AS & A-Level Physics

Lecture Notes

(Draft)

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Contents

CHAPTER 1

Circular Motion

1.1 angular quantities

movement or rotation of an object along a circular path is called **circular motion**

to describe a circular motion, we can use *angular quantities*, which turn out to be more useful than linear displacement , linear velocity , etc.

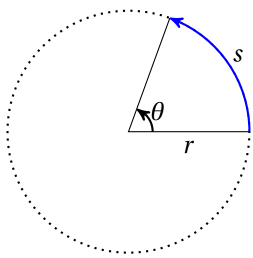
1.1.1 angular displacement

angular displacement is angle swiped out by object moving along circular

- unit: $[\theta] =$ rad (natural unit of measurement for angles)

  conversion rule: 2π rad $= 360°$
- if two radii form an angle of θ, then length of arc: $s = r\theta$

  two radii subtending an arc of same length as radius form an angle of one **radian**



1.1.2 angular velocity

angular velocity describes how fast an object moves along a circular path

angular velocity is defined as angular displacement swiped out per unit time: $\boxed{\omega = \frac{\Delta\theta}{\Delta t}}$

- unit of: $[\omega] =$ rad s^{-1}, also in radian measures
- angular velocity is a *vector* quantity

  this vector points in a direction normal to the plane of circular motion but in A-level course, we treat angular velocity as if it is a scalar

angular velocity and angular speed may be considered to be the same idea

➢ relation with linear velocity

in interval Δt, distance moved along arc $\Delta s = v\Delta t = r\Delta\theta \quad \Rightarrow \quad \omega = \dfrac{\Delta\theta}{\Delta t} = \dfrac{v}{r} \quad \Rightarrow \quad \boxed{v = \omega r}$

this relation between linear speed and angular speed holds at any instant

1.1.3 uniform circular motion

when studying linear motion, we started from motion with constant velocity v

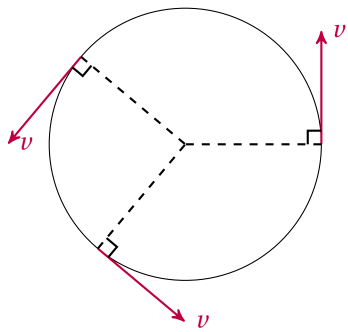
consider the simplest possible circular motion $\longrightarrow$ circular motion with constant ω

analogy with linear motion with constant v

uniform linear motion: $s = vt$

displacement $s \leftrightarrow \theta$, velocity $v \leftrightarrow \omega$

for uniform circular motion, one has: $\boxed{\theta = \omega t}$



➢ time taken for one complete revolution is called **period** T

in one T, angle swiped is 2π, so $\boxed{\omega = \dfrac{2\pi}{T}}$

➢ uniform circular motion is still *accelerated* motion

speed is unchanged, but *velocity* is changing

direction of velocity always *tangential* to its path, so direction of velocity keeps changing

in general, any object moving along circular path is accelerating

Example 1.1 An object undergoes a uniform motion around a circular track of radius 2.5 m in 40 s, what is its angular speed and linear speed?

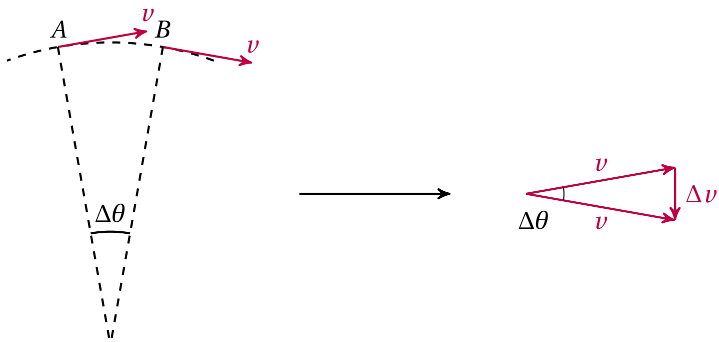
✍ $$\omega = \frac{2\pi}{T} = \frac{2\pi}{40} \approx 0.157 \text{ rad s}^{-1} \qquad v = \omega r = 0.157 \times 2.5 \approx 0.39 \text{ m s}^{-1}$$ □

1.1.4 centripetal acceleration

> **centripetal acceleration** is the acceleration due to the change in direction of velocity vector, it points toward the centre of circular path

consider motion along a circular path from A to B with constant speed v

under small (infinitesimal) duration of time Δt[1]



change in velocity: $\Delta v = 2v\sin\frac{\Delta\theta}{2} \approx v\Delta\theta$ (as $\Delta\theta \to 0$, $\sin\Delta\theta \approx \Delta\theta$)

acceleration: $a = \frac{\Delta v}{\Delta t} \approx v\frac{\Delta\theta}{\Delta t} = v\omega$ (as $\omega = \frac{\Delta\theta}{\Delta t}$)

recall relation $v = \omega r$, we find centripetal acceleration: $\boxed{a_c = \frac{v^2}{r} = \omega^2 r}$

- direction of centripetal acceleration: always towards centre of circular path
- centripetal acceleration is only responsible for the change in *direction* of velocity

  change in *magnitude* of velocity will give rise to *tangential acceleration*

  this is related to *angular acceleration*[2] , which is beyond the syllabus

1.2 centripetal force

circular motion must involve change in velocity, so object is not in equilibrium

there must be a *net force* on an object performing circular motion

> **centripetal force** (F_c) is the resultant force acting on an object moving along a circular path, and it is always directed towards centre of the circle

- centripetal force causes centripetal acceleration

  using Newton's 2nd law: $\boxed{F_c = m\frac{v^2}{r} = m\omega^2 r}$
- F_c is not a new force by nature

[1] A more rigorous derivation can be given by using differentiation techniques

[2] Angular acceleration is analogous to linear acceleration α, defined as rate of change of angular velocity: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ ($\star$). Similar to $v = \omega r = \frac{ds}{dt}$, the relation $a = \alpha r = \frac{dv}{dt}$ also holds.

F_c is a resultant of forces you learned before (weight, tension, contact force, friction, etc.)

➢ F_c acts at right angle to direction of velocity

or equivalently, if $F_{\text{net}} \perp v$, then this net force provides centripetal force for circular motion

➢ effect of F_c: change *direction* of motion, or maintain circular orbits

to change *magnitude* of velocity, there requires a *tangential* component for the net force

again the idea of tangential force is beyond the syllabus

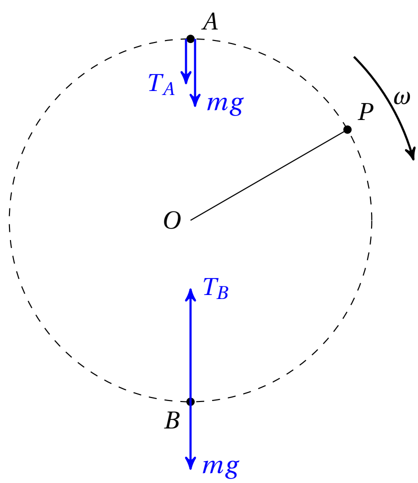
Example 1.2 A rock is able to orbit around the earth near the surface of the earth. Given that radius of the earth $R = 6400$ km and air resistance is ignored, (a) what is its orbital speed? (b) What is the orbital period?

✍ weight of object provides centripetal force: $mg = \frac{mv^2}{R}$

orbital speed: $v = \sqrt{gR} = \sqrt{9.81 \times 6.4 \times 10^6} \approx 7.9 \times 10^3$ m s^{-1}

period: $T = \frac{2\pi R}{v} = \frac{2\pi \times 6.4 \times 10^6}{7.9 \times 10^3} \approx 5.1 \times 10^3$ s ≈ 85 min □

Example 1.3 Particle P of mass $m = 0.40$ kg is attached to one end of a light inextensible string of length $r = 0.80$ m. The particle is whirled at a constant angular speed ω in a vertical plane. (a) Given that the string never becomes slack, find the minimum value of ω. (b) Given instead that the string will break if the tension is greater than 20 N, find the maximum value of ω.



✍ at top of circle (point A): $F_c = T_A + mg = m\omega^2 r \quad \Rightarrow \quad T_A = m\omega^2 r - mg$

at bottom of circle (point B): $F_c = T_B - mg = m\omega^2 r \quad \Rightarrow \quad T_B = m\omega^2 r + mg$

tension is minimum at A, but string being taut requires $T \geq 0$ at any point, so $T_A \geq 0$

$$m\omega^2 r - mg \geq 0 \quad \Rightarrow \quad \omega^2 \geq \frac{g}{r}$$
$$\omega_{\min} = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.81}{0.80}} \approx 3.5 \text{ rad s}^{-1}$$

tension is maximum at B, but string does not break requires $T \leq T_{\max}$, so $T_B \leq T_{\max}$

$$m\omega^2 r + mg \leq T_{\max} \quad \Rightarrow \quad \omega^2 \leq \frac{T_{\max}}{m} - \frac{g}{r}$$
$$\omega_{\max} = \sqrt{\frac{T_{\max}}{m} - \frac{g}{r}} = \sqrt{\frac{20}{0.40} - \frac{9.81}{0.80}} \approx 6.1 \text{ rad s}^{-1} \qquad \square$$

Example 1.4 A turntable can rotate freely about a vertical axis through its centre. A small object is placed on the turntable at distance $d = 40$ cm from the centre. The turntable is then set to rotate, and the angular speed of rotation is slowly increased. The coefficient of friction between the object and the turntable is $\mu = 0.30$. If the object does not slide off the turntable, find the maximum number of revolutions per minute.

✍ if object stays on turntable, friction provides the centripetal force required: $f = m\omega^2 d$

increasing ω requires greater friction to provide centripetal force

but maximum limiting friction possible is: $f_{\lim} = \mu N = \mu mg$, therefore

$$f \leq f_{\lim} \quad \Rightarrow \quad m\omega^2 d \leq \mu mg \quad \Rightarrow \quad \omega^2 \leq \frac{\mu g}{d} \quad \Rightarrow \quad \omega_{\max} = \sqrt{\frac{0.30 \times 9.81}{0.40}} \approx 2.71 \text{ rad s}^{-1}$$

period of revolution: $T_{\min} = \dfrac{2\pi}{\omega_{\max}} = \dfrac{2\pi}{2.71} \approx 2.32$ s

number of revolutions in one minute: $n_{\max} = \dfrac{t}{T_{\min}} = \dfrac{60}{2.32} \approx 25.9$ $\square$

Example 1.5 A pendulum bob of mass 120 g moves at constant speed and traces out a circle or radius $r = 10$ cm in a horizontal plane. The string makes an angle $\theta = 25^\circ$ to the vertical. (a) What is the tension in the string? (b) At what speed is the bob moving?

✍ vertical component of tension T_y equals weight

$$T_y = mg \quad \Rightarrow \quad T\cos\theta = mg$$
$$T = \frac{mg}{\cos\theta} = \frac{0.12 \times 9.81}{\cos 25^\circ} \approx 1.3 \text{ N}$$

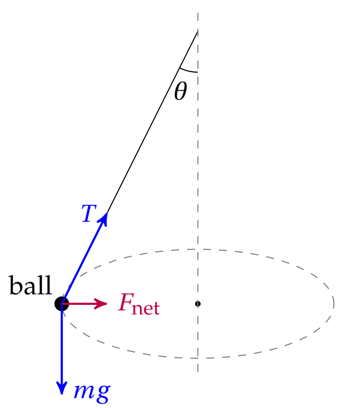
net force equals horizontal component of tension T_x

so component T_x provides centripetal force

$$F_c = T_x \quad \Rightarrow \quad T\sin\theta = \frac{mv^2}{r}$$

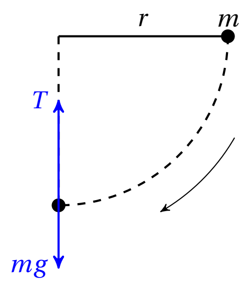
by eliminating T and m, one can find

$$v^2 = \frac{r\tan\theta}{g} = \frac{0.10 \times \tan 25^\circ}{9.81} \quad \Rightarrow \quad v \approx 0.069 \text{ m s}^{-1} \qquad \square$$



Example 1.6 A small ball of mass m is attached to an inextensible string of length l. The ball is held with the string taut and horizontal and is then released from rest.

When the ball reaches lowest point, find its speed and the tension in the string in terms of m and l.



✍ energy conservation: G.P.E. loss = K.E. gain

$$mgr = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gr}$$

at lowest point: $F_c = T - mg = m\dfrac{v^2}{r}$

$$T = mg + m\frac{v^2}{r} = mg + m\frac{2gr}{r} = 3mg \qquad \square$$

Question 1.1 A turntable that can rotate freely in a horizontal plane is covered by dry mud. When the angular speed of rotation is gradually increased, state and explain whether the mud near edge of the plate or near the mud will first leave the plate?

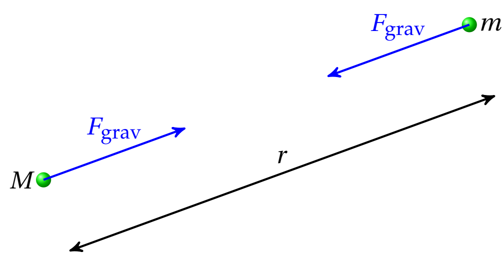
Question 1.2 A bucket of water is swung at a constant speed and the motion describes a circle of radius $r = 1.0$m in the vertical plane. If the water does not pour down from the bucket even when it is at the highest position, how fast do you need to swing the bucket?

Chapter 2
Gravitational Fields

2.1 gravitational forces

2.1.1 Newton's law of gravitation

any object attracts any other object through the gravitational force



gravitational attraction between M and m

Newton's law of gravitation states that gravitational force between two *point* masses is proportional to the product of their masses and inversely proportional to the square of their distance $\left(F_{\text{grav}} \propto \dfrac{Mm}{r^2}\right)$

this law was formulated in *Issac Newton*'s work 'The Principia', or 'Mathematical Principles of Natural Philosophy', first published in 1687

mathematically, gravitational force takes the form: $\boxed{F_{\text{grav}} = \dfrac{GMm}{r^2}}$

$G = 6.67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}$ is the *gravitational constant*

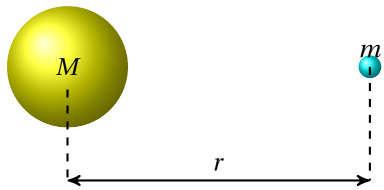
- gravitational force is always *attractive*
- gravity is *universal*, i.e., gravitational attraction acts between *any* two masses
- Newton's law of gravitation refers to *point masses*

i.e., particles with no size, therefore distance r can be easily defined

➢ a sphere with uniform mass distribution (e.g., stars, planets) can be treated as a *point model*

distance r is taken between centres of the spheres [3]

(see Example 2.8, field lines around a planet *seem* to point towards centre of planet)

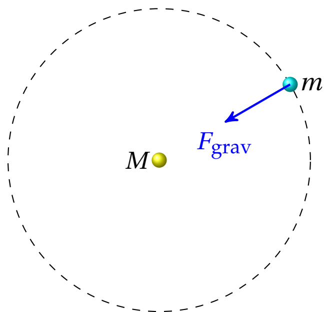


Example 2.1 The Earth can be thought as a uniform sphere of radius $R = 6.4 \times 10^6$ m and mass $M = 6.0 \times 10^{24}$ kg. Estimate the gravitational force on a man of 60 kg at sea level.

✍ $$F = \frac{GMm}{R^2} = \frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 60}{(6.4 \times 10^6)^2} \approx 586 \text{ N}$$ □

Question 2.1 Estimate the gravitational force between you and your deskmate.

2.1.2 planetary motion



a planet/satellite orbiting around a star/earth

a planet/satellite can move around a star/earth in circular orbit

circular motion requires centripetal force

for these objects, gravitational force provides centripetal force

$$F_{\text{grav}} = F_c \quad \Rightarrow \quad \boxed{\frac{GMm}{r^2} = \frac{mv^2}{r}} \quad \text{or} \quad \boxed{\frac{GMm}{r^2} = m\omega^2 r}$$

[3]This is known as *shell theorem*: a spherically symmetric shell (i.e., a hollow ball) affects external objects gravitationally as though all of its mass were concentrated at its centre, and it exerts no net gravitational force on any object inside, regardless of the object's location within the shell. (⋆)

Example 2.2 GPS (Global Positioning System) satellites move in a circular orbits at about 20000 km above the earth's surface. The Earth has a radius $R = 6.4 \times 10^6$ m and mass $M = 6.0 \times 10^{24}$ kg. (a) Find the speed of GPS satellites. (b) Find its orbital period.

✍ $$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6 + 2.0 \times 10^7}} \approx 3.9 \times 10^3 \text{ m s}^{-1}$$

$$v = \frac{2\pi r}{T} \quad \Rightarrow \quad T = \frac{2\pi r}{v} = \frac{2\pi \times (6.4 \times 10^6 + 2.0 \times 10^7)}{3.9 \times 10^3} \approx 4.3 \times 10^4 \text{ s} \approx 11.8 \text{ hours}$$ □

Example 2.3 A **geostationary satellite** moves in a circular orbit that appears motionless to ground observers. The satellite follows the Earth's rotation, so the satellite rotates from west to east above equator with an orbital period of 24 hours. Find the radius of this orbit.

✍ $$\frac{GMm}{r^2} = m\omega^2 r \quad \Rightarrow \quad \frac{GMm}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r \quad \Rightarrow \quad r^3 = \frac{GMT^2}{4\pi^2}$$

$$r = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = \left(\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times (24 \times 3600)^2}{4\pi^2}\right)^{1/3} \approx 4.23 \times 10^7 \text{ m}$$ □

Example 2.4 Assuming the planets in the solar system all move around the sun in circular orbits, show that the square of orbital period is proportional to the cube of orbital radius. [4]

✍ $$\frac{GMm}{r^2} = m\omega^2 r \quad \Rightarrow \quad \frac{GMm}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM} \cdot r^3$$

G is gravitational constant, M is mass of the sun, so $\dfrac{4\pi^2}{GM}$ is a constant, so $T^2 \propto r^3$ □

Question 2.2 Given that it takes about 8.0 minutes for light to travel from the sun to the earth. (a) What is the mass of the sun? (b) At what speed does the earth move around the sun?

2.1.3 apparent weight

an object's *actual weight* is the gravitational attraction exerted by the earth's gravity

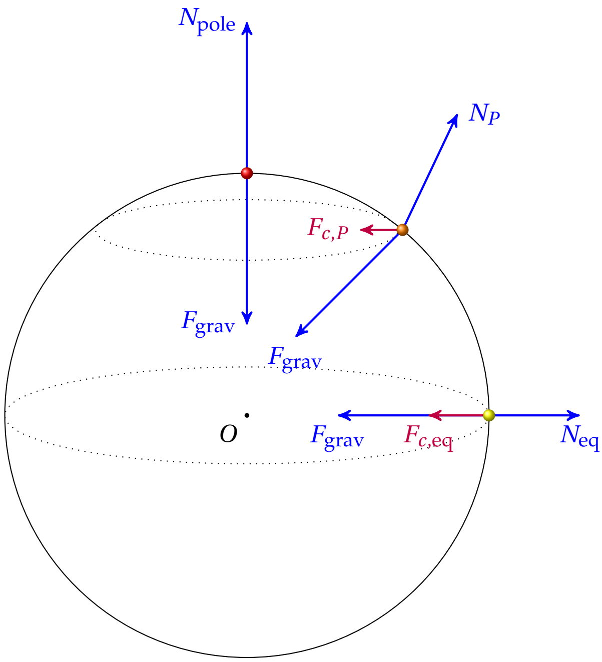
an object's *apparent weight* is the upward force (e.g., normal contact force exerted by ground, tension in a spring balance, etc.) that opposes gravity and prevents the object from falling

apparent weight can be different from actual weight due to vertical acceleration or buoyancy

but if we consider rotation of the earth, this also causes apparent weight to be lessened

[4] This is known as *Kepler's 3rd law* for planetary motions. In the early 17th century, German astronomer Johannes Kepler discovered three scientific laws which describes how planets move around the sun. This $T^2 \propto r^3$ relation not only holds for circular orbits but are also correct for elliptical orbits.

Isaac Newton proved that Kepler's laws are consequences of his own law of universal gravitation, and therefore explained why the planets move in this way. (⋆)



apparent weight at various positions near earth's surface (not to scale)

object resting on ground is actually rotating together with earth

resultant of gravitational force and contact force should provide centripetal force

for object on equator: $F_{c,\text{eq}} = m\omega^2 R \quad \Rightarrow \quad F_{\text{grav}} - N_{\text{eq}} = m\omega^2 R \quad \Rightarrow \quad N_{\text{eq}} = \dfrac{GMm}{R^2} - m\omega^2 R$

for object at poles: $F_{c,\text{pole}} = 0 \quad \Rightarrow \quad F_{\text{grav}} - N_{\text{pole}} = 0 \quad \Rightarrow \quad N_{\text{pole}} = \dfrac{GMm}{R^2}$

at lower latitudes, object describe larger circles, hence requires greater centripetal force

this offsets the balancing normal force, so apparent weight decreases near the equator

Example 2.5 A stone of mass 5.0 kg is hung from a newton-meter near the equator. The Earth can be considered to be a uniform sphere of radius $R = 6370$ km and mass $M = 5.97 \times 10^{24}$ kg. (a) What is the gravitational force on the stone? (b) What is the reading on the meter?

✍ gravitational force: $F_{\text{grav}} = \dfrac{GMm}{R^2} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 5.0}{(6.37 \times 10^6)^2} \approx 49.07$ N

centripetal force required: $F_c = m\omega^2 R = m\left(\dfrac{2\pi}{T}\right)^2 R = 5.0 \times \dfrac{4\pi^2}{(24 \times 3600)^2} \times 6.37 \times 10^6 \approx 0.17$ N

apparent weight, or reading on meter: $N = F_{\text{grav}} - F_c = 49.07 - 0.17 \approx 48.90$ N □

Question 2.3 Why astronauts in space stations are said to be *weightless*?

Question 2.4 How do you find the apparent weight of an object at an arbitrary latitude P?

Does the apparent weight act vertically downwards? Give your reasons.

2.2 gravitational fields

to explain how objects exert gravitational attraction upon one another at a distance, we introduce the concept of *force fields*

> **gravitational field** is a region of space where a mass is acted by a force

any mass M (or several masses) can produce a gravitational field around it

a test mass m within this field will experience a gravitational force

to describe the effect on a small mass m in the field, we will further introduce

- *gravitational field strength*, to help us compute gravitational force on objects
- *gravitational potential*, to help us compute gravitational potential energy between objects

2.3 gravitational field strength

2.3.1 gravitational field strength

> **gravitational field strength** is defined as gravitational force per unit mass: $\boxed{g = \frac{F_{\text{grav}}}{m}}$

➢ unit of g: $[g] = \text{N kg}^{-1} = \text{ m s}^{-2}$, same unit as acceleration

➢ field strength due to an isolated source of mass M

at distance r from the source, a test mass m is acted by a force: $F_{\text{grav}} = \frac{GMm}{r^2}$

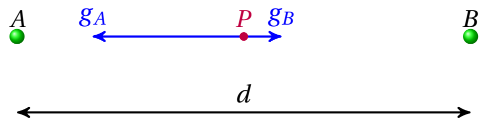
field strength at this position: $g = \frac{F_{\text{grav}}}{m} = \quad \Rightarrow \quad \boxed{g = \frac{GM}{r^2}}$

note that the field is produced by the source M, so field strength g depends on M, not m

➢ field strength g is a *vector* quantity, it has a direction

gravitation is *attractive*, so g points towards source mass to compute combined field strength due to several sources, should perform *vector sum* of contributions from each individual

Example 2.6 Star A of mass 6.0×10^{30} kg and star B of mass 1.5×10^{30} kg are separated by a distance of 2.0×10^{12} m. (a) What is the field strength at the mid-point P of the two stars? (b) If a comet of mass 4.0×10^{6} kg is at the mid-point, what force does it experience?



✍ g_A acts towards A, g_B acts towards B, they are in opposite directions

$$g_P = g_A - g_B = \frac{GM_A}{r_A^2} - \frac{GM_B}{r_B^2} = 6.67 \times 10^{-11} \times \left[\frac{6.0 \times 10^{30}}{(1.0 \times 10^{12})^2} - \frac{1.5 \times 10^{30}}{(1.0 \times 10^{12})^2}\right] \approx 3.0 \times 10^{-4}\ \mathrm{N\,kg^{-1}}$$

force on comet: $F = mg = 4.0 \times 10^{6} \times 3.0 \times 10^{-4} \approx 1.2 \times 10^{3}$ N □

2.3.2 acceleration of free fall

if field strength g is known, gravitational force on an object of mass m is: $F_{\text{grav}} = mg$

if the object is acted by gravity only, then $F_{\text{net}} = F_{\text{grav}} \quad \Rightarrow \quad ma = mg \quad \Rightarrow \quad \boxed{a = g}$ [5]

this shows gravitational field strength gives the acceleration of free fall!

Example 2.7 The earth has a radius of 6370 km. (a) Find the mass of the earth. [6] (b) Find the acceleration of free fall at the top of Mount Everest. (height of Mount Everest $H \approx 8.8$ km)

[5] Rigorously speaking, the two m's are different concepts. There is the *inertia* mass, decribing how much an object resists the change of state of motion. There is also the *gravitational* mass, describing the effect produced and experienced by the object in gravitational fields. Yet no experiment has ever demonstrated any significant difference between the two. The reason why the two masses are identical is very profound. We have shown here acceleration of free fall equals gravitational field strength, but Albert Einstein's *equivalence principle* suggests that it is actually impossible to distinguish between a uniform acceleration and a uniform gravitational field. This idea lies at the heart of the *general theory of relativity*, where I should probably stop going further.

[6] British scientist Henry Cavendish devised an experiment in 1798 to measure the gravitational force between masses in his laboratory. He was the first man to yield accurate values for the gravitational constant G. Then he was able to carry out this calculation, referred by himself as 'weighing the world'.

✍ consider acceleration of free fall near surface of earth:

$$g_s = \frac{GM}{R^2} \quad \Rightarrow \quad 9.81 = \frac{6.67\times 10^{-11}\times M}{(6.37\times 10^6)^2} \quad \Rightarrow \quad M \approx 5.97\times 10^{24}\text{ kg}$$

at top of Mount Everest:

$$g_{\text{ME}} = \frac{GM}{(R+H)^2} = \frac{6.67\times 10^{-11}\times 5.97\times 10^{24}}{(6.37\times 10^6 + 8.8\times 10^3)^2} \approx 9.78\text{ N kg}^{-1} \quad \Rightarrow \quad a_{\text{ME}} \approx 9.78\text{ m s}^{-2}$$ □

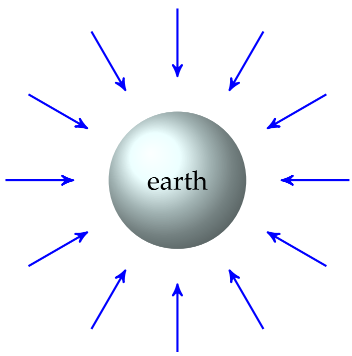
2.3.3 gravitational field lines

gravitational field lines are drawn to graphically represent the pattern of field strength

- *direction* of field lines show the *direction* of field strength in the field
- *spacing* between field lines indicates the *strength* of the gravitational field
- gravitational field lines always end up at a mass

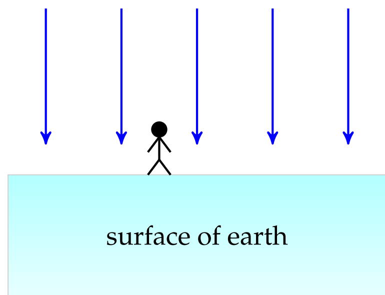
this arises from the attractive nature of gravitation

Example 2.8 field around the earth



radial field (field lines normal to surface)

Example 2.9 field near earth's surface



almost a *uniform* field

(field lines are parallel and equally spaced)

2.4 gravitational potential & potential energy

2.4.1 potential energy

potential energy is the energy possessed by an object due to its position in a force field

work done *by* force field decreases P.E., and work done *against* a force field increases P.E.

let W be work by the force field, then we have: $\boxed{W = -\Delta E_p}$

to define potential energy of an object at a specific point X, we can

(1) choose a position where potential energy is defined to be zero

(2) find work done by force field to bring the object from zero P.E. point to X

(3) consider change in P.E.: $\Delta E_p = E_{p,X} - E_{p,\text{initial}} = E_{p,X} - 0 = E_{p,X}$

but $\Delta E_p = -W$, so P.E. at point X is found: $E_{p,X} = -W$

so potential energy is equal to (negative) work done to move the object to a specific position

gravitational potential energy near earth's surface

we may choose a zero G.P.E. point, for example, $E_p(0) = 0$ at sea level

if mass m is moved up for a height h, work done by gravity is $W = -mgh$[7]

this causes a change in gravitational potential energy $\Delta E_p = -W = mgh$

then at altitude h, G.P.E. can be given by $E_p(h) = mgh$

2.4.2 gravitational potential energy

we are now ready to derive an expression for G.P.E. between two masses M and m

we define $E_p = 0$ at $r = \infty$ (choice of zero potential energy, no force so no G.P.E.), then

gravitation potential energy is equal to the work done by gravitational force to bring a mass to a specific position from *infinity*

consider a mass m at infinity with zero energy and a source mass M at origin

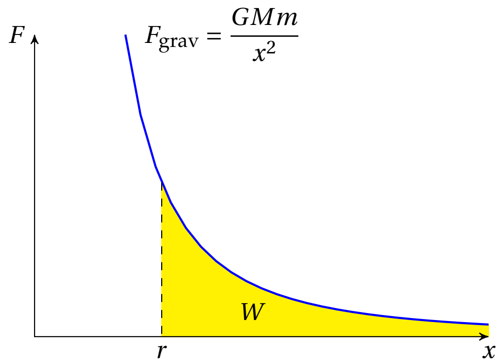
let's find out how much work is done by gravitational force to pull m towards the origin

but F_{grav} varies as inverse square of separation x

so here we need to evaluate work done by a non-constant force

we can plot a F-x graph, then magnitude of work done equals area under the graph

[7]Negative sign because this is actually work against gravity.



integrate[8] to evaluate the area: $W = \int_r^\infty \frac{GMm}{x^2} \mathrm{d}x = -\frac{GMm}{x}\Big|_r^\infty = \frac{GMm}{r}$

$\Delta E_p = -W \quad \Rightarrow \quad E_p(r) - E_p(\infty) = -\frac{GMm}{r}$

but we have defined $E_p(\infty) = 0$, therefore: $\boxed{E_p(r) = -\frac{GMm}{r}}$

$E_p(r)$ gives G.P.E between masses M and m when they are at distance r from each other

- as $r \to \infty$, $E_p \to 0$, this agrees with our choice of zero G.P.E. point
- potential energy is a *scalar* quantity (negative sign cannot be dropped)
- G.P.E. is always *negative*, this is due to *attractive* nature of gravity

  to separate masses, work must be done to overcome attraction

  so G.P.E. increases with separation r, i.e., G.P.E. is maximum at infinity, which is zero

  G.P.E. between masses at finite separation must be less than zero
- $E_p = mgh$ is only applicable near earth's surface where field is almost *uniform*

  $E_p = -\frac{GMm}{r}$ is a more *general* formula for gravitational potential energy [9]

Example 2.10 A meteor is travelling towards a planet of mass M. When it is at a distance of r_1 from centre of M, it moves at speed v_1. When it is r_2 from M, it moves at speed v_2. Assume

---

[8]In general, work done by a non-constant force over large distance is $W = \int_{\text{initial}}^{\text{final}} F \mathrm{d}x$.

For our case, x is the displacement away from the source, but gravitational force tends to pull the mass towards the source. F and x are in opposite directions, a negative sign is needed for F. Therefore the work done by gravity to bring mass m from infinity is: $W = \int_\infty^r F \mathrm{d}x = \int_\infty^r \left(-\frac{GMm}{x^2}\right) \mathrm{d}x = +\frac{GMm}{x}\Big|_\infty^r = \frac{GMm}{r}$.

[9]One can recover $\Delta E_p = mg\Delta h$ from $E_p = -\frac{GMm}{r}$. Near the earth's surface, if $r_1 \approx r_2 \approx R$, and $r_1 > r_2$, then we have: $\Delta E_p = E_p(r_1) - E_p(r_2) = -GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = GMm\frac{r_1 - r_2}{r_1 r_2} \approx m\frac{GM}{R^2}\Delta r \overset{g=GM/R^2}{=\!=\!=} mg\Delta h$.

only gravitational force applies, establish a relationship between these quantities.

✍ energy conservation: K.E. + G.P.E. = const $\Rightarrow \frac{1}{2}mv_1^2 + \left(-\frac{GMm}{r_1}\right) = \frac{1}{2}mv_2^2 + \left(-\frac{GMm}{r_2}\right)$ □

Example 2.11 If an object is thrown from the surface of a planet at sufficiently high speed, it might escape from the influence of the planet's gravitational field. The minimum speed required is called the *escape velocity*. Using the data from previous examples, find the escape velocity from the surface of earth.

✍ assuming no energy loss to air resistance, then total energy is conserved

$$\text{K.E.} + \text{G.P.E. at surface of planet} = \text{K.E.} + \text{G.P.E. at infinity}$$

$$\frac{1}{2}mu^2 + \left(-\frac{GMm}{R}\right) = \frac{1}{2}mv^2 + 0 \quad \xrightarrow{v\geq 0} \quad u^2 \geq \frac{2GM}{R} \quad \Rightarrow \quad u_{\text{min}} = \sqrt{\frac{2GM}{R}}$$

for earth, escape velocity $u_{\text{min}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6}} \approx 1.12 \times 10^4 \text{ m s}^{-1}$ □

Question 2.5 A planet of uniform density distribution is of radius R and mass M. A rock falls from a height of $3R$ above the surface of the planet. Assume the planet has no atmosphere, show that the speed of the rock when it hits the ground is $v = \sqrt{\frac{3GM}{4R}}$.

Question 2.6 A space probe is travelling around a planet of mass M in a circular orbit of radius r. (a) Show that the total mechanical energy (sum of kinetic energy and gravitational energy) of the space probe is $E_{\text{total}} = -\frac{2GMm}{r}$. (b) If the space probe is subject to small resistive forces, state the change to its orbital radius and its orbiting speed.

Question 2.7 A *black hole* is a region of spacetime where gravitation is so strong that even light can escape from it. For a star of mass M to collapse and form a black hole, it has to be compressed below a certain radius. (a) Show that this radius is given by $R_S = \frac{2GM}{c^2}$, known as the *Schwarzschild radius*[10]. (b) Show that the Schwarzschild radius for the sun is about 3 km.

[10] When you deal with very strong gravitational fields, Newton's law of gravitation breaks down and effects of Einstein's *general theory of relativity* come into play. The radius of a *Newtonian* black hole being equal to the radius of a Schwarzschild black hole is a mere coincidence.

2.4.3 gravitational potential

it is useful to introduce a quantity called *potential* at a specific point in a gravitational field

gravitational potential can be considered as the potential energy per unit mass: $\varphi = \dfrac{E_P}{m}$

gravitational potential at a point is defined as the work done to bring *unit* mass from *infinity* to that point

- unit: $[\varphi] = \mathrm{J\ kg^{-1}}$
- gravitational potential due to an isolated source M
$$\varphi = \frac{E_P}{m} = \frac{-\frac{GMm}{r}}{m} \quad \Rightarrow \quad \boxed{\varphi = -\frac{GM}{r}}$$
- potential at infinity is zero: $\varphi_\infty = 0$

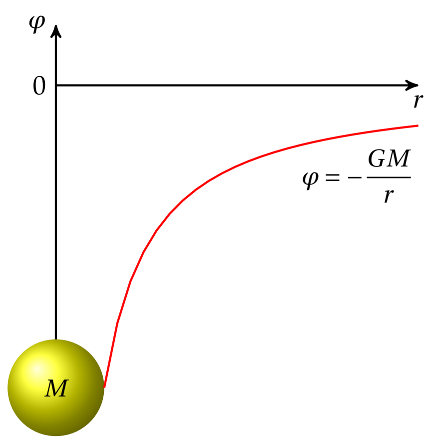
  this is our choice of zero potential point
- gravitational potential is a *scalar*

  combined potential due to several masses equals scalar sum of potential of each individual
- gravitational potential is always *negative*

  again this arises from attractive nature of gravity

  work is done to pull unit mass away from source

  farther from source means higher potential



Example 2.12 A star A of mass $M_A = 1.5 \times 10^{30}$ kg and a planet B of mass $M_B = 6.0 \times 10^{26}$ kg form an isolated astronomical system. Point P is between A and B, and is at distance $r_A = 2.0 \times 10^{12}$ m from A, and distance $r_B = 8.0 \times 10^{10}$ m from B. (a) Find the gravitational potential at P. (b) A meteor is initially at very large distance from the system with negligible speed. It then travels towards point P due to the gravitational attraction. Find its speed when it reaches P.

✍ gravitational potential at P: $\varphi_P = \varphi_A + \varphi_B = \left(-\dfrac{GM_A}{r_A}\right) + \left(-\dfrac{GM_B}{r_B}\right)$
$$\varphi_P = -6.67 \times 10^{-11} \times \left(\frac{1.5 \times 10^{30}}{2.0 \times 10^{12}} + \frac{6.0 \times 10^{26}}{8.0 \times 10^{10}}\right) \approx -5.05 \times 10^7 \mathrm{\ J\ kg^{-1}}$$

gain in K.E. = loss in G.P.E.: $\dfrac{1}{2}mv^2 = m\Delta\varphi \quad \Rightarrow \quad v^2 = 2(\varphi_\infty - \varphi_P) = -2\varphi_P$
$$v = \sqrt{-2 \times (-5.05 \times 10^7)} \approx 1.01 \times 10^4 \mathrm{\ m\ s^{-1}}$$
□

Example 2.13 The Moon may be considered to be an isolated sphere of radius $R = 1.74 \times 10^3$ km. The gravitational potential at the surface of the moon is about -2.82×10^6 J kg^{-1}. (a) Find the mass of the moon. (b) A stone travels towards the moon such that its distance from the centre of the moon changes from $3R$ to $2R$. Determine the change in gravitational potential. (c) If the stone starts from rest, find its final speed.

✍ at surface: $\varphi(R) = -\dfrac{GM}{R} \quad \Rightarrow \quad -2.82 \times 10^6 = -\dfrac{6.67 \times 10^{-11} \times M}{1.74 \times 10^6} \quad \Rightarrow \quad M = 7.36 \times 10^{22}$ kg

from $3R$ to $2R$: $\Delta\varphi = \varphi_{(3R)} - \varphi_{(2R)} = \left(-\dfrac{GM}{3R}\right) - \left(-\dfrac{GM}{2R}\right) = \dfrac{GM}{6R} = \dfrac{2.82 \times 10^6}{6} \approx 4.70 \times 10^5$ J kg^{-1}

note this change is a *decrease* in gravitational potential

gain in K.E. = loss in G.P.E.: $\dfrac{1}{2}mv^2 = m\Delta\varphi \quad \Rightarrow \quad v = \sqrt{2\Delta\varphi} = \sqrt{2 \times 4.70 \times 10^5} \approx 970$ m s^{-1} □

Question 2.8 Given that the moon is of radius 1700 km and mass 7.4×10^{22} kg. (a) Find the change in gravitational potential when an object is moved from moon's surface to 800 km above the surface. (b) If a rock is projected vertically upwards with an initial speed of 1800 m s^{-1} from surface, find the rock's speed when it reaches a height of 800 km. (c) Suggest whether the rock can escape from the moon's gravitational field completely.

