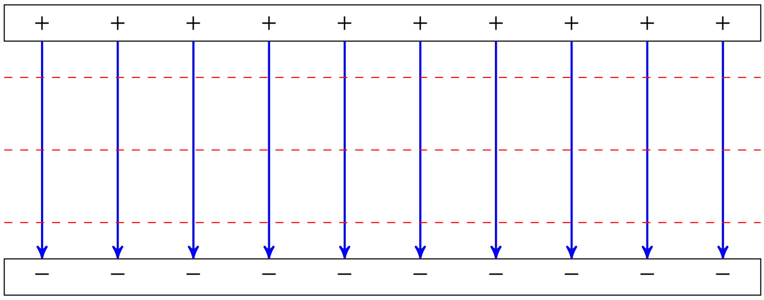
i.e., equipotential lines are *contour lines* of equal electric potential

➢ for a field near a point charge, equipotential lines are a set of *concentric* circles



➢ for uniform fields, equipotential lines are a set of *parallel* straight lines

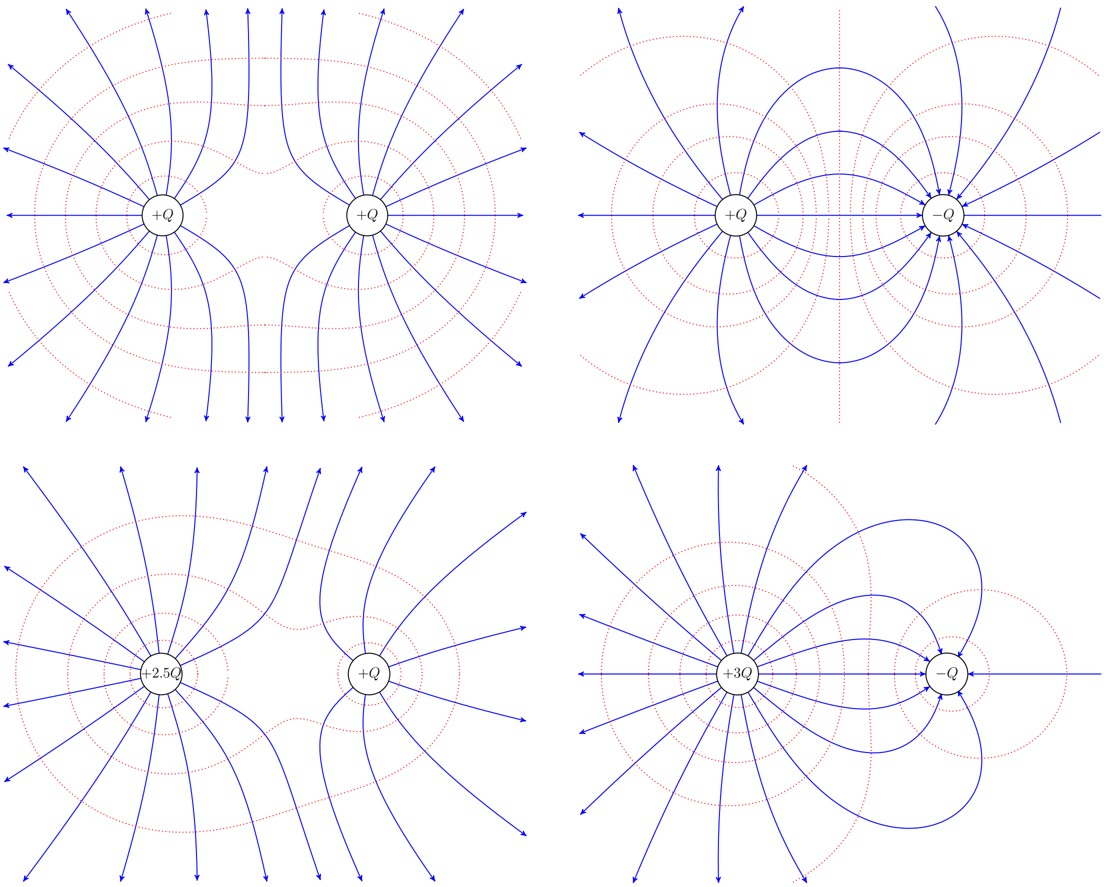
[35] In three dimensions, these lines form equipotential *surfaces*.

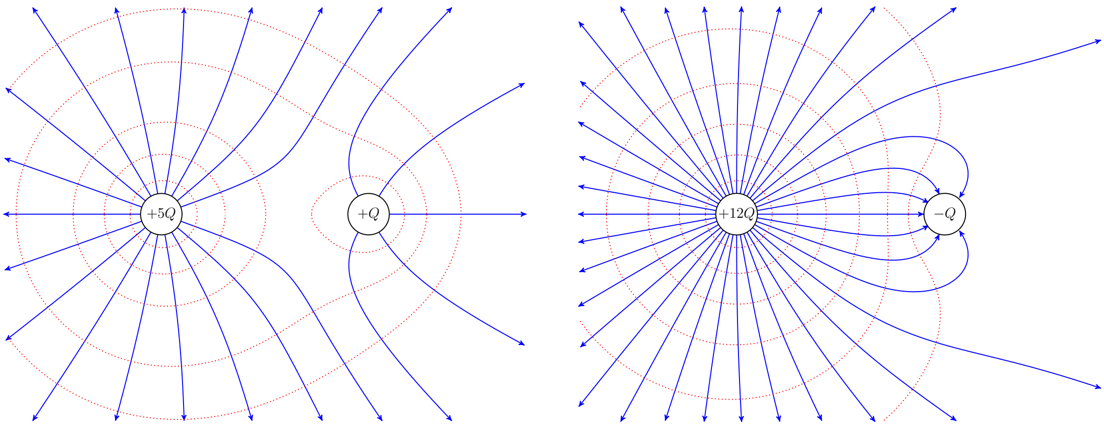


➢ equipotential lines are always perpendicular to the electric field lines

moving along an equipotential line requires no work done

Example 6.11 Field lines and equipotential lines due to two charges of various magnitudes