Chapter 3
Oscillation

3.1 oscillatory motion

oscillation refers to a repetitive back and forth motion about its *equilibrium position*

the equilibrium position is a point where all forces on oscillator are balanced

release an object from its equilibrium position from rest, it will stay at rest

examples of oscillation includes pendulum of a clock, vibrating string, swing, etc.

3.1.1 amplitude, period, frequency

to describe motion of an oscillator, we define the following quantities:

- **displacement** (x): distance from the equilibrium position
- **amplitude** (x_0): maximum displacement from the equilibrium position
- **period** (T): time for one complete oscillation
- **frequency** (f): number of oscillations per unit time

  frequency is related to period as: $\boxed{f = \frac{1}{T}}$

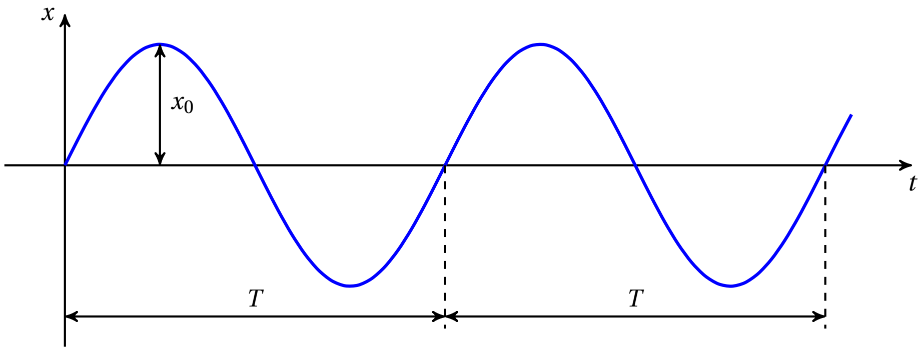
displacement x varies with time t repetitively, for which we can plot an x-t graph

amplitude x_0 and period T are labelled on the graph

3.1.2 phase angle

the point that an oscillator has reached within a complete cycle is called **phase angle** (ϕ)

- unit of phase angle: $[\phi] = \text{rad}$



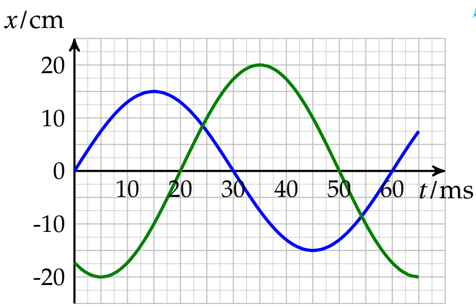
displacement-time graph for a typical oscillator

it looks like an angle, but better think of it as a number telling fraction of a complete cycle

➢ we use **phase difference** $\Delta\phi$ to compare how much one oscillator is ahead of another

$\Delta\phi$ is found in terms of fraction of an oscillation: $\Delta\phi = \frac{\Delta t}{T} \times 2\pi$ (also measured in radians)

Example 3.1 Compare the two oscillations from the x-t graph below.



✍ both have period $T = 60$ ms

frequency $f = \frac{1}{60 \times 10^{-3}} \approx 16.7$ Hz

they are of different amplitudes

one has $x_0 = 15$ cm, the other has $x_0 = 20$ cm

time difference: $\Delta t = 20$ ms

phase difference: $\Delta\phi = \frac{\Delta t}{T} \times 2\pi = \frac{20}{60} \times 2\pi = \frac{2\pi}{3}$ rad

3.1.3 acceleration & restoring force

for any oscillatory motion, consider its velocity and acceleration at various positions

its acceleration must be always pointing towards the equilibrium position

resultant force always acts in the direction to restore the system back to its equilibrium point, this net force is known as the **restoring force**

if at equilibrium position, then no acceleration or restoring force

3.2 simple harmonic oscillation

> if an oscillator has an acceleration always proportional to its displacement from the equilibrium position, and acceleration is in opposite direction to displacement, then the oscillator is performing **simple harmonic motion**

many phenomena can be approximated by simple harmonics

examples are motion of a pendulum, molecular vibrations, etc.

complicated motions can be decomposed into a set of simple harmonics

simple harmonic motion provides a basis for the study of many complicated motions [11]

3.2.1 equation of motion

defining equation for simple harmonics can be written as $\boxed{a = -\omega^2 x}$

ω is some constant, so a is proportional to x

the minus sign shows a and x are in opposite directions

general solution to this this equation of motion[12] takes the form: $\boxed{x = x_0 \sin(\omega t + \phi)}$

x_0 represents the amplitude, ω is called the angular frequency, ϕ is the phase angle

angular frequency

- **angular frequency** satisfies the relation: $\boxed{\omega = \dfrac{2\pi}{T} = 2\pi f}$
- unit of angular frequency: $[\omega] = \text{rad} \cdot \text{s}^{-1}$

---

[11]This can be done through a mathematical technique known as *Fourier analysis*. For example, a uniform circular motion can be considered as the combination of two simple harmonic motion in x- and y-directions.

[12]You probably know that acceleration can be written as the second derivative of displacement: $a = \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, so $a = -\omega^2 x$ is equivalent to $\dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0$, which a *second-order differential equation*. If you do not know how to solve it, you may have the chance to study this in an advanced calculus course.

- angular frequency ω is determined by the system's *physical constants* only

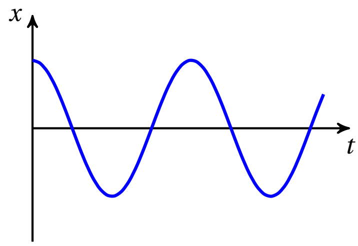
  if an object is set to oscillate *freely* with no external force, its period will always be the same

  frequency of an free oscillatory system is called the **natural frequency**

phase angle

- phase angle ϕ is dependent on *initial conditions* (e.g. initial position and initial speed at $t = 0$?)
- in many cases, phase angle term can be avoided if a suitable trigonometric function is chosen

Example 3.2 A simple harmonic oscillator is displaced by 6.0 cm from its rest position and let go at $t = 0$. Given that the period of this system is 0.80 s, state an equation for its displacement-time relation.



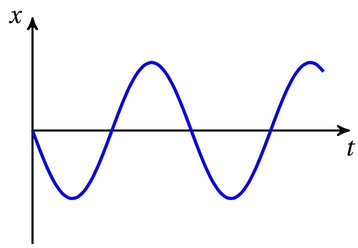
✍ angular frequency: $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80} = \frac{5\pi}{2}\pi$ rad s^{-1}

initial displacement $x(0) = +x_0 = 6.0$ cm

for displacement-time relation, we use cosine function

$$x(t) = x_0 \cos\omega t \quad \Rightarrow \quad x = 6.0\cos\left(\frac{5\pi}{2}t\right) \qquad \square$$

Example 3.3 A simple harmonic oscillator is initially at rest. At $t = 0$, it is given an initial speed in the negative direction. Given that the frequency is 1.5 Hz and the amplitude is 5.0 cm, state an equation for its displacement-time relation.



✍ angular frequency: $\omega = 2\pi f = 2\pi \times 1.5 = 3\pi$ rad s^{-1}

initial displacement $x(0) = 0$

for displacement-time relation, we use sine function

$$x(t) = -x_0 \cos\omega t \quad \Rightarrow \quad x = -5.0\sin(3\pi t) \qquad \square$$

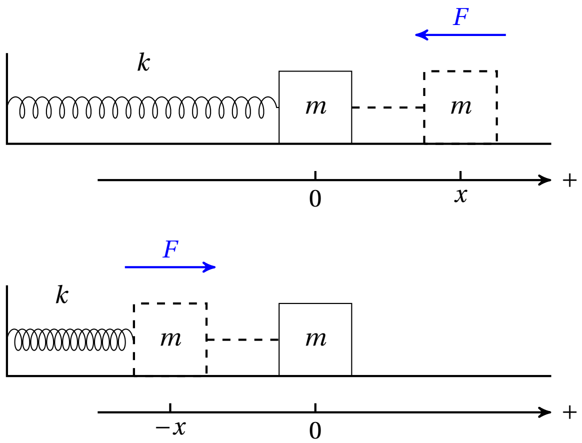
3.2.2 examples of simple harmonics

mass-spring oscillator

a mass-spring oscillator system consists of a block of mass m and an ideal spring

when a spring is stretched or compressed by a mass, the spring develops a restoring force

magnitude of this force obeys *Hooke's law*: $F = kx$



restoring force acting on the ideal mass-spring oscillator

direction of this force is in opposite direction to displacement x

take vector nature of force into account, we find

$$F_{\text{net}} = ma \quad \Rightarrow \quad -kx = ma \quad \Rightarrow \quad a = -\frac{k}{m}x$$

spring constant k and mass m are constants, so $a \propto x$

negative sign shows a and x are in opposite directions

so mass-spring oscillator executes simple harmonic motion

compare with $a = -\omega^2 x \quad \Rightarrow \quad \omega^2 = \dfrac{k}{m} \quad \Rightarrow \quad \boxed{\omega = \sqrt{\dfrac{k}{m}}}$

period of mass-spring oscillator: $T = \dfrac{2\pi}{\omega} \quad \Rightarrow \quad T = 2\pi\sqrt{\dfrac{m}{k}}$

➢ period and frequency are solely determined by mass of oscillator m and spring constant k

identical mass-spring systems will oscillate at same frequency no matter what amplitude

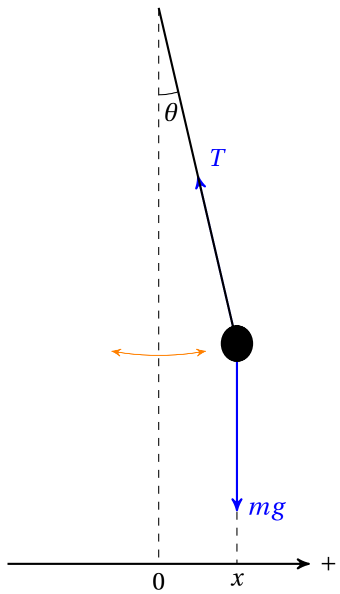
➢ $m \nearrow \Rightarrow T \nearrow$, greater mass means greater inertia, oscillation becomes slower

➢ $k \nearrow \Rightarrow T \searrow$, greater k means stiffer spring, greater restoring force makes oscillation go faster

simple pendulum

a simple pendulum is set up by hanging a bob on a light cord from a fixed point

displace the bob by some angle and release from rest, it can swing freely



one can show this performs simple harmonic motion for *small-angle* oscillation

if angular displacement θ is small, then the pendulum has almost no vertical displacement, the motion can be considered to be purely horizontal

vertically: $T\cos\theta \approx mg \xRightarrow{\cos\theta\approx 1 \text{ as } \theta\to 0} T \approx mg$

horizontally: $-T\sin\theta = ma \xRightarrow{\sin\theta = x/L} a \approx -\frac{g}{L}x$

this shows simple pendulum undergoes simple harmonics

compare with defining equation for simple harmonics:

$$a = -\omega^2 x \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{L}}$$

period for a simple pendulum: $\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$

- ➢ period and frequency of a pendulum are determined by length of the string L only

  as long as angular displacement remains small, frequency does not depend on amplitude

  fix length L, then simple pendulum oscillates at same frequency no matter what amplitude
- ➢ $L \nearrow \Rightarrow T \nearrow$, longer pendulums oscillate more slowly
- ➢ $g \searrow \Rightarrow T \nearrow$, if there is no gravity ($g = 0$), then the bob will not move at all ($T \to \infty$)

Question 3.1 A cylindrical tube of total mass m and cross sectional area A floats upright in a liquid of density ρ. When the tube is given a small vertical displacement and released, the magnitude of the resultant force acting on the tube is related to its vertical displacement y by the expression: $F_{\text{net}} = \rho g A y$. (a) Show that the tube executes simple harmonic motion. (b) Find an expression for the frequency of the oscillation.

Question 3.2 A small glider moves along a horizontal air track and bounces off the buffers at the ends of the track. Assume the track is frictionless and the buffers are perfectly elastic, state and explain whether the glider describes simple harmonic motion.

3.2.3 velocity & acceleration

displacement of simple harmonic oscillator varies with time as: $x = x_0 \sin(\omega t + \phi)$

from this displacement-time relation, we can find velocity and acceleration relations

velocity

to find velocity-time relation, let's recall that velocity v is rate of change of displacement x

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} x_0 \sin(\omega t + \phi) \quad \Rightarrow \quad \boxed{v(t) = \omega x_0 \cos(\omega t + \phi)}$$

by taking $v^2 + \omega^2 x^2$, the sine and cosine terms can be eliminated, we find:

$$v^2 + \omega^2 x^2 = \omega^2 x_0^2 \cos^2(\cdots) + \omega^2 x_0^2 \sin^2(\cdots) = \omega^2 x_0^2$$

this gives velocity-displacement relation: $\boxed{v(x) = \pm\omega\sqrt{x_0^2 - x^2}}$

- at equilibrium position $x = 0$, speed is maximum: $\boxed{v_{\max} = \omega x_0}$
- when $x = \pm x_0$, oscillator is momentarily at rest: $v = 0$

acceleration

acceleration-time relation is found by further taking rate of change of velocity v

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \omega x_0 \cos(\omega t + \phi) \quad \Rightarrow \quad \boxed{a(t) = -\omega^2 x_0 \sin(\omega t + \phi)}$$

this is actually unnecessary, if we compare this with $x(t) = x_0 \sin(\omega t + \phi)$, we have: $a = -\omega^2 x$

we have recovered the definition for simple harmonics

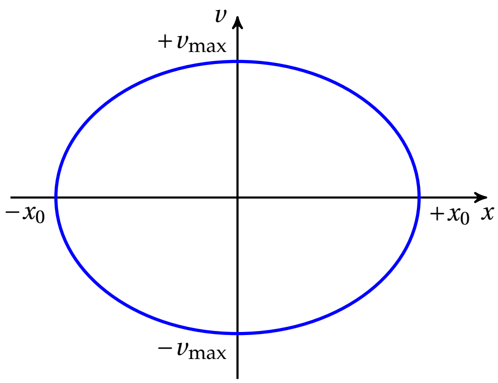
(if $a \propto x$ and in opposite directions to x, then simple harmonic motion)

so acceleration-displacement relation is given by the defining equation explicitly $\boxed{a(x) = -\omega^2 x}$

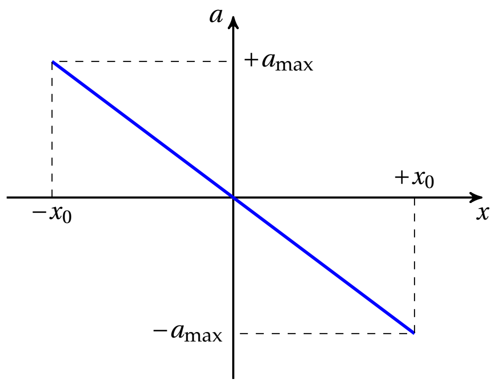
- at equilibrium position $x = 0$, zero acceleration
- when $x = \pm x_0$, acceleration is greatest: $\boxed{a_{\max} = \omega^2 x_0}$

let's take $x = x_0 \sin\omega t$ as example, changes of x, v, a over time are listed below

time t	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
displacement: $x = x_0 \sin\omega t$	0	+max	0	−max	0
velocity: $v = \omega x_0 \cos\omega t$	+max	0	−max	0	+max
acceleration: $a = -\omega^2 x = -\omega^2 x_0 \sin\omega t$	0	−max	0	+max	0



velocity-displacement graph



acceleration-displacement graph

Example 3.4 The motion of a simple pendulum is approximately simple harmonic. As the pendulum swings from one side to the other end, it moves through a distance of 6.0 cm and the time taken is 1.0 s. (a) State the period and amplitude. (b) Find the greatest speed during the oscillation. (c) Find its speed when displacement $x = 1.2$ cm.

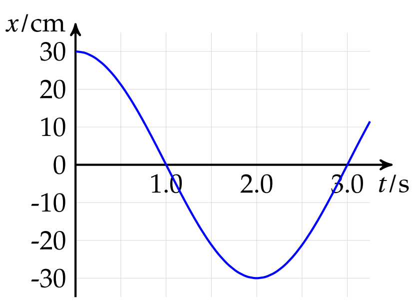
✍ period: $T = 2 \times 1.0 = 2.0$ s, and amplitude: $x_0 = \dfrac{1}{2} \times 6.0 = 3.0$ cm

angular frequency: $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{2.0} = \pi \text{ rad s}^{-1}$

greatest speed: $v_{\max} = \omega x_0 = \pi \times 3.0 \approx 9.4 \text{ cm s}^{-1}$

speed at 1.2 cm: $v = \omega\sqrt{x_0^2 - x^2} = \pi \times \sqrt{3.0^2 - 1.2^2} \approx 8.6 \text{ cm s}^{-1}$ □

Example 3.5 Given the x-t graph of a simple harmonic oscillator. (a) Find its speed at $t = 0$. (b) Find its greatest speed. (b) Find its acceleration at $t = 1.0$ s.



✍ at $t = 0$, $x = +x_0 \quad \Rightarrow \quad v = 0$ (zero gradient)

from graph: amplitude $x_0 = 30$ cm, period $T = 4.0$ s

angular frequency: $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{4} = \dfrac{\pi}{2} \text{ rad s}^{-1}$

greatest speed: $v_{\max} = \omega A = \dfrac{\pi}{2} \times 30 \approx 47 \text{ cm s}^{-1}$

at $t = 1.0$ s, $x = 0 \quad \Rightarrow \quad a = 0$

(equilibrium position so no acceleration) □

Question 3.3 Assume the motion of a car engine piston is simple harmonic. The piston completes 3000 oscillations per minute. The amplitude of the oscillation is 4.0 cm. (a) Find the greatest speed. (b) Find the greatest acceleration.

3.2.4 vibrational energy

consider the *ideal* mass-spring oscillator, its vibrational energy consists of two parts:

- kinetic energy of the mass: $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t \xlongequal{v=\pm\omega\sqrt{x_0^2-x^2}} \frac{1}{2}m\omega^2(x_0^2 - x^2)$
- (elastic) potential energy in the spring: $E_p = \frac{1}{2}kx^2 \xlongequal{\omega=\sqrt{\frac{k}{m}}} \frac{1}{2}m\omega^2 x^2$

total energy of the oscillator: $E = E_k + E_p \quad \Rightarrow \quad \boxed{E = \frac{1}{2}m\omega^2 x_0^2}$

➢ although this formula is derived from the mass-spring model

$E = \frac{1}{2}m\omega^2 x_0^2$ can be used to compute vibrational energy of all simple harmonic oscillators

➢ for an ideal system, total energy remains constant

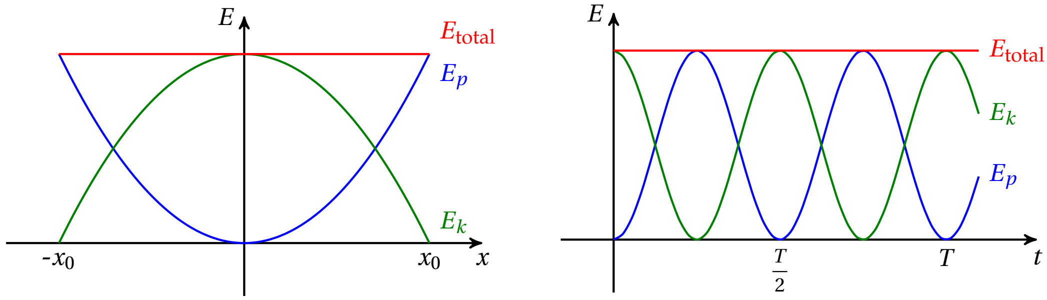
E_k and E_p keep changing, one transfers into another, but total energy is *conserved*

➢ when $x = 0$, $E_k = \max$, $E_p = 0$, vibrational energy is purely kinetic

$$E = E_{k,\max} = \frac{1}{2}mv_{\max}^2 \xlongequal{v_{\max}=\omega x_0} \frac{1}{2}m\omega^2 x_0^2$$

➢ when $x = \pm x_0$, $E_k = 0$, $E_p = \max$, vibrational energy is purely potential

$$E = E_{p,\max} = \frac{1}{2}kx_0^2 \xlongequal{\omega=\sqrt{\frac{k}{m}}} \frac{1}{2}m\omega^2 x_0^2$$



vibrational energy of a mass-spring oscillator

Example 3.6 A block of mass 150 g at the end of a spring oscillates with a period of 0.80 s. The maximum displacement from its rest position is 12 cm. Find the energy of the vibration.

✍ $$E = \frac{1}{2}m\omega^2 x_0^2 = \frac{1}{2}m\left(\frac{2\pi}{T}\right)^2 x_0^2 = \frac{1}{2} \times 0.15 \times \frac{4\pi^2}{0.80^2} \times 0.12^2 \approx 6.7 \times 10^{-2}\ \text{J}$$ □

Question 3.4 An oscillator is given an energy of 20 mJ and starts to oscillate, it reaches an amplitude of 8.0 cm. If we want to double the amplitude, find the vibrational energy required.

3.3 damped oscillations

total vibrational energy stays constant for an ideal system

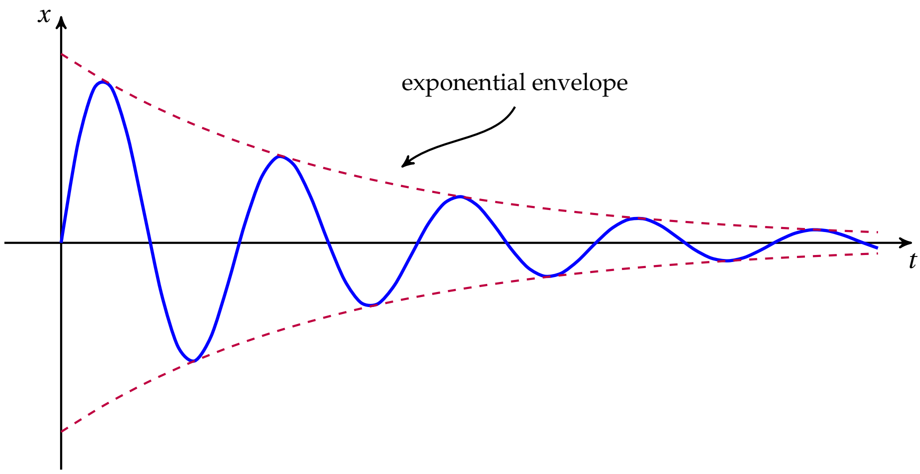
but in reality, there are friction, resistance and viscous forces that oppose motion

amplitude of an oscillator decreases due to energy loss to friction, this is called **damping**

3.3.1 light damping

for a **lightly-damped oscillator**, amplitude decreases *gradually*

oscillator will not stop moving back and forth after quite a few oscillations



- decrease in amplitude is *non-linear* in time (exponential decay in many cases)
- frequency and period are (almost) unchanged

Example 3.7 An oscillator is composed of a block of mass $m = 250$ g and a spring of $k = 1.6$ N/cm. It is displaced by 5.0 cm from its rest position and set free. (a) What is its angular frequency? (b) what is the initial vibrational energy? (c) After a few oscillations, 40% of its energy is lost due to damping. What is its new amplitude?

✍ angular frequency: $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{160}{0.25}} \approx 25.3 \text{ rad s}^{-1}$

energy of oscillator: $E = \dfrac{1}{2} m\omega^2 x_0^2 = \dfrac{1}{2} \times 0.25 \times 25.3^2 \times 0.050^2 = 0.20$ J[13]

[13] An easier approach: $E = \dfrac{1}{2} k x_0^2 = \dfrac{1}{2} \times 160 \times 0.050^2 = 0.20$ J.

since $E \propto x_0^2$, so: $\dfrac{E'}{E} = \dfrac{x_0'^2}{x_0^2} \quad \Rightarrow \quad 60\% = \dfrac{x_0'^2}{x_0^2} \quad \Rightarrow \quad x_0' = \sqrt{0.6}x_0 = \sqrt{0.6} \times 5.0 \approx 3.9$ cm □

Question 3.5 A small toy boat of mass 360 g floats on surface of water. It is gently pushed down and then released. During the fist four complete cycles of its oscillation, its amplitude decreased from 5.0 cm to 2.0 cm in a time of 6.0 s. Find the energy loss.

3.3.2 heavy damping

if resistive forces are too strong, there will be no oscillatory motion

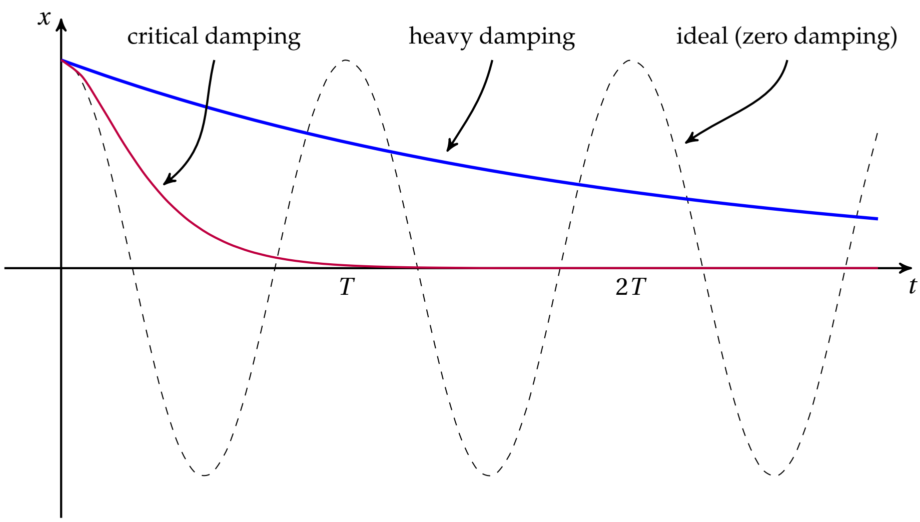
the system will return to the equilibrium position very slowly

this system is said to be **heavily damped**

3.3.3 critical damping

critical damping is the border between light damping and heavy damping

it occurs when system returns to equilibrium in *shortest* time without any oscillation



➢ critical damping is desirable in many engineering designs [14]

examples include door-closing mechanism, shock absorbers in vehicles and artillery, etc.

[14]When a damped oscillator is required, critically-damped system provides the quickest approach to equilibrium without overshooting, while lightly-damped system reaches the zero position quickly but continues to oscillate, and heavily-damped system reaches zero position in very long time.

3.4 forced oscillations

3.4.1 free & forced oscillation

an oscillator moving on its own with no gain or loss of energy is called **free oscillation**

amplitude of the oscillation is constant, its frequency called **natural frequency**

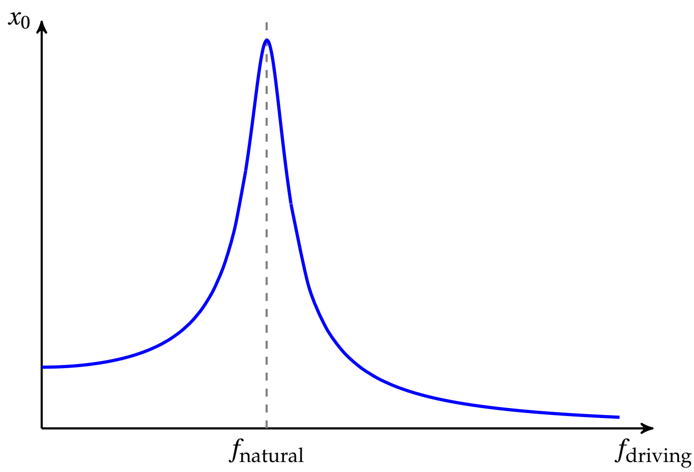
an oscillator may also move under an external driving force, it is **forced oscillation**

frequency of forced oscillator tends to driving frequency after sufficiently long time

3.4.2 resonance

for a forced oscillation system, when frequency of driving force f_{driving} is close to natural frequency f_{natural}, amplitude of oscillator increases rapidly

when driving frequency of external force equals natural frequency of the system, amplitude of the system becomes maximum, this phenomenon is called **resonance**



resonance is achieved when $f_{\text{driving}} = f_{\text{natural}}$

(amplitude tends to infinity if no damping)

- practical application of resonance
  - microwave oven – water molecules resonate at microwave frequency and vibrate greatly

- MRI (magnetic resonance imaging) — precession of nuclei resonate at radio frequency, signals are processed to image nuclei of atoms inside a human body in detail
- radio/TV — *RLC* tuning circuits resonate at frequency of signals being received

➢ possible problems caused by resonance

- buildings during earthquake – resonate at frequency of shockwaves and collapse
- car suspension system – going over bumps may give large amplitude vibrations
- bridges and skyscrapers – resonance due to wind conditions

3.4.3 damping & resonance

an oscillation system can be subject to both driving force and resistive force

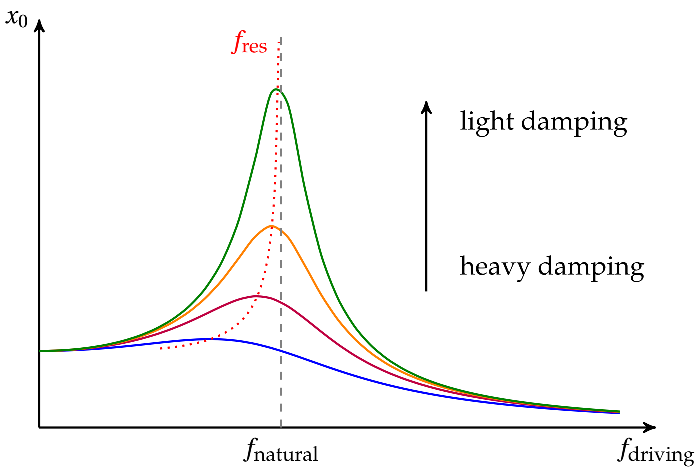
resonance behaviour will be changed by damping effects

➢ damping decreases amplitude of oscillation at all frequencies

greater damping causes resonance peak to become *flatter*

engineering systems are often deliberately damped to minimise resonance effect

➢ damping also shifts resonance frequency (slightly reduced for light damping)



resonance effect for various damping conditions

Chapter 4
Ideal Gases

4.1 gas molecules

4.1.1 motion of gas particles

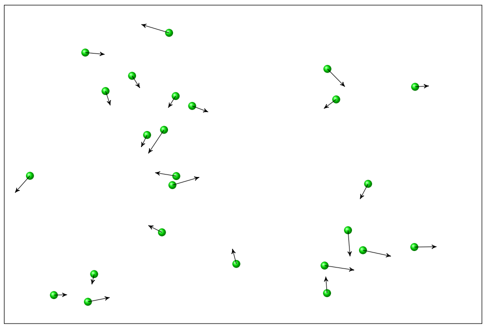
gas consists of a large number of molecules

gas molecules move *randomly* at high speeds

➢ randomness results from *collisions* of fast-moving molecules in the gas

for an individual molecule, its velocity changes constantly as it collides with other molecules

for the gas at any instant, there is a range of velocities for molecules



motion of gas molecules in a container

➢ experimental evidence of random motion: **Brownian motion**

dust or smoke particles in air undergo jerky random motion (viewed through microscope)

this is due to collisions with gas molecules that move randomly

➢ speed of gas molecules depend on temperature

molecules move faster at higher temperature[15]

4.1.2 amount of molecules

there are a huge number of molecules in a gas

we introduce **amount of substance** to measure the size of a collection of particles

➢ unit of amount of substance: $[n] = \mathrm{mol}$

[15]We will prove this statement later in this chapter.

one **mole** is defined as the amount carbon-12 atoms in a sample of 12 grams

➢ 1 mole of substance contains 6.02×10^{23} particles

this number is called **Avogadro constant**: $N_A = 6.02 \times 10^{23}\ \text{mol}^{-1}$ [16]

conversion between number of molecules and amount of substance: $\boxed{N = nN_A}$

➢ it is useful to introduce the notion of molar mass M

molar mass of a substance is defined as the mass of a given sample divided by the amount of substance: $M = \dfrac{m}{n}$

– amount of substance $= \dfrac{\text{mass of sample}}{\text{molar mass}}$, or $n = \dfrac{m}{M}$

– mass of single molecule $= \dfrac{\text{molar mass}}{\text{Avogadro constant}}$, or $m_0 = \dfrac{M}{N_A}$

Example 4.1 Find the number of molecules in 160 grams of argon-40 gas.

✍ amount of gas: $n = \dfrac{m}{M} = \dfrac{160\ \text{g}}{40\ \text{g mol}^{-1}} = 4.0\ \text{mol}$

number of gas molecules: $N = nN_A = 4.0\ \text{mol} \times 6.02 \times 10^{23}\ \text{mol}^{-1} \approx 2.41 \times 10^{24}$ □

Question 4.1 Find the mass of a sample of uranium-235 that contains 6.0×10^{20} atoms.

4.1.3 pressure (qualitative view)

when gas molecules collide with walls of container and rebound, they are acted by a force

by Newton's third law, gas molecules must exert a reaction force on container in return

contributions from many molecules give rise to a pressure

Example 4.2 If a gas is heated with its volume fixed, how does the pressure change?

✍ at higher temperature, gas molecules move faster

they will collide *harder* and produce a greater force upon each collision

they will also collide more *frequently* with the container

[16] In 2018, IUPAC suggested a new definition of the mole, which is defined to contain exactly 6.02×10^{23} particles. This new definition fixed numerical value of the Avogadro constant, and emphasized that the quantity 'amount of substance' is concerned with counting number of particles rather than measuring the mass of a sample.

so pressure of the gas will increase □

Question 4.2 If you pump gas into a bicycle tyre, state and explain how the pressure changes.

Question 4.3 A fixed amount of gas is allowed to expand at constant temperature, state and explain how the pressure changes.

4.2 ideal gas

4.2.1 ideal gas equation

a gas that satisfies the equation $\boxed{pV = nRT}$ or $\boxed{pV = NkT}$ at any pressure p, any volume V, and thermodynamic temperature T is called an **ideal gas**

molar gas constant: $R = 8.31\ \mathrm{J\ mol^{-1}\ K^{-1}}$

Boltzmann constant: $k = 1.38 \times 10^{-23}\ \mathrm{J\ K^{-1}}$

values of R and k apply for any ideal gas, i.e., they are *universal* constants

➢ recall conversion between number of molecules and amount of substance: $\boxed{N = nN_A}$

we have relation between the constants: $R = kN_A$, or $k = \dfrac{R}{N_A}$

➢ one must use *thermodynamic temperature* in the equation

thermodynamic temperature is measured in kelvins (K), so it is also called the *Kelvin scale*[17]

conversion between Kelvin temperature and Celsius temperature: $\boxed{T_K(\mathrm{K}) \underset{+273}{\overset{-273}{\rightleftharpoons}} T_C(°\mathrm{C})}$

real gases

real gas behaves ideally at sufficiently high temperature and low pressure

- at very low temperatures, real gas will condense into liquid or solid
- at very high pressures, intermolecular forces become important

however, under normal conditions (room temperature $T \approx 300$ K and standard atmospheric pressure $p \approx 1.0 \times 10^5$ Pa), there is no significant difference between a real gas and an ideal gas

[17]We will discuss in details about Kelvin scale in §5.1.1 and §5.3.2.

so ideal gas approximation can be used with good accuracy for most of our applications

Example 4.3 A sealed cylinder of volume of 0.050 m^3 contains 75 g of air. The molar mass of air is 29 g mol^{-1}. (a) Find the air pressure when its temperature is 30°C. (b) The gas is allowed to expand with its pressure fixed. Find the temperature of the gas when the volume doubles.

✍ amount of gas: $n = \dfrac{m}{M} = \dfrac{75}{29} \approx 2.59$ mol

pressure at 30°C: $p = \dfrac{nRT_1}{V_1} = \dfrac{2.59 \times 8.31 \times (30+273)}{0.050} \approx 1.30 \times 10^5$ Pa

pressure fixed, so $V \propto T \quad \Rightarrow \quad \dfrac{T_2}{T_1} = \dfrac{V_2}{V_1} = 2 \quad \Rightarrow \quad T_2 = 2 \times (30+273) = 606\ \text{K} = 333°\text{C}$ □

Example 4.4 A gas cylinder holding 5000 cm^3 of air at a temperature of 27 °C and a pressure of 6.0×10^5 Pa is used to fill balloons. Each balloon contains 1000 cm^3 of air at 27 °C and 1.0×10^5 Pa when filled. (a) Find the initial amount of gas in the cylinder. (b) Find the number of balloons that can be filled.

✍ initial amount of gas in cylinder: $n_0 = \dfrac{p_0 V}{RT} = \dfrac{9.0 \times 10^5 \times 5000 \times 10^{-6}}{8.31 \times (27+273)} \approx 1.203$ mol

final amount of gas in cylinder: $n_{\text{remain}} = \dfrac{pV}{RT} = \dfrac{1.0 \times 10^5 \times 5000 \times 10^{-6}}{8.31 \times (27+273)} \approx 0.201$ mol[18]

amount of gas in each balloon: $n_b = \dfrac{pV_b}{RT} = \dfrac{1.0 \times 10^5 \times 1000 \times 10^{-6}}{8.31 \times (27+273)} \approx 0.040$ mol

number of balloons: $N = \dfrac{n_0 - n_{\text{remain}}}{n_b} = \dfrac{1.203 - 0.201}{0.040} \approx 25$ □

Example 4.5 A storage cylinder has a volume of 5.0×10^{-4} m^3. The gas is at a temperature of 300 K and a pressure of 4.0×10^6 Pa. (a) Find the number of molecules in the cylinder. (b) The gas molecules slowly leak from the cylinder at a rate of 1.6×10^{16} s^{-1}. Find the time, in days, after which the pressure will reduce by 5.0%.

✍ initial number of molecules: $N_0 = \dfrac{p_0 V}{kT} = \dfrac{4.0 \times 10^6 \times 5.0 \times 10^{-4}}{1.38 \times 10^{-23} \times 300} \approx 4.83 \times 10^{23}$

volume fixed, so $N \propto p \quad \Rightarrow \quad \dfrac{\Delta N}{N_0} = \dfrac{\Delta p}{p_0} = 5.0\%$

number of molecules escaped: $\Delta N = 0.05 \times 4.83 \times 10^{23} \approx 2.42 \times 10^{22}$

time needed: $t = \dfrac{2.42 \times 10^{22}}{1.6 \times 10^{16}} \approx 1.51 \times 10^6\ \text{s} \approx 17.4$ days □

[18] Air will leave the cylinder to fill balloons only if pressure inside the cylinder is higher than pressure of the balloon. When the two pressures become equal, no more balloons can be filled, there will be some air remain in cylinder.

Question 4.4 Containers A has a volume of 2.5×10^{-2} m^3 contains a gas at a temperature of 17°C and pressure of 1.3×10^5 Pa and . Another container B of same size holds a gas at same temperature and a pressure of 1.9×10^5 Pa. The two containers are initially isolated from each another. (a) Find the total amount of molecules. (b) The two containers are now connected through a tube of negligible volume. Assume the temperature stays unchanged, find the final pressure of the gas.

Question 4.5 The air in a car tyre can be assumed to have a constant volume of 3.0×10^{-2} m^3 . The pressure of this air is 2.8×10^5 Pa at a temperature of 25°C. The pressure is to be increased using a pump. On each stroke 0.015 mol of air is forced into the tyre. If gas has a final pressure of 3.6×10^5 Pa and final temperature of 28°C. Find the number of strokes of the pump required.