6.4 further discussions on electric fields

6.4.1 comparison with gravitational fields

both gravitational and electric force are described by inverse square law, so it follows that the mathematical language for both theories are very similar

➢ physical quantities that describe gravitational/electric fields

	vector description	scalar description
interaction between two masses/charges	force F	potential energy E_p
effect of source mass/charge	field g/E	potential φ/V

➢ comparing gravitational field with electric field [36]

	gravitational field	electric field	meaning
force	$F = (-)G\dfrac{Mm}{r^2}$	$F = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r^2}$	force between masses/charges
field strength	$g = \dfrac{F}{m} = (-)G\dfrac{M}{r^2}$	$E = \dfrac{F}{q} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$	force per unit mass/charge
potential energy	$E_p = -G\dfrac{Mm}{r}$	$E_p = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r}$	related to work done by force
potential	$\varphi = \dfrac{E_p}{m} = -G\dfrac{M}{r}$	$V = \dfrac{E_p}{q} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$	energy per unit mass/charge

[36]Note that there is no negative mass, gravitational force always interacts attractively. This is the fundamental difference between gravitational fields and electric fields.

➢ similarities between gravitational field and electric field

- force and field strength both obey *inverse square laws*
- potential energy and potential is inversely proportional to separation
- no potential energy and no potential at infinite separation

➢ differences between gravitational field and electric field

mass (source of gravity) is always positive, but electric charges can be positive or negative

this fact leads to many fundamental differences between the two force fields

- electric force can be repulsive or attractive, but gravitational force is always attractive
- electric potential can take both signs, but gravitational potential is always negative

6.4.2 electric field inside conductors

consider electric field *inside* a metal conductor carrying charge Q

conductor means there are free charge carriers that can move around

but charge distribution should be stable for a charged conductor (no circulating currents)

so charge carriers must experience no force, i.e., field strength inside conductor is zero

put it the other way round, if there is an excess field, it will push free charge carriers to move around, until they are distributed so that the field inside the conductor becomes zero

moreover, there shall be no potential difference between any two points inside the conductor, otherwise charge carriers would flow, so electric potential must be constant

> electric field strength is everywhere zero inside a conductor: $E = 0$
>
> electric potential is everywhere constant inside a conductor: $V = \text{const}$

Example 6.12 Consider the electric field due to a metal sphere of radius R carrying charge Q. Plot the variation with the distance r from sphere's centre of the field strength, and the variation with r of the electric potential.

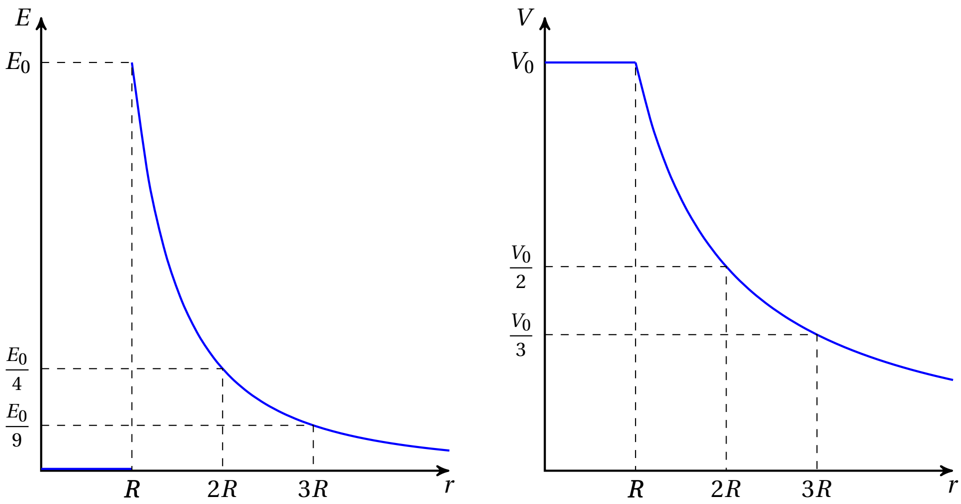
✍ charge Q is uniformly spread out on *surface* of sphere

viewed from *outside*, the sphere appears to have all of its charge concentrated at the centre

so it can be modelled as a point charge due to its symmetric distribution of charges electric field strength at distance r from sphere's centre is: $E = \dfrac{Q}{4\pi\epsilon_0 r^2}$ for $r > R$

electric potential at distance r from sphere's centre is: $V = \dfrac{Q}{4\pi\epsilon_0 r}$ for $r > R$

inside the sphere, i.e., for $r < R$, we have $E = 0$, and $V = \dfrac{Q}{4\pi\epsilon_0 R} = \text{const}$ □



Example 6.12: field strength and potential due to a charged metal sphere

Question 6.8 State whether the two spheres in Example 6.10 are conductors. State what feature of the potential graph supports your answer.

Question 6.9 You might have the experience that your mobile phone signal gets much weaker when you get into an elevator. Explain why this happens.

6.4.3 field strength & potential

for a small displacement Δr in an electric field, change in potential ΔV is

$$\Delta V = \frac{\Delta E_p}{q} \xlongequal{\Delta E_p = -W} -\frac{\Delta W}{q} = -\frac{F\Delta r}{q} \xlongequal{F = Eq} -E\Delta r \quad \Rightarrow \quad E = -\frac{\Delta V}{\Delta r}$$

as change in displacement $\Delta r \to 0$, we have $\boxed{E = -\dfrac{\mathrm{d}V}{\mathrm{d}r}}$

therefore we have the following theorem:

> field strength is negative gradient of potential with respect to displacement

can also consider change in potential ΔV for large distance due to work done in a field

$$\Delta V = \int \mathrm{d}V \xlongequal{E=-\mathrm{d}V/\mathrm{d}r} \int (-)E\mathrm{d}r = -\int E\mathrm{d}r$$ [37]

this gives the inverse relation: $\boxed{\Delta V = -\int E\mathrm{d}r}$ [38]

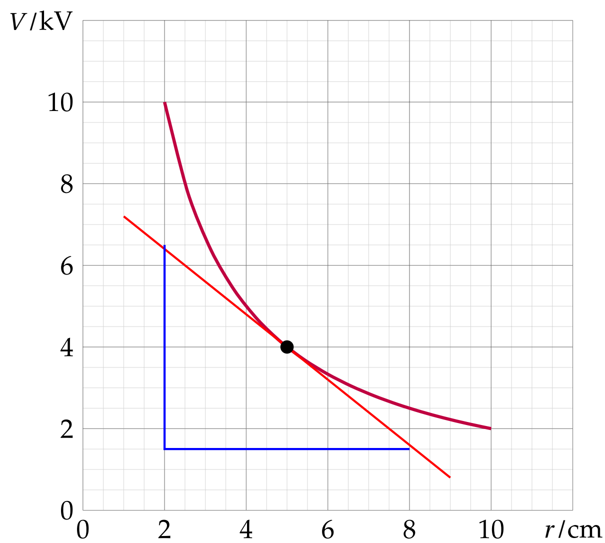
➢ given a V-r graph, gradient of curve gives field strength

conversely, given a E-r graph, area under curve gives change in potential

➢ can also write $F = -\dfrac{\mathrm{d}U}{\mathrm{d}r}$ and $\Delta U = -\int F\mathrm{d}r$

force always acts in a direction to lower the potential energy of an object [39]

Example 6.13 The variation of electric potential near a charged object is shown on the graph. Calculate the electric field strength at 5.0 cm from the centre of the object.



✍ draw tangent to the graph at $r = 5.0$ cm (red line), gradient of tangent gives field strength:

$$\text{gradient} = \frac{\Delta V}{\Delta r} = \frac{(1.5-6.5)\times 10^3}{(8.0-2.0)\times 10^{-2}} \approx -8.3\times 10^4\ \mathrm{V\ m^{-1}} \quad\Rightarrow\quad E = -\frac{\Delta V}{\Delta r} = 8.3\times 10^4\ \mathrm{V\ m^{-1}}$$ □

Question 6.10 Show that the charged object in Example 6.13 behaves like a point charge.

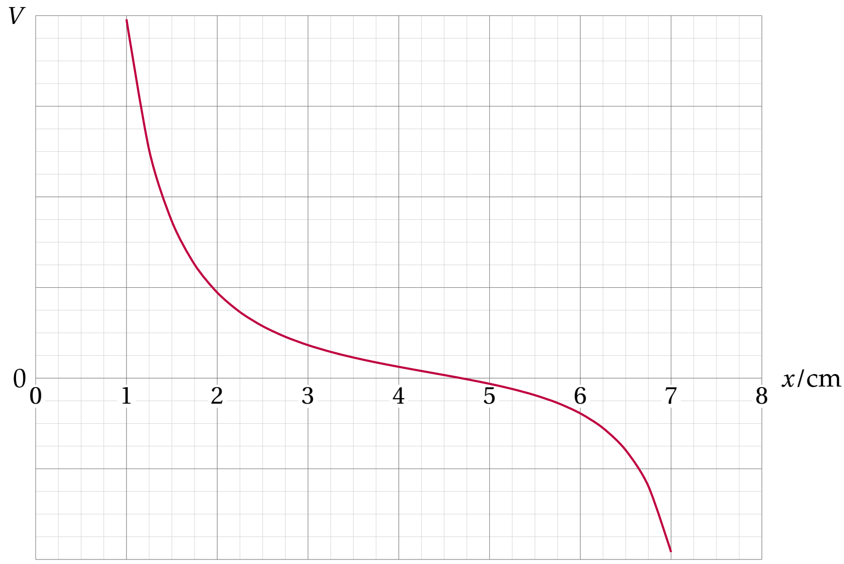
[37]The expression is implicitly integrated from initial position to final position.