4.2.2 empirical laws

historically, the ideal gas law was first stated by *Émile Clapeyron* in 1834:

for a fixed amount of gas, $\boxed{\dfrac{PV}{T}=\text{const}}$

his work was based on the empirical Boyle's law, Charles's law, and Gay-Lussac's law

we will next recover these laws from the ideal gas equation

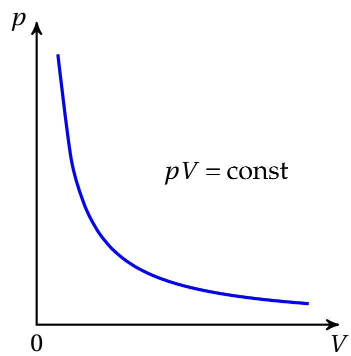
Boyle's law

Boyle's law was discovered by *Robert Boyle* in 1662, based on experimental observations

if temperature T remains constant, then

$$\boxed{pV=\text{const}},\ \text{or}\ \boxed{p\propto\frac{1}{V}}$$

i.e., pressure p of gas is inversely proportional to volume V

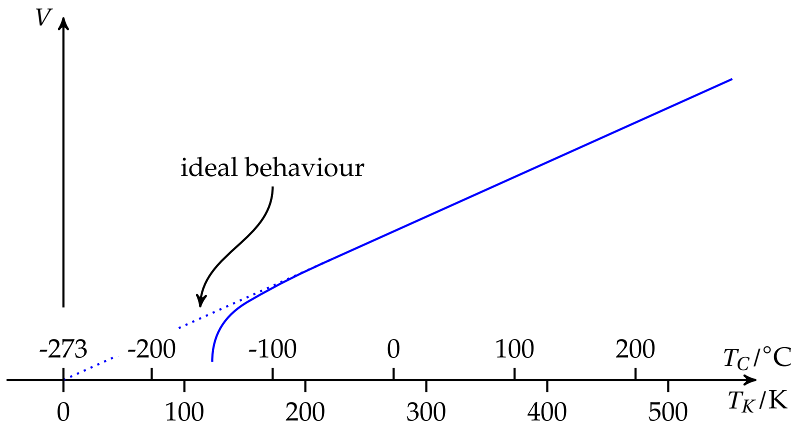


- for a gas with fixed temperature: $p_1V_1=p_2V_2$
- a thermodynamic process for which temperature is kept constant is called an *isothermal* process

p-V relation for an isothermal process is shown

Charles's law

Charles's law was discovered by *Jacques Charles* in 1787, based on experimental observations



if pressure p remains constant, then: $\boxed{\frac{V}{T} = \text{const}}$, or $\boxed{V \propto T}$

i.e., volume V of gas is directly proportional to its temperature T

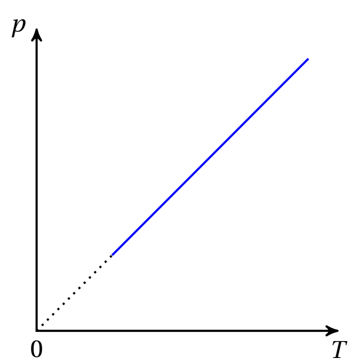
- ➢ proportionality relation only applies if Kelvin scale is used
- ➢ a thermodynamic process for which pressure is kept constant is called an *isobaric* process

  V-T relation for an isobaric process is shown
- ➢ Charles's law implies that volume of gas tends to zero at a certain temperature

  historically this is how the idea of *absolute zero* first arose
- ➢ as $T \to 0$, a real gas condenses into solid

  there will be deviation from ideal behaviour (dotted line)

Gay-Lussac's law



Gay-Lussac's law was discovered by *Joseph Louis Gay-Lussac* between 1800 and 1802

if volume V remains constant, then

$$\boxed{\frac{p}{T} = \text{const}}, \text{ or } \boxed{p \propto T}$$

i.e., pressure p is directly proportional to temperature T

➢ a thermodynamic process for which volume is kept constant

is called an *isochoric* process, or *isometric* process

p-T relation for an isochoric process is shown

➢ behaviour of real gas again deviates from ideal behaviour (dotted line) as $T \to 0$

4.3 kinetic theory of ideal gases

kinetic model of gases: a theory based on microscopic motion of molecules of a gas that explains its macroscopic properties

4.3.1 assumptions of ideal gas model

kinetic theory of the ideal gas model is based on the following assumptions:

- gas molecules are in constant *random* motion
- *intermolecular separation* is much greater than size of molecules

  volume of molecules is negligible compared to volume occupied by gas
- *intermolecular forces* are negligible
- collisions between molecules are perfectly *elastic*, i.e., no kinetic energy lost
- molecules travel in straight line between collisions

Example 4.6 A mass of 20 g helium-4 at a temperature of 37°C has a pressure of 1.2×10^5 Pa. Each helium-4 atom has a diameter of 280 pm. (a) Find the volume occupied by the gas. (b) Find the volume of atoms in this gas. (c) Compare the two volumes, suggest whether this gas can be considered as an ideal gas.

✍ number of helium molecules: $N = nN_A = \frac{m}{M} \times N_A = \frac{20}{4.0} \times 6.02 \times 10^{23} \approx 3.01 \times 10^{24}$

volume of gas: $V_{\text{gas}} = \frac{NkT}{p} = \frac{3.01 \times 10^{24} \times 1.38 \times 10^{-23} \times (37 + 273)}{1.2 \times 10^5} \approx 0.107 \text{ m}^3$

volume of one atom: $V_{\text{atom}} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (140 \times 10^{-12})^3 \approx 1.15 \times 10^{-29} \text{ m}^3$

volume of all atoms: $V_{\text{atoms}} = NV_{\text{atom}} = 3.01 \times 10^{24} \times 1.15 \times 10^{-29} \text{ m}^3 \approx 3.46 \times 10^{-5} \text{ m}^3$

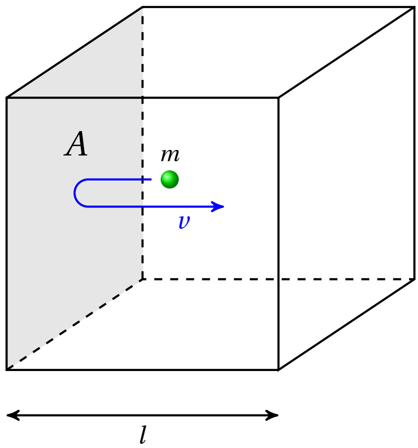
$V_{\text{gas}} \gg V_{\text{atoms}}$, so negligible volume of molecules compared to volume of gas this gas can approximate to an ideal gas □

4.3.2 pressure (quantitative view)

we are ready to derive a formula for pressure due to ideal gas

pressure of gas is due to collision of gas molecules with container

let's first consider the effect of one single molecule moving in one dimension only, and then generalise the result to a gas containing N molecules moving in all three dimensions



one gas molecule moving in 1-D

let's assume this single molecule only moves in x-direction (see figure)

change in momentum when colliding with wall: $\Delta P_x = mv_x - (-mv_x) = 2mv_x$[19]

time interval between collisions: $\Delta t = \dfrac{2l}{v_x}$

average force acting: $F_x = \dfrac{\Delta P_x}{\Delta t} = \dfrac{2mv_x}{\frac{2l}{v_x}} = \dfrac{mv_x^2}{l}$

average pressure: $p_x = \dfrac{F}{A} = \dfrac{mv_x^2}{lA} = \quad \Rightarrow \quad p_x = \dfrac{mv_x^2}{V}$

generalisation to N molecules moving in 3-D

- N molecules so N times the contributions to pressure

  but there is a *distribution* of speeds for N molecules, so should take average of v^2

[19]In this section we use P for momentum of a particle and p for pressure of a gas to avoid confusion.

- in three-dimensional space, we have: $v^2 = v_x^2 + v_y^2 + v_z^2$

  but molecules have no preference in any specific direction, so: $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle = \dfrac{\langle v^2\rangle}{3}$

  pressure should be shared equally among three dimensions: $p = p_x = p_y = p_z$

therefore we find the pressure of an ideal gas is given by: $\boxed{p = \dfrac{Nm\langle v^2\rangle}{3V}}$

➢ $\langle v^2\rangle$ is the *mean square velocity* of gas molecules

we can further define r.m.s. (root mean square) velocity: $v_{\mathrm{rms}} = \sqrt{\langle v^2\rangle}$

gas molecules in random motion so there exists a range of velocities

we cannot tell exact velocity of a specific molecule, but can only tell mean values

➢ N is number of molecules, m is mass of one molecule

then Nm gives total mass of the gas, and $\dfrac{Nm}{V}$ gives gas density ρ

we can rewrite the pressure formula as: $\boxed{p = \dfrac{1}{3}\rho\langle v^2\rangle}$

(pressure depends only on density and mean square speed of molecules)

➢ physical interpretation of the formula

- $N \nearrow \Rightarrow$ more molecules, more collisions $\Rightarrow p \nearrow$
- $m \nearrow \Rightarrow$ greater mass, greater force upon collision $\Rightarrow p \nearrow$
- $v \nearrow \Rightarrow$ strike container harder, also more often $\Rightarrow p \nearrow$
- $V \nearrow \Rightarrow$ spend more time in gas, less frequent collision with container $\Rightarrow p \searrow$

4.3.3 kinetic energy

we now have two equations for ideal gases:

$$\begin{cases} pV = nRT\text{, or } pV = NkT & \text{ideal gas law} \\ p = \dfrac{Nm\langle v^2\rangle}{3V} & \text{pressure law} \end{cases}$$

compare the two equations: $pV = \dfrac{1}{3}Nm\langle v^2\rangle = NkT \quad \Rightarrow \quad m\langle v^2\rangle = 3kT$

mean kinetic energy of a single molecule in a gas is: $\boxed{\langle E_k\rangle = \dfrac{1}{2}m\langle v^2\rangle = \dfrac{3}{2}kT}$

mean K.E. of ideal gas molecules is *proportional* to its thermodynamic temperature

➢ useful relation for molecular speeds: $\boxed{v_{\mathrm{rms}}^2 \propto T}$

recall our statement in §4.1.1, higher temperature means higher speed for molecules

- we only talk about *translational* K.E. here
  molecules have this energy because they are moving through space
  total kinetic energy may also include *rotational* K.E. and *vibrational* K.E. [20]
- $\langle E_k \rangle = \frac{3}{2}kT$ gives the *mean*, or *average* K.E. per molecule
  gas molecules exchange energies with each other upon collisions
  for an individual molecule, its K.E. is not a constant
  but mean K.E. is constant, which depends on temperature T only
- in a mixture of several gases, K.E. is shared *equally* among its components
  this is because of repeated collisions between particles
  though all molecules have same K.E., heavier molecules will move more slowly

Example 4.7 Air consists of oxygen (O_2, molar mass 32 g mol^{-1}) and nitrogen (N_2, molar mass 28 g mol^{-1}). (a) Calculate the mean translational kinetic energy of these molecules at 300 K. (b) Estimate the typical speed for each type of the molecule.

✍ mean K.E. of single molecule: $\langle E_k \rangle = \frac{3}{2}kT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 \approx 6.21 \times 10^{-21}$ J

$$\langle E_k \rangle = \frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}kT \quad \Rightarrow \quad \frac{1}{2}\frac{M}{N_A}\langle v^2 \rangle = \frac{3}{2}kT \quad \Rightarrow \quad \langle v^2 \rangle = \frac{3kN_AT}{M} = \frac{3RT}{M}$$

for oxygen molecule: $v_{O_2} \approx \sqrt{\frac{3 \times 8.31 \times 300}{0.032}} \approx 483$ m s^{-1}

for nitrogen molecule: $v_{N_2} \approx \sqrt{\frac{3 \times 8.31 \times 300}{0.028}} \approx 517$ m s^{-1} □

Example 4.8 A cylinder container initially holds a gas of helium-4 at a temperature of 54°C. (a) Find the mean square speed of these helium atoms. (b) If the temperature is raised to 540°C, find the r.m.s. speed of the atoms.

✍ mass of one helium-4 atom: $m = 4\text{u} = 4 \times 1.66 \times 10^{-27} \approx 6.64 \times 10^{-27}$ kg

[20]There is an important result in classical thermal physics, known as the *equipartition of energy theorem*. It states that the average energy per molecule is $\frac{1}{2}kT$ for each independent *degree of freedom*. A molecule can move in three directions, corresponding to three translational degrees of freedom, thus its mean translational kinetic energy is $\frac{3}{2}kT$. For a polyatomic gas (each molecule consists of several atoms), apart from translational motion , it has additional rotational degrees of freedom and different vibrational modes, so its average energy can be calculated by counting the total number of degrees of freedom.

$$\text{at } 54^\circ\text{C:}\ \frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \quad \Rightarrow \quad \langle v^2\rangle = \frac{3kT}{m} = \frac{3\times 1.38\times 10^{-23}\times(54+273)}{6.64\times 10^{-27}} \approx 2.04\times 10^6\ \text{m}^2\ \text{s}^{-2}$$

$$\text{note relation between } v \text{ and } T\text{: } \langle v^2\rangle \propto T \quad \Rightarrow \quad \frac{\langle v'^2\rangle}{\langle v^2\rangle} = \frac{T'}{T} \quad \Rightarrow \quad v'_{\text{rms}} = \sqrt{\frac{T'}{T}}\times v_{\text{rms}}$$

$$\text{at } 540^\circ\text{C:}\ v'_{\text{rms}} = \sqrt{\frac{540+273}{54+273}}\times\sqrt{2.04\times 10^6} \approx 2.25\times 10^3\ \text{m s}^{-1}$$

□

Question 4.6 A fixed mass of gas expands to twice its volume at constant temperature. (a) How does its pressure change? (b) How does mean kinetic energy change?

Question 4.7 In order for a molecule to escape from the gravitational field of the earth, it must have a speed of 1.1×10^6 m s^{-1} at the top of the atmosphere. (a) Estimate the temperature at which helium-4 atoms could have this speed. (b) Helium atom actually escape from top of the atmosphere at much lower temperatures, explain how this is possible.

Chapter 5

Thermodynamics

5.1 thermal physics basics

5.1.1 temperature scales

- Celsius scale (unit: °C)

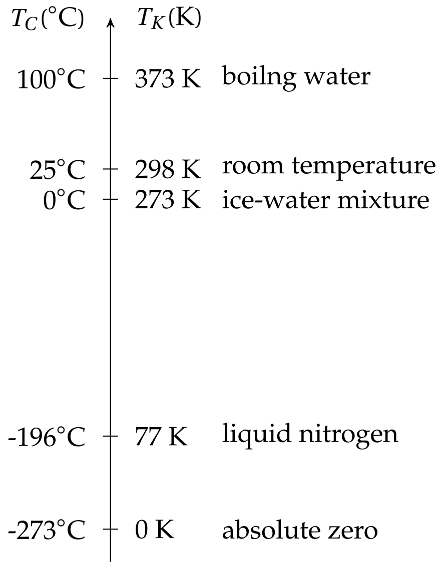
  0°C defined as temperature of ice-water mixture

  100°C defined as temperature of boiling water

- Kelvin scale (unit: K)

  0 K (*absolute zero*) is lowest temperature possible

- conversion rule: $T_K(\mathrm{K}) \underset{+273}{\overset{-273}{\rightleftharpoons}} T_C(^\circ\mathrm{C})$
- change of 1°C equals change of 1 K



5.1.2 kinetic theory of matter

there are three common states of matter: solid, liquid and gas

they have very different physical properties (density, compressibility, fluidity, etc.)

but deep down, they are all composed of a large number of small molecules

in the **kinetic theory of matter**, we look at microscopic behaviour at molecular level (arrangement, motion, intermolecular forces, separation, etc.)

microscopic behaviour of molecules cause differences in *macroscopic* properties of matter

- solid: molecules close together, tightly bonded, vibrate about their positions
- liquid: molecules quite close together, vibrate but has some freedom to move about
- gas: molecules widely separated, free from neighbours, move rapidly

5.1.3 specific latent heat

it requires heat energy to *melt* a solid or *boil* a liquid

melting and boiling usually occur at a fixed temperature

thermal energy to cause the change of state at a constant temperature is called *latent heat*

amount of latent heat needed depends on mass of substance: $\boxed{Q = Lm}$

we define **specific latent heat** (L) as the thermal energy required to change the state of *unit* mass of substance with no change in temperature is called

- unit of specific latent heat: $[L] = \mathrm{J \cdot kg^{-1}}$
- specific latent heat is an *intensive* property

  i.e., L does not depend on size or shape of sample, L depends on type of substance only
- for melting, L is called *specific latent heat of fusion*

  for boiling, L is called *specific latent heat of vaporisation*
- latent heat is related to breaking bonds and increasing intermolecular separation

  vaporisation requires larger increase in particle separation than fusion

  for a given substance, $L_{\text{vapour}} > L_{\text{fuse}}$

Example 5.1 A 3.0 kW electric kettle contains 0.5 kg of water already at its boiling point. Neglecting heat losses, determine how long it takes to boil dry. ($L_{\text{water}} = 2.26 \times 10^6\ \mathrm{J\ kg^{-1}}$)

✍ heat required: $Q = mL = 0.50 \times 2.26 \times 10^6 = 1.13 \times 10^6$ J

time needed: $t = \dfrac{Q}{P} = \dfrac{1.13 \times 10^6}{3.0 \times 10^3} \approx 380\ \mathrm{s} \approx 6.3\ \mathrm{min}$ □

Example 5.2 A student measures specific latent heat of fusion for ice. He uses an electric heater to melt ice but the insulation is not perfect. The experiment is carried out twice, with the heater operating at different powers. Use the data table to calculate specific latent heat of fusion.

	Power (W)	time interval (min)	mass of ice melted (g)
test 1	60	3.0	40.4
test 2	90	3.0	56.6

✍ there exists heat gain from surroundings, so effective power $P_{\text{eff}} = P_{\text{heater}} + P_{\text{sur}}$

heat energy to melt ice: $Q = mL = (P_{\text{heater}} + P_{\text{sur}})t$

$$\begin{cases} 40.4 \times L = (60 + P_{\text{sur}}) \times 3.0 \times 60 \\ 56.6 \times L = (90 + P_{\text{sur}}) \times 3.0 \times 60 \end{cases} \Rightarrow \begin{cases} L \approx 333 \text{ J g}^{-1} \\ P_{\text{sur}} \approx 14.8 \text{ W} \end{cases}$$ □

Question 5.1 A student designs an experiment to determine the specific latent heat of fusion L of ice. Some ice at 0°C is heated with an electric heater. The experiment is carried out twice and the following data are obtained.

	energy supply from heater (J)	time interval (min)	mass of ice melted (g)
heater off	0	10.0	14.3
heater on	21000	5.0	70.0

(a) Suggest why two sets of readings are taken. (b) Find specific latent heat of fusion for ice.

5.1.4 specific heat capacity

heating a substance could cause an increase in its temperature

heat required is proportional to its mass m and temperature change ΔT: $\boxed{Q = cm\Delta T}$

we define **specific heat capacity** (c) as the thermal energy required per unit mass of substance to cause an increase of one unit in its temperature

- unit of specific heat capacity: $[c] = \text{J kg}^{-1}\text{ K}^{-1}$ or $\text{J kg}^{-1}\ {}^\circ\text{C}^{-1}$
- c is also an *intensive* property, i.e., independent of size or shape of the sample

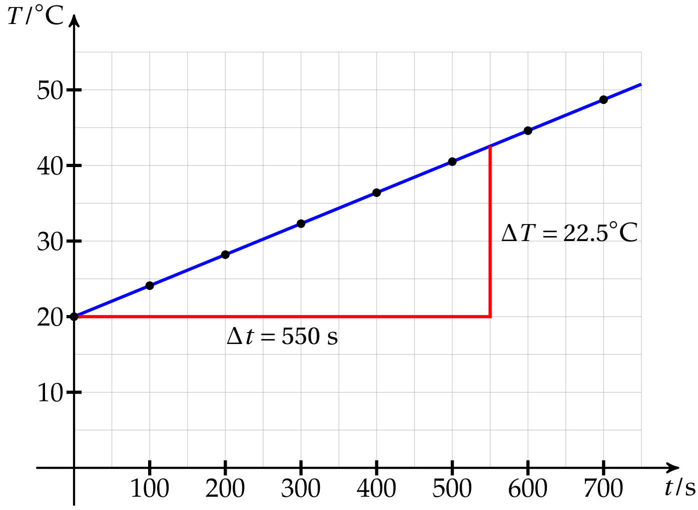
Example 5.3 A 1.00 kg aluminium block is heated using an electrical heater. The current in the heater is 4.2 A and the p.d. across is 12 V. Measurements of the rising temperature are represented by the graph. Determine specific heat capacity of aluminium.

✍ energy supplied: $Q = cm\Delta T \quad\Rightarrow\quad IV\Delta t = cm\Delta T \quad\Rightarrow\quad c = \dfrac{IV}{m\frac{\Delta T}{\Delta t}}$

$\dfrac{\Delta T}{\Delta t}$ is gradient of fitting line: $\dfrac{\Delta T}{\Delta t} = \dfrac{22.5}{550} \approx 4.09 \times 10^{-2}\ {}^\circ\text{C s}^{-1}$

specific heat capacity: $c = \dfrac{4.2 \times 12}{1.00 \times 4.09 \times 10^{-2}} \approx 1230 \text{ J kg}^{-1}\ {}^\circ\text{C}^{-1}$ □

Example 5.4 A block of 30 g ice at −20°C is added to a large cup of 270 g water at 80°C. Assume there is no energy lost, what is the final temperature of the mixture? (data: specific



heat capacity of water is 4200 J kg^{-1} K^{-1}, specific heat capacity of ice is 2100 J kg^{-1} K^{-1}, specific latent heat of ice is 3.3×10^5 J kg^{-1}.

✍ energy lost by hot water = energy gain by ice cube

$$\underbrace{4200 \times 0.27 \times (80 - T)}_{95\,^\circ\text{C water} \to T\,^\circ\text{C water}} = \underbrace{2100 \times 0.030 \times [0 - (-20)]}_{-20\,^\circ\text{C ice} \to 0\,^\circ\text{C ice}} + \underbrace{3.3 \times 10^5 \times 0.030}_{0\,^\circ\text{C ice} \to 0\,^\circ\text{C water}} + \underbrace{4200 \times 0.030 \times (T - 0)}_{0\,^\circ\text{C water} \to T\,^\circ\text{C water}}$$

$$90720 - 1134T = 1260 + 9900 + 126T$$

$$T = \frac{90720 - 1260 - 9900}{1134 + 126} \approx 63^\circ\text{C}$$

□

Question 5.2 A mixture contains 5% silver and 95% of gold by weight. Some gold is melted and the correct weight of silver is added. The initial temperature of silver is 20°C. Use the data to calculate the initial temperature of gold so that the final mixture is at melting point of gold.

	silver	gold
melting point (K)	1240	1340
specific heat capacity (solid or liquid) (J kg^{-1} K^{-1})	235	129
specific latent heat of fusion (kJ kg^{-1})	105	628

5.2 internal energy

we now consider the total energy within a thermodynamic system

molecules in a system undergo random motion, so they have kinetic energy there are potential energy between molecules due to intermolecular interaction

internal energy is defined as the sum of random kinetic energy of molecules and potential energy between molecules: $\boxed{U = E_k + E_p}$

- internal energy is a *state function* of the system

  it only depends on current state of system, not on process to arrive at this state

5.2.1 kinetic energy

- internal energy counts K.E. due to random motion at molecular level

  K.E. of macroscopic motion of the system as a whole is not included
- mean K.E. of molecules is directly proportional to temperature: $\boxed{E_k \propto T}$

  K.E. of molecules depends on temperature only

  higher temperature means molecules move faster, vibrate more intensively, etc.

5.2.2 potential energy

- internal energy counts P.E. due to force fields *within* the system

  P.E. of the system as a whole due to *external* force fields is not included
- P.E. between molecules depends on intermolecular separation and chemical bonding

  in general, greater intermolecular separation means greater P.E.[21]: $\boxed{r \nearrow \Leftarrow E_p \nearrow}$

  breaking intermolecular bonds also causes an increase in P.E.

  mean P.E. of gas > mean P.E. of liquid > mean P.E. of liquid solid

5.2.3 internal energy of ideal gas

for ideal gas, mean K.E. of one molecule: $E_k = \dfrac{3}{2}kT$

[21]Intermolecular separation does not necessarily increase during melting processes. A typical counter example is melting of ice into water, for which intermolecular separation actually decreases (density of water > density of ice), but potential energy of the system will still increase because hydrogen bonds between H_2O molecules are broken.

there is no intermolecular force, so P.E. of ideal gas is defined to be zero: $E_p = 0$

internal energy per molecule: $U = E_k + E_p = \frac{3}{2}kT$

hence internal energy of ideal gas is purely kinetic and directly proportional to temperature

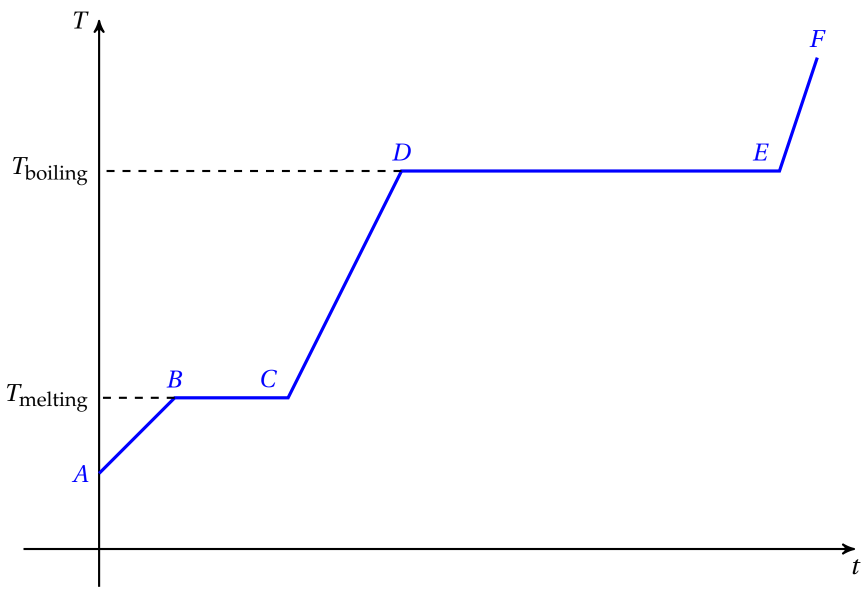
total internal energy of the gas: $U_{\text{gas}} = NU \quad \Rightarrow \quad \boxed{U_{\text{gas}} = \frac{3}{2}NkT}$

5.2.4 change of states

consider a substance being heated from its solid state

it will melt into a liquid and further vaporise into its gaseous state

we now look into the changes of internal energy during each stage



- AB (solid state): $T \nearrow \Rightarrow E_k \nearrow$, greater vibration for solid particles

  but no (significant) change in mean separation[22] $\Rightarrow$ no change in P.E.

- BC (*melting*): latent heat goes into breaking intermolecular bonds, $r \nearrow \Rightarrow E_p \nearrow$

  but melting occurs at constant temperature[23] $\Rightarrow$ no change in K.E.

[22]A typical solid material expands when it is heated, so intermolecular separation will increase slightly.

[23]Here we talk about *pure substance*, which changes from solid into liquid at a particular temperature, called the *melting point*. But for a *mixture* of substances, melting may occur over a *range* of temperatures. It is also possible for a substance to decompose before they change states.

- *CD* (liquid state): $T \nearrow \Rightarrow E_k \nearrow$, greater vibration and free motion

  but no (significant) change in mean separation $\Rightarrow$ no change in P.E.
- *DE* (*boiling*): molecules break free, $r \nearrow \Rightarrow E_p \nearrow$

  boiling occurs at constant temperature[24] $\Rightarrow$ no change in K.E.
- *EF* (gas state): $T \nearrow \Rightarrow E_k \nearrow$ particles move even faster

  particles completely separated, no intermolecular force, so constant $E_p = 0$

Question 5.3 For a particular substance, why is the specific latent heat of vaporisation much greater than the specific latent heat of fusion?

evaporation

liquid changes into gas without boiling $\longrightarrow$ **evaporation**

particles move randomly, i.e., they move at various speeds

some molecules move fast enough to break free

➢ *cooling effect*: evaporation causes a decrease in temperature of the liquid

most energetic molecules escaped, those remain in the liquid have less energy, $E_k \searrow \Rightarrow T \searrow$

➢ rate of evaporation increases with temperature, surface area of liquid

➢ different between boiling and evaporation

	boiling	evaporation
occurrence	throughout the liquid	at surface only
temperature	occur at boiling point	occur at any temperature
bubble formation	bubbles formed	no bubbles
rate of process	fast	slow

5.2.5 first law of thermodynamics

internal energy of a system changes upon heat transfer or doing work

[24] Again we only concern pure substances. Mixtures that boil over a range of temperatures or substance decompose before phase transition are not considered here.

first law of thermodynamics states that the increase in internal energy equals sum of heat supply to the system and work done on the system: $\boxed{\Delta U = Q + W}$

- first law of thermodynamics is an extension of the law of conservation of energy
- sign conventions for Q and W
  - $Q > 0$ if heat supplied to system
  - $Q < 0$ if heat released by system to surroundings
  - $W > 0$ if work done *on* system by external

    i.e., if system is compressed and volume decreases, then $W > 0$
  - $W < 0$ if system does work *against* surroundings

    i.e., system expands and volume increases, then $W < 0$
- amount of heat energy: $Q = \begin{cases} cm\Delta T & \text{(if no change of state)} \\ Lm & \text{(during change of state)} \end{cases}$
- amount of work is related to pressure and change of volume

  if volume changed at *constant* pressure, then $W = F\Delta s = pA\Delta s \Rightarrow \boxed{W = p\Delta V}$[25]

  if no change of volume, then no work is done

Example 5.5 A gas is heated by supplying it with 25 kJ of energy. The gas expands so that the volume increases by 0.10 m^3. Assume the gas has a fixed pressure of 150 kPa during the process. Calculate the change in internal energy.

✍ amount of work done: $W = p\Delta V = 150 \times 0.10 = 15$ kJ

but gas expands means work is done against surroundings, so this is negative work

change in internal energy: $\Delta U = Q + W = (+25) + (-15) = +10$ kJ □

Example 5.6 Use the idea of internal energy and the first law of thermodynamics, explain why boiling water requires heat supply.

✍ boiling occurs at constant temperature, so $\Delta E_k = 0$

but separation between molecules increased, so $\Delta E_p > 0$

[25]If pressure changes with volume during a thermodynamics process, then work done $W = \int p\mathrm{d}V$. Alternatively, we can evaluate the area under a p-V graph to find the work done.

by definition, internal energy $U = E_k + E_p$, so $\Delta U > 0$

during boiling, there is an increase in volume, so work against surroundings, $W < 0$

recall first law of thermodynamics $\Delta U = Q + W$, must have $Q > 0$

this means heat must be supplied for boiling processes □

Example 5.7 When you pump up a bicycle tyre, the temperature of air inside the tyre will go up. Explain why this happens using the first law of thermodynamics.

✍ pumping up tyre involves compressing gas, so positive work is done: $W > 0$

for each stroke, there is little time for heat transfer, so $Q \approx 0$

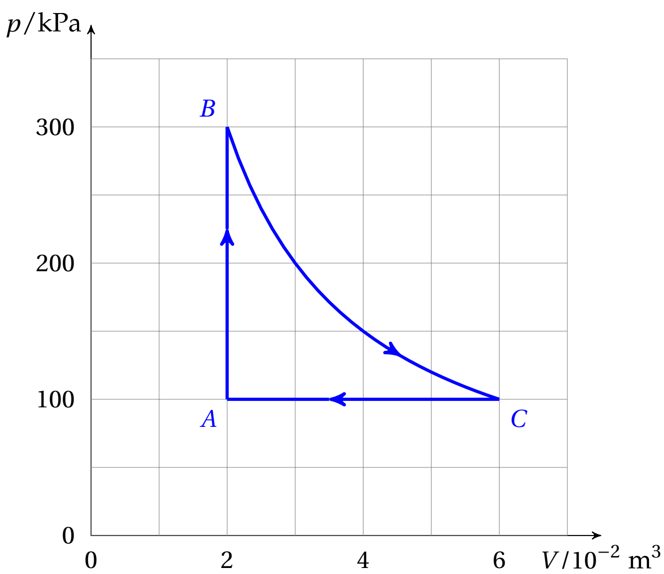
according to first law of thermodynamics $\Delta U = Q + W \quad \Rightarrow \quad \Delta U > 0$

by definition, internal energy $U = E_k + E_p \quad \Rightarrow \quad \Delta U = \Delta E_k + \Delta E_p > 0$

but for gas, there is negligible intermolecular force, so $\Delta E_p = 0$, then must have $\Delta E_k > 0$

K.E. of molecules is proportional to temperature, higher K.E. so higher temperature □

Example 5.8 An ideal gas of 0.080 mol is initially at state A and then undergoes a cycle $ABCA$. The variation of its pressure p with its volume V is shown on the graph.



Temperature of state A is 300 K. The magnitude of work on gas from state B to C is 6570 J.

For each stage $A \to B$, $B \to C$ and $C \to A$ during the cycle, determine work done and heat supply to the gas, and also find the change in internal energy.

✍ work done depends on change in volume

$A \to B$: no change in volume, so $W_{AB} = 0$

$B \to C$: $|W_{BC}| = 6570$ J, but expansion implies $W < 0$, so $W_{BC} = -6570$ J

$C \to A$: $|W_{CA}| = p\Delta V_{CA} = 1 \times 10^5 \times (6-2) \times 10^{-2} \quad \Rightarrow \quad W_{CA} = +4000$ J (compression so $W > 0$)

change in internal energy of ideal gas depends on change in temperature

$A \to B$: same V but $p_B = 3p_A$, so $T_B = 3T_A = 900$ K

$$\Delta U_{AB} = \frac{3}{2}Nk\Delta T_{AB} = \frac{3}{2} \times 0.80 \times 6.02 \times 10^{23} \times 1.38 \times 10^{-23} \times (900-300) \approx +5980 \text{ J}$$

$B \to C$: note that $p_B V_B = p_C V_C$, so $T_B = T_C$, no change in temperature, so $\Delta U_{BC} = 0$

$C \to A$: $\Delta U_{CA} = \frac{3}{2}Nk\Delta T_{CA} = \frac{3}{2} \times 0.80 \times 6.02 \times 10^{23} \times 1.38 \times 10^{-23} \times (300-900) \approx -5980$ J

for cycle $ABCA$, same initial and final state, so total change in internal energy must be zero

one can check that $\Delta U_{\text{cycle}} = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA} = 0$

to find supply of thermal energy, we apply first law of thermodynamics: $\Delta U = Q + W$

$A \to B$: $+5980 = Q_{AB} + 0 \quad \Rightarrow \quad Q_{AB} = +5980$ J

$B \to C$: $0 = Q_{BC} + (-6570) \quad \Rightarrow \quad Q_{BC} = +6570$ J

$C \to A$: $-5980 = Q_{CA} + (+4000) \quad \Rightarrow \quad Q_{CA} = -9980$ J

the table below summarises all energy changes during the cycle $ABCA$

change	W/J	Q/J	ΔU/J
$A \to B$	0	+5980	+5980
$B \to C$	-6570	+6570	0
$C \to A$	+4000	-9980	-5980

□

Question 5.4 Show that when n mol of gas is heated at a fixed volume, thermal energy required to raise the temperature by 1.0 K is nR.

Question 5.5 Two identical balloons A and B hold the same amount of gas at the same initial temperature. They are given the same amount of heat. Suppose volume of A is fixed, while B is allowed to expand, compare the final temperatures of the gases in the two balloons.

5.3 temperature

5.3.1 temperature & thermal energy

➢ temperature can be considered as a *relative* measure of thermal energy

temperature can tell the *direction* of thermal energy flow

heat always (spontaneously) flows from high temperature regions to colder regions[26]

➢ if two objects in contact have the same temperature, then there is no net heat transfer

the two objects are said to be in **thermal equilibrium**

➢ if two systems A and B are each in thermal equilibrium with a third system C, A and B are also in thermal equilibrium, this is called the **zeroth law of thermodynamics** [27]

Question 5.6 A student thinks that temperature measures the amount of heat in an object. Suggest why this statement is incorrect with examples.

5.3.2 absolute zero

mean K.E. of molecules is a microscopic description of temperature T

minimum K.E. occurs if molecules do not move at all (completely frozen)[28]

this corresponds to the lowest possible temperature, called **absolute zero**

➢ **Kelvin scale of thermodynamic temperature** is defined based on absolute zero as 0 K[29]

[26]This is the consequence of the *second law of thermodynamics*, which is concerned with the direction of natural processes. The law states that the total *entropy*, a quantity that counts the number of microstates of a system, of an isolated system can never decrease over time. You will learn more about entropy if you study A-Level chemistry.

[27]This law is important for the formulation of thermal physics. The physical meaning of the law was expressed by Maxwell: "All heat is of the same kind." The zeroth law allows us to give the mathematical definition of temperature.

[28]In the *classical* description, there is no reason not allowing a molecule to cease motion. However, due to *quantum mechanical effects*, kinetic energy of a system cannot be zero even at absolute zero.

[29]More precisely, 0 K for absolute zero and 273.16 K for water triple point.

- conversion rule between Celsius scale and Kelvin scale: $\boxed{T_K(\mathrm{K}) \underset{+273.15}{\overset{-273.15}{\rightleftharpoons}} T_C(^\circ\mathrm{C})}$ [30]
- Kelvin scale is said to be an *absolute scale*

  zero of Kelvin scale does not depend on property of a specific substance

  in contrast, zero of Celsius scale is based on properties of water
- it is impossible to remove any more energy from a system at 0 K (or -273.15 °C)

  but there is no practicable means to bring a physical system to exactly 0 K [31]

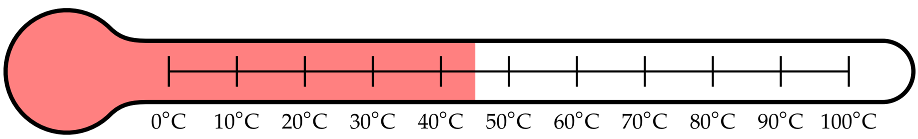
5.3.3 thermometer

a **thermometer** is a device which can be used to measure temperature

liquid-in-glass thermometer

basic principle: liquid expands in volume at higher temperature

examples include alcohol thermometer, mercury-in-glass thermometer, etc.



resistance temperature detectors (RTD)

basic principle: resistance of electronic element changes with temperature

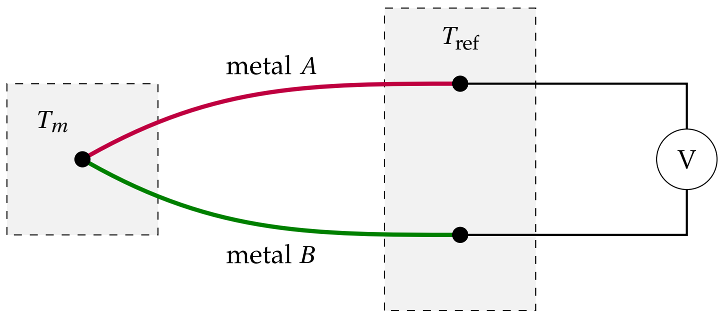
metal wires and thermistor are both used in RTD elements

thermocouple

basic principle: difference in temperature can produce a *thermoelectric voltage* across junctions, thermocouple measures temperatures by means of this voltage

[30]The numerical value of 273.15 will only be quoted in this section. For everywhere else in the notes, we will use the less precise value 273 for simplicity.

[31]This is known as the *third law of thermodynamics*, which states that it is impossible, no matter how idealized, to reduce the temperature of any closed system to absolute zero in a finite number of operations.



typical configuration of a thermocouple unit

two pieces of different metal wires are joined at their ends

if there exists a temperature difference between the ends, a *thermoelectric voltage* is developed

this voltage depends on temperature difference, captured by some characteristic function

in practice, we place the *measurement* junction in an environment of unknown temperature

the other end, or the *reference* junction, is at a known temperature

temperature difference is deduced from the voltage reading

hence the desired temperature can be determined

➢ features of a thermometer:

- *range*: whether the thermometer can measure very low or very high temperatures
- *sensitivity*: whether a small change in temperature can be detected
- *response time*: whether changes in temperature can be immediately measured
- *linearity*: whether changes in temperature are proportional to changes in output

	liquid-in-glass	RTD wire	thermistor	thermocouple
valid range	narrow	wide	narrow	very wide
sensitivity	low	fair	high	high
response time	slow	fast	fast	very fast
linearity	good	good	limited	non-linear

Chapter 6

Electrostatics

6.1 electric forces

6.1.1 Coulomb's law

charged objects will attract or repel one another through the electric force

Coulomb's law states that the electric force between two electrically charged particles is proportional to their charges and inversely proportional to the square of their separation: $\boxed{F = \dfrac{Qq}{4\pi\epsilon_0 r^2}}$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$ is the *permittivity of free space*

$k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \mathrm{N\ m^2\ C^{-2}}$ is a useful constant for calculations

this law was first published by French physicist *Charles Augustin de Coulomb* in 1785

- charges Q, q in Coulomb's law are *point charges*
- for uniformly charged spheres, they can be thought as point charges

  separation r is taken to be centre-to-centre distance
- symbolically, sign of F can tell direction of the electric force

  for like charges (both positive or both negative), $Q_1 Q_2 > 0 \Rightarrow F > 0 \Rightarrow$ *repulsion*

  for opposite charges (one positive and one negative), $Q_1 Q_2 < 0 \Rightarrow F < 0 \Rightarrow$ *attraction*

Example 6.1 The hydrogen atom has a radius of about 53 pm. Estimate the electric force between the proton and the orbiting electron.

✍ $$F = \frac{Q_p Q_e}{4\pi\epsilon_0 r^2} = 8.99 \times 10^9 \times \frac{(1.60 \times 10^{-19})^2}{(53 \times 10^{-12})^2} \approx 8.2 \times 10^{-8}\ \mathrm{N}$$ □

Question 6.1 Two protons are separated by a distance r. Find the ratio of the electric force to the gravitational force between them.

6.1.2 electric fields

to explain how charges affect each other at a distance, we introduce notion of *electric fields*

electric field is a region of space where a charged object is acted by a force

any charge Q (or several charges) can produce an electric field

any test charge q within the field will experience an electric force

Next, we will introduce the concepts of *electric field strength* and *electric potential*, and see how they are related to the force acting on a charged object and the potential energy it possesses.

You might have noticed that Coulomb's law for electrostatic forces and Newton's law of gravitation are both *inverse square laws*, it turns out that electric fields are very similar to gravitational fields in various aspects.

6.2 electric field strength

6.2.1 electric field strength

electric field strength is defined as electric force per unit positive charge: $\boxed{E = \frac{F}{q}}$

- unit of E: $[E] = \mathrm{N\,C^{-1}} = \mathrm{V\,m^{-1}}$[32]
- field strength due to an isolated source of *point* charge Q

  a small test charge q at distance r is acted by a force: $F = \frac{Qq}{4\pi\epsilon_0 r^2}$

  field strength at this point: $E = \frac{F}{q} \quad \Rightarrow \quad \boxed{E = \frac{Q}{4\pi\epsilon_0 r^2}}$

  the field is produced by Q, so field strength only depends on the source Q
- if the source is a charged *sphere* of radius R with uniform charge distribution

  viewed from *outside* the sphere, it acts like a point charge concentrated at the centre [33]

[32]You will later find in §6.4.3 the deeper reason why $\mathrm{V\,m^{-1}}$ is also a reasonable unit for field strength.

[33]A brief explanation is given in Example 6.4.

therefore, $E = \dfrac{Q}{4\pi\epsilon_0 r^2}$ also holds for field strength at $r > R$[34]

where r is the distance from the point of interest to *centre* of the sphere

➢ field strength E is a *vector* quantity, it has a direction

to compute combined field strength due to several sources, should perform *vector sum* of contributions from each individual

➢ direction of field strength depends on the source charge Q

for positive source ($Q > 0$): field points away from the source

for negative source ($Q < 0$): field points towards the source

➢ electric force on a charge q can be found if field strength E is known

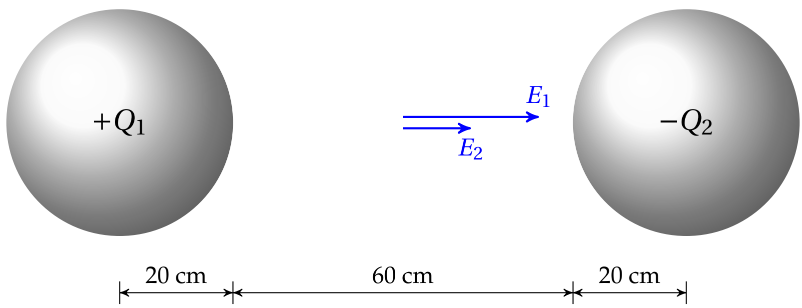
magnitude of electric force: $F = Eq$

direction of force: same direction as E if $q > 0$, but opposite to E if $q < 0$

Example 6.2 A *Van de Graaff generator* produces sparks when its surface electric field strength 4.0×10^4 V cm^{-1}. If the diameter of the sphere is 40 cm, what is the charge on it?

✍ $$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad\Rightarrow\quad Q = 4\pi\epsilon_0 E r^2 = 4\pi \times 8.85 \times 10^{-12} \times 4.0 \times 10^6 \times 0.20^2 \approx 1.8 \times 10^{-5}\ \text{C}$$ □

Example 6.3 Two identical metal spheres of radius 20 cm carry charges +2.0 μC and −1.0 μC respectively. There is a 60 cm gap between them. (a) Find the electric field strength midway along the line joining their centres. (b) A dust particle carrying a charge of -1.3×10^{-8} C is at this position. Find the electric force it experiences.



✍ field strengths due to the two spheres are in same direction

$$E = E_1 + E_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{r_1^2} + \frac{Q_2}{r_2^2}\right) = 8.99 \times 10^9 \times \left(\frac{2.0 \times 10^{-6}}{0.25^2} + \frac{1.0 \times 10^{-6}}{0.25^2}\right) \approx 4.32 \times 10^5\ \text{N C}^{-1}$$

[34]For electric field strength *inside* a conducting sphere, detailed discussions are given in §6.4.2.

field strength points to the right

force on dust particle: $F = Eq = 4.32 \times 10^5 \times 1.4 \times 10^{-8} \approx 5.6 \times 10^{-3}$ N

dust particle is negatively-charged means force is opposite to field strength

so force on dust particle acts to the left □

Question 6.2 When the charge on the Van de Graaff generator is 4.0×10^{-7} C, the electric field strength at the sphere's surface is 2.4×10^6 V m^{-1}. Determine the additional charge added to the sphere if the field strength at the surface becomes 3.0×10^6 V m^{-1}.

Question 6.3 Two positively charged particles A and B are situated in a vacuum. Point P lies on the line joining the centres of the two spheres and is a distance x from A. Sketch the variation with x of electric field strength E due to the two particles.

6.2.2 electric field lines

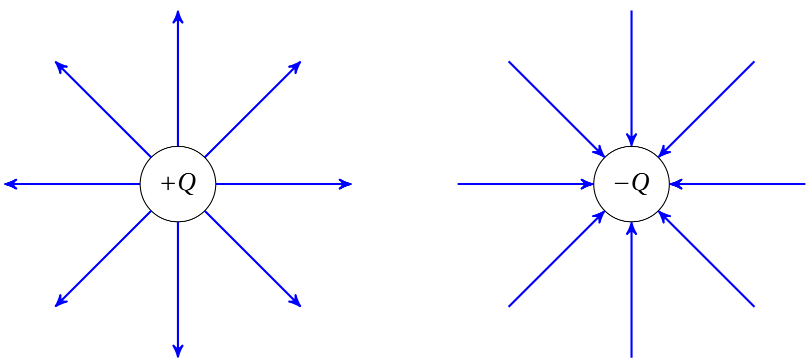
we can use **electric field lines** to visualize an electric field

- *arrows* of field lines show direction of the field

field lines always tend to leave positive charge, and end up at negative charges

- *density* or spacing of lines show strength of the field

Example 6.4 Sketch the electric field around a positively-charged sphere or a negatively charged sphere, and explain why they can be considered as point charges.

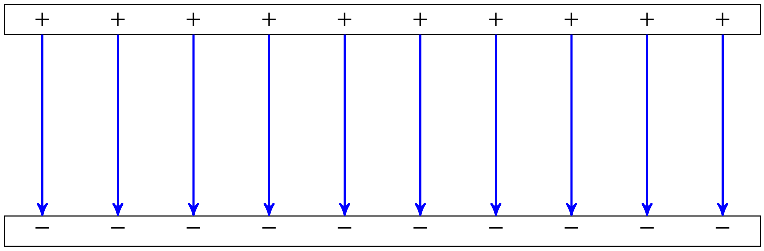


field lines of either case are *radial*, i.e., perpendicular to surface

field lines appear to start from or converge towards centre of the sphere

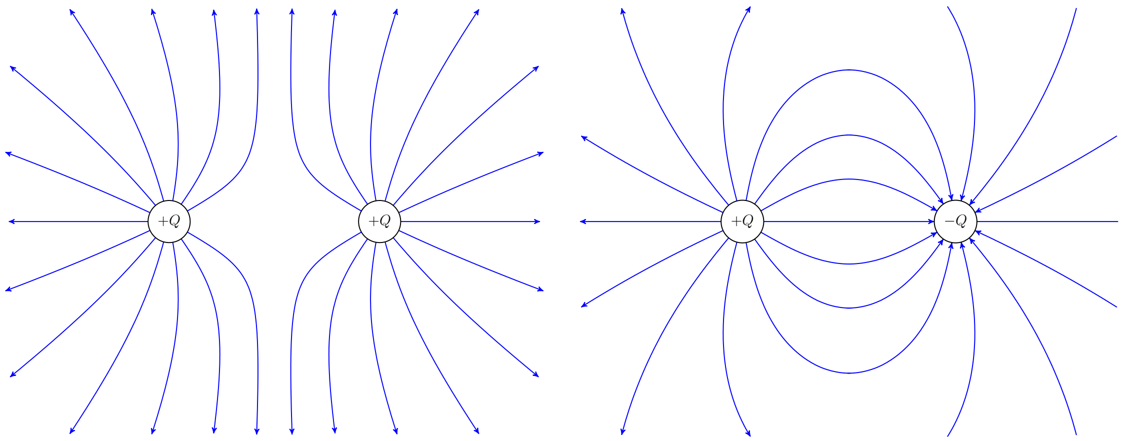
so charged spheres act like point charges □

Example 6.5 Field lines between two oppositely-charged large metal plates.



field lines are *parallel* and equally spaced, so this is a *uniform* electric field □

Example 6.6 Field pattern due to two charges of equal magnitude.



two positive charges two opposite charges □

6.3 potential & potential energy

6.3.1 electric potential energy

gain/loss in **electric potential energy** is defined as work done against/by electric force

(compare everything in this section with what you have learned about gravitational P.E.!)

let's start to derive the electrical P.E. between two charges Q and q separated by r

again we define $E_p = 0$ at $r = \infty$ (choice of zero potential energy, no force so no P.E.), then

> **electric potential energy** is equal to the work done by electric force to bring a charge to a specific position from *infinity*

moving a test charge q from $r = \infty$ to a distance of r from Q

work done by electric force: $W = \int_{\infty}^{r} F dr = \int_{\infty}^{r} \frac{Qq}{4\pi\epsilon_0} \frac{dx}{x^2} = -\frac{Qq}{4\pi\epsilon_0 x}\Big|_{\infty}^{r} = -\frac{Qq}{4\pi\epsilon_0 r}$

since $\Delta E_p = -W$, we find: $E_p(r) - E_p(\infty) = \dfrac{Qq}{4\pi\epsilon_0 r}$

but $E_p(\infty) = 0$, so electric P.E. between two charges Q and q is $\boxed{E_p(r) = \dfrac{Qq}{4\pi\epsilon_0 r}}$

➢ as $r \to \infty$, $E_p \to 0$, this agrees with our definition for zero P.E. point

➢ for like charges, $Qq > 0$, so $E_p > 0$

to bring like charges closer, work must be done to overcome their *repulsion*, P.E. increases

minimum P.E. $E_p(\infty) = 0$ at infinity, so positive P.E. at finite r

➢ for opposite charges, $Qq < 0$, so $E_p < 0$

to pull opposite charges apart, work must be done to overcome their *attraction*, P.E. increases

maximum P.E. $E_p(\infty) = 0$ at infinity, so negative P.E. at finite r

➢ electric P.E. is a *scalar quantity*, sign is important

repulsion implies positive P.E., and attraction implies negative P.E.

sign of P.E. is hidden in polarities of charges

Example 6.7 To gain information about the gold nucleus ($^{197}_{79}$Au), we fire α-particles ($^4_2\alpha$) towards a thin gold foil. The size of a typical nucleus is about 10^{-14}m, what is the minimum initial speed for α-particles so that radius of gold nucleus can be determined?

✍ as α-particle approaches the nucleus, it slows down due to the repulsive interaction

kinetic energy decreases and electric potential energy increases

if it gets close enough to the nucleus before coming to a stop, nuclear radius can be estimated

$$\text{K.E. loss} = \text{P.E. gain} \quad \Rightarrow \quad \frac{1}{2}mu^2 - \underbrace{\frac{1}{2}mv^2}_{0} = E_p(r) - \underbrace{E_p(\infty)}_{0} \quad \Rightarrow \quad \frac{1}{2}mu^2 = \frac{Qq}{4\pi\epsilon_0 r}$$

$$\frac{1}{2} \times 4 \times 1.66 \times 10^{-27} \times u^2 = \frac{79 \times 1.60 \times 10^{-19} \times 2 \times 1.60 \times 10^{-19}}{4\pi \times 8.85 \times 10^{-12} \times 10^{-14}} \quad \Rightarrow \quad u \approx 3.3 \times 10^7 \text{ m s}^{-1} \qquad \square$$

Question 6.4 A metal sphere of radius 20 cm carries a charge of 5.0×10^{-7} C. A proton is sent towards the sphere at a speed of 1.8×10^{6} m s^{-1}. Can the proton reach the surface of the sphere?

6.3.2 electric potential

it is also useful to define the *electric potential* for any specific point in a field

electric potential can be thought as the electric potential energy per unit charge: $V = \dfrac{E_p}{q}$

electric potential is the work needed to bring a unit positive charge from infinity

- unit: $[V] = \text{J C}^{-1} = \text{V}$
- electric potential due to an isolated source Q: $V = \dfrac{E_p}{q} = \dfrac{\frac{Qq}{4\pi\epsilon_0 r}}{q} \quad \Rightarrow \quad \boxed{V = \dfrac{Q}{4\pi\epsilon_0 r}}$
- potential at infinity vanishes: $V_\infty = 0$
- electric potential can take both signs

  the sign depends on whether unit positive charge is repelled or attracted by the source

  for positively-charged sources $V > 0$, while for negatively-charged sources $V < 0$
- electric potential is a *scalar* quantity

  to find combined potential due to multiple charges, add up contributions of each charge

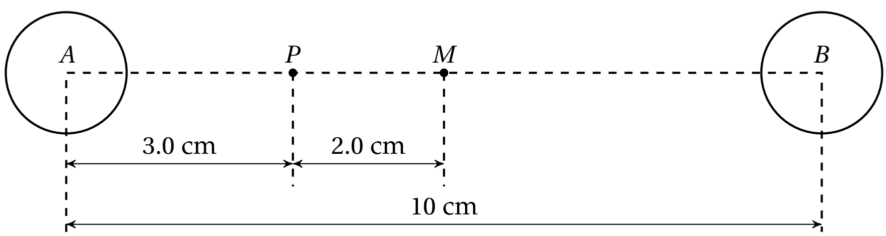
Example 6.8 An electron is accelerated from rest through a potential difference of 600 V.

Find the final speed of the electron.

✍ gain in K.E. = change in electric P.E. $\quad \Rightarrow \quad \frac{1}{2}mv^2 = q\Delta V$

$$v = \sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2.60 \times 10^{-19} \times 600}{9.11 \times 10^{-31}}} \approx 1.45 \times 10^{7} \text{ m s}^{-1}$$ □

Example 6.9 Two small metal spheres A and B are in a vacuum. Sphere A has charge +20 pC and sphere B has charge +84 pC. The arrangement is shown below.



(a) Find the electric potential at point P and point M respectively.

(b) Find the work done to move an α-particle from P to M.

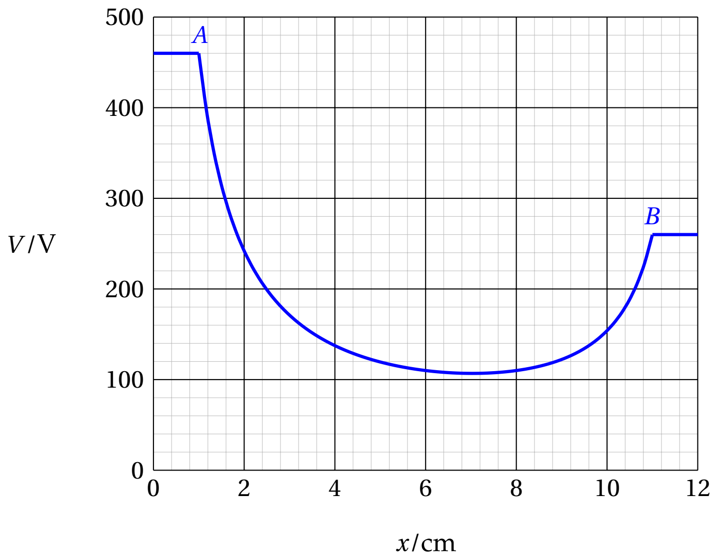
✍ combined electric potential: $V = V_A + V_B = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{Q_A}{r_A} + \dfrac{Q_B}{r_B}\right)$

at P: $V_P = 8.99 \times 10^9 \times \left(\dfrac{+20 \times 10^{-12}}{0.030} + \dfrac{+84 \times 10^{-12}}{0.070}\right) \approx 16.8$ V

at M: $V_M = 8.99 \times 10^9 \times \left(\dfrac{+20 \times 10^{-12}}{0.050} + \dfrac{+84 \times 10^{-12}}{0.050}\right) \approx 18.7$ V

from P to M: $W = \Delta E_p = q\Delta V = q(V_M - V_P) = 2 \times 1.60 \times 10^{-19} \times (18.7 - 16.8) \approx 6.2 \times 10^{-19}$ J □

Example 6.10 A and B are two positively-charged spheres of radius 1.0 cm. A proton P initially at rest on the surface of A moves along the line joining the centres of the two spheres. The variation with distance x from the centre of A of electric potential V at point P is given.



(a) Find the maximum speed as the proton moves from A to B.

(b) Find the speed when the proton reaches surface of B.

✍ increase in K.E. = loss in P.E., so: $\dfrac{1}{2}mv^2 - 0 = q\Delta V \quad \Rightarrow \quad \dfrac{1}{2}mv^2 = q(V_A - V_P)$

maximum speed when ΔV is maximum, or $V_P = 107$ V becomes minimum (at $x = 7.2$ cm)

$$\frac{1}{2} \times 1.67 \times 10^{-27} \times v_{\text{max}}^2 = 1.60 \times 10^{-19} \times (460 - 107) \quad \Rightarrow \quad v_{\text{max}} \approx 2.60 \times 10^5 \text{ m s}^{-1}$$

at surface of B, $V_P = 260$ V (at $x = 11.0$ cm)

$$\frac{1}{2} \times 1.67 \times 10^{-27} \times v_B^2 = 1.60 \times 10^{-19} \times (460 - 260) \quad \Rightarrow \quad v_B \approx 1.96 \times 10^5 \text{ m s}^{-1}$$ □

Question 6.5 Electrical breakdown occurs when electric field strength at surface of a metal sphere exceeds 5.0×10^6 N C^{-1}. Given that the radius of the sphere is 16 cm. What is the electric potential at the surface when electrical breakdown occurs?

Question 6.6 Two charged particles A and B are separated by 20 cm. P is a point on the line AB. Given that particle A carries charge $+7.2$ μC, and electric potential is zero where $AP = 5.0$ cm. Find the electric charge of B.

Question 6.7 A particle with specific charge (ratio of its electric charge to its mass) $+9.58 \times 10^7$ C kg^{-1} is moving towards a fixed metal sphere. The sphere has a potential of $+500$ V. The initial speed of the particle is 3.0×10^5 m s^{-1} when it is a large distance from the sphere. Determine whether the particle can reach the surface of the sphere.

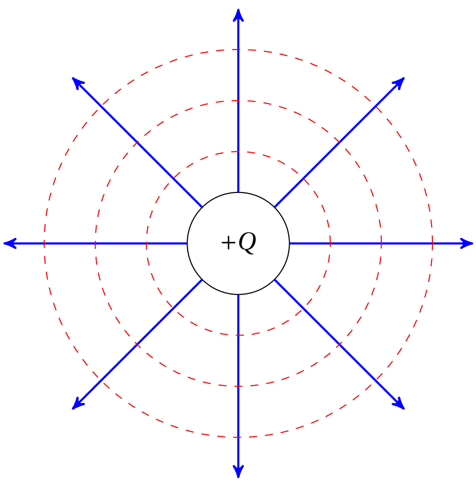
6.3.3 equipotential lines

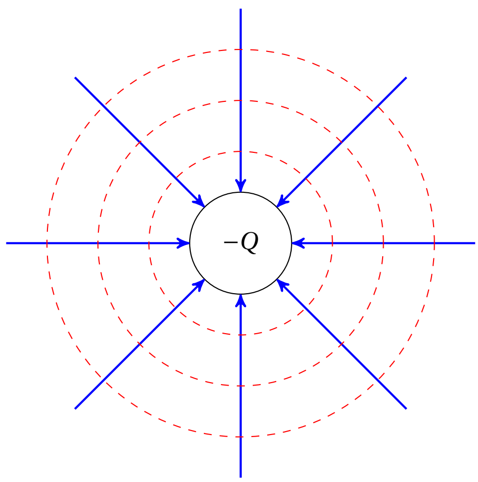
to show potential distributions, we draw **equipotential lines** [35]

points on same equipotential line have constant electric potential

i.e., equipotential lines are *contour lines* of equal electric potential

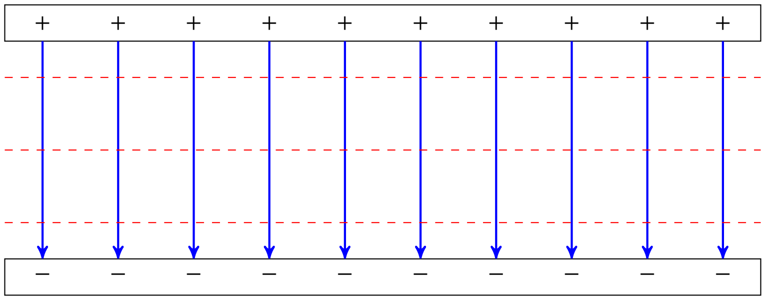
- for a field near a point charge, equipotential lines are a set of *concentric* circles





[35] In three dimensions, these lines form equipotential *surfaces*.

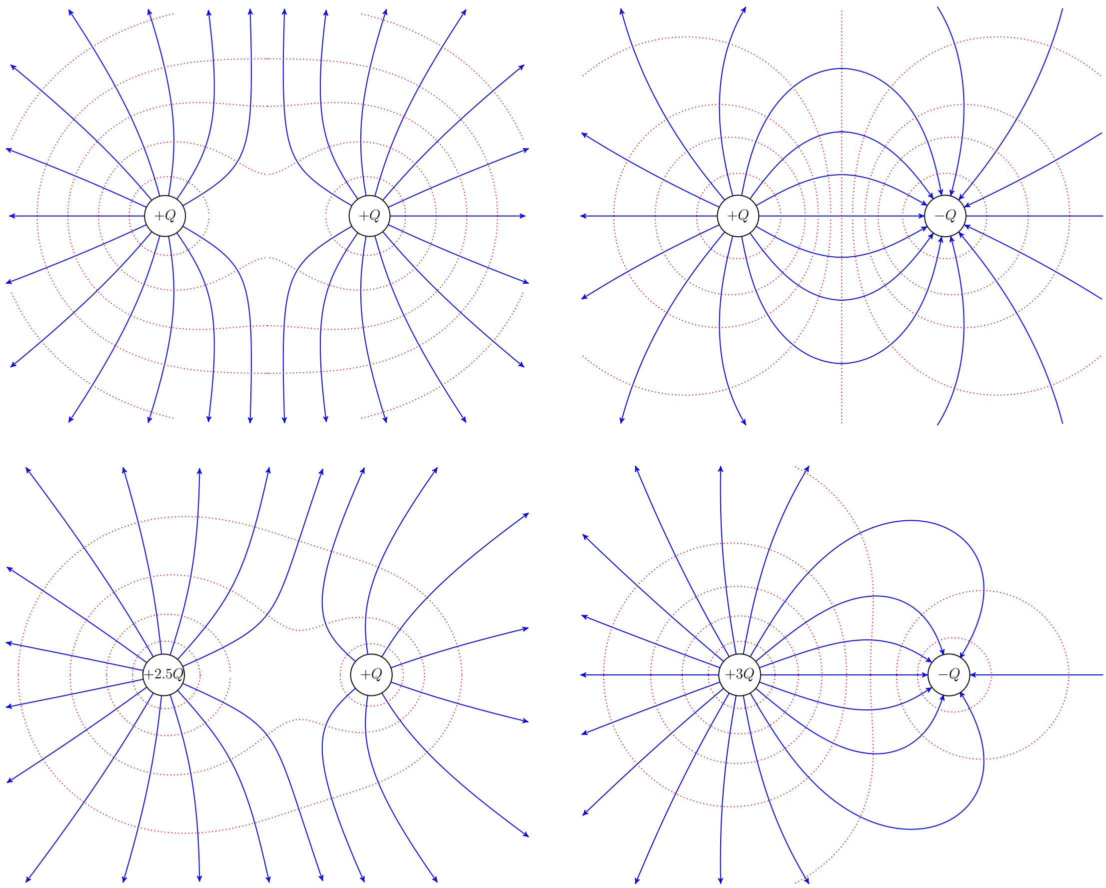
➢ for uniform fields, equipotential lines are a set of *parallel* straight lines

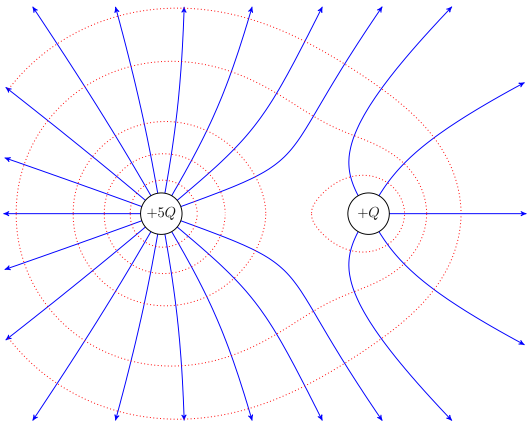


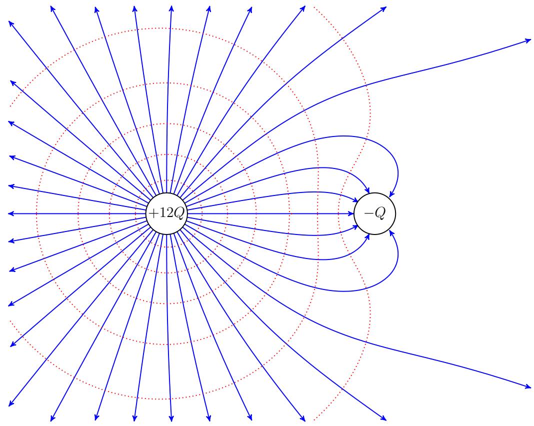
➢ equipotential lines are always perpendicular to the electric field lines

moving along an equipotential line requires no work done

Example 6.11 Field lines and equipotential lines due to two charges of various magnitudes







6.4 further discussions on electric fields

6.4.1 comparison with gravitational fields

both gravitational and electric force are described by inverse square law, so it follows that the mathematical language for both theories are very similar

➢ physical quantities that describe gravitational/electric fields

	vector description	scalar description
interaction between two masses/charges	force F	potential energy E_p
effect of source mass/charge	field g/E	potential φ/V

➢ comparing gravitational field with electric field [36]

	gravitational field	electric field	meaning
force	$F=(-)G\dfrac{Mm}{r^2}$	$F=\dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r^2}$	force between masses/charges
field strength	$g=\dfrac{F}{m}=(-)G\dfrac{M}{r^2}$	$E=\dfrac{F}{q}=\dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$	force per unit mass/charge
potential energy	$E_p=-G\dfrac{Mm}{r}$	$E_p=\dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r}$	related to work done by force
potential	$\varphi=\dfrac{E_p}{m}=-G\dfrac{M}{r}$	$V=\dfrac{E_p}{q}=\dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$	energy per unit mass/charge

[36]Note that there is no negative mass, gravitational force always interacts attractively. This is the fundamental difference between gravitational fields and electric fields.

➢ similarities between gravitational field and electric field

- force and field strength both obey *inverse square laws*
- potential energy and potential is inversely proportional to separation
- no potential energy and no potential at infinite separation

➢ differences between gravitational field and electric field

mass (source of gravity) is always positive, but electric charges can be positive or negative

this fact leads to many fundamental differences between the two force fields

- electric force can be repulsive or attractive, but gravitational force is always attractive
- electric potential can take both signs, but gravitational potential is always negative

6.4.2 electric field inside conductors

consider electric field *inside* a metal conductor carrying charge Q

conductor means there are free charge carriers that can move around

but charge distribution should be stable for a charged conductor (no circulating currents)

so charge carriers must experience no force, i.e., field strength inside conductor is zero

put it the other way round, if there is an excess field, it will push free charge carriers to move around, until they are distributed so that the field inside the conductor becomes zero

moreover, there shall be no potential difference between any two points inside the conductor, otherwise charge carriers would flow, so electric potential must be constant

> electric field strength is everywhere zero inside a conductor: $E = 0$
>
> electric potential is everywhere constant inside a conductor: $V = \text{const}$

Example 6.12 Consider the electric field due to a metal sphere of radius R carrying charge Q. Plot the variation with the distance r from sphere's centre of the field strength, and the variation with r of the electric potential.

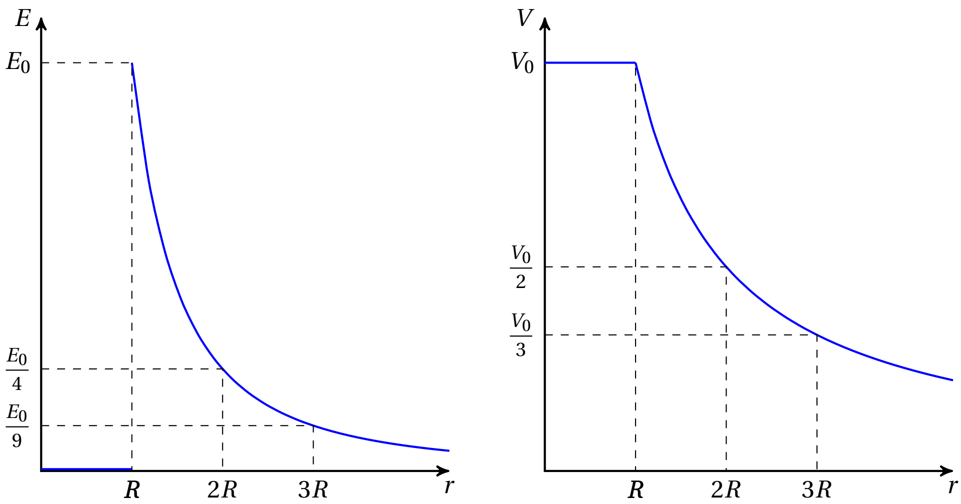
✍ charge Q is uniformly spread out on *surface* of sphere

viewed from *outside*, the sphere appears to have all of its charge concentrated at the centre

so it can be modelled as a point charge due to its symmetric distribution of charges electric field strength at distance r from sphere's centre is: $E = \dfrac{Q}{4\pi\epsilon_0 r^2}$ for $r > R$

electric potential at distance r from sphere's centre is: $V = \dfrac{Q}{4\pi\epsilon_0 r}$ for $r > R$

inside the sphere, i.e., for $r < R$, we have $E = 0$, and $V = \dfrac{Q}{4\pi\epsilon_0 R} = \text{const}$ □



Example 6.12: field strength and potential due to a charged metal sphere

Question 6.8 State whether the two spheres in Example 6.10 are conductors. State what feature of the potential graph supports your answer.

Question 6.9 You might have the experience that your mobile phone signal gets much weaker when you get into an elevator. Explain why this happens.

6.4.3 field strength & potential

for a small displacement Δr in an electric field, change in potential ΔV is

$$\Delta V = \frac{\Delta E_p}{q} \overset{\Delta E_p = -W}{=\!=\!=} -\frac{\Delta W}{q} = -\frac{F\Delta r}{q} \overset{F=Eq}{=\!=\!=} -E\Delta r \quad \Rightarrow \quad E = -\frac{\Delta V}{\Delta r}$$

as change in displacement $\Delta r \to 0$, we have $\boxed{E = -\dfrac{\mathrm{d}V}{\mathrm{d}r}}$

therefore we have the following theorem:

field strength is negative gradient of potential with respect to displacement can also consider change in potential ΔV for large distance due to work done in a field

$$\Delta V = \int \mathrm{d}V \xlongequal{E=-\mathrm{d}V/\mathrm{d}r} \int (-)E\mathrm{d}r = -\int E\mathrm{d}r$$ [37]

this gives the inverse relation: $\boxed{\Delta V = -\int E\mathrm{d}r}$ [38]

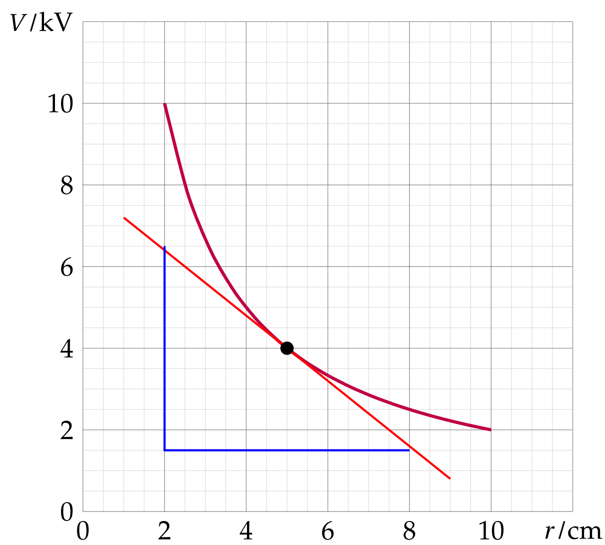
➢ given a V-r graph, gradient of curve gives field strength

conversely, given a E-r graph, area under curve gives change in potential

➢ can also write $F = -\frac{\mathrm{d}U}{\mathrm{d}r}$ and $\Delta U = -\int F\mathrm{d}r$

force always acts in a direction to lower the potential energy of an object [39]

Example 6.13 The variation of electric potential near a charged object is shown on the graph. Calculate the electric field strength at 5.0 cm from the centre of the object.



✍ draw tangent to the graph at $r = 5.0$ cm (red line), gradient of tangent gives field strength:

$$\text{gradient} = \frac{\Delta V}{\Delta r} = \frac{(1.5-6.5)\times 10^3}{(8.0-2.0)\times 10^{-2}} \approx -8.3\times 10^4 \text{ V m}^{-1} \quad \Rightarrow \quad E = -\frac{\Delta V}{\Delta r} = 8.3\times 10^4 \text{ V m}^{-1}$$ □

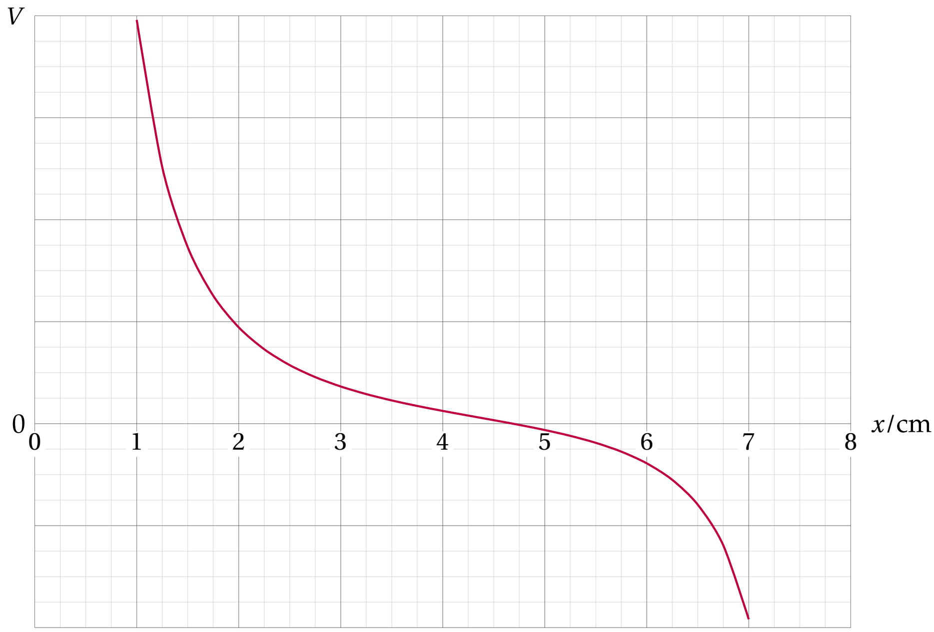
[37]The expression is implicitly integrated from initial position to final position.

[38]These equations are correct if the charge is moving in the parallel direction to the field, i.e., the motion is along the field lines. But an object can move in all directions in the field. More rigorously, if we take the vector nature of electric field into account, we should write $\Delta V = -\int \mathbf{E}\cdot\mathrm{d}\mathbf{r}$, and $\mathbf{E} = -\frac{\partial V}{\partial \mathbf{r}}$. (⋆)

[39]This result can be generalised to a very important principle of physical laws called the *least action principle*. It states that any motion of a system tends to minimise the action, a physical quantity related to the energy of the system. This fundamental law plays a crucial role in the study of theoretical physics. (⋆)

Question 6.10 Show that the charged object in Example 6.13 behaves like a point charge. Determine the charge it carries, and hence calculate the field strength at $r = 5.0$ cm.

Example 6.14 The variation of electrical potential along a certain line is shown. State and explain where in the field an electron will experience the greatest force.



✍ greatest force means greatest field strength, which means maximum potential gradient

largest gradient of V-x curve at $x = 1$ cm, so greatest force at $x = 1$ cm □

Example 6.15 electric field due to an isolated point charge

we have learned that the electric potential due to a point charge is: $V = \dfrac{Q}{4\pi\epsilon_0 r}$

using $E = -\dfrac{\mathrm{d}V}{\mathrm{d}r}$, we have $E = -\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{Q}{4\pi\epsilon_0 r}\right) = -\dfrac{Q}{4\pi\epsilon_0}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(\dfrac{1}{r}\right) = \dfrac{Q}{4\pi\epsilon_0 r^2}$

this agrees with the expression for field strength due to an isolated charge □

Question 6.11 For the electric field due to a charged metal sphere (see Example 6.12), convince yourself that the field strength equals negative gradient of potential at any point.