[38]These equations are correct if the charge is moving in the parallel direction to the field, i.e., the motion is along the field lines. But an object can move in all directions in the field. More rigorously, if we take the vector nature of electric field into account, we should write $\Delta V = -\int \mathbf{E}\cdot\mathrm{d}\mathbf{r}$, and $\mathbf{E} = -\dfrac{\partial V}{\partial \mathbf{r}}$. (⋆)

[39]This result can be generalised to a very important principle of physical laws called the *least action principle*. It states that any motion of a system tends to minimise the action, a physical quantity related to the energy of the system. This fundamental law plays a crucial role in the study of theoretical physics. (⋆)

Determine the charge it carries, and hence calculate the field strength at $r = 5.0$ cm.

Example 6.14 The variation of electrical potential along a certain line is shown. State and explain where in the field an electron will experience the greatest force.



✍ greatest force means greatest field strength, which means maximum potential gradient

largest gradient of V-x curve at $x = 1$ cm, so greatest force at $x = 1$ cm □

Example 6.15 electric field due to an isolated point charge

we have learned that the electric potential due to a point charge is: $V = \frac{Q}{4\pi\epsilon_0 r}$

using $E = -\frac{\mathrm{d}V}{\mathrm{d}r}$, we have $E = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{Q}{4\pi\epsilon_0 r}\right) = -\frac{Q}{4\pi\epsilon_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = \frac{Q}{4\pi\epsilon_0 r^2}$

this agrees with the expression for field strength due to an isolated charge □

Question 6.11 For the electric field due to a charged metal sphere (see Example 6.12), convince yourself that the field strength equals negative gradient of potential at any point.

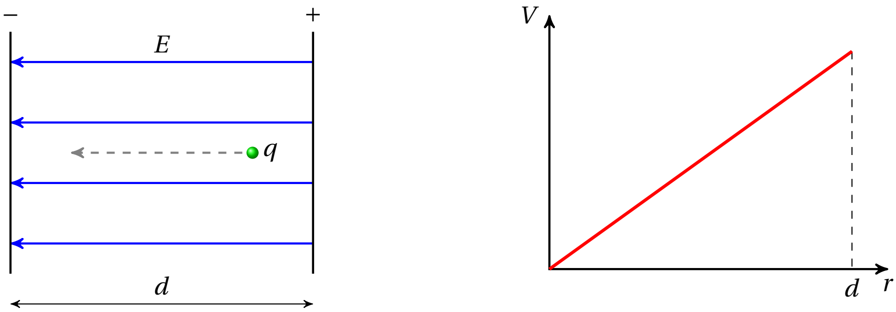
Question 6.12 We have seen the statement field strength equals negative potential gradient holds for electric fields. Does it also hold for gravitational fields?

uniform fields revisited

given two oppositely-charged metal plates separated by a distance of d

if p.d. between the plates is V, then electric field strength between is given by $E = \frac{V}{d}$[40]

we will derive this result using the theorem introduced in the last section



moving a test charge in a uniform electric field

moving a test charge q in a uniform field, work done by electric force: $W = Fd = Eqd$

change in P.E.: $\Delta E_p = -W = -Eqd$

change in potential: $\Delta V = \frac{\Delta E_p}{q} = -Ed$, or $E = -\frac{\Delta V}{d}$

plotting V-r graph, gradient of line $= -E$

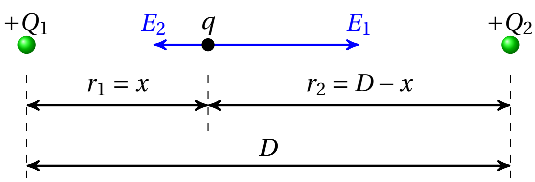
the minus sign means field strength points in the direction such that potential decreases

i.e., electric field acts from high potential to low potential

electric field due to two positive point charges

two point charges $+Q_1$, $+Q_2$ are separated by a distance of D

let's look into the electric field along the segment joining the two charges



combined potential: $V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x} + \frac{Q_2}{D-x}\right)$

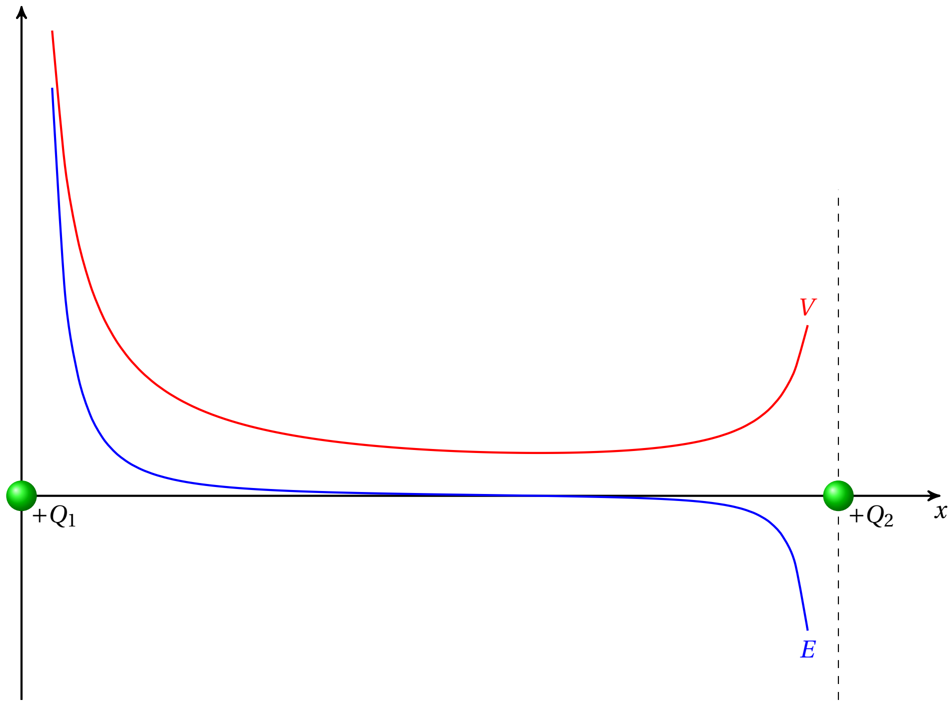
combined field strength: $E = E_1 - E_2 = \frac{Q_1}{4\pi\epsilon_0 r_1^2} - \frac{Q_2}{4\pi\epsilon_0 r_2^2} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x^2} - \frac{Q_2}{(D-x)^2}\right)$