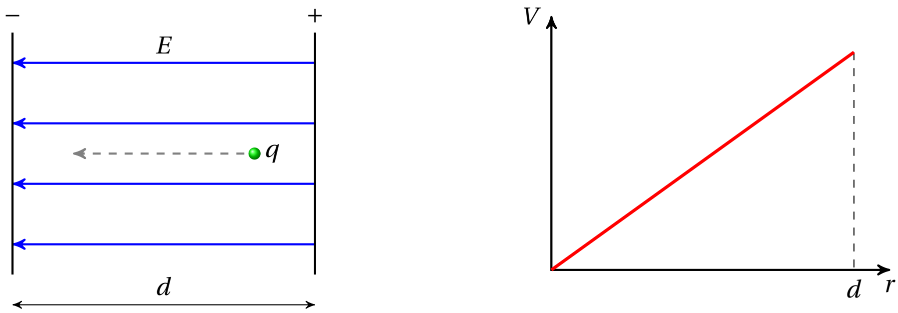
Question 6.12 We have seen the statement field strength equals negative potential gradient holds for electric fields. Does it also hold for gravitational fields?

uniform fields revisited

given two oppositely-charged metal plates separated by a distance of d

if p.d. between the plates is V, then electric field strength between is given by $E = \dfrac{V}{d}$ [40]

we will derive this result using the theorem introduced in the last section



moving a test charge in a uniform electric field

moving a test charge q in a uniform field, work done by electric force: $W = Fd = Eqd$

change in P.E.: $\Delta E_p = -W = -Eqd$

change in potential: $\Delta V = \dfrac{\Delta E_p}{q} = -Ed$, or $E = -\dfrac{\Delta V}{d}$

plotting V-r graph, gradient of line $= -E$

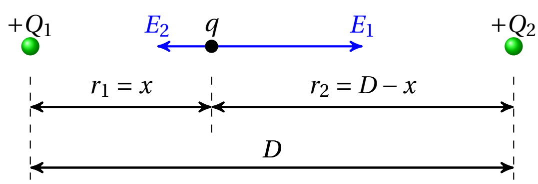
the minus sign means field strength points in the direction such that potential decreases

i.e., electric field acts from high potential to low potential

electric field due to two positive point charges

two point charges $+Q_1$, $+Q_2$ are separated by a distance of D

let's look into the electric field along the segment joining the two charges



[40] You should have learned this in AS-level physics.

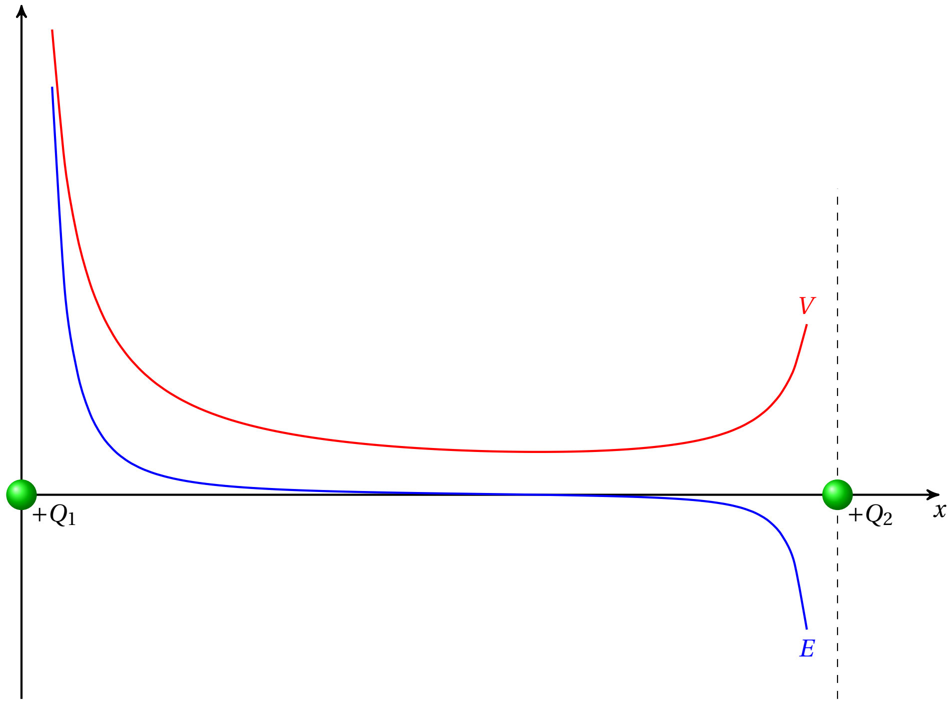
combined potential: $V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x} + \frac{Q_2}{D-x}\right)$

combined field strength: $E = E_1 - E_2 = \frac{Q_1}{4\pi\epsilon_0 r_1^2} - \frac{Q_2}{4\pi\epsilon_0 r_2^2} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x^2} - \frac{Q_2}{(D-x)^2}\right)$

notice that when computing V, we carry out *scalar sum*

but for E, we carry out *vector sum*, i.e., directions of E_1 and E_2 become important

V-x graph and E-x graph for the case where $Q_1 = 3Q_2$ are sketched



Question 6.13 Convince yourself that field strength is indeed given by negative potential gradient. You may interpret it either graphically (think about gradient of tangent along the curve) or algebraically (think about the derivative of V).

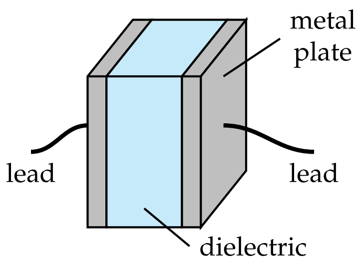
Question 6.14 We have looked into the electric field between two positively-charged particles. Discuss the cases where (a) both particles are negatively charged, (b) the two particles carry opposite charges.

CHAPTER 7
Capacitors

7.1 capacitors: an introduction

7.1.1 capacitors

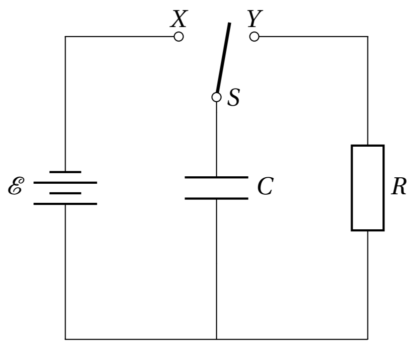
capacitors are elementary electrical units widely used in electrical and electronic engineering



a typical capacitor has two conductive plates

between the plates there is usually an insulating material called *dielectric*

circuit symbol for a capacitor is ⊣⊢



if we construct an electric circuit as shown

when contact S is moved to X, capacitor is connected to a voltage supply and becomes charged

positive and negative charges are separated onto two plates, and they will stay where they are even if we disconnect the capacitor from the voltage supply

if we then move S to Y, the charged capacitor discharges and drives a current through resistor R, i.e., it can act as a temporary power source[41]

so capacitors can be used to store and release energy[42]

[41]Details on charging and discharging processes will be gone through in §7.4.

[42]Other important functions of capacitors in electronic circuits include smoothing output voltage of power supplies, blocking direct current while allowing alternating current to pass, etc.

7.1.2 mutual capacitance

to describe ability of a capacitor to store charges, we define the notion of capacitance

> **(mutual) capacitance** of a parallel-plate capacitor is defined as the ratio of the charge stored on one plate to the potential difference across the two plates.

in a word equation: mutual capacitance $C = \dfrac{\text{charge on one plate } Q}{\text{p.d. } V \text{ across the plates}} \quad \Rightarrow \quad \boxed{C = \dfrac{Q}{V}}$

➢ unit of capacitance: **farad** [43] : $[C] = \text{F}$

farad a derived unit: $1\ \text{F} = 1\ \text{C} \cdot \text{V}^{-1}$

farad is a large unit, more common subunits of capacitance in use are sub-multiples of farad:

$$1\ \mu\text{F} = 10^{-6}\ \text{F}, \quad 1\ \text{nF} = 10^{-9}\ \text{F}, \quad 1\ \text{pF} = 10^{-12}\ \text{F}$$

➢ for a parallel-plate capacitor, charges on the two plates are equal but opposite

net charge on the capacitor: $Q_{\text{net}} = (+Q) + (-Q) = 0$

so we should emphasise on the notion of charge on *one* plate in the definition

➢ capacitance depends on *geometry* of the device and permittivity of the dielectric material

capacitance does not depend on electric field or potential[44]

for example, capacitance between two metal plates is: $C = \dfrac{\epsilon_0 A}{d}$ [45]

A is area of plate, d is distance between plates, both are geometrical quantities

7.1.3 self-capacitance

there are two closely related notions of capacitance: *mutual* capacitance and *self* capacitance

[43]The unit is named after Michael Faraday, a British physicist who developed the concept of capacitance. Faraday's other main discoveries include electromagnetic induction and electrolysis. He established the basis for the concept of the electromagnetic field in physics.

[44]Recall the resistance of an electrical component. Resistance is defined as the ratio of p.d. to current, but the value of resistance is essentially dependent on the length, cross-sectional area and material of the component, instead of the p.d. applied or the current flowing through it.

[45]If there is *dielectric* in between, the formula should be rewritten as $C = \dfrac{\epsilon A}{d}$, where ϵ is permittivity of dielectric. These formulae are not examinable by the syllabus.

the definition for capacitance given in the previous section, is actually *mutual* capacitance[46]

on the other hand, all bodies are able to store electrical charge

any object that can be electrically charged exhibits capacitance

we define **self-capacitance** of an object as the amount of charge that must be added to increase per unit electrical potential

in a word equation, self capacitance $C = \dfrac{\text{charge of object } Q}{\text{electric potential of object } V} \quad \Rightarrow \quad \boxed{C = \dfrac{Q}{V}}$ [47]

Example 7.1 Self-capacitance of a charged metal sphere in a vacuum

consider a metal sphere of radius R and carries an electric charge of Q

its electric potential: $V = \dfrac{Q}{4\pi\epsilon_0 R}$

self-capacitance of the sphere: $C_{\text{sphere}} = \dfrac{Q}{V} = Q \times \dfrac{4\pi\epsilon_0 R}{Q} \quad \Rightarrow \quad \boxed{C_{\text{sphere}} = 4\pi\epsilon_0 R}$

note that capacitance is only dependent on its geometrical property (radius R) □

Example 7.2 A conducting sphere of radius 1.0 m is situated in free space. (a) Find its capacitance. (b) In order to raise its potential to 5000 V, find the amount of charge needed.

✍ capacitance of the sphere: $C = 4\pi\epsilon_0 R = 4\pi \times 8.85 \times 10^{-12} \times 1.0 \approx 1.11 \times 10^{-10}$ F

(here you can see farad being an impractically huge unit)

charge on sphere: $Q = CV = 1.11 \times 10^{-10} \times 5000 \approx 5.56 \times 10^{-7}$ C □

analogy with ideal gases

an interesting analogy can be made between capacitors and ideal gases

recall an ideal gas is described by equation $pV = nRT$ [48]

$$\text{compare} \begin{cases} \text{amount of charge:} & Q = CV \\ \text{amount of substance:} & n = \left(\dfrac{V}{RT}\right) p \end{cases}$$

volume V of a container has a certain space *capacity*, at fixed T, pumping more gas (increase n) into system, pressure p increases

[46]In many cases, the term capactiance is a shorthand for mutual capacitance.

[47]For either mutual capacitance or self capacitance, defining equation $C = \frac{Q}{V}$ takes the same form, but you should keep in mind that Q and V represent different things in different contexts.

[48]Don't confuse voltage V with volume V!

similarly, a capacitor has a certain charge capacity, adding more charge Q increases p.d. V, so the quantity C is naturally called *capacitance*

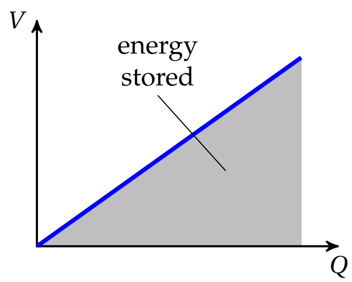
also for a container, there exists a maximum pressure which it can withstand

for a capacitor, there exists a *breakdown voltage*, or *withstand voltage*, beyond which there could be sparkling across the capacitor

7.2 energy stored in a capacitor

to charge a capacitor, need to push electrons off one plate and onto the other

separation of positive and negative charges requires work done ⇒ energy is stored



since charge Q varies with p.d. V, we shall use the V-Q graph to find work done W

area under V-Q graph is equal to work done W

$$W = \frac{1}{2}QV$$

substitute $Q = CV$, energy stored in capacitor is[49]:

$$\boxed{W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}}$$

Example 7.3 When the p.d. across a capacitor of 1.8×10^{-4} F is increased from 10 V to 20 V, how much additional energy is stored?

✍ energy change: $\Delta W = W_f - W_i = \frac{1}{2}CV_f^2 - \frac{1}{2}CV_i^2 = \frac{1}{2} \times 1.8 \times 10^{-4} \times (20^2 - 10^2) = 2.7 \times 10^{-2}$ J □

Question 7.1 A capacitor of 2500 μF is charged to a working voltage of 18 V. (a) What is the magnitude of positive charge on its plate? (b) What is the energy stored?

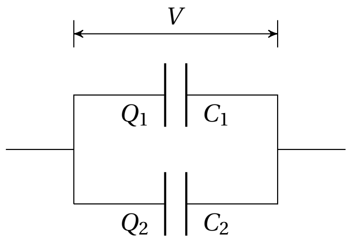
Question 7.2 A capacitor initially charged to a potential difference of 16 V discharges and loses 40% of its energy. What is its new p.d.?

Question 7.3 For an isolated metal sphere of radius 30 cm situated in vacuum, what is the electric potential energy stored when charged to a potential of 120 kV?

[49]Rigorously speaking, this electrical potential energy is stored within the *electric fields* between metal plates of capacitor.

7.3 capacitor networks

7.3.1 capacitors in parallel



consider two capacitors connected in parallel

same p.d. V across the network: $V = V_1 = V_2$

but charge Q is shared: $Q_{total} = Q_1 + Q_2$

$$\frac{Q_{total}}{V} = \frac{Q_1}{V} + \frac{Q_2}{V}$$

$$C_{total} = C_1 + C_2$$

if three of more capacitors in parallel: $\boxed{C_{total} = C_1 + C_2 + C_3 + \cdots}$ $\boxed{Q_{total} = Q_1 + Q_2 + Q_3 + \cdots}$

➢ adding extra capacitor in parallel to a network, total capacitance will increase

explanation: when several capacitors connected in parallel, equivalent to a single capacitor with larger plates, so more charge on the plates $\Rightarrow C \nearrow$

Example 7.4 A capacitor with capacitance C_0 is charged to a p.d. V_0. It is disconnected from the power supply, and then connected across an identical capacitor. Discuss the change in p.d., and change in energy stored in the system.

✍ initial charge $Q = C_0 V_0$, initial energy stored $W_0 = \frac{1}{2} C V_0^2$

combined capacitance: $C = C_0 + C_0 = 2C_0$

charge is conserved, so final p.d. across: $V = \frac{Q}{C} = \frac{C_0 V_0}{2C_0} = \frac{1}{2} V_0$

charge shared between capacitors, the first capacitor loses half its charge and p.d.

final energy stored in system: $W = \frac{1}{2} C V^2 = \frac{1}{2} \times 2C_0 \times (\frac{1}{2} V_0)^2 = \frac{1}{4} C_0 V_0^2 \Rightarrow W = \frac{1}{2} W_0$

half of stored energy is lost as heat when electrons flow between two capacitors □

Question 7.4 A capacitor A of capacitance C and a second capacitor B of capacitance $3C$ are connected in parallel. If a voltage is applied across the network, what is the ratio of energy stored in A to that in B? What about the ratio of electric charge?

7.3.2 capacitors in series

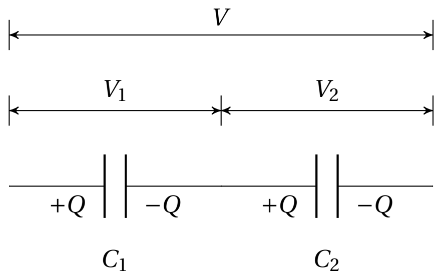
consider next two series capacitors

same charge Q on each plate: $Q = Q_1 = Q_2$[50]

p.d. is shared: $V_{\text{total}} = V_1 + V_2$

$$\frac{V_{\text{total}}}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q}$$

$$\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

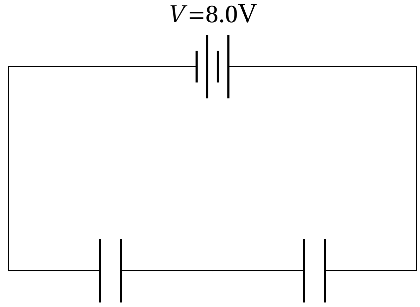


if three of more capacitors in series: $\boxed{\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots}$ $\boxed{V_{\text{total}} = V_1 + V_2 + V_3 + \cdots}$

➢ adding extra capacitor in series to a network, total capacitance will decrease

explanation: when several capacitors connected in series, equivalent to a parallel-plate capacitor with greater separation, so more charge on the plates $\Rightarrow C \searrow$

Example 7.5 For the circuit shown below, find the p.d. across each of the capacitor.



✍ using result for combined capacitance, we have: $C_{\text{total}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \left(\frac{1}{4.0} + \frac{1}{12}\right)^{-1} = 3.0\ \mu\text{F}$

charge for the network: $Q = C_{\text{total}} V_{\text{total}} = 3.0 \times 8.0 = 24\mu\text{C}$

series network so all capacitors have same Q, so: $Q_1 = Q_2 = 24\mu\text{C}$

p.d. across each individual capacitor: $V_1 = \frac{Q_1}{C_1} = \frac{24}{4.0} = 6.0$ V, $\quad V_2 = \frac{Q_2}{C_2} = \frac{24}{12} = 2.0$ V

alternatively, we can use properties of series network: $Q_1 = Q_2$ and $V_{\text{total}} = V_1 + V_2$

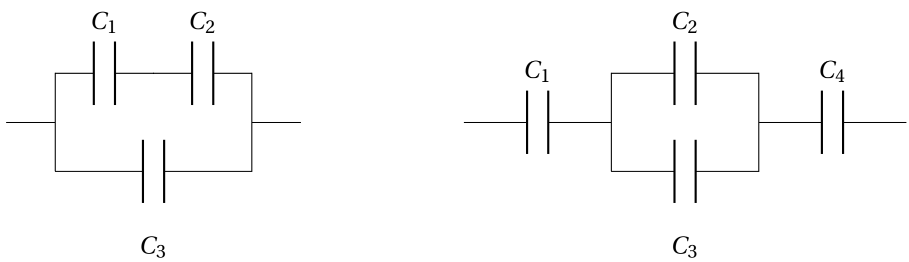
[50] Initially, the H-shaped isolated section between capacitors is uncharged. But no charge can enter or leave this section, its net charge must remain zero. Since a capacitor carries equal and opposite charges on its plates, so every capacitor in series carries same charge Q

we can solve simultaneous equations: $\begin{cases} 4.0V_1 = 12V_2 \\ V_1 + V_2 = 8.0 \end{cases} \Rightarrow \begin{cases} V_1 = 6.0\text{ V} \\ V_2 = 2.0\text{ V} \end{cases}$ □

7.3.3 capacitor networks

more complicated capacitor networks can be considered as a combination of some smaller networks with capacitors in parallel or in series

Example 7.6 $C_1 = C_2 = C_3 = C_4 = 10\ \mu$F, calculate the capacitance of the network (a) and (b).



(a) (b)

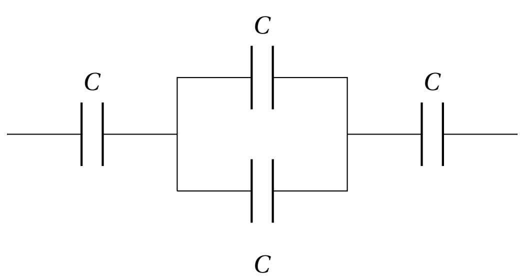
(a) $C_{12} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{10}\right)^{-1} = 5.0\ \mu\text{F}$

$C_{\text{total}} = C_{12} + C_3 = 5 + 10 = 15\ \mu\text{F}$

(b) $C_{23} = C_2 + C_3 = 10 + 10 = 20\ \mu\text{F}$

$C_{\text{total}} = \left(\dfrac{1}{C_1} + \dfrac{1}{C_{23}} + \dfrac{1}{C_4}\right)^{-1} = \left(\dfrac{1}{10} + \dfrac{1}{20} + \dfrac{1}{10}\right)^{-1} = 4.0\ \mu\text{F}$ □

Example 7.7 Four identical capacitors are arranged as shown. Each capacitor can withstand a maximum p.d. of 12 V, what is the maximum safe p.d. to be applied between the terminals?



suppose charge on the leftmost capacitor is Q, then charge of rightmost capacitor is also Q

but for the two capacitors in parallel, charge is shared, so each has charge $\dfrac{Q}{2}$

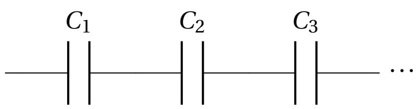
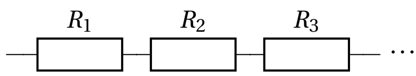
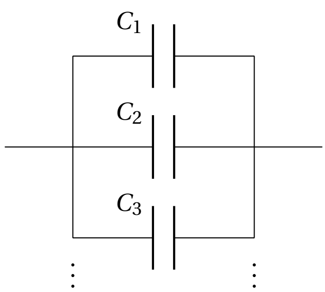
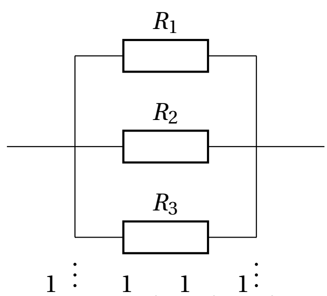
p.d. across the parallel network is half of the p.d. across the two capacitors near the ends so maximum p.d. across the terminals: $V_{\text{max}} = 12 + 6 + 12 = 30$ V □

Question 7.5 Using at most four capacitors of 24 μF, design a network that has a combined capacitance of (a) 72 μF, (b) 8 μF, (c) 36 μF, and (d) 18 μF.

Question 7.6 If the two networks in Example 7.6 are both connected to a supply voltage of 15 V, determine the p.d. across each individual capacitor.

7.3.4 capacitors & resistors

as a quick review, we compare capacitor networks with resistor networks

	capacitors	resistors
in series	same charge  $\frac{1}{C_{\text{total}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$	same current  $R_{\text{total}} = R_1 + R_2 + R_3 + \cdots$
in parallel	same p.d. across  $C_{\text{total}} = C_1 + C_2 + C_3 + \cdots$	same p.d. across  $\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$

7.4 charging & discharging capacitors (⋆)

in this section, we will investigate how the p.d. across a capacitor changes with time when it is being charged, and we will also look into discharging processes[51]

7.4.1 charging phase

initial state: no charge in capacitor: $Q(0) = 0, V_C(0) = 0$

at any instant: $\mathrm{d}Q = C\mathrm{d}V_C$

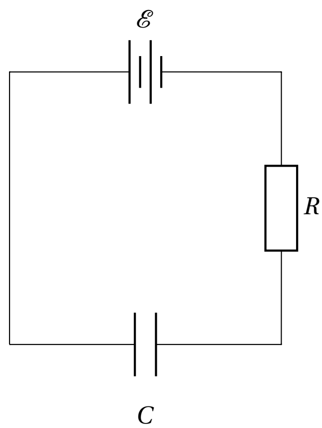
current in circuit: $I = \dfrac{V_R}{R} = \dfrac{\mathscr{E} - V_C}{R}$

change of charge: $\mathrm{d}Q = I\mathrm{d}t = \dfrac{\mathscr{E} - V_C}{R}\mathrm{d}t = C\mathrm{d}V_C$

$$\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}V_C}{\mathscr{E} - V_C}$$

$$\int_0^t \frac{\mathrm{d}t}{RC} = \int_0^{V_C} \frac{\mathrm{d}V_C}{\mathscr{E} - V_C}$$

$$\left.\frac{t}{RC}\right|_0^t = -\ln(\mathscr{E} - V_C)\Big|_0^{V_C}$$



simplify everything, we get: $\boxed{V_C = \mathscr{E}\left(1 - \mathrm{e}^{-t/RC}\right)}$

➢ p.d. of capacitor increases at a decreasing rate when it is being charged

as electric charges are separated onto the two plates, pushing more $+Q/-Q$ onto +ve/−ve plate requires more work done to overcome the repulsion ⇒ increase in p.d. slows down

➢ p.d of capacitor eventually tends to the battery e.m.f.

charge will continue to flow if there exists a potential difference

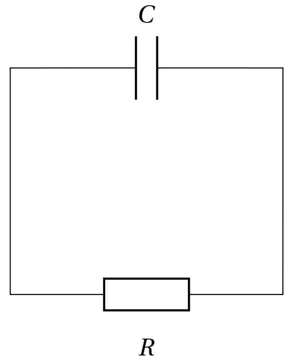
when $V_C = \mathscr{E}$, no charge flow, hence charging current gradually drops to zero

7.4.2 discharging phase

capacitor initially charged with $Q(0) = Q, V_C(0) = V_0$

at any instant, $\mathrm{d}Q = -C\mathrm{d}V_C$ (minus sign because charge decreases during discharging)

[51] This section is not required by the CIE A-Level exams. However, the contents introduced here may appear in the A-Level syllabus of other examination board.



also $V_C = V_R$ because in parallel

charge change: $\mathrm{d}Q = I\mathrm{d}t = \dfrac{V_C}{R}\mathrm{d}t = -C\mathrm{d}V_C$

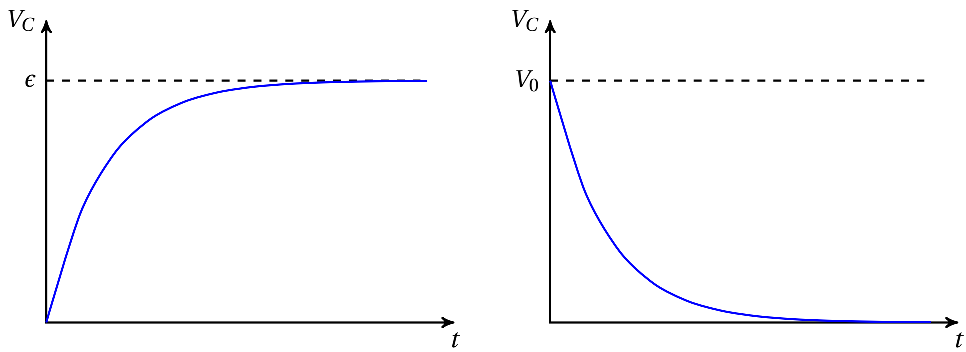
$$-\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}V_C}{V_C}$$

$$-\int_0^t \frac{\mathrm{d}t}{RC} = \int_{V_0}^{V_C} \frac{\mathrm{d}V_C}{V_C}$$

$$-\frac{t}{RC}\Big|_0^t = \ln(V_C)\Big|_{V_0}^{V_C}$$

simplify everything, we get: $\boxed{V_C = V_0 \mathrm{e}^{-t/RC}}$

- p.d. of capacitor gradually drops to zero during discharging
- discharging current also gradually approaches zero



charging phase of capacitor discharging phase of capacitor

7.4.3 time constant

$\tau \equiv RC$ is called *time constant*, which determines charging and discharging rate of a capacitor

$R \nearrow \Rightarrow$ smaller charging/discharging current $\Rightarrow$ takes longer to charge/discharge

$C \nearrow \Rightarrow$ more charge to be charged/discharged $\Rightarrow$ takes longer

charging or discharging of capacitors is not instantaneous, always a certain time delay

as a rule of thumb, after a time $t = 3 \sim 5\tau$, charging or discharging is almost complete

time delay for common RC circuits is usually small, but the delay could hinder further increasing of speed in integrated circuits

Chapter 8
Magnetic Fields

8.1 magnetism

8.1.1 magnets

magnetic effects are commonly seen in *magnets*

a magnet creates a magnetic field that attracts or repels other magnets

➢ polarity of magnets

a magnet has two poles, the *north pole* and the *south pole*

freely suspend a magnet, the north pole points towards earth's geographic north pole

when two magnets are brought near each other, like poles repel, and opposite poles attract

➢ we use **magnetic field lines** to graphically represent how magnetic field permeate space

by convention, fields lines emerge from north pole and go into south pole

density of field lines shows strength of magnetic field

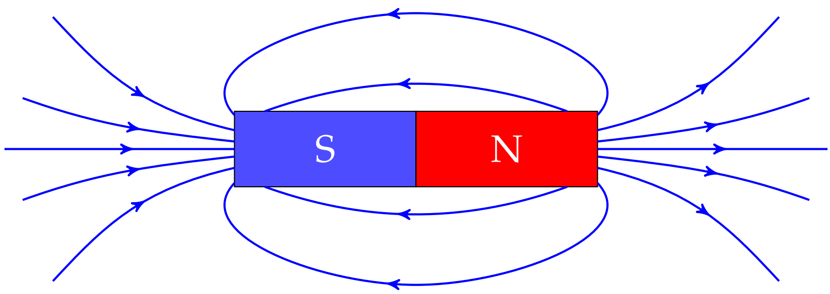
direction of field line tells how the north pole of a small compass will line up at that point

➢ strength of the field is described by a quantity called **magnetic flux density**, denoted by B

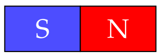
when we draw field lines, we are actually drawing the pattern of flux density B

the notion of flux density will be defined later in details in §8.2.2

Example 8.1 magnetic field around a bar magnet



Question 8.1 For two identical bar magnets placed side by side as shown, what does the magnetic field look like? Try sketching the magnetic field lines.



(a) two attracting bar magnets

(b) two repelling bar magnets

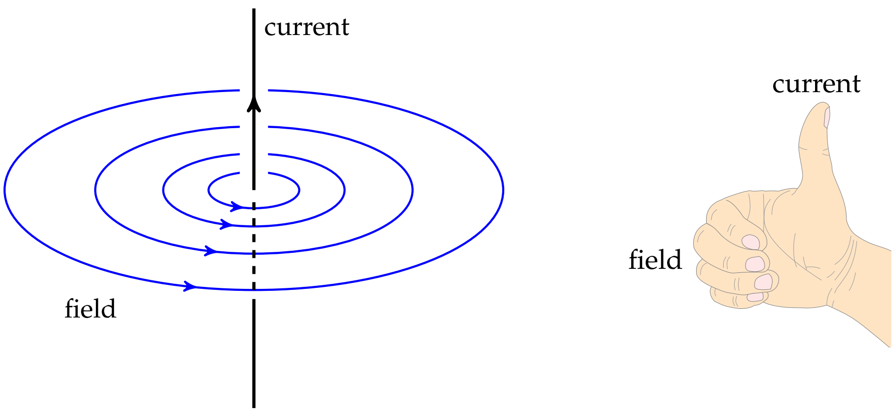
8.1.2 magnetic field due to currents

an electric current also induces a magnetic field around it

this was first discovered by Danish physicist *Hans Christian Orsted* in 1820, when he noticed the turning of a compass needle placed next to a wire carrying current.

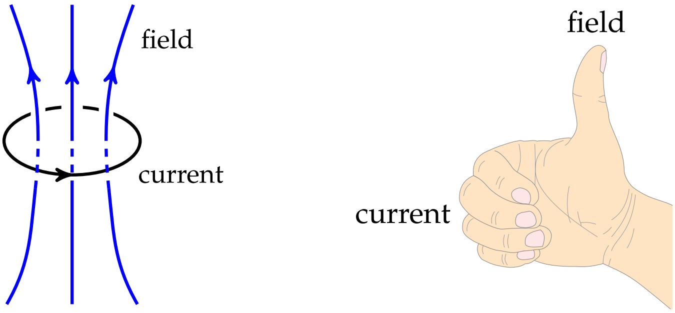
we will look into what happens when a current flows through a straight wire or a coil

➢ pattern of the field can be determined using **right-hand (grip) rule**[52]



field due to a long straight current-carrying wire

[52]The awesome right-hand rule illustrations below are copied from the PSTricks web site: `http://tug.org/PSTricks/main.cgi?file=examples`. The credits for these figures are attributed to CTAN community member *Thomas Söll*.



field due to a circulating current

- strength of the field is proportional to the current: $B \propto I$

  for both straight wires [53] and coils[54] , greater current means stronger field
- strength of magnetic field can be increased with *soft iron*

  this is because *ferromagnetic materials* (iron, cobalt, nickel) can attract magnetic field lines[55]

8.1.3 solenoids & electromagnets

strength and polarity of the field due to a coil can be changed easily by tuning currents, so coils are widely used to create magnetic fields where needed

a current-carrying coil is also called a **solenoid**

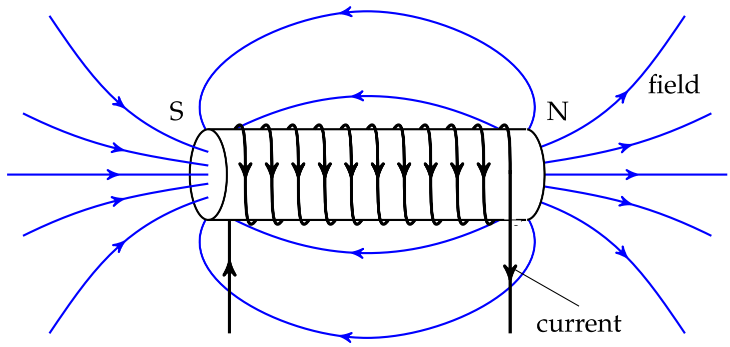
- a solenoid generates a magnetic field similar to that of a bar magnet

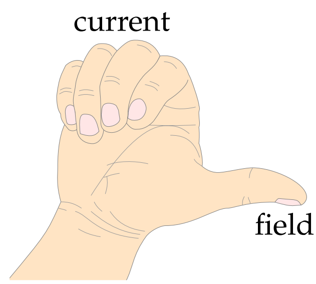
  useful to talk about the north and south poles of a solenoid

[53]The magnetic flux density generated by a current flowing in an infinitely long wire in free space is given by: $\boxed{B = \frac{\mu_0 I}{2\pi r}}$, I is the electric current, r is the perpendicular distance from the current, and $\mu_0 = 4\pi \times 10^{-7}$ T m A^{-1} is a fundamental physical constant called the *vacuum permeability*. (⋆)

[54]The magnetic flux density at the centre of a current-carrying coil in free space is given by: $\boxed{B = \frac{\mu_0 N I}{L}}$, where N is the number of turns, I is the electric current flowing through the coil, L is the length of coil, and μ_0 is the vacuum permeability. (⋆)

[55]If the straight wire is immersed in a material with *relative permeability* μ_r, then the field becomes: $B = \frac{\mu_0 \mu_r I}{2\pi r}$. Similarly, if a material with relative permeability μ_r is present, then the magnetic flux density inside a coil becomes: $B = \frac{\mu_0 \mu_r N I}{L}$. A good magnetic material (high permeability material), such as iron, has large μ_r, and therefore can greatly intensify the magnetic field. (⋆)





magnetic field around a solenoid

- direction of magnetic field in a solenoid is given by the **right-hand (grip) rule**
- magnetic field produced by a solenoid can be controlled

  current $I \nearrow \Rightarrow$ stronger field, also number of turns $N \nearrow \Rightarrow$ stronger field
- inserting an *iron core* inside greatly strengthens the field, this makes an *electromagnet*

Question 8.2 Describe the magnetic field due to an alternating current through a solenoid.

8.2 magnetic force on current-carrying conductor

8.2.1 magnetic force on current-carrying conductor

a current-carrying wire produces its own magnetic field, when it is surrounded by an external magnetic field, the two fields would interact $\Rightarrow$ a **magnetic force** on the conductor

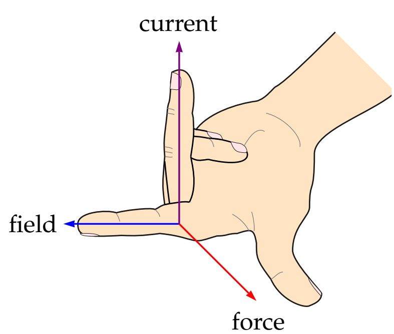
- direction of magnetic force on current can be worked out with **Fleming's left-hand rule**[56]
- magnitude of magnetic force: $\boxed{F = BIL\sin\theta}$

  B is the magnetic flux density to be specified later

  I is the current, θ is the angle between B and I
- when $B \perp I$, magnetic force $F = BIL$

  when $B \parallel I$, there is no magnetic force

  when B forms angle θ with I, only the *perpendicular*



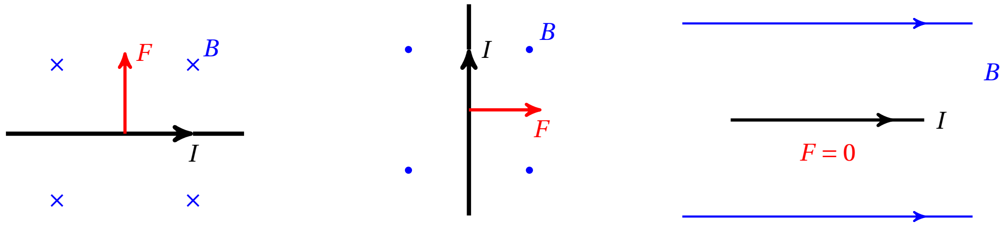
Fleming's left-hand rule

[56]Figure credit again goes to *Thomas Söll* from the CTAN community.

component contributes to the force, giving rise to the $\sin\theta$ factor in the formula

➢ magnetic force is perpendicular to both B and I [57]

Example 8.2 Determine the direction of the magnetic force acting. Check yourself.



(a) (b) (c)

8.2.2 magnetic flux density

rewrite $F = BIL\sin\theta$ as $B = \dfrac{F}{IL\sin\theta}$,we can now give a formal definition for flux density B:

> **magnetic flux density** B at a point is defined as the force acted per unit length on a conductor carrying a unit current at right angle to the magnetic field

➢ flux density describes the strength of a magnetic field[58]

➢ B is measured in **tesla** (T)[59]: $[B] = \mathrm{T}$, $1\ \mathrm{T} = 1\ \mathrm{N}\cdot\mathrm{A}^{-1}\cdot\mathrm{m}^{-1}$

> if a wire of 1 m normal to the magnetic field that carries a current of 1 A experiences a force of 1 N, then the magnetic flux density is 1 T

➢ B is a *vector* quantity, with both magnitude and direction

Example 8.3 A wire of 0.80 m carrying a current of 5.0 A is lying at right angles to a magnetic field, it experiences a force of 0.60 N, what is the flux density?

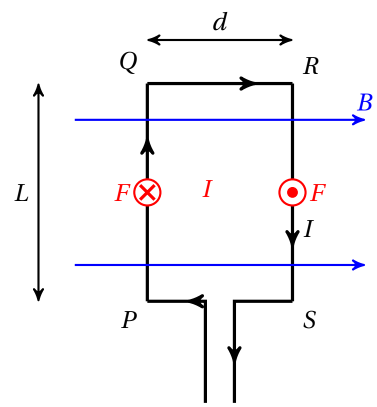
✍ $B = \dfrac{F}{IL} = \dfrac{0.60}{5.0 \times 0.80} = 0.15\ \mathrm{T}$ (a field of over 0.1 T is actually quite strong!) □

[57]Vector form of the formula: $\vec{F} = l\vec{I} \times \vec{B}$. (⋆)

[58]Magnetic field strength is a different quantity, defined by $H = \dfrac{B}{\mu}$, with $\mu = \mu_0\mu_r$ being the *magnetic permeability* of material. The naming of field strength H and flux density B are due to historical reasons.

[59]Telsa is a very large unit. For example, the strength of a typical refrigerator magnet is of about 10^{-3} T, even the very strong superconducting coils used in MRI are of around $1 \sim 3$ T.

Example 8.4 Torque on a rectangular metal frame in uniform magnetic field



a rectangular frame lies in parallel with the field

$QR, PS \parallel B$, so no force acting on these two sides

$PQ, RS \perp B$, so there is magnetic force $F = BIL$

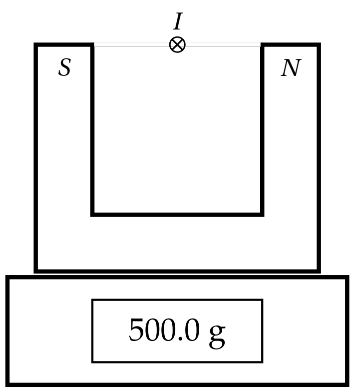
using left-hand rule, we find F_{PQ} acts into the paper, F_{RS} acts out of the paper

this is a pair of equal but opposite forces

they produce a torque about central axis of frame: $\tau = Fd = BILd$, causing the frame to rotate[60] □

Question 8.3 If the plane of the rectangular frame is at right angle to the magnetic field, describe the magnetic forces acting on each side and hence state what happens to the frame.

Example 8.5 A U-shaped magnet is placed on a balance with a wire suspended above it as shown. Magnetic flux density between the poles is about 0.50 T. The part of the wire that is in the field is of length 8.0 cm. The balance initially shows a reading of 500.0 g. When a current of 10 A flowing into the paper is switched on, what is the new reading on the balance?



✍ use left-hand rule, force on wire acts upwards

from *Newton's third law*, reaction force on magnet acts downwards

so there will be an increase in the balance reading

magnitude of the force: $F = BIl = 0.50 \times 10 \times 8.0 \times 10^{-2} = 0.40$ N

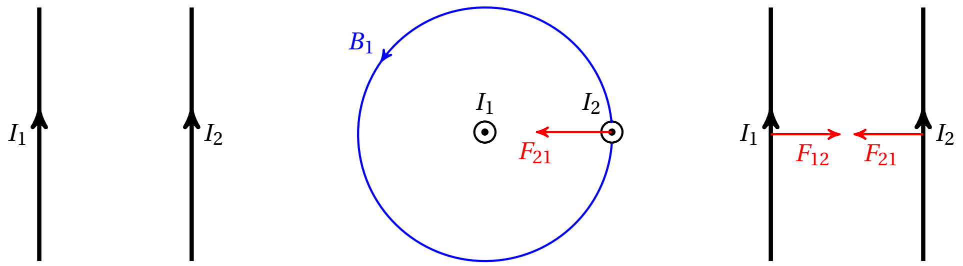
change in the mass reading: $\Delta m = \frac{F}{g} = \frac{0.40}{9.81} \approx 0.0407 \text{ kg} \approx 40.7 \text{ g}$

new reading on balance: $m_{\text{new}} = 500.0 + 40.7 = 540.7$ g □

Question 8.4 If the current in Example 8.5 is replaced by a current of 6.0 A flowing out of the plane of the paper. What is the reading on the balance?

[60]This effect allows engineers to build *electric motors*, such as those in electric vehicles, washing machines, blenders, etc.

force between long straight current-carrying wires



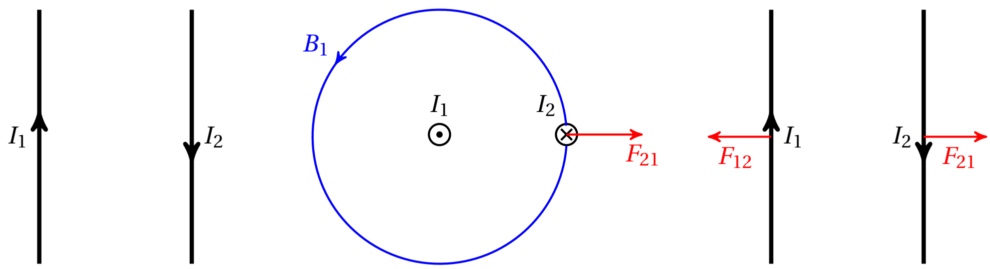
if we have two straight parallel wires carrying currents in the same direction

force acting on I_2 is due to magnetic field B_1 generated by I_1

from top view, I_1 creates a counter-clockwise B_1, so I_2 experiences an upward field

using left-hand rule, force acting on I_2 by I_1, denoted F_{21}[61], points to the left

since F_{21} and F_{12} are a pair of action-reaction, so F_{12} points to the right



similar discussions apply for I_1 and I_2 flowing in opposite directions (left as exercise)

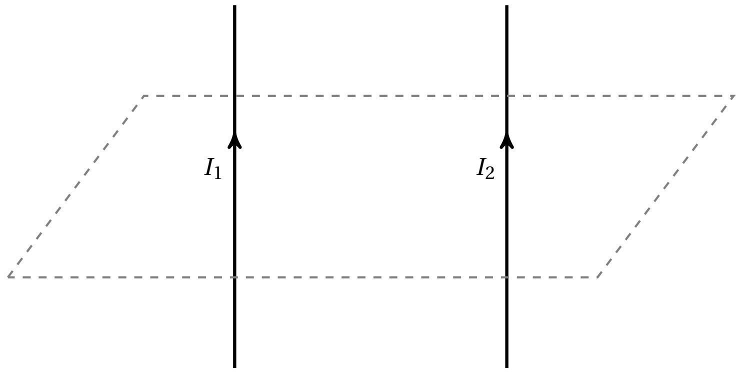
hence we come to an conclusion: *parallel currents attract, and anti-parallel currents repel*[62]

Question 8.5 A light coil of wire of several loops is suspended from a fixed point. When an electric current is switched on in the coil, state and explain the change in the separation between the loops.

Question 8.6 Two long parallel wires carry currents of I_1 =3.0 A and I_2 =5.0 A in the same direction as shown. The flux density at a point at perpendicular distance r from a straight long wire carrying current I is given by $B = \dfrac{\mu_0 I}{2\pi r}$.

[61] F_{AB} denotes the force acted on object A by object B.

[62] The S.I. unit for current, ampere, is defined using two long wires parallel to each other carrying the same current in the same direction. If the wires are separated by 1 m and the magnetic force experienced per metre is 2.0×10^{-7} N, then the current is of 1 A.



(a) Draw at least three lines to show the magnetic field due to I_1.

(b) State and explain the direction of force acting on I_2.

(c) Given that I_1 and I_2 are separated by 4.0 cm, what is flux density due to I_1 at I_2?

(d) What is the magnetic force per unit length acting on I_2?

(e) What is the magnitude and the direction of the force per unit length acting on I_1?

Question 8.7 If the current in one of the two wires in Question 8.6 is replaced by an alternating current, then the two wires should begin to vibrate. However, the wires are not observed to move. Make some reasonable estimates and explain why the vibration is not observed.

8.3 magnetic force on charged particles

electric currents are formed by moving charges, since current-carrying wires in a magnetic field experience force, charged particle moving in magnetic field should also experience force

8.3.1 magnetic force on charged particles

starting from $F = BIl\sin\theta$, let's substitute $I = nAqv \quad \Rightarrow \quad F = B(nAqv)l\sin\theta$

notice n is the number density of charged particles, so nAl together gives the total number

if we look at the force acting on *one* charged particle, then $nAl = 1$, so: $F = Bqv\sin\theta$

➢ magnitude of magnetic force on charged particle: $\boxed{F = Bqv\sin\theta}$

B is magnetic flux density, q is electric charge of the particle, v is its velocity

θ refers to the angle between B and v

➢ direction of magnetic force can be determined using *Fleming's left-hand rule*

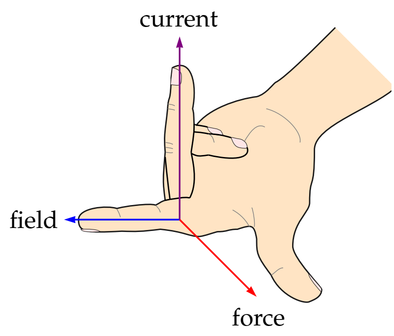
for $+q$, current in same direction as v

for $-q$, current in opposite direction as v

➢ force depends on *perpendicular component* of B

if $v \perp B$, magnetic force $F = Bqv$

if $v \parallel B$, there is no magnetic force

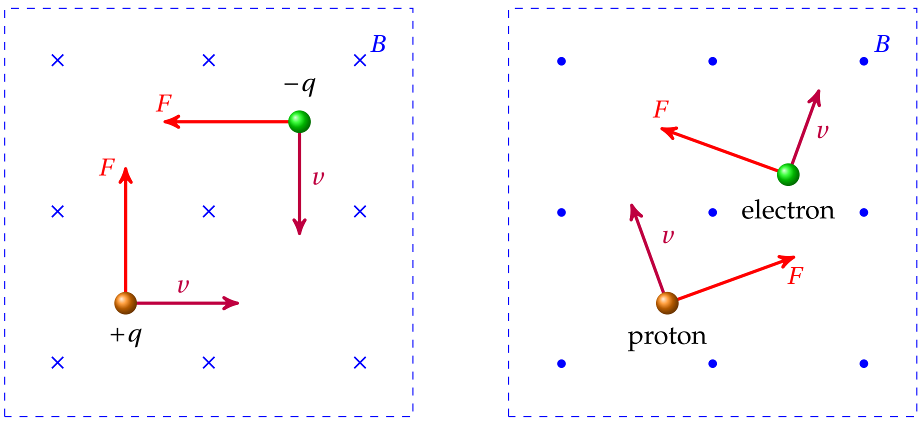


Fleming's left-hand rule

when particle moves at angle θ to B, contribution to the force only comes from the *perpendicular component*, giving rise to the $\sin\theta$ factor

➢ magnetic force always perpendicular to both velocity v and magnetic field B[63]

Example 8.6 Use the left-hand rule to find the direction of the magnetic forces acting on the following moving charges. Check yourself.



(a) (b)

charged particles in electric & magnetic fields

in either electric field or magnetic field, charged particle may experience a force[64]

in this section, we will compare the difference between electric field and magnetic field

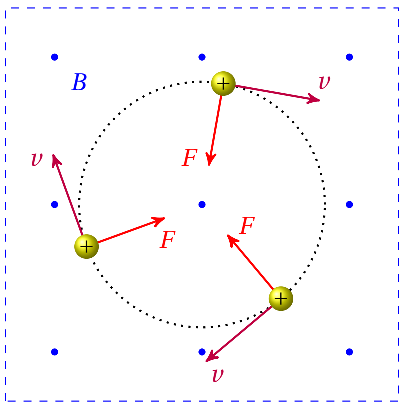
[63]Vector form of the formula: $\vec{F} = q\vec{v} \times \vec{B}$. ($\star$)

[64]The combination of electric and magnetic force on the charge due to electromagnetic fields is called the *Lorentz force*: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. If you have a good knowledge in maths, everything we cover in this section can be recovered from this vector equation. ($\star$)

	electric field	magnetic field
strength of the field	electric field strength E	magnetic flux density B
magnitude of force	$F_E = Eq$	$F_B = Bqv\sin\theta$
whether force depends on velocity	no dependence, acts equally on stationary or moving charges	depend on perpendicular component of velocity
direction of force	$F_E \parallel E$ same as/opposite to E for $+q/-q$	$F_B \perp B$ and $F_B \perp v$ use left-hand rule
effect of force on motion of particle	can change both magnitude and direction of velocity	only changes direction of velocity, cannot change magnitude of velocity
work done by force	work can be done, can define notions of potential and P.E.	magnetic force does no work for charged particles

8.3.2 charged particle in uniform magnetic fields

suppose a charged particle is moving at right angle to a uniform magnetic field



circular motion of a charged particle in uniform magnetic field

magnetic force F_B always at right angle to motion, so F_B keeps changing direction of velocity

uniform field means a constant force, so F_B deflects the particle at the same rate

the particle should describe a *circular* path!

for a charged particle moving in a uniform magnetic field, *magnetic force provides centripetal force for circular motion*: $\boxed{Bqv = \frac{mv^2}{r}}$, or $\boxed{Bqv = m\omega^2 r}$

➢ can solve for radius of the orbiting particles: $r = \frac{mv}{Bq}$

- $v \nearrow \Rightarrow r \nearrow$, faster particles take larger circles
- $B \nearrow \Rightarrow r \searrow$, stronger magnetic field, larger centripetal force, smaller circles
- $m \nearrow \Rightarrow r \nearrow$, larger mass, larger inertia, so larger circles

➢ radius of curvature relates to charge-to-mass ratio (also called *specific charge*) of the particle

rearrange the equation we have $\frac{q}{m} = \frac{v}{Br}$, which can be computed using experimental data

Example 8.7 An α-particle travelling at 2.5×10^4 m s^{-1} enters a region of uniform magnetic field. The field has flux density of 5.4 mT and is normal to direction of particle's velocity. What is the radius of α-particle's path?

✍ radius of circular arc: $r = \frac{mv}{Bq} = \frac{4 \times 1.66 \times 10^{-27} \times 2.5 \times 10^4}{5.4 \times 10^{-3} \times 2 \times 1.60 \times 10^{-19}} \approx 0.096 \text{ m} \approx 9.6 \text{ cm}$ □

Question 8.8 For an α-particle and a β-particle entering the same uniform magnetic field at a same speed, compare the radius of their paths.

Question 8.9 For a charged particle undergoing circular motion in a uniform magnetic field, show that its angular velocity is independent of the radius of its path.

Question 8.10 If a charged particle enters a uniform magnetic field at an angle $\theta \neq 90°$, state and explain the path of this particle. (Hint: think about components of its velocity.)

8.3.3 mass spectrometer

a **mass spectrometer** is a device to measure the charge-to-mass ratio of charged particles

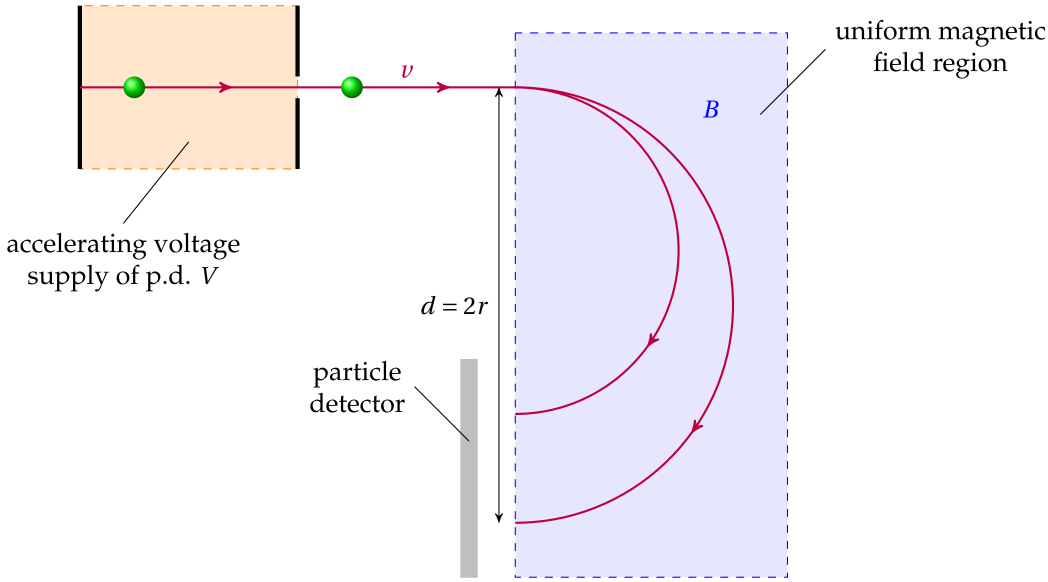
charged particles accelerated through an electric field: $\frac{1}{2}mv^2 = qV$

they then enter a uniform magnetic field: $Bqv = \frac{mv^2}{r} \Rightarrow v = \frac{Bqr}{m}$

eliminating v: $v^2 = \frac{2qV}{m} = \frac{B^2q^2r^2}{m^2} \Rightarrow \frac{q}{m} = \frac{2V}{B^2r^2}$

we can measure V, B, r in practice, so the charge-to-mass ratio $\frac{q}{m}$ is worked out

since different particles usually have different values of $\frac{q}{m}$, so this technique is widely used



deflection of two charged particles in a mass spectrometer

to identify unknown particles

Question 8.11 In the figure above, two paths of deflected particles are shown. Give reasons why the radius of the circular path can be different.

Question 8.12 If the particles sent into the mass spectrometer are positively-charged, in which direction should the magnetic field be applied?

Question 8.13 A particle carrying a charge of $+e$ enters a uniform magnetic field of 8.8 mT at right angles with an initial speed of 1.4×10^5 m s^{-1}. It describes a semi-circle with diameter of 66 cm. (a) Find the mass of the particle. (b) Suggest possible composition of this particle.

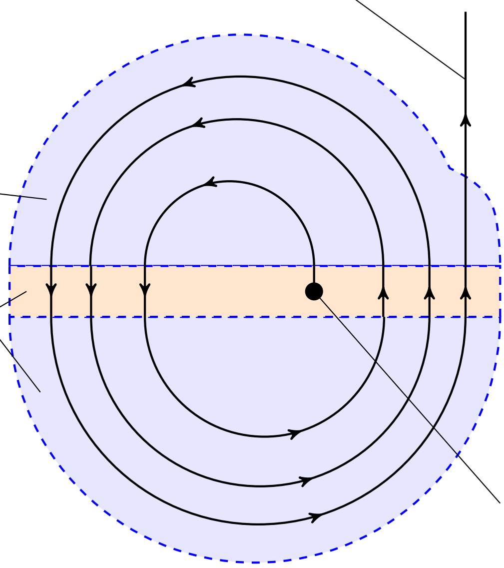
8.3.4 cyclotron

a **cyclotron** is a type of *particle accelerator*

the idea is to makes use of magnetic force to guide moving charges into a spiral path between accelerations by an electric field

as the particle enters and leaves the region of electric field, it gains extra energy of qV

it then follows a semi-circular path under magnetic force and re-enters the electric field



protons being accelerated in a cyclotron

polarity of the electric field is reversed so the particle continues to accelerate across the gap

energy of particle increases by qV, it then moves in a larger semi-circle in magnetic field

repeat this process, the particle leaves the exit port with very high speed

➢ cyclotron frequency

for charged particles circulating in a uniform magnetic field: $Bqv = m\omega^2 r$

time to complete one full turn is: $T = \frac{2\pi r}{v} = \frac{2\pi}{v} \times \frac{mv}{Bq} \Rightarrow T = \frac{2\pi m}{Bq}$

this shows the period is independent of radius of the circular path!

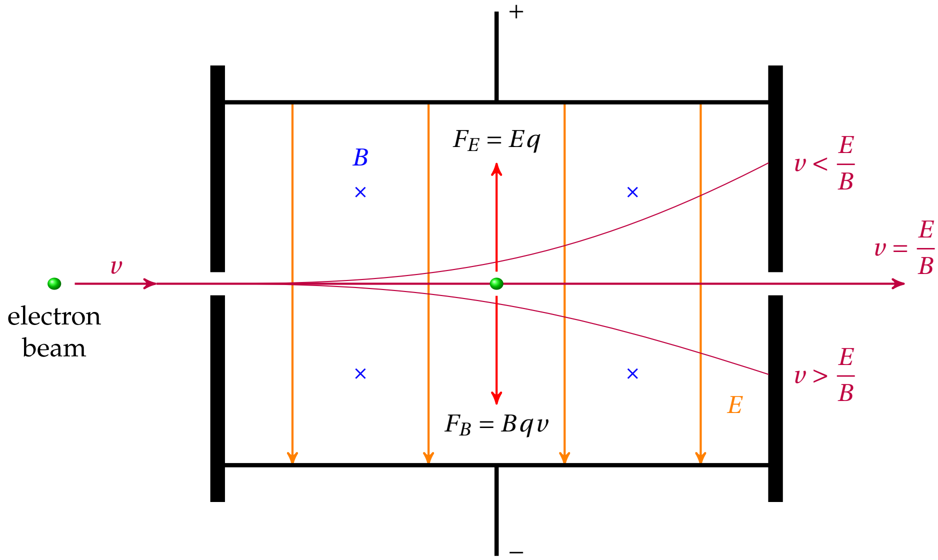
if an alternating voltage is applied at frequency $f = \frac{Bq}{2\pi m}$, particles can be accelerated continuously at just the right time when they cross the gap

this frequency is known as the *cyclotron frequency*

Question 8.14 Protons are accelerated in a cyclotron where the uniform magnetic field region has a flux density of 1.5 T and the voltage between the gap is 20 kV. (a) Find the frequency of the voltage supply needed. (b) Find the number of circles the protons must make in order to gain an energy of 10 MeV.

8.3.5 velocity selector

velocity selector is a device to produce a beam of charged particles all with same speed v



velocity selector

let's consider a beam of electrons passing through a region where both uniform electric field and magnetic field are applied, electrons at desired speed v are *undeflected*

no net force acting on these electrons, so equilibrium between electric and magnetic force

$$F_E = F_B \quad \Rightarrow \quad Eq = Bqv \quad \Rightarrow \quad \boxed{v = \frac{E}{B}}$$

for particles entering the same region with higher speed, $F_B > F_E$, they deflect upwards

for particles entering the same region with lower speed, $F_B < F_E$, they deflect downwards

Question 8.15 If electrons at speed v are undeflected when they pass through the velocity selector, what about a beam of α-particles entering the same region at the same speed v?

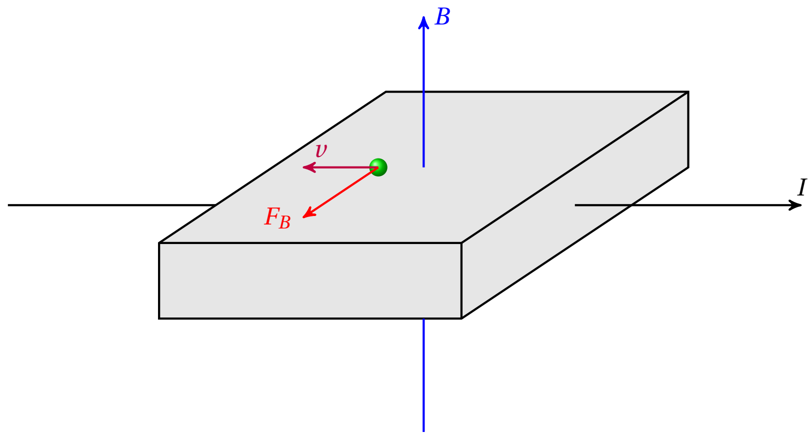
Question 8.16 A uniform magnetic field with flux density 6.0×10^{-2} T is applied out of the plane of the paper. A beam of protons are travelling into this region at a speed of 3.5×10^4m s^{-1}. (a) In which direction do the protons deflect? (b) A uniform electric field is now applied in the same region so that the protons become undeviated. What is the strength and the direction of this electric field?

8.4 Hall effect

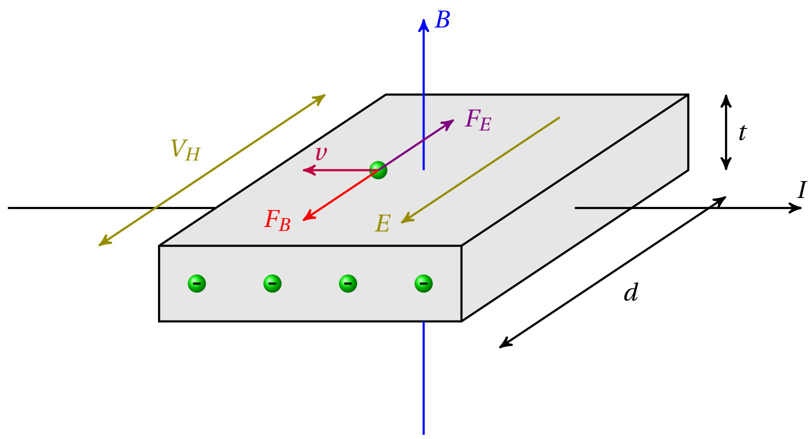
when an electric current passes through a conductor surrounded by an external magnetic field perpendicular to the current, a transverse potential difference is produced

this phenomenon is known as the **Hall effect** (discovered by American physicist *Edwin Hall* in 1879), the voltage difference is called the **Hall voltage**

suppose magnetic field goes upward, current flows to the right (see figure), then charge carriers (assumed to be electrons for now) experience a magnetic force pointing out of paper



this magnetic force causes a build-up of extra charge on front side of conductor, which produces an internal electric field E inside conductor, i.e., a potential difference V_H is formed



as charge carriers accumulate on one side, Hall voltage opposes further migration of charges

steady V_H is established when magnetic force and electric force are balanced: $Eq = Bqv$

- strength of internal electric field is related to Hall voltage: $E = \dfrac{V_H}{d}$

  where d is width of the conductor

- drift velocity v of particles is related to current I: $I = nAqv$

  where A is cross section of conductor and n is number density of charge carriers

we now have: $\dfrac{V_H}{d}q = Bq\dfrac{I}{nAq} \Rightarrow V_H = \dfrac{BId}{nAq}$

note that cross section $A = dt$, where t is thickness of conductor as shown

we therefore obtain an useful expression for Hall voltage: $V_H = \dfrac{BId}{n(td)q} \Rightarrow \boxed{V_H = \dfrac{BI}{ntq}}$

➢ to produce noticeable V_H, we want smaller n and smaller t

- $n \downarrow \Rightarrow$ few free charge carriers $\Rightarrow$ *semi-conductors* are preferable
- $t \downarrow \Rightarrow$ small thickness $\Rightarrow$ *thin slice* of component is preferable

➢ polarity of V_H is determined by nature of charge carries

charge carriers can be *free electrons* (negatively charged) or *holes* (positively charged)

then Hall voltage induced would have opposite polarity

➢ if current I forms angle θ with flux density B, again only perpendicular component matters

expression for Hall voltage becomes: $V_H = \dfrac{BI}{ntq}\sin\theta$

➢ apply a fixed current in a conductor, V_H is proportional to B

so magnetic flux density can be calculated once we find V_H

one can make use of Hall effect to build a *Hall probe*, a device used to measure flux density B

when using a Hall probe, one should rotate and record the greatest voltage reading to ensure the current applied is at right angle to the external magnetic field

Question 8.17 Why is it difficult to detect Hall voltage in a thin slice of copper?

Question 8.18 A Hall probe is placed near to one end of a strong magnet. State and explain the variation in the Hall voltage as the probe is rotated for one complete revolution.

Chapter 9

Electromagnetic Induction

9.1 magnetic flux

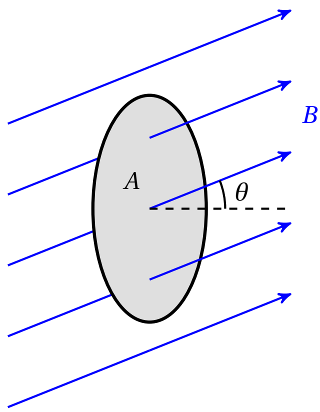
magnetic flux is defined as the product of a closed area and the magnetic flux density going through it at right angles: $\boxed{\phi = BA\cos\theta}$

- unit for magnetic flux: $[\phi] = \mathrm{Wb}$, $1\ \mathrm{WB} = 1\ \mathrm{T}\cdot\mathrm{m}^2$
- if magnetic field is perpendicular to the area

  magnetic flux simply becomes: $\boxed{\phi = BA}$
- for a coil with N turns, total magnetic flux is

$$\boxed{\Phi = N\phi = NBA}$$

  this is called **magnetic flux linkage**



- magnetic flux ϕ can be graphically thought as the number of field lines through area A

  cutting of field lines means a change in flux
- change in flux can be caused by many processes, for example
  - moving a magnet towards/away from a coil
  - varying current in a solenoid/electromagnet
  - inserting an iron core into a solenoid/electromagnet
  - pushing a straight conductor through a magnetic field
  - rotating a coil in a magnetic field
  - ...

Question 9.1 Explain why the processes mentioned above would give rise to a change in magnetic flux.

9.2 laws of electromagnetic induction

electromagnetic induction is the phenomena that magnetism produces electricity

the laws of electromagnetic induction were discovered by *Michael Faraday* and *Heinrich Lenz* in the 1830s, and later described mathematically by *James Clerk Maxwell*

we will study in what conditions voltages and currents could be induced, and how to find their magnitudes and polarities

9.2.1 Faraday's law

Faraday's law states that induced e.m.f./voltage is proportional to rate of change in magnetic flux (linkage): $\boxed{\mathscr{E} \propto \frac{\Delta\Phi}{\Delta t}}$

➢ Faraday's law gives the *magnitude* of the induced e.m.f.

the key here is the *change* in flux: as long as flux changes, e.m.f. will be induced

➢ if $\mathscr{E}$, Φ, t are all given in SI units, this proportional relation becomes an identity: $\boxed{\mathscr{E} = \frac{\Delta\Phi}{\Delta t}}$

Example 9.1 A coil of 80 turns is wound tightly around a solenoid with a cross-sectional area of 35 cm^2. The flux density at the centre of the solenoid is 75 mT. (a) What is the flux linkage in the coil? (b) The current in the solenoid is *reversed* in direction in a time of 0.40 s, what is the average e.m.f. induced?

✍ flux linkage: $\Phi = NBA = 80 \times 75 \times 10^{-2} \times 35 \times 10^{-4} = 0.21$ Wb

average e.m.f. induced: $\mathscr{E} = \frac{\Delta\Phi}{\Delta t} = \frac{(+\Phi) - (-\Phi)}{\Delta t} = \frac{2 \times 0.21}{0.40} = 1.05$ V □

9.2.2 Lenz's law

Lenz's law states that induced e.m.f. or current is always in the direction to produced effects that *oppose* the change in flux that produced it

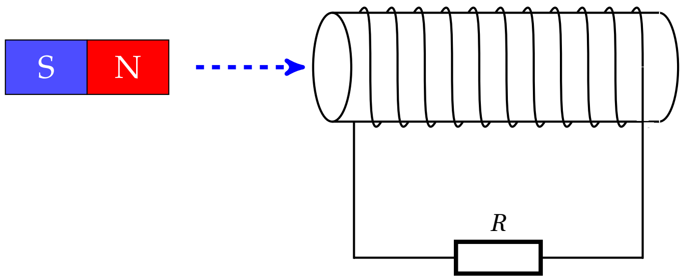
➢ Lenz's law gives the *polarity* of the induced current the key here is the nature dislikes any change in flux: effects of induced current always *oppose* the cause of its production

➢ Lenz's law is a consequence of the conservation of energy

since induced current dissipates electrical energy as heat, it must cause loss of original forms of energy possessed by the system

if the change in flux is caused by motion, then magnetic force on/by the induced current must *resist* this motion

Example 9.2 A coil of 200 turns is connected to a resistor $R = 4.0\ \Omega$. The coil has a diameter 10 cm. Initially a bar magnet is at great distance from the coil. The magnet is then inserted into the coil and field inside coil becomes 0.40 T. The process occurs within a duration of 2.0 s.



(a) What is average induced e.m.f. in the coil? (b) What is average induced current through resistor R? (c) In which direction does this current flow?

✍ change in flux linkage: $\Delta\Phi = NBA - 0 = 200 \times 0.40 \times \pi \times 0.050^2 \approx 0.628$ Wb

average e.m.f. induced: $\mathscr{E} = \dfrac{\Delta\Phi}{\Delta t} = \dfrac{0.628}{2.0} \approx 0.314$ V

average current induced: $I = \dfrac{\mathscr{E}}{R} = \dfrac{0.314}{4.0} \approx 0.0785$ A

as magnet approaches, coil experiences an increasing flux to the right

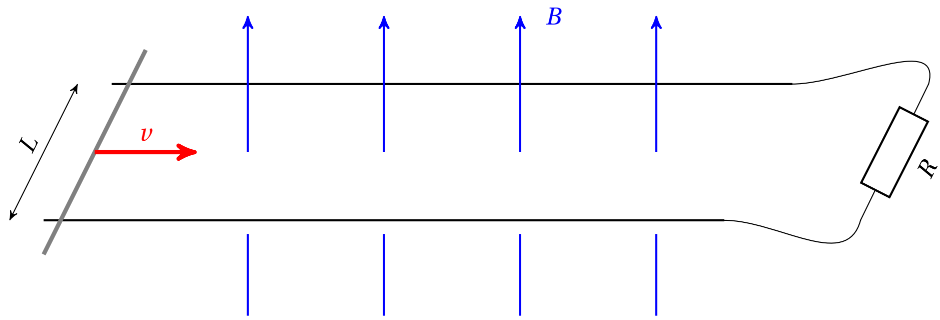
according to Lenz's law, field due to induced current acts to the left to oppose the change

alternatively, left side of the coil should behave as a north pole to oppose motion of magnet

use right-hand rule, we find induced current flows through R to the right

Question 9.2 If the magnet is now pulled away from the coil, state ad explain the direction of the induced current.

Example 9.3 Two parallel metal tracks are separated by a distance of $L = 45$ cm and placed in a uniform magnetic field of 0.80 T. The tracks are connected to a resistor $R = 6.0\ \Omega$. A long metal rod is being pushed under an external force at $v = 3.0$ m s^{-1} along the tracks as shown.



(a) What is the induced current through resistor? (b) In which direction does this current flow? (c) For the rod to travel at constant speed, what is magnitude of external force required?

✍ change of flux in time interval Δt is: $\Delta\phi = \Delta(BA) = B\Delta A = BLv\Delta t$

e.m.f. induced: $\mathscr{E} = \dfrac{\Delta\phi}{\Delta t} \Rightarrow \boxed{\mathscr{E} = BLv}$

induced current: $I = \dfrac{\mathscr{E}}{R} = \dfrac{BLv}{R} = \dfrac{0.80 \times 0.45 \times 3.0}{6.0} = 0.18A$

according to Lenz's law, magnetic force on induced current opposes the rod's motion of cutting field lines, so magnetic force acts to the left

using Fleming's left-hand rule, we find induce current flows in anti-clockwise direction

if rod travels at constant speed, then equilibrium between external push and magnetic force

external force required: $F_{\text{ext}} = F_B = BIL = 0.80 \times 0.18 \times 0.45 \approx 0.065$ N □

Question 9.3 In Example 9.3, can you find alternative arguments to determine the direction of the current induced as the rod cuts through the magnetic field?

Hall voltage & induced voltage in a coil

in this section, we compare readings on voltmeter connected to a Hall probe and a coil

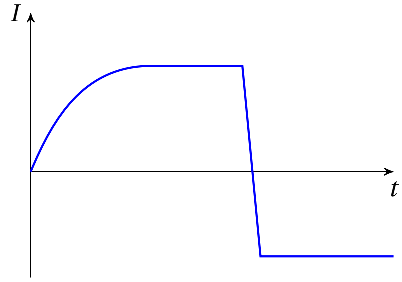
Hall probe picks up voltage proportional to flux density: $V_H \propto B$

coil picks up induced e.m.f. proportional to *rate of change* in flux: $\mathscr{E} \propto \dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$

Question 9.4 If a Hall probe is placed near a permanent magnet, what voltage do you measure?

If a small coil is placed in the same field, state and explain whether you can measure a non-zero voltage in the coil. If not, state three different ways in which you can produce a voltage.

Question 9.5 A Hall probe is placed near one end of a solenoid that carries a varying current as shown in the graph. (a) Sketch the variation of Hall voltage with time. (b) If Hall probe is replaced by a small coil parallel to the solenoid's end, sketch the variation of the voltage induced in the coil.



9.3 applications of electromagnetic induction

9.3.1 magnetic braking

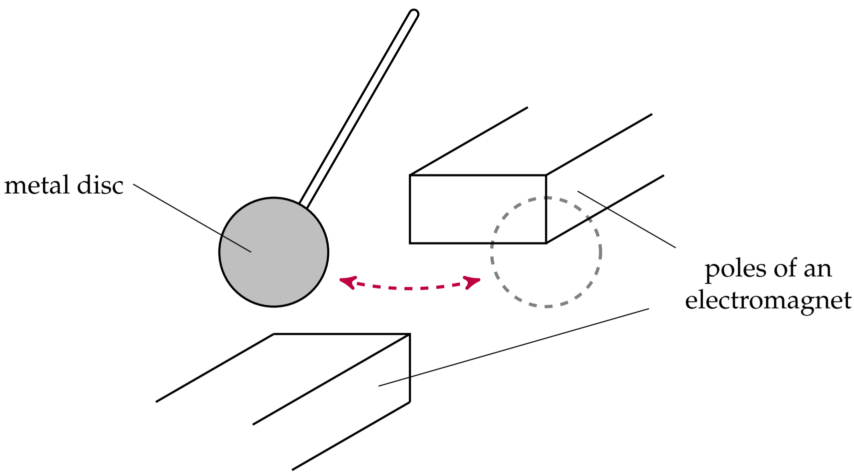
induction brakes make use of induced current to dissipate kinetic energy of a moving object

in this section, we will look at two simple demonstrations, but the principle can be used in braking system of high-speed trains, roller-coasters, etc.

the damped pendulum

a metal disc can swings freely between the poles of an electromagnet

when the electromagnet is switched on, the disc comes to rest very quickly



as disc moves in and out of field, change in flux gives rise to e.m.f induced

note that different parts of the disc experience different rates of change in flux, different e.m.f. is induced in different parts for the disc

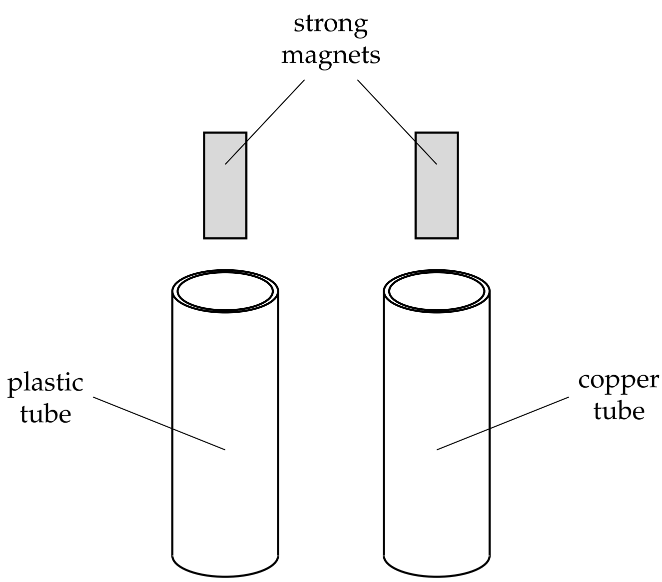
this causes circulating currents flowing in the disc, called **eddy currents**

- vibrational energy is lost as heat due to the eddy currents induced
- magnetic force on induced current causes damping

so amplitude of oscillation decreases quickly

falling magnet

suppose we drop two strong magnets down a plastic tube and a copper tube



as magnet falls, tube experiences change in flux, so e.m.f is induced

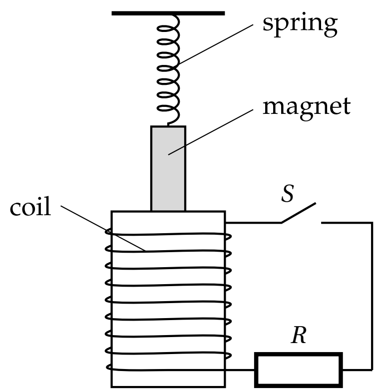
since plastic is an insulator, so no current flows in plastic tube, magnet undergoes free fall

but in the conducting copper tube, eddy current is induced in the tube

- energy is lost as heat due to induced current, not all G.P.E. converted into K.E.
- induced current exerts magnetic force on magnet to oppose its motion, acceleration $a < g$

so magnet falls more slowly through the copper tube

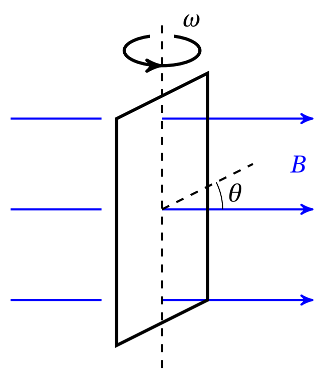
Question 9.6 A magnet is suspended from the free end of a spring. When displaced vertically and released, the magnet can oscillate in and out of a coil (see diagram). The switch S is initially open, there is negligible change in the amplitude. However, when the switch is closed, the amplitude is seen to decrease quickly. Explain the reasons.