[40]You should have learned this in AS-level physics.

notice that when computing V, we carry out *scalar sum*

but for E, we carry out *vector sum*, i.e., directions of E_1 and E_2 become important

V-x graph and E-x graph for the case where $Q_1 = 3Q_2$ are sketched



Question 6.13 Convince yourself that field strength is indeed given by negative potential gradient. You may interpret it either graphically (think about gradient of tangent along the curve) or algebraically (think about the derivative of V).

Question 6.14 We have looked into the electric field between two positively-charged particles. Discuss the cases where (a) both particles are negatively charged, (b) the two particles carry opposite charges.

Chapter 7
Capacitors

7.1 capacitors: an introduction

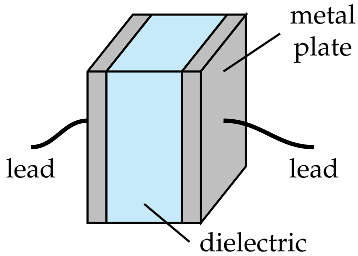
7.1.1 capacitors

capacitors are elementary electrical units widely used in electrical and electronic engineering

a typical capacitor has two conductive plates

between the plates there is usually an insulating material called *dielectric*

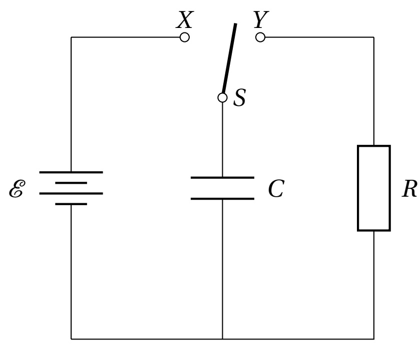
circuit symbol for a capacitor is ⊣⊢



if we construct an electric circuit as shown

when contact S is moved to X, capacitor is connected to a voltage supply and becomes charged

positive and negative charges are separated onto two plates, and they will stay where they are even if we disconnect the capacitor from the voltage supply



if we then move S to Y, the charged capacitor discharges and drives a current through resistor R, i.e., it can act as a temporary power source[41]

so capacitors can be used to store and release energy[42]

[41]Details on charging and discharging processes will be gone through in §7.4.

[42]Other important functions of capacitors in electronic circuits include smoothing output voltage of power supplies, blocking direct current while allowing alternating current to pass, etc.

7.1.2 mutual capacitance

to describe ability of a capacitor to store charges, we define the notion of capacitance

(mutual) capacitance of a parallel-plate capacitor is defined as the ratio of the charge stored on one plate to the potential difference across the two plates.

in a word equation: mutual capacitance $C = \dfrac{\text{charge on one plate } Q}{\text{p.d. } V \text{ across the plates}}$ $\Rightarrow$ $\boxed{C = \dfrac{Q}{V}}$

- unit of capacitance: **farad** [43] : $[C] = \mathrm{F}$

  farad a derived unit: $1\ \mathrm{F} = 1\ \mathrm{C \cdot V^{-1}}$

  farad is a large unit, more common subunits of capacitance in use are sub-multiples of farad:

  $1\ \mu\mathrm{F} = 10^{-6}\ \mathrm{F}, \quad 1\ \mathrm{nF} = 10^{-9}\ \mathrm{F}, \quad 1\ \mathrm{pF} = 10^{-12}\ \mathrm{F}$

- for a parallel-plate capacitor, charges on the two plates are equal but opposite

  net charge on the capacitor: $Q_{\text{net}} = (+Q) + (-Q) = 0$

  so we should emphasise on the notion of charge on *one* plate in the definition

- capacitance depends on *geometry* of the device and permittivity of the dielectric material

  capacitance does not depend on electric field or potential[44]

  for example, capacitance between two metal plates is: $C = \dfrac{\epsilon_0 A}{d}$ [45]

  A is area of plate, d is distance between plates, both geometrical quantities

7.1.3 self-capacitance

there are two closely related notions of capacitance: *mutual* capacitance and *self* capacitance

[43]The unit is named after Michael Faraday, a British physicist who developed the concept of capacitance. Faraday's other main discoveries include electromagnetic induction and electrolysis. He established the basis for the concept of the electromagnetic field in physics.

[44]Recall the resistance of an electrical component. Resistance is defined as the ratio of p.d. to current, but the value of resistance is essentially dependent on the length, cross-sectional area and material of the component, instead of the p.d. applied or the current flowing through it.

[45]If there is *dielectric* in between, the formula should be rewritten as $C = \dfrac{\epsilon A}{d}$, where ϵ is permittivity of dielectric. These formulae are not examinable by the syllabus.

the definition for capacitance given in the previous section, is actually *mutual* capacitance[46]

on the other hand, all bodies are able to store electrical charge

any object that can be electrically charged exhibits capacitance

we define **self-capacitance** of an object as the amount of charge that must be added to increase per unit electrical potential

in a word equation, self capacitance $C = \dfrac{\text{charge of object } Q}{\text{electric potential of object } V} \quad \Rightarrow \quad \boxed{C = \dfrac{Q}{V}}$ [47]

Example 7.1 Self-capacitance of a charged metal sphere in a vacuum

consider a metal sphere of radius R and carries an electric charge of Q

its electric potential: $V = \dfrac{Q}{4\pi\epsilon_0 R}$

self-capacitance of the sphere: $C_{\text{sphere}} = \dfrac{Q}{V} = Q \times \dfrac{4\pi\epsilon_0 R}{Q} \quad \Rightarrow \quad \boxed{C_{\text{sphere}} = 4\pi\epsilon_0 R}$

note that capacitance is only dependent on its geometrical property (radius R) □

Example 7.2 A conducting sphere of radius 1.0 m is situated in free space. (a) Find its capacitance. (b) In order to raise its potential to 5000 V, find the amount of charge needed.

✍ capacitance of the sphere: $C = 4\pi\epsilon_0 R = 4\pi \times 8.85 \times 10^{-12} \times 1.0 \approx 1.11 \times 10^{-10}$ F

(here you can see farad being an impractically huge unit)

charge on sphere: $Q = CV = 1.11 \times 10^{-10} \times 5000 \approx 5.56 \times 10^{-7}$ C □

analogy with ideal gases

an interesting analogy can be made between capacitors and ideal gases

recall an ideal gas is described by equation $pV = nRT$ [48]

$$\text{compare} \begin{cases} \text{amount of charge:} & Q = CV \\ \text{amount of substance:} & n = \left(\dfrac{V}{RT}\right) p \end{cases}$$

volume V of a container has a certain space *capacity*, at fixed T, pumping more gas (increase n) into system, pressure p increases

[46]In many cases, the term capactiance is a shorthand for mutual capacitance.

[47]For either mutual capacitance or self capacitance, defining equation $C = \dfrac{Q}{V}$ takes the same form, but you should keep in mind that Q and V represent different things in different contextsãĂĆ

[48]Don't confuse voltage V with volume V!

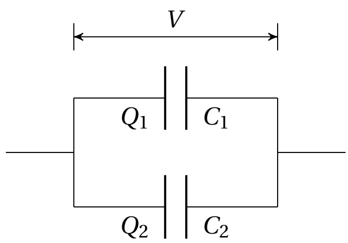
similarly, a capacitor has a certain charge capacity, adding more charge Q increases p.d. V, so the quantity C is naturally called *capacitance*

also for a container, there exists a maximum pressure which it can withstand

for a capacitor, there exists a *breakdown voltage*, or *withstand voltage*, beyond which there could be sparkling across the capacitor

7.2 capacitor networks

7.2.1 capacitors in parallel



consider two capacitors connected in parallel

same p.d. V across the network: $V = V_1 = V_2$

but charge Q is shared: $Q_{\text{total}} = Q_1 + Q_2$

$$\frac{Q_{\text{total}}}{V} = \frac{Q_1}{V} + \frac{Q_2}{V}$$

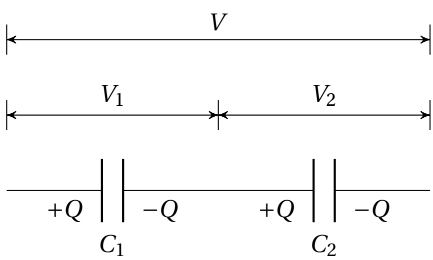
$$C_{\text{total}} = C_1 + C_2$$

if three of more capacitors in parallel: $\boxed{C_{\text{total}} = C_1 + C_2 + C_3 + \cdots}$ $\boxed{Q_{\text{total}} = Q_1 + Q_2 + Q_3 + \cdots}$

➢ adding extra capacitor in parallel to a network, total capacitance will increase

explanation: when several capacitors connected in parallel, equivalent to a single capacitor with larger plates, so more charge on the plates $\Rightarrow C \nearrow$

7.2.2 capacitors in series



consider next two series capacitors

same charge Q on each plate: $Q = Q_1 = Q_2$[49]

p.d. is shared: $V_{\text{total}} = V_1 + V_2$

$$\frac{V_{\text{total}}}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q}$$

$$\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

[49]Initially, the H-shaped isolated section between capacitors is uncharged. But no charge can enter or leave this section, its net charge must remain zero. Since a capacitor carries equal and opposite charges on its plates, so every capacitor in series carries same charge Q

if three of more capacitors in series: $\boxed{\dfrac{1}{C_{\text{total}}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \cdots}$ $\boxed{V_{\text{total}} = V_1 + V_2 + V_3 + \cdots}$

➢ adding extra capacitor in series to a network, total capacitance will decrease

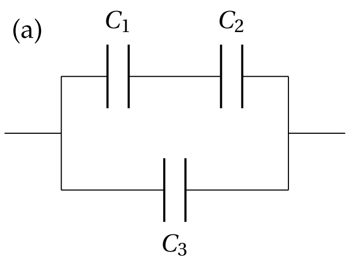
explanation: when several capacitors connected in series, equivalent to a parallel-plate capacitor with greater separation, so more charge on the plates $\Rightarrow C \searrow$