9.3.2 the generator

imagine a coil rotates with constant angular speed ω in a uniform magnetic field B

let's assume that the coil initially lies in parallel to the field, i.e., $\theta = 0$ at $t = 0$



at time t, coil forms an angle $\theta = \omega t$ with the magnetic field (see diagram)

magnetic flux linkage through coil is:

$$\Phi = NBA \sin\theta = NBA \sin\omega t$$

magnitude of induced e.m.f. is:

$$\mathscr{E} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \omega NBA \cos\omega t$$

this is a sinusoidal voltage with maximum value: $\mathscr{E}_{\text{max}} = \omega NBA$

for a coil rotating in a magnetic field like this, an *alternating current* is generated

this is basically how a *generator* works

in practice, it is a strong electromagnet rotating inside a large coil to generate electricity

Question 9.7 State and explain the effect on the voltage generated if the coil rotates faster.

Question 9.8 For the coil rotating at a uniform angular speed in a uniform magnetic field, is the magnetic flux in phase with the e.m.f. generated? If not, what is the phase difference?

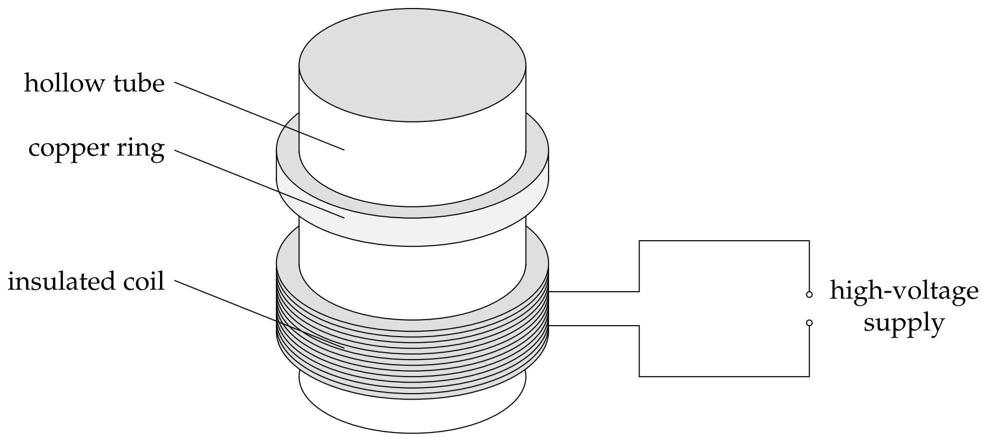
Question 9.9 Does the generator output the maximum voltage when the rotating coil is in parallel to the magnetic field or when the coil is at right angle to the field?

9.3.3 electromagnetic gun

a coil of wire of many turns is wound on a hollow tube

a light copper ring that can move freely along the tube is placed on the coil

let's find out what happens to the ring when a high-voltage supply is switched on



when supply is switched on, flux density in coil greatly increases

sudden change in magnetic flux could induce a huge e.m.f. in copper ring

induced current in ring would experience a repulsive magnetic force

this repulsive force could be way larger that ring's weight, ring would jump out from tube

if the ring is replaced by a small metal sphere placed inside the tube, the same strong magnetic force arising from a sudden change in flux could fire the sphere at very high speed

Question 9.10 The coil is now connected to a stable d.c. voltage supply. If we quickly insert an iron core into the tube, what might happen to the light copper ring?

other applications

as a final remark, electromagnetic induction is widely used in many other areas as well

for those who are interested, you may research on the principles behind wireless charging, induction cooking, contactless payment technology, smart pencils for tablets or computers, etc.

Chapter 10

Alternating Currents

a **direct current** (d.c.) flows in one direction only

an **alternating current** (a.c.) reverses its flow direction from time to time

a.c. has certain advantages than d.c., as you will see in this chapter

we will study the mathematics of a.c. and the transmission process for a.c.

10.1 sinusoidal a.c.

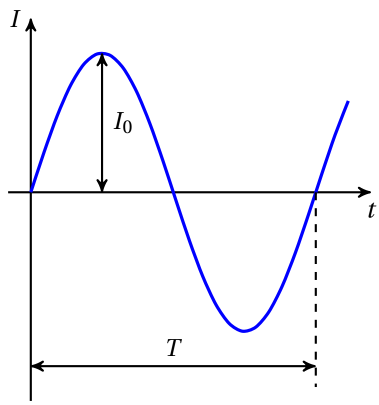
as we have seen in §9.3.2, currents produced from generators are naturally sinusoidal

sinusoidal a.c. is one of the most important types of a.c. waveform in electrical engineering.

for most cases in this course, we focus on a.c. that varies like a sine wave

10.1.1 sinusoidal waveform

current: $\boxed{I = I_0 \sin \omega t}$ / voltage: $\boxed{V = V_0 \sin \omega t}$



➢ I_0, V_0 are called *peak current* and *peak voltage* (amplitudes of the a.c. signal)

➢ ω is **angular frequency**, which describes how fast a.c. signal oscillates (same idea as for simple harmonic motion, see §3)

frequency and period of the signal are given by:

$$\omega = 2\pi f, \quad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

➢ mean current: $\langle I \rangle = 0$, mean voltage: $\langle V \rangle = 0$

this is because a.c. signal fluctuates with time, positive and negative bits cancel out

Question 10.1 An alternating voltage has a peak value of 16 V and period of 0.10 s. Write down a mathematical equation that describes the variation of this voltage.

Question 10.2 An alternating voltage is produced from a simple generator. If the rotating speed of the coil in the generator doubles, describe quantitatively the change in the peak value and frequency of the alternating voltage.

Question 10.3 A student argues that when an alternating current is driven through a resistor, the mean current is zero, so an alternating current does not produce heating power on the resistor. State and explain whether this is correct.

10.1.2 power

electrical power dissipated in a resistor: $P = I^2 R$, or, $P = \dfrac{V^2}{R}$

since $I^2, V^2 \geq 0$, P can never be negative, so an a.c. can produce effective power in a resistor

note that for an a.c., P keeps changing with time, as I and V are both varying with time

in everyday life, we are more concerned about the *mean power*

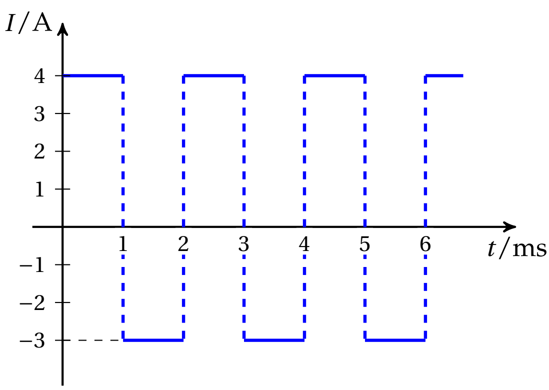
mean power output for an a.c. is: $\langle P \rangle = \langle I^2 \rangle R = \dfrac{\langle V^2 \rangle}{R}$

so we see the necessity to introduce mean square values $\langle I^2 \rangle$ and $\langle V^2 \rangle$

let's further introduce root mean square (r.m.s.) values: $I_{\text{rms}} = \sqrt{\langle I^2 \rangle}$, and $V_{\text{rms}} = \sqrt{\langle V^2 \rangle}$

we can now write the mean power for a.c. as: $\boxed{\langle P \rangle = I_{\text{rms}}^2 R = \dfrac{V_{\text{rms}}^2}{R}}$

Example 10.1 The variation with time of an alternating current in a resistor of 120Ω is shown. (a) What is the value of its r.m.s. current? (b) What is the mean power dissipated by the resistor?



✍ $\langle I^2 \rangle = \dfrac{4^2 + (-3)^2}{2} = 12.5\ \text{A}^2$

$I_{\text{rms}} = \sqrt{\langle I^2 \rangle} = \sqrt{12.5} \approx 3.54$ A

$\langle P \rangle = I_{\text{rms}}^2 R = 12.5 \times 120 = 1500$ W □

10.1.3 r.m.s. current & r.m.s. voltage

in last section, we studied the mathematical aspect of the r.m.s. value

but we still need a definition for r.m.s. current and r.m.s. voltage from a physical viewpoint

physically, r.m.s. value of an alternating current is defined as follows

> **r.m.s. current/voltage** of an a.c. equals a steady d.c. current/voltage that delivers same average power to a resistive load

➢ for *sine waves*, r.m.s values are related to peak values by: $\boxed{I_{\text{r.m.s}} = \frac{1}{\sqrt{2}} I_0}$ and $\boxed{V_{\text{r.m.s}} = \frac{1}{\sqrt{2}} V_0}$

proof: total energy dissipation in one period is: $W_T = \int_0^T P \mathrm{d}t$

so mean power can be given by: $\langle P \rangle = \frac{W_T}{T} = \frac{1}{T} \int_0^T P \mathrm{d}t$

substitute $P = I^2 R \xlongequal{I = I_0 \sin \omega t} I_0^2 R \sin^2 \omega t$, we have: $\langle P \rangle = \frac{I_0^2 R}{T} \int_0^T \sin^2 \omega t \mathrm{d}t$

the integral is carried out: $\int_0^T \sin^2 \omega t \mathrm{d}t = \frac{1}{2} \int_0^T (1 - \cos 2\omega t) \mathrm{d}t = \frac{1}{2} \left(t - \frac{\sin \omega t}{2\omega} \right) \Big|_0^T = \frac{1}{2}$

now we have: $\langle P \rangle = I_{\text{rms}}^2 R = \frac{1}{2} I_0^2 R \quad \Rightarrow \quad I_{\text{rms}} = \frac{1}{\sqrt{2}} I_0$

a similar calculation for voltage would show: $V_{\text{rms}} = \frac{1}{\sqrt{2}} V_0$ □

➢ it is worth pointing out that the $\frac{1}{\sqrt{2}}$-relation for r.m.s. values only holds for sine waves

for other waveforms, e.g., square waves or triangle waves, numerical constant is different

➢ value of voltages stated for mains electricity supply usually refers to the r.m.s. value[65]

Example 10.2 An a.c. power supply produces a sinusoidal output across a resistor of 30 Ω. The maximum voltage is found to be 75 V. Find energy dissipated in the resistor in 2.0 minutes.

✍ r.m.s voltage: $V_{\text{rms}} = \frac{1}{\sqrt{2}} V_0 = \frac{75}{\sqrt{2}} \approx 53.0$ V

mean power output: $\langle P \rangle = \frac{V_{\text{rms}}^2}{R} = \frac{53.0^2}{30} \approx 93.8$ W

energy dissipation: $W = \langle P \rangle t = 93.8 \times 2.0 \times 60 \approx 1.13 \times 10^4$ J □

[65]Different countries have different standards. For example, China mains electricity supplies a voltage of 220 V. UK uses a 230 V distribution system. USA has national standard of a 110 V voltage. These are all r.m.s. voltages.

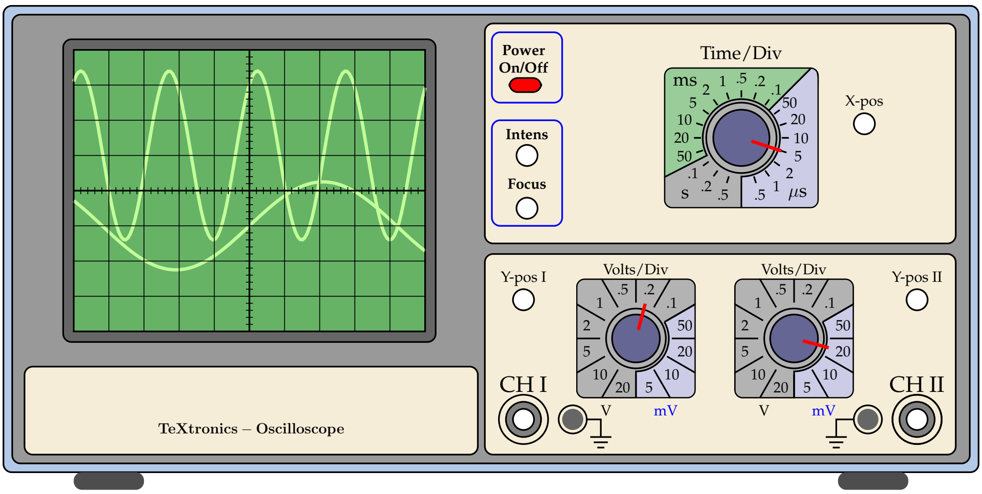
Question 10.4 An alternating voltage V is represented by the equation: $V = 310\sin(100\pi t)$, where V and t are measured in SI units. For this voltage, find (a) the peak voltage, (b) the r.m.s voltage, (c) the frequency, (d) the mean power when it is applied across a resistor of 250 Ω.

Question 10.5 What is the average power dissipated in a resistor when the alternating supply has a peak current of 5.0 A and a peak voltage of 8.0 V?

Question 10.6 The peak value of a sinusoidal alternating current is equal to a steady direct current. When they are applied to the same load, what is the ratio of power dissipation $\frac{P_{\text{d.c.}}}{P_{\text{a.c.}}}$?

10.1.4 measurement of a.c. with an oscilloscope

to measure a varying voltage, we can use a **cathode-ray oscilloscope** (c.r.o.) (see figure[66])



an oscilloscope is basically an electron deflection tube that uses a beam of electrons to trace the input voltage as a function of time.

electron beam is controlled by two sets of parallel plates

horizontal plates control how fast beam moves across screen, giving a horizontal time axis

a.c. voltage is applied across vertical plates, causing beam to bend upwards or downwards

when beam hits fluorescent screen, entire trace of the beam can be seen across the screen

[66]The illustration of the oscilloscope was created by *Hugues Vermeiren*. The source code is downloaded from TeXample: http://www.texample.net/tikz/examples/textronics-oscilloscope/.

➢ to take readings from oscilloscope, remember it displays a voltage against time graph

- **voltage gain**, or **Y-gain**, tells the number of volts per vertical division
- **time base** gives the time unit per horizontal division

Example 10.3 An oscilloscope displays an a.c. voltage signal as shown.

suppose the time base setting is 10 ms/div, and the voltage gain is 5 V/div.

4 vertical divisions from highest to lowest, so

$$\text{peak voltage: } V_0 = \frac{1}{2} \times 4 \times 5 = 10 \text{ V}$$

3 horizontal divisions between peak to peak, so

$$\text{period: } T = 3 \times 10 = 30 \text{ ms}$$

$$\text{frequency } f = \frac{1}{T} = \frac{1}{30 \times 10^{-3}} \approx 33.3 \text{ Hz} \qquad \square$$

Question 10.7 In Example 10.3, if the time-base setting is 5 ms/div and the voltage gain is 2 V/div, write down an equation that represents this alternating voltage.

10.2 power supply systems

10.2.1 high-voltage transmission

electricity is sent from power stations to consumers around the country

for long-distance transmission, we need minimise energy losses

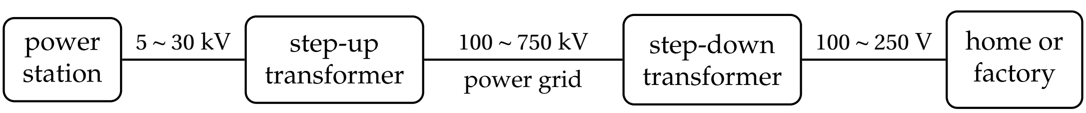
power dissipated due to resistance in cables is I^2R, should transmit at low currents

since output power of a power station is fixed, low current means high voltage

so transmission at high voltages minimises energy loss in power grids

but for reasons of safety and efficiency, desirable to have low voltages at both generating end (power station) and receiving end (home or factory)

this requires converting a.c. into higher or lower voltages ⇒ need for *transformers*

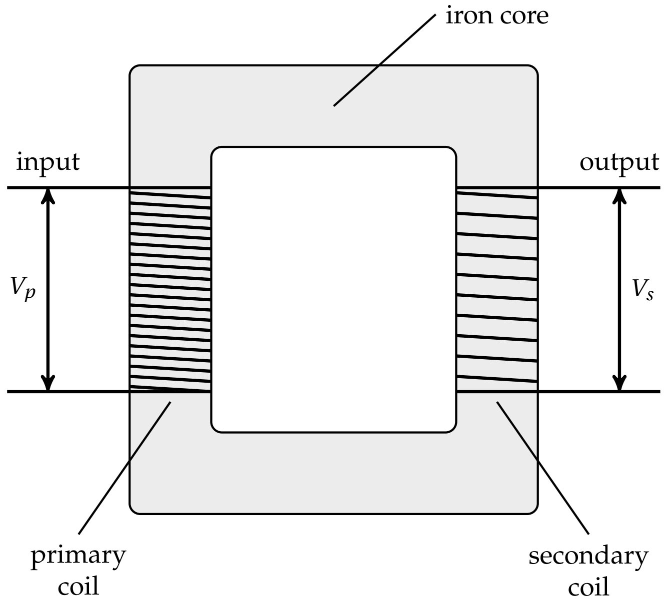


10.2.2 transformers

transformer is a device that can change values of voltage for an alternating current

transformers are an important part of the national power grid

transformers are essential for transmission, distribution, and utilization of electrical energy



structure of a typical transformer

- principle of transformers

  a.c. flowing in **primary coil** (input) produces a changing magnetic field, i.e. a changing flux

  this flux is linked with **secondary coil** through iron core

  an a.c. voltage with same frequency is then induced in secondary coil (output)

- **turns-ratio equation** for a transformer

  assume no loss in magnetic flux, i.e., all field inside transformer's iron core

  from Faraday's law: $V_p = N_p \frac{d\phi}{dt}$, $V_s = N_s \frac{d\phi}{dt}$, where N_p, N_s are number of turns of coils

  cancel out $\frac{d\phi}{dt}$, we find: $\boxed{\frac{V_s}{V_p} = \frac{N_s}{N_p}}$

  if $N_p < N_s$, output voltage is increased → *step-up transformers*

  if $N_p > N_s$, output voltage is decreased → *step-down transformers*

➢ if transformer is 100% efficient, then input power equals output power

for ideal transformers: $\boxed{I_p V_p = I_s V_s}$

➢ in practice, there always exists losses of energy from the transformer

causes of energy loss from the transformer include

- heat produced by *eddy currents* induced in iron core

  this is reduced by *laminating* the core with insulate layers

- heat generated in coils due to resistance

  can use thick copper wire to minimise resistance

- leakage of magnetic flux into surroundings

  transformer's core is made of a continuous loop of iron to minimise this effect

Example 10.4 An ideal transformer has 200 turns on the primary coil and 5000 turns on the secondary coil. The r.m.s. input voltage to the primary coil is 8.0 V. What is the *peak* voltage across a resistor connected to the secondary coil?

✍ $$\frac{V_{s,0}}{V_{p,0}} = \frac{N_s}{N_p} \quad\Rightarrow\quad \frac{V_{s,0}}{\sqrt{2}V_{p,\text{rms}}} = \frac{N_s}{N_p} \quad\Rightarrow\quad \frac{V_{s,0}}{8.0\times\sqrt{2}} = \frac{5000}{200} \quad\Rightarrow\quad V_{s,0} \approx 2830 \text{ V}$$ □

Question 10.8 An ideal transformer has 6000 turns on its primary coil. It converts a mains supply of 220 V r.m.s. to an a.c. voltage with a peak value of 12.0 V. Find the number of turns on the secondary coil.

Question 10.9 If a *steady* d.c. voltage is applied to the input of a simple transformer, what is the output voltage produced?

Question 10.10 State and explain whether the current in the primary coil of a transformer is in *phase* with the voltage induced in the secondary coil.

10.2.3 rectifiers

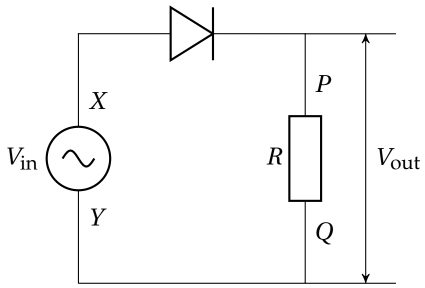
some electronic equipments (e.g., your smartphone, laptop, etc.) must work with d.c.

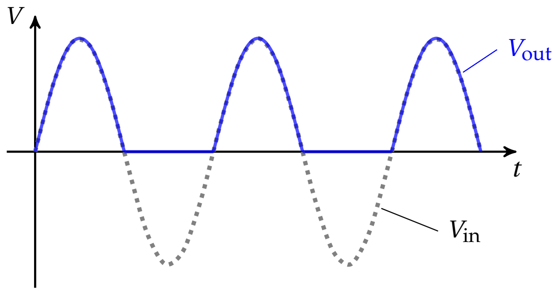
for these appliances require **rectification**, a process that converts an a.c. into a d.c.

rectification uses *diodes*, electronic components that only allow current in one direction

half-wave rectification

half-wave rectification uses a single diode





when input terminal X is positive, current can flow through diode, terminal P is positive with respect to Q for load resistor R

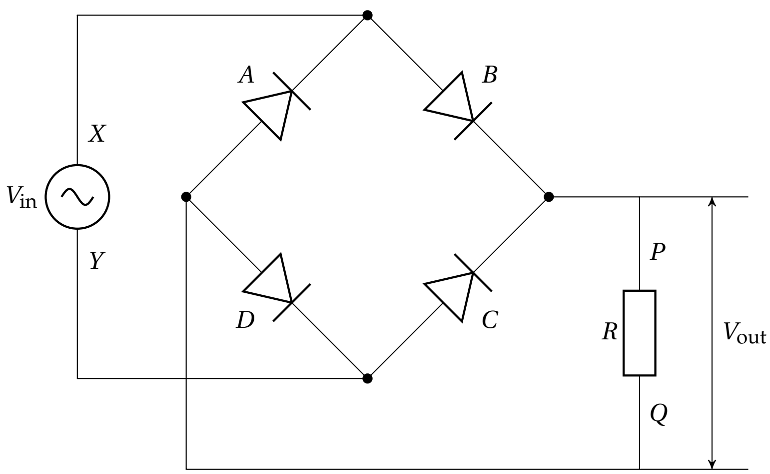
when input terminal Y is negative, flow of current is blocked, so zero output voltage on R

V_{out} across load R is in one direction only, i.e., it becomes a d.c.

but power available from half-wave rectifier is only half of supply power

full-wave rectification

full-wave rectification requires using a combination of four-diode bridge structure



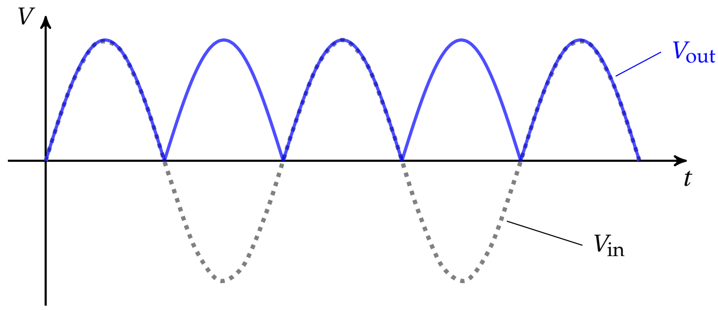
when input terminal X is positive, current flow: $X \to B \to R \to D \to Y$

when input terminal Y is positive, current flow: $Y \to C \to R \to A \to X$

either case, resulting output current in load resistor R always flows in same direction

terminal P is always positive with respect to Q for R no matter what polarity for V_{in}

output voltage across R is now full-wave rectified



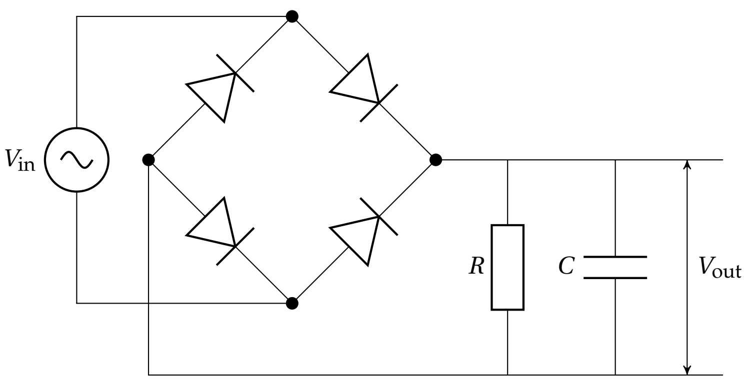
Question 10.11 If we want to have terminal Q to be positive with respect to P for load resistor R, how should we rebuild the bridge rectifier with the same four diodes?

10.2.4 smoothing

note that d.c. resulting from rectification still varies with time

to produce steady d.c. from fluctuating d.c., a process called **smoothing** is carried out

smoothing uses *capacitors*, which are connected in *parallel* with the load resistor (see figure)

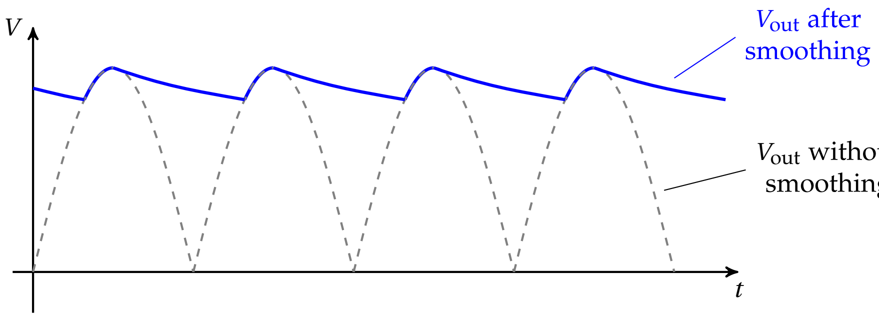


capacitor can store electrical energy as voltage output from rectifier rises

as voltage from rectifier drops, capacitor slowly discharges and feeds energy to the load

output voltage V_{out} across load will have less ripples over the cycles

less fluctuation in V_{out} so we say V_{out} is now smoothed



- effect of smoothing is controlled by choice of load resistor R and smoothing capacitor C

  rate of charging and discharging depend on *time constant* RC (see §7.4)

  greater RC, smoothing capacitor discharges more slowly, giving less ripples

  a larger capacitor for a fixed resistor usually gives better smoothing effect

  but if RC is too large, charging would become very slow, capacitor might not be completely charged after each cycle, this could also result into undesirable effects

Question 10.12 If a second capacitor is added in *series* with the first smoothing capacitor, use sketches to show the changes you would expect for the output voltage.

Chapter 11
Quantum Physics Basics

In this chapter, we will study the laws of nature at very small scales.

We start by reviewing classical concepts like particles or waves, and then see how they break down when being applied to the microscopic world. For those observations that could not be explained with classical physics, we need *quantum physics*. The theory of quantum physics is very deep and profound, so surely way beyond the scope of our course. What we will study in this A-Level course are merely four tips of an enormous iceberg, namely

- photon theory / photoelectric effect
- matter waves / electron diffraction
- electron energy levels / atomic spectra
- band theory / electrical conductivity of solids

11.1 classical theories

in classical physics, both particle models and the wave models have been very useful

though both being successful, the two model are distinct in many aspects

to understand a phenomenon, we take either the particle picture or the wave picture

it seems that there is no way particle models would reconcile with wave models, or is it not?

11.1.1 particle models

in particle model, any system is considered to consist of particles governed by *Newtonian mechanics*, physical properties of this system are predicted by studying behaviour of particles

➢ areas of science where particle models are used to interpret and make predictions include:

- *electricity*: electric current formed by motion of charge carriers

- *ideal gas*: pressure caused by collisions of gas molecules
- *solid*: elasticity due to interaction between solid atoms
- *radioactivity*: decay interpreted as the emission of α-/β-particles from the nucleus
- *chemistry*: chemical reactions due to exchange of electrons between atoms and molecules
- ⋯

11.1.2 wave models

in the wave picture, energy is transferred via the vibration of medium or force fields

➢ phenomena that can be explained in terms of wave models include:

- *water waves*: variation in the vertical displacement of water surface
- *propagation of sound*: variation in the pressure and density of medium
- *light*: variation of electric and magnetic fields
- ⋯

➢ a key feature that makes waves distinct from particles is that waves can *superpose*

when two or more waves meet, they add up or cancel out to give a resultant wave

this gives rise to the characteristic properties of waves:

- *interference*: waves superpose to form a resultant wave of greater or lower amplitude
- *diffraction*: bending of waves around obstacles, or spreading out of waves through slits

11.1.3 history of light

efforts to understand the nature of light can be traced all the way back to ancient Greeks

we are not going to examine the ideas of early thinkers well over 3,000 years ago

let's skip a couple of years and jump to the ideas developed since the Scientific Revolution

➢ particle theory of light (*Issac Newton*, 1671)

key idea: light rays is comprised of a stream of massless particles called *corpuscles*

– explains straight-line propagation
– explains reflection and refraction
– explains colours of light seen in dispersion in prism (corpuscles have different colours)

the problems with the particle model include:

- does not agree with observations on refraction
- cannot predict the interference and diffraction of light

➢ wave theory of light (*Christian Huygens*, 1678)

key idea: light is a wave that transfers energy within a medium known as *aether*

- follows laws of reflection and refraction
- explains colours of light (light have different wavelengths)
- explains interference (Thomas Young's double slit experiment)
- explains diffraction (Poisson spot experiment)

only problem is that aether, the medium in which light lives, was not experimentally found

➢ electromagnetic theory (*James Maxwell*, 1865)

key idea: light is an *electromagnetic wave* [67]

- electric and magnetic fields travel through space in the form of waves at speed of light
- propagation of electromagnetic wave does not require medium, hence no need for aether

all behaviour of light known at that time could be explained with Maxwell's theory

so scientists were convinced that light travelled through space as an electromagnetic wave

11.2 photon theory

11.2.1 photoelectric effect

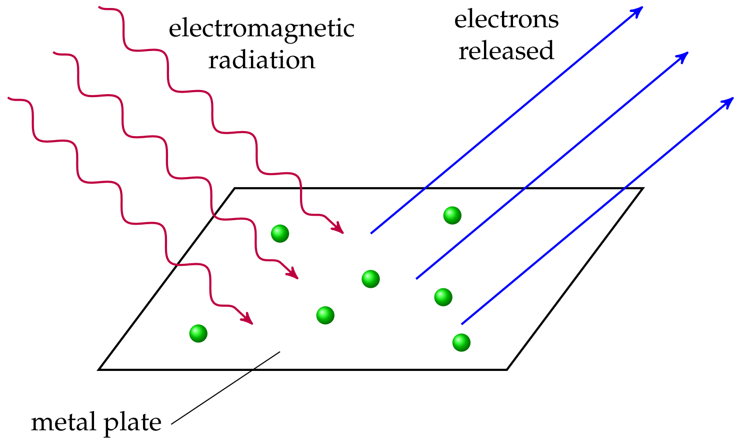
electromagnetic radiation incident upon a metal can cause emission of electrons from the metal surface, this is called **photoelectric effect**, [68] emitted electrons are called **photoelectrons**

[67] *Maxwell's equations* fully describe the behaviour of electric and magnetic fields. Based on the four equations that Maxwell established, he found the vibration of electric and magnetic field can propagate in space as a classical wave. The speed of electromagnetic wave is given by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{8.85 \times 10^{-12} \times 4\pi \times 10^{-7}}} = 3.00 \times 10^8 \text{ m s}^{-1},$$

which is the same as speed of light. This gave Maxwell the intuition that light is an electromagnetic wave.

[68] Photoelectric effect was first observed in 1887 by *Heinrich Hertz*, who found electrodes illuminated with ultraviolet light create electric sparks. In school labs, one can shine ultraviolet radiation from a



simply put, conduction electrons in metal gain additional energy by absorbing the incoming radiation, if this energy is sufficient for electrons to overcome the electrostatic attraction from the positive metal ions, they break free from the metal surface

- some detailed experimental observations on the photoelectric effect are:
  - there exists a minimum **threshold frequency** f_0 for incident radiation
    when $f < f_0$, no electrons released from metal surface
  - emission of electrons is immediate when radiation is incident as long as $f > f_0$
    even low-intensity light is effective
  - increasing intensity has no effect on energies of electrons
  - increasing intensity of incident light causes number of photoelectrons emitted to increase
  - increasing radiation frequency increases electron energies
- wave theory of light fails to explain any of these properties, according to wave model:
  - electrons could gradually build up energies by absorbing wave energy over time
    so radiation at any frequency should all work
  - need very intense light to have immediate effect
  - greater intensity mean higher energy, electrons released should have greater K.E.
  - varying frequency of radiation should have no effect on energy of electrons released

photoelectric effect sees the breakdown of the wave model, some new ideas are needed!

mercury lamp onto a zinc plate to cause photo-emission.

11.2.2 photon theory

photoelectric effect was explained by *Albert Einstein* in 1905 [69]

Einstein's revolutionary idea: when radiation delivers energy to matter, the transfer of energy is not continuous but carried in *discrete* packets, called *photons*

> a **photon** is a packet (*quanta*) of electromagnetic energy

- energy of one photon is given by: $\boxed{E = hf}$, where $h = 6.63 \times 10^{-34}$ J s is the **Planck constant**

  since $E \propto f$, higher/lower radiation frequency means greater/smaller photon energies
- wave equation $c = \lambda f$ relates frequency f of an electromagnetic wave to its wavelength λ

  so energy of one photon is also given by: $\boxed{E = \frac{hc}{\lambda}}$

  since $E \propto \frac{1}{\lambda}$, longer/shorter wavelength means lower/greater photon energies
- the equation $E = hf$, or $E = \frac{hc}{\lambda}$, relates a particle property with a wave property

  E is the energy of one photon, treated like a single particle

  but f and λ are both introduced to describe a wave, not a particle
- intensity of a beam of radiation is: $I = \frac{P}{A}$

  total energy incident on a given area per unit time determines the radiation intensity

  so intensity depends on the product of the number of photons arriving per unit time and energy of each photon: $\boxed{I \propto nhf}$, or more precisely, $I = \frac{nhf}{A}$
- when considering energy of a photon, **electronvolt** is a useful energy unit

  one electronvolt (1 eV) is work needed to make an electron travel through a p.d. of 1 V

  conversion between electronvolt and joule is: $\boxed{1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}}$

Example 11.1 A laser emits red light of 650 nm at a power rating of 2.0 mW. (a) What is the energy carried by one photon? (b) How many photons are emitted per second?

[69] Albert Einstein was awarded the Nobel Prize in physics in 1921 for 'his discovery of the law of the photoelectric effect'. This discovery led to the quantum revolution of modern physics.

✍ energy of one photon: $E = \dfrac{hc}{\lambda} = \dfrac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{650 \times 10^{-9}} \approx 3.06 \times 10^{-19}$ J

number of photons: $N = \dfrac{\text{total energy output from laser}}{\text{energy of one photon}} = \dfrac{2.0 \times 10^{-3}}{3.06 \times 10^{-19}} \approx 6.54 \times 10^{15}$ □

Question 11.1 Calculate the energy, in eV, of a photon of light of wavelength 440 nm.

Question 11.2 If the light source in Example 11.1 is replaced by a green laser with the same power output, how does the number of emitted photons per unit time change?

11.2.3 photoelectric effect explained

from the viewpoint of photon theory, photoelectric effect can be explained easily

to release an electron from metal, it requires a minimum energy Φ, called the **work function**, for the electron to overcome attraction due to metal ions to escape from metal surface

as radiation shines upon metal, photon energies are absorbed by electrons

since photon energies are *discrete*, or *quantised*, the absorption is sort of all or nothing

if photon energy is greater than Φ, electrons break free from metal

excess energy, if any, would become kinetic energy of the free electron

this is summarised in **Einstein's photoelectric equation**: $\boxed{hf = \Phi + E_{k,\max}}$

➢ note that we are talking about *maximum* K.E. of emitted electrons

electron emitted from the *surface* would have greatest K.E.

for electrons to be released from *below* the surface, they require more energy than work function, so less K.E. than maximum value

➢ at critical condition, incoming photon has just enough energy to release electron

so at threshold frequency f_0, photon energy equals work function: $\boxed{hf_0 = \Phi}$

➢ experimental observations mentioned in §11.2.1 can now be understood

- below threshold frequency f_0, not enough photon energy available to electron to overcome work function, so no effect for $f < f_0$
- interaction between photon and electron is *one-to-one*, so no time delay
- greater radiation intensity means more photons per unit time, so more electrons released
- greater frequency means higher photon energy, so greater K.E. for emitted electron

Example 11.2 Given that work function energy of gold is 4.9 eV. Find the longest wavelength of electromagnetic wave that could release electrons from gold.

✍ threshold frequency: $f_0 = \dfrac{\Phi}{h} = \dfrac{4.9 \times 1.60 \times 10^{-19}}{6.63 \times 10^{-34}} \approx 1.18 \times 10^{18}$ Hz

threshold wavelength: $\lambda_0 = \dfrac{c}{f_0} = \dfrac{3.00 \times 10^{8}}{1.18 \times 10^{15}} \approx 2.54 \times 10^{-7}$ m (ultraviolet light) □

Question 11.3 Sodium has a work function of 3.8×10^{-19} J. (a) Find the threshold frequecy for sodium. (b) If a light of 500 nm is incident on sodium, determine whether electrons can be emitted from the surface.

Question 11.4 When electromagnetic radiation of wavelength 1200 nm is incident on a metal surface, the maximum kinetic energy of the electrons released is found to be 5.4×10^{-20} J. What is the work function of this metal?