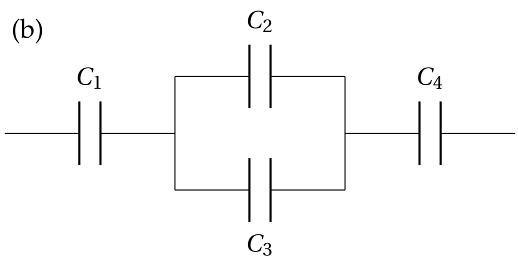
7.2.3 capacitor networks

more complicated capacitor networks can be considered as a combination of some smaller networks with capacitors in parallel or in series

(recall how you work out the *effective resistance* of a resistance network before)

Example 7.3 $C_1 = C_2 = C_3 = C_4 = 10\ \mu\text{F}$, calculate the capacitance of the network (a) and (b).





(a) $C_{12} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{10}\right)^{-1} = 5.0\ \mu\text{F}$

$C_{\text{total}} = C_{12} + C_3 = 5 + 10 = 15\ \mu\text{F}$

(b) $C_{23} = C_2 + C_3 = 10 + 10 = 20\ \mu\text{F}$

$C_{\text{total}} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_{23}} + \dfrac{1}{C_4}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{20} + \dfrac{1}{10}\right)^{-1} = 4.0\ \mu\text{F}$ □

Example 7.4 $C_1 = C_2 = C_3 = C_4 = 10\ \mu\text{F}$, calculate the capacitance of the network (a) and (b).

(a) $C_{12} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{10}\right)^{-1} = 5.0\ \mu\text{F}$

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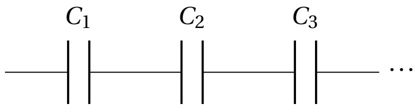
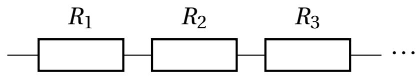
(b) $C_{23} = C_2 + C_3 = 10 + 10 = 20\ \mu\text{F}$

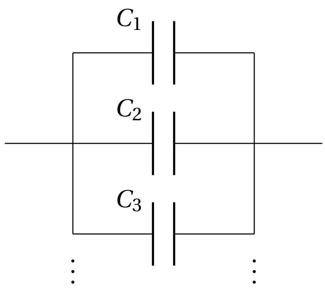
$C_{\text{total}} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_{23}} + \dfrac{1}{C_4}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{20} + \dfrac{1}{10}\right)^{-1} = 4.0\ \mu\text{F}$ □

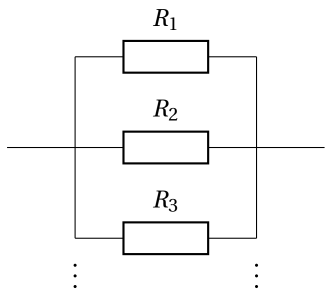
7.2.4 capacitors & resistors

- comparing capacitor networks and resistor networks

<table>
<tr><th></th><th>capacitors</th><th>resistors</th></tr>
<tr><td>in series</td><td>same charge


$\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$</td><td>same current


$R_{\text{total}} = R_1 + R_2 + R_3 + \cdots$</td></tr>
<tr><td>in parallel</td><td>same p.d. across


$C_{\text{total}} = C_1 + C_2 + C_3 + \cdots$</td><td>same p.d. across


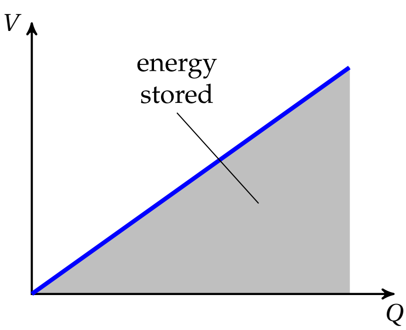
$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$</td></tr>
</table>

7.3 energy stored in a capacitor

to charge a capacitor, need to push electrons off one plate and onto the other

separation of positive and negative charges requires work done

this is how (electrical) energy is stored in capacitors



can use the graph of p.d. V against charge Q to find work done W

area under V-Q graph is equal to work done W

(recall how one can calculate displacement from a v-t graph, or work done from a F-s graph)

$$W = \frac{1}{2}QV \xlongequal{Q=CV} \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

work done in charging up a capacitor $\longrightarrow$ energy stored energy stored in a capacitor: $\boxed{W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}}$

- comment 1. this energy is an *electric potential energy*
- comment 2. this energy is stored within the *electric fields* between metal plates of capacitor

Example 7.5 One capacitor with capacitance C_0 is charged to a p.d. V_0. It is disconnected from the power supply, and then connected across an identical capacitor. Discuss the change in p.d., and change in energy stored in the system.

✍ initial charge $Q = C_0 V_0$, initial energy stored $W_0 = \frac{1}{2}CV_0^2$

combined capacitance: $C = C_0 + C_0 = 2C_0$

charge is conserved, so final p.d. across: $V = \frac{Q}{C} = \frac{C_0 V_0}{2C_0} = \frac{1}{2}V_0$

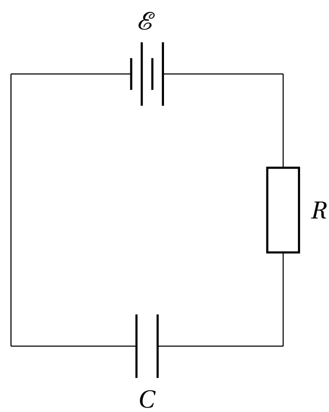
charge shared between capacitors, original one losing half its charge and p.d.

final energy stored in system: $W = \frac{1}{2}CV^2 = \frac{1}{2} \times 2C_0 \times (\frac{1}{2}V_0)^2 = \frac{1}{4}C_0 V_0^2 \quad \Rightarrow \quad W = \frac{1}{2}W_0$

half of stored energy is lost as heat when electrons flow between two capacitors □

7.4 charging & discharging capacitors (⋆)[50]

7.4.1 charging phase



initial state: no charge in capacitor: $Q(0) = 0, V_C(0) = 0$

at any instant: $\mathrm{d}Q = C\mathrm{d}V_C$

current in circuit: $I = \frac{V_R}{R} = \frac{\mathscr{E} - V_C}{R}$

change of charge: $\mathrm{d}Q = I\mathrm{d}t = \frac{\mathscr{E} - V_C}{R}\mathrm{d}t = C\mathrm{d}V_C$

$$\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}V_C}{\mathscr{E} - V_C}$$

$$\int_0^t \frac{\mathrm{d}t}{RC} = \int_0^{V_C} \frac{\mathrm{d}V_C}{\mathscr{E} - V_C}$$

$$\left.\frac{t}{RC}\right|_0^t = -\ln(\mathscr{E} - V_C)\Big|_0^{V_C}$$

simplify everything, we get: $\boxed{V_C = \mathscr{E}\left(1 - \mathrm{e}^{-t/RC}\right)}$

➢ p.d. of capacitor increases at a decreasing rate when it is being charged

[50]This section is not required by the CIE A-Level exams. However, the contents introduced here may appear in the A-Level syllabus of other examination board.

as electric charges are separated onto the two plates, pushing more $+Q$ ($-Q$) onto +ve (−ve) plate requires more work done to overcome the repulsion $\Rightarrow$ increase in p.d. slows down

➢ p.d of capacitor eventually tends to the battery e.m.f.

charge will continue to flow if there exists a potential difference

when $V_C = \mathscr{E}$, no charge flow, hence charging current gradually drops to zero

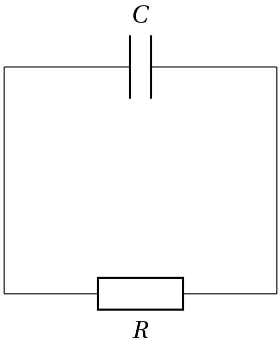
7.4.2 discharging phase

capacitor initially charged with $Q(0) = Q, V_C(0) = V_0$

at any instant, $\mathrm{d}Q = -C\mathrm{d}V_C$

minus sign because charge decreases during discharging

also $V_C = V_R$ because in parallel



charge change: $\mathrm{d}Q = I\mathrm{d}t = \dfrac{V_C}{R}\mathrm{d}t = -C\mathrm{d}V_C$

$$-\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}V_C}{V_C}$$

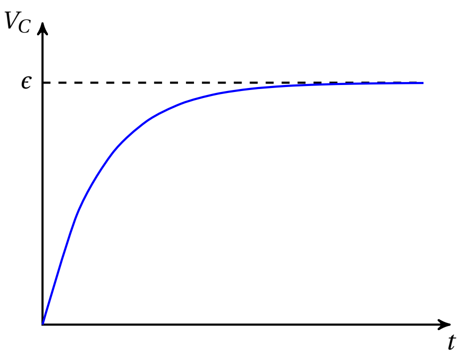
$$-\int_0^t \frac{\mathrm{d}t}{RC} = \int_{V_0}^{V_C} \frac{\mathrm{d}V_C}{V_C}$$

$$-\frac{t}{RC}\bigg|_0^t = \ln(V_C)\bigg|_{V_0}^{V_C}$$

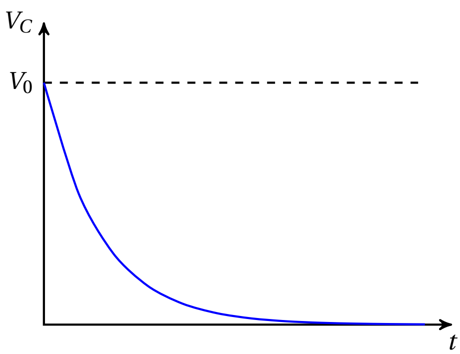
simplify everything, we get: $\boxed{V_C = V_0 \mathrm{e}^{-t/RC}}$

➢ p.d. of capacitor gradually drops to zero during discharging

➢ discharging current also gradually approaches zero



charging phase of capacitor



discharging phase of capacitor

7.4.3 time constant

RC is called *time constant*, which determines charging and discharging rate of a capacitor

$R \nearrow \Rightarrow$ smaller charging/discharging current $\Rightarrow$ takes longer to charge/discharge

$C \nearrow \Rightarrow$ more charge to be charged/discharged $\Rightarrow$ takes longer

charging and discharging of capacitors is never instant, always exists a certain time delay

time delay for common RC circuits is usually small, but the delay could hinder further increasing of speed in integrated circuits

Index