Question 11.5 When a beam of light of a particular frequency and intensity is shone onto a metal surface, electrons are released. If another beam of light of same intensity but higher frequency is used, what is the effect on the rate of emission of electrons from this surface?

measurement of the Planck constant and work function energy

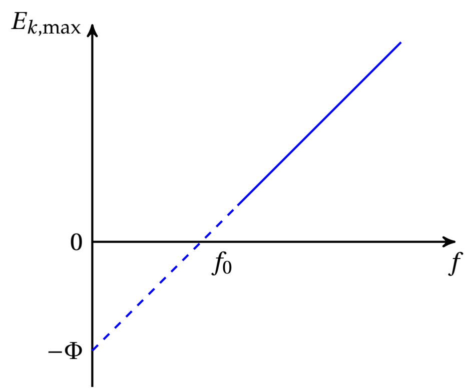
when radiation with different frequencies is incident onto a metal, we measure maximum K.E. of the electrons emitted from the surface, a set of readings $(f, E_{k,\max})$ can be found

note that the photoelectric equation can be rearranged as: $E_{k,\max} = hf - \Phi$

if we plot a graph of $E_{k,\max}$ against f, data points shall fall on a straight line

information about the Planck constant h, threshold frequency f_0, work function Φ can all be computed with the best-fit line

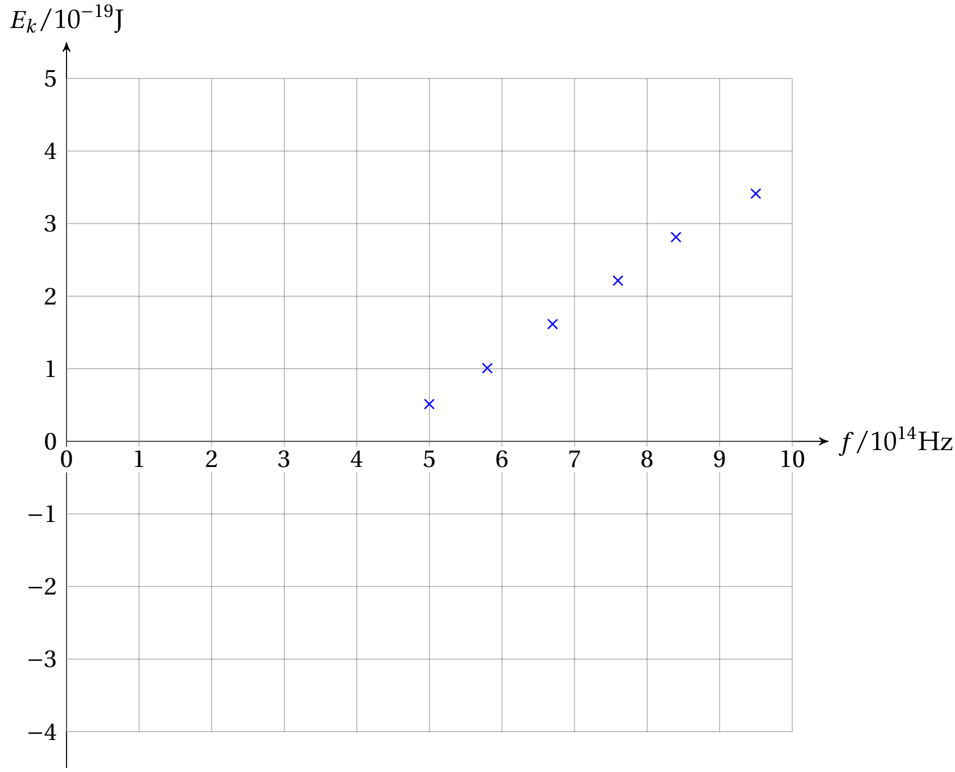
- gradient $= h$
- y-intercept $= -\Phi$
- x-intercept $= \dfrac{\Phi}{h} = f_0$



Question 11.6 If a different metal with a greater work function energy is used, describe the

change for the line that shows the variation with f of $E_{k,\max}$ for this metal.

Question 11.7 Electromagnetic adiation is incident upon a metal plate. The graph shows how maximum kinetic energy E_k of emitted electrons varies with frequency f of the radiation. Use the graph to find (a) the threshold frequency, (b) a value of Planck constant.



11.2.4 photon momentum

in photoelectric effect, photons can knock electrons out of a metal, this suggests that photons could have *momentum*, even though they do not have mass

earliest experimental evidence of photon momentum was came from *Arthur Compton* in 1923, who studied the scattering of X-ray photons by electrons in substances[70]

it can be shown[71] that photon momentum is given by: $\boxed{p = \frac{h}{\lambda}}$

[70]Arthur Compton was awarded the Nobel Physics Prize in 1929 for the discovery of this scattering effect, now known as *Compton scattering*.

[71]This derives from Einstein's theory of *special relativity*, which states that energy and momentum are

➢ photon momentum a *relativistic* momentum, as photons move at speed of light

definition for classical momentum $p = mv$ does not apply for photons

➢ due to exchange of momentum, electromagnetic wave can exert *radiation pressure*

forces generated by radiation pressure are negligible under everyday circumstances, but they could have noticeable effects on spacecraft in outer space and comet tails

Example 11.3 A beam of light has wavelength 600 nm, cross-sectional area 0.16 cm^2 and power 5.0 mW. The beam is normally incident onto a surface and is completely absorbed. Calculate, for a time of 1.0 s, (a) the number of photons incident onto the surface, (b) the change of total momentum of the photons, (c) the light pressure on the surface.

✍ energy of one photon: $E = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{600 \times 10^{-9}} \approx 3.32 \times 10^{-19}$ J 1.5083

number of photons arriving in 1.0 s: $N = \frac{5.0 \times 10^{-3} \times 1.0}{3.32 \times 10^{-19}} \approx 1.51 \times 10^{16}$

momentum for one photon: $p = \frac{h}{\lambda} = \frac{6.63 \times 10^{-34}}{600 \times 10^{-9}} \approx 1.11 \times 10^{-27}$ kg m s^{-1}

change of total momentum: $\Delta P = Np = 1.51 \times 10^{16} \times 1.11 \times 10^{-27} \approx 1.67 \times 10^{-11}$ kg m s^{-1}

average force due to these photons: $F = \frac{\Delta P}{\Delta t} = \frac{1.67 \times 10^{-11}}{1.0} \approx 1.67 \times 10^{-11}$ N

light pressure on surface: $p = \frac{F}{A} = \frac{1.67 \times 10^{-11}}{0.16 \times 10^{-4}} \approx 1.04 \times 10^{-6}$ Pa □

Question 11.8 A laser of power P is incident normally on a spot of area A, show that the pressure caused by the beam can be given by: $p = \frac{P}{cA}$.

Question 11.9 When an electron and a positron meet together, they will annihilate and produce two γ-photons: ${}^{0}_{-1}e + {}^{0}_{+1}e \longrightarrow \gamma + \gamma$. Assume the electron and the positron have negligible kinetic energy before the interaction, explain why the two photons produced must move off in opposite directions with equal energies.

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related by the equation: $E^2 = m_0^2c^4 + p^2c^2$. Photons have zero rest mass, i.e., $m_0 = 0$. This relativistic relation becomes $E = pc$ for photons. Now recall photon energy is given by $E = hf$. Rearrange the terms, we can show: $p = \frac{hf}{c} = \frac{h}{\lambda}$.

11.3 wave-particle duality

11.3.1 matter waves

inspired by photon theory, which shows electromagnetic waves have a particulate nature, *Louis de Broglie* suggested in his 1924 PhD thesis that all matter has a wave-like nature [72]

wave characteristic of a particle can be represented by a wavelength

de Broglie wavelength of a matter particle is given by: $\boxed{\lambda = \dfrac{h}{p}}$, where $p = mv$ is the particle's momentum

Example 11.4 What is the wavelength of a human of 70 kg walking at around 2.0 m s^{-1}?

✍ $\lambda = \dfrac{h}{mv} = \dfrac{6.63 \times 10^{-34}}{75 \times 2} \approx 4.7 \times 10^{-36}$ m

this wavelength is too small compared with any obstacle we encounter in everyday lives

so human bodies do not exhibit noticeable wave behaviour □

Example 11.5 An electron is accelerated from rest through a potential difference of 50 V. What is the de Broglie wavelength of this electron?

K.E. of electron equals change in electric P.E., so

$$\frac{1}{2}mv^2 = qV \quad \Rightarrow \quad v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-19} \times 50}{9.11 \times 10^{-31}}} \approx 4.19 \times 10^6 \text{ m s}^{-1}$$

wavelength of electron: $\lambda = \dfrac{h}{mv} = \dfrac{6.63 \times 10^{-34}}{9.11 \times 10^{-31} \times 4.19 \times 10^6} \approx 1.74 \times 10^{-10}$ m

this wavelength is comparable to scale of atomic spacing (also around 10^{-10} m)

so these electron can be *diffracted* by solid crystals □

Question 11.10 An α-particle is moving with a kinetic energy of 2.4×10^{-15} J. Find its speed, and hence find its de Broglie wavelength.

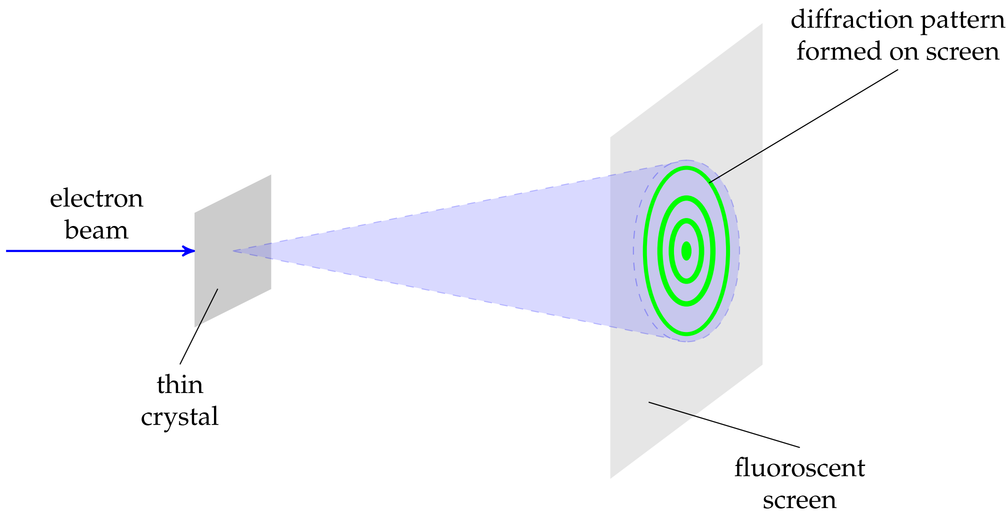
Question 11.11 If a proton and an electron are accelerated through the same voltage, (a) how would their energy compare? (b) how would their wavelength compare?

[72]Louis de Broglie was awarded the 1929 Nobel Physics Prize 'for his discovery of the wave nature of electrons'. *Schröedinger's equation* and *Bohr's atomic model* was heavily influenced by ideas of de Broglie.

11.3.2 electron diffraction

wave property of electron was confirmed by *Clinton Davisson* and *George Thomson* in 1927

they showed experimentally electrons could be diffracted by metal crystals [73]



electron diffraction experiment

➢ electron diffraction experiment shows that particles do have wave-like properties

if electrons behaved like particles, we would see a round spot of *uniform* distribution

but the actual pattern formed on *fluorescent screen* is a set of *concentric rings*

this is a typical diffraction pattern, hence proves wave properties of electrons

➢ each metal has a different lattice structure, so each produces a different pattern

this allows investigation of structure of matter (explore arrangements of atoms, structures of complex molecules, structure of atomic nuclei, etc.) using electron diffraction

➢ since wavelength of an electron is much shorter than visible light

this allows *electron microscope* to have much higher resolving power than optical microscopes

Question 11.12 If a higher p.d. is applied to accelerate the electron beam in the electron diffraction experiment, how would the pattern change?

[73]The Nobel Prize in Physics 1937 was awarded jointly to Clinton Davisson and George Thomson 'for their experimental discovery of the diffraction of electrons by crystals'

11.3.3 wave-particle duality

we have seen both particle-like and wave-like behaviour in light and electrons

	electromagnetic radiation	electron
particle-like behaviour	photoelectric effect, radiation pressure	deflection in electric/magnetic fields, decay, scattering
wave-like behaviour	interference, diffraction, Doppler effect, reflection, refraction	electron diffraction

when light/electron move through space, they behave like a wave

when light/electron interact with each other, they behave like particles

they show a two-sided nature, or a *dual* nature of being both a wave and a particle, described by either particle model or wave model under different circumstances

in fact, all matter (protons, neutrons, atoms, cells, basketballs, human body, earth, etc.) has this universal dual nature, called **wave-particle duality**

wave-particle duality addresses breakdown of classical concepts like particle or wave

to fully describe the behaviour of microscopic objects, we need *quantum mechanics* [74]

11.4 quantisation of electron energy levels

in a simple atomic model, electrons move around the nucleus in circular orbits

but now we understand electrons have wave properties, as an electron moves in its orbit as a wave, it can superpose with itself

only orbits in which the electron can superpose constructively with itself are preferable

so electrons are only allowed to move in certain orbits in an atom

this means can only take certain values of energy, called *energy levels*

[74]Being a central concept of quantum theory, wave-particle duality is deeply embedded into the foundations of quantum physics. In non-relativistic quantum mechanics, all information about a particle is encoded in its *wave function*, whose evolution with time is described by the famous *Schrödinger's equation*.

11.4.1 hydrogen atom (⋆)

theoretical explanation for electron energy levels was developed in 1913 by Danish physicist *Niels Bohr* in his theory of hydrogen atom [75]

let's look at the hydrogen atom – the simplest possible atom in nature

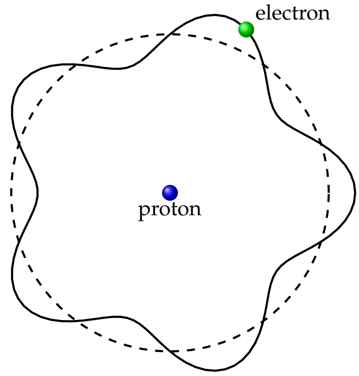
electrostatic attraction by proton provides centripetal force for electron to move in circles

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r} \quad \Rightarrow \quad v^2 = \frac{e^2}{4\pi\epsilon_0 m_e r}$$

for electron to *constructively* superpose with itself, the perimeter of its orbit must be an integer multiple of its de Broglie wavelength

$$2\pi r = n\lambda \quad \Rightarrow \quad 2\pi r = \frac{nh}{m_e v} \quad \Rightarrow \quad v = \frac{nh}{2\pi m_e r}$$

where n is positive integer $1, 2, 3, \cdots$



the hydrogen atom

compare the two equations, we can eliminate v

$$\frac{e^2}{4\pi\epsilon_0 m_e r} = \frac{n^2 h^2}{4\pi^2 m_e^2 r^2}$$

we hence find radius of allowed orbits satisfies:

$$\boxed{r_n = n^2 a_0} \quad \text{where } a_0 = \frac{h^2\epsilon_0}{\pi m_e e^2} \approx 5.29 \times 10^{-11}\ \text{m}$$

so we see all allowed orbits must have a radius equal to integer multiple of a_0

total energy possessed by the electron consist of K.E. and electric P.E.:

$$E = E_k + E_p = \frac{1}{2} m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r} = \frac{1}{2} m_e \frac{e^2}{4\pi\epsilon_0 m_e r} - \frac{e^2}{4\pi\epsilon_0 r} \quad \Rightarrow \quad E = -\frac{e^2}{8\pi\epsilon_0 r}$$

substitute the allowed orbital radius, we find an expression for electron energy:

$$\boxed{E_n = -\frac{R_E}{n^2}} \quad \text{where } R_E = \frac{m_e e^4}{8h^2\epsilon_0^2} \approx 2.18 \times 10^{-18}\ \text{J} \approx 13.6\ \text{eV}$$

this shows electron can only have specific energies in the hydrogen atom

[75]Niels Bohr was surely one of the greatest physicists of the 20th century. He made foundational contributions to understanding atomic structure and quantum mechanics, for which he received the Nobel Physics Prize in 1922. He was also the founder of the Institute of Theoretical Physics at the University of Copenhagen, which soon became the centre of pioneering researches on quantum theory in the world.

11.4.2 electron energy levels

it can be shown that for any atom, electrons can only have certain fixed values of energy

we say energy of electrons in an atom is *discrete*, or **quantised**

these allowed values of energies are called **energy levels**[76]

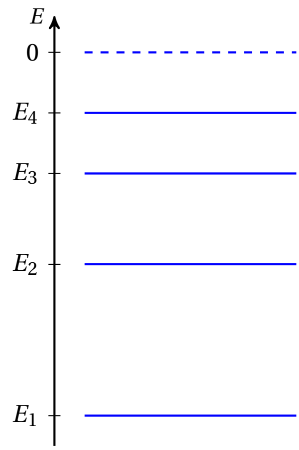
➢ *quantisation* means an electron can have only specific values of energy in an atom

no intermediate value between levels is allowed

➢ the lowest energy level is called the **ground state**

any level higher than the ground state is called an **excited state**

in our illustration, E_1 is the ground state, while E_2, E_3, E_4 are all excited states



electron energy levels of some atom

➢ electron energy levels are all *negative*

to pull an electron away from the nucleus, work must be done to overcome electrostatic attraction

an electron at infinity would have greatest P.E., which is defined to be zero

so orbiting electrons have energies less than zero

an electron that has zero energy would become a *free* electron

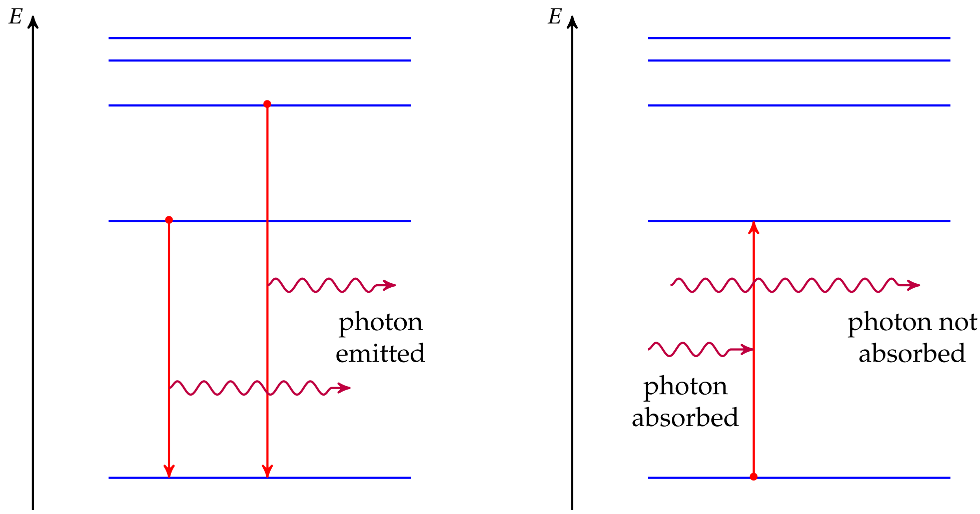
➢ electrons can jump, or *transit*, between energy levels

electron transition is associated with emission or absorption of photon with the right energy

11.4.3 atomic spectrum

passing light through a *prism* or a *diffraction grating*, different wavelengths are separated, a pheonomenon known as **dispersion**, this produces a **spectrum** which shows distribution of energy emitted from the source in order of wavelengths

[76]If you study chemistry, you shall recall this idea of discrete energy levels is related to the concepts of energy shells and sub-shells of atoms.



emission and absorption of photons as a result of electron transitions

for example, white light consists of a range of wavelengths from around 400 nm to 700 nm, so it gives a *continuous spectrum* with rainbow of colours

continuous spectrum of white light

electron transition between different levels causes emission or absorption of photons

this leads to the emission spectrum and absorption spectrum of atoms

emission spectrum

hot gases of an element can release photons, giving an **emission spectrum**

this happens when an electron transits from a high energy level to a lower level

energy of the photon emitted is equal to energy difference between the two levels

since electron energy levels are discrete, only specific changes of energies are possible, so only photons with specific energies can be emitted

this means photons emitted only have specific wavelengths, or specific frequencies so a collection of sharp and bright lines are seen in emission spectrum

emission spectrum of the hydrogen atom

- ➢ emission spectrum is a *discrete* spectrum, also called a *line* spectrum
- ➢ atoms of each element has a unique electron energy level structure

so each element produces a unique emission spectrum, leading to quite a few applications:

- explains the colour of flames when a particular chemical element is present
- allows the identification of elements in an unknown substance in chemical analysis
- explains varied colours of electric signs lighted by gas-discharge tubes

absorption spectrum

pass white light through a *cool gas*, photons with the right energies can be absorbed, giving rise to an **absorption line spectrum**

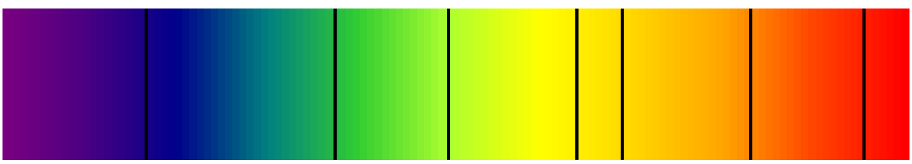
by absorbing photon, electron can transit from a low energy level to a higher level

energy of the photon absorbed must equal energy difference between the two levels

since electron energy levels are discrete, so only photons with specific energies can be absorbed, while other photons are unaffected as they pass through the gas

wavelengths of these absorbed photons will be missing in the emergent spectrum

so we would observe a set of *dark lines* appearing in a background of continuous spectrum



absorption spectrum for some element

➢ absorption spectrum is also a *discrete* spectrum, or a *line* spectrum

➢ as photons with the right energies are absorbed, they can be re-emitted through *de-excitation*

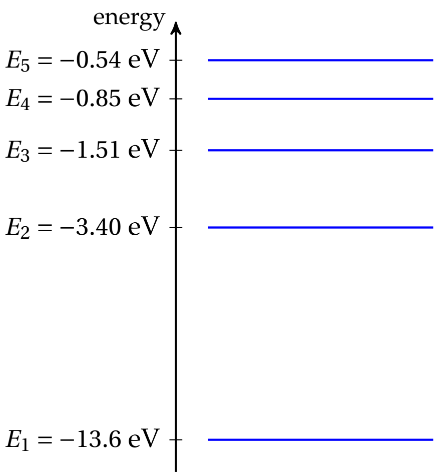
but these photons are re-emitted in *random* directions

so they would still appear dark compared with those unaffected wavelengths

➢ absorption spectrum is widely used in many areas

- determine the composition of a particular substance in analytical chemistry
- determine chemical compositions of stars in astronomical spectroscopy
- explain colour of chemicals in terms of the complementary colour of photons absorbed

Example 11.6 Some electron energy levels of the hydrogen atom is shown. Find the longest and the shortest wavelength produced by electron transitions between the energy levels given.



energy levels of the hydrogen atom

✍ energy of photon emitted equals change of electron energy level: $\frac{hc}{\lambda} = \Delta E$, or $\lambda = \frac{hc}{\Delta E}$

transition with least/greatest energy change gives rise to longest/shortest wavelength, so

$$E_5 \to E_4 \Rightarrow \lambda_{\max} = \frac{hc}{E_5 - E_4} = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^8}{(0.85 - 0.54) \times 1.60 \times 10^{-19}} \approx 4.0 \times 10^{-6} \text{ m} \quad \text{(infra-red)}$$

$$E_5 \to E_1 \Rightarrow \lambda_{\min} = \frac{hc}{E_5 - E_1} = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^8}{(13.6 - 0.54) \times 1.60 \times 10^{-19}} \approx 9.5 \times 10^{-8} \text{ m} \quad \text{(ultraviolet)} \qquad \square$$

Example 11.7 The emission spectrum of the hydrogen atom consists of a number of wavelengths in the visible spectrum. Given that the visible spectrum consists of light of wavelengths within the range from 380 nm to 740 nm, also use data in Example 11.6, find the energies of the photons that can be produced by transitions between the energy levels shown.

✍ let's first find the range of energies, in eV, for visible photons

$$E_{\text{max}} = \frac{hc}{\lambda_{\text{min}}} = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{380 \times 10^{-9}} \approx 5.23 \times 10^{-19}\ \text{J} = \frac{5.23 \times 10^{-19}}{1.60 \times 10^{-19}}\ \text{eV} \approx 3.27\ \text{eV}$$

$$E_{\text{min}} = \frac{hc}{\lambda_{\text{max}}} = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{740 \times 10^{-9}} \approx 2.69 \times 10^{-19}\ \text{J} = \frac{2.69 \times 10^{-19}}{1.60 \times 10^{-19}}\ \text{eV} \approx 1.68\ \text{eV}$$

we shall find the combinations of electron energy levels such that their difference fall within the range between 1.68 eV and 3.27 eV, by trial and error, we find three possible combinations:

(1) $E_3 \to E_2 \Rightarrow 3.40 - 1.51 = 1.89$ eV

(2) $E_4 \to E_2 \Rightarrow 3.40 - 0.85 = 2.55$ eV

(3) $E_5 \to E_2 \Rightarrow 3.40 - 0.85 = 2.85$ eV

these emisssion lines are called H_α (656 nm), H_β (486 nm) and H_γ (435 nm) lines[77]

they are three of the four the hydrogen emission lines that are visible to human eyes[78] □

Example 11.8 A white light is incident on a cloud of cool hydrogen gas. In the emergent spectrum, a dark line is observed at a wavelength of 435 nm. Determine the energy change that gives rise to this dark line.

✍ photon absorbed: $E = \dfrac{hc}{\lambda} = \dfrac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{435 \times 10^{-9}} \approx 4.57 \times 10^{-19}\ \text{J} = \dfrac{4.57 \times 10^{-19}}{1.60 \times 10^{-19}}\ \text{eV} \approx 2.86\ \text{eV}$

note that $3.40 - 0.54 = 2.86$ eV, so this dark line is due to the electron transition: $E_2 \to E_5$ □

Question 11.13 If we only consider the electron energy levels given in Example 11.6, how many different wavelengths can be detected in the emission spectrum of the hydrogen atom ?

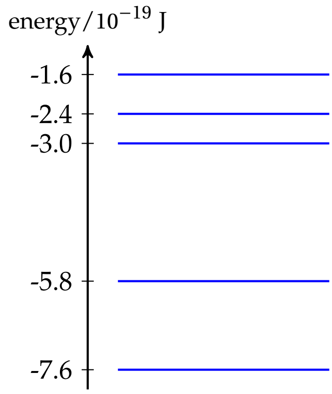
Question 11.14 Three lines are observed at wavelengths 486 nm , 656 nm, and 1880 nm in the emission spectrum of hydrogen atoms. (a) Calculate the photon energies for these wavelengths.

[77]The H_α line is the brightest hydrogen line in the visible range. It plays an important role in astronomy, as it can be used to study a star's surface temperature, the velocity of a distant stellar object, etc.

[78]These lines belong to a family of the spectral lines of the hydrogen atom, known as the *Balmer lines*, named after Johann Balmer. Balmer discovered an empirical equation to predict the series in 1885, but the reason why the equation worked was eventually clarified by Neils Bohr with the atomic model which now bears his name. We now understand the Balmer lines correspond to emissions of photons by electrons jumping to the second level from higher energy states.

(b) Draw a diagram with *three* labelled energy levels, and show the energy changes for the three wavelengths produced with arrows in your diagram.

Question 11.15 The relative cool atmosphere of the sun could give rise to dark lines in the spectrum of sunlight. One particular dark spectral line has a wavelength of 590 nm. By reference to the energy levels of the helium atom, suggest how this dark line provides evidence of the presence of helium in sun's atmosphere. You may draw an arrow to show the possible electron transition that gives rise to this dark line.



energy levels of the helium atom

11.5 band theory

11.5.1 energy bands

in an isolated atom, electrons have discrete energy levels

in solids, interaction between neighbouring atoms causes change of energies

original energy levels split into a band with many sub-levels

number of sub-levels in each band equals number of atoms in solid in general

due to very large number of atoms, sub-levels (seem to) form *continuous* **energy bands**

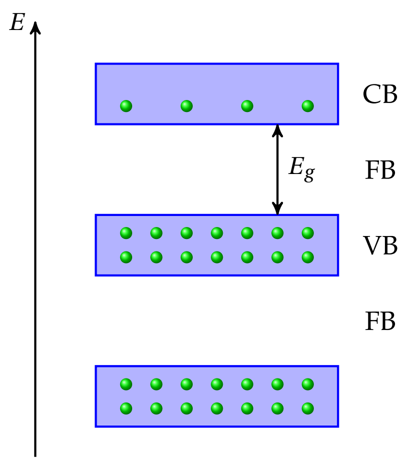
in solids, electrons fill up from lowest energy bands available (see figure for a rough idea)

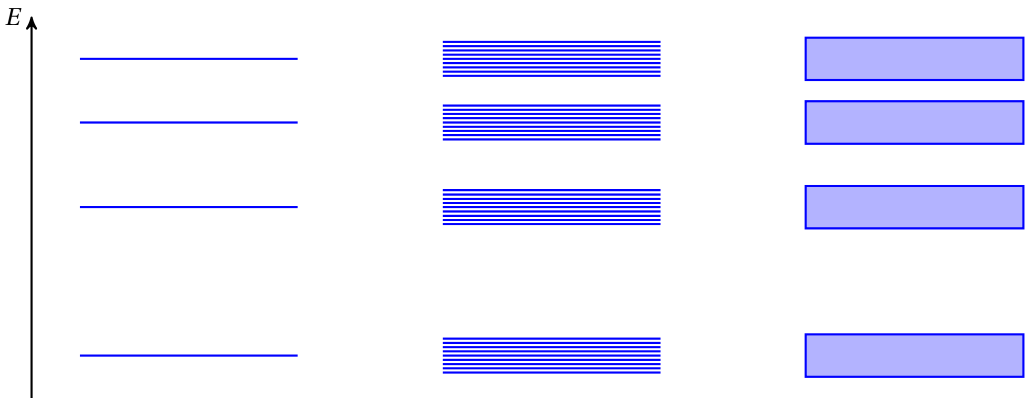
forbidden bands (FB) represent energies that electrons are not allowed to take

highest fully filled band is called **valence band** (VB)

next partially filled or empty band is called **conduction band** (CB)

separation between valence band and conduction band defines **band gap** (E_g) of material





dicrete energy levels in isolated atom

split energy sub-levels due to interactions

energy bands formed in many-particle system

11.5.2 band theory & electrical conductivity

electrical conductivity of material can be explained by its band structure

low-energy bands below VB are usually fully filled at all times, all states are occupied, electrons cannot move freely to form electric currents, so we are not interested in low-energy bands

conductivity of material will depend on whether there are free *charge carriers* in CB and VB

metallic conductors

for typical metals, CB *overlaps* with VB, i.e., there is no band gap

there exist vacant states for conduction electrons to occupy at ease

this means conduction electrons are free to move around to form currents

as temperature rises, there is no significant change in number of free electrons

but *lattice vibration* of atoms increases, electrons become more likely to collide with vibrating atoms, so resistance of metal increases with temperature

➢ metals are *opaque* to visible light or other low-frequency radiation

width of CB for a typical metal is about 2 ~ 3 eV

low-energy photons can be absorbed by conduction electrons

visible light, infra-red, microwave therefore cannot pass through metal

➢ metals are *transparent* to high-frequency radiation such as X-rays

if X-ray photons were absorbed, electrons would take energies in forbidden band

but this is not allowed, so high-frequency photons penetrate through metal

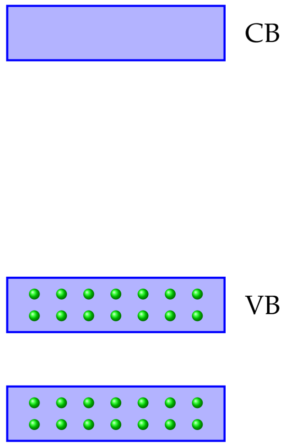
insulators

examples of insulators include glass, diamond, etc.

in normal conditions, CB of insulator is empty, energy bands up to VB are fully filled, so no movement of electron is possible, material has poor conductivity

insulators also have huge band gap[79][80], so thermal excitations cannot easily make VB electrons jump into CB

insulators remain poor conductors as temperature rises



band structure of insulator

Question 11.16 Diamond has a band gap $E_g \approx 6.0$ eV. By reference to its electronic band structure, explain why diamond appears *transparent* to visible light?

intrinsic semiconductors

examples of an intrinsic semiconductor are silicon and germanium

band structure of an intrinsic semiconductor is similar to that of insulators

at low temperature, no free electron from the empty CB and completely occupied VB, so semi-conductors are not conductive in normal conditions

however, semiconductors have much narrower band gaps

as temperature rises, VB electrons gain thermal energy to cross band gap and enter CB

the electrons jumped into CB surely can move around to form electric currents

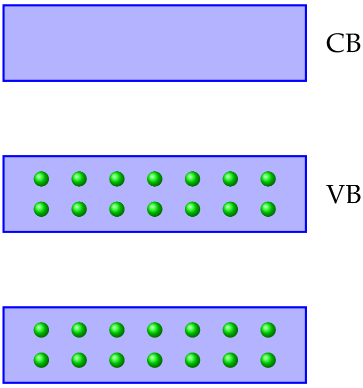
[79]To be a little more precise, materials with a band gap $E_g \approx 5$ eV or greater are regarded as insulators.

[80]There exist a class of insulators called *Mott insulators* which do not have large band gaps. Their poor conductivity at low temperatures is due to electron-electron interactions, which are not considered under conventional band theories.

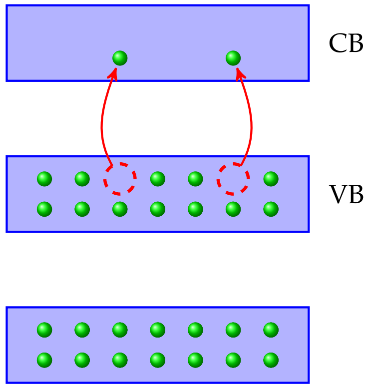
at the same time, VB is no longer completely filled, *holes* are formed

a hole is a site where an electron is missing, neighbouring electrons can move to fill the hole and leave a new hole, but this can be thought of as the hole moving about

there are now more *charge carriers* (negatively-charge electrons and positively-charged holes) available, so semiconductor has better conductivity, resistance of material would decrease



band structure of an intrinsic semiconductor at low temperature



at higher temperature VB electrons cross band gap and enter CB

➢ *lattice vibration* could also affect resistance of semiconductors

as temperature rises, vibration of atoms increases, charge carrier are more likely to get scattered, causing a decrease in conductivity

but the effect of more charge carriers is far greater than effect due to lattice vibration

so resistance of semiconductor decreases at higher temperature

Example 11.9 Silicon has a band gap $E_g \approx 1.1$ eV. State and explain whether it becomes conducting when exposed to red light of 600 nm.

✍ energy of radiation: $E = \dfrac{hc}{\lambda} = \dfrac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{600 \times 10^{-9}} \approx 3.32 \times 10^{-19}\ \text{J} \approx 2.1\ \text{eV} > E_g$

this energy is sufficient to make valence electrons in silicon to cross band gap

silicon now possesses conducting electrons and holes therefore becomes conducting □

Question 11.17 Use band theory to explain why an LDR (light-dependent resistor) has a very high resistance in the dark, but its resistance drops dramatically when being exposed to light.

Chapter 12
Nuclear Physics

12.1 nuclear energy

12.1.1 mass-energy equivalence

mass-energy equivalence principle states that mass and energy are equivalent to one another

this idea is represented mathamatically by **Einstein's mass-energy relation**: $\boxed{E = mc^2}$ [81]

m is **rest mass** of the object, $c = 3.00 \times 10^8$ m s^{-1} is speed of light in vacuum

➢ mass-energy equivalence implies mass can be converted into pure energy

conversely, mass can be created out of energy

➢ $\Delta E = \Delta mc^2$ applies to *all* energy changes

if energy is supplied to a system, then mass of this system increases

for example, an object will have a greater mass when it is set to motion or is heated

mass is not a converved quantity, it is *mass-energy* that is conserved in all processes

➢ if an object is stationary, its mass is called the *rest mass* m_0

mass-energy equivalence implies an object at rest still has an intrinsic *rest energy*: $E_0 = m_0c^2$

➢ energy transformations such as chemical reactions can cause a system to lose some mass content, but this change in mass is usually negligible

[81] This might be the most famous equation in all physical sciences. When Albert Einstein wrote down the theory of *special relativity* in 1905, he found the mass m of an object with rest mass m_0 is related to the speed v at which it is moving by: $m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. Total energy is given by $E = mc^2 = m_0c^2\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. At low speeds $v \ll c$, $E \approx m_0c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = m_0c^2 + \frac{1}{2}m_0v^2$. The second piece clearly gives the kinetic energy term, so the first piece can be thought as the rest energy stored within the mass. This is how the idea of mass-energy relation came to the great mind of Einstein.

Example 12.1 When an antiproton (antiparticle of a proton) collides with a proton, they are annihilated and two photons of equal energy are formed. (a) What is the energy of each photon? (b) What is the photon frequency?

✍ rest energy of proton becomes photon energy:

$$E_\gamma = m_p c^2 = 1.67 \times 10^{-27} \times (3.0 \times 10^8)^2 \approx 1.50 \times 10^{-10} \text{ J}$$

photon frequency: $f = \frac{E_\gamma}{h} = \frac{1.50 \times 10^{-10}}{6.63 \times 10^{-34}} \approx 2.27 \times 10^{23}$ Hz (high-frequency γ-photon) □

Question 12.1 The combustion of one mole of solid carbon to form carbon dioxide (CO_2) at standard condition is 394 kJ. Find the change in mass for this amount of energy, and hence compare this mass change with the mass of carbon before combustion.

Question 12.2 Given that the specific heat capacity of copper is 380 J kg^{-1} K^{-1}. When a copper block is heated from 300 K to 1000 K, what is the additional mass as a fraction of its rest mass?

12.1.2 binding energy

nucleons (protons and neutrons) in a nucleus bind together through the *strong nuclear force*

to pull nucleons apart, work must be done to overcome the attraction

so free nucleons at infinity have greater potential energy than a single nucleus

since energy is equivalent to mass, free nucleons would appear more massive than when they are held together in a nucleus

this statement is supported by experimental data

useful notions in nuclear physics related to this idea can now be introduced

> difference between mass of a nucleus and total mass of its consistituent nucleons when separated to infinity is called the **mass defect**

> energy needed to separate the nucleons in a nucleus to infinity, or equivalently, energy released when free individual nucleons combine to form a nucleus, is called the **nuclear binding energy** (E_B)

➢ for a nucleus Z with proton number Z and nucleon number A (represented by ${}^A_Z X$)

its mass defect is given by: $\Delta m = Zm_\text{p} + (A-Z)m_\text{n} - m_X$

where $m_\text{p} = 1.673\times10^{-27}$ kg is mass of proton, $m_\text{n} = 1.675\times10^{-27}$ kg is mass of neutron

➢ by definition, binding energy is equivalent to mass defect: $\boxed{E_B = \Delta mc^2}$

➢ nucleons in a nucleus have *negative* P.E., while free nucleons have zero P.E.

E_B is the energy required to fill this gap in order to pull nucleons apart

i.e., E_B of a nucleus equals *loss* of potential energy during its formation

➢ nuclear mass is often measured in *unified atomic mass* u, where $\boxed{1\text{ u} = 1.66\times10^{-27}\text{ kg}}$

nuclide ${}^A_ZX$ has a nuclear mass of about Au

➢ binding energy is often measured in MeV, where $\boxed{1\text{ MeV} = 10^6\text{ eV} = 1.60\times10^{-13}\text{ J}}$

Example 12.2 An iron-56 nucleus (${}^{56}_{26}$Fe) has a mass of 9.288×10^{-26} kg. Calculate the nuclear binding energy per nucleon, in MeV, for ${}^{56}_{26}$Fe.

✍ mass defect: $\Delta m = 26m_\text{p} + (56-26)m_\text{n} - m_\text{Fe} = 26\times1.673\times10^{-27} + 30\times1.675\times10^{-27} - 9.288\times10^{-26}$

so we find $\Delta m = 8.68\times10^{-28}$ kg

binding energy: $E_B = \Delta mc^2 = 8.68\times10^{-28}\times(3.00\times10^8)^2 \approx 7.812\times10^{-11}$ J

binding energy per nucleon: $\epsilon_b = \dfrac{E_b}{A} = \dfrac{7.812\times10^{-11}}{56} \approx 1.395\times10^{-12}$ J

convert into MeV: $\epsilon_b = \dfrac{1.395\times10^{-12}}{1.60\times10^{-13}}$ MeV ≈ 8.72 MeV □

Question 12.3 Show that the energy equivalent of 1.0 u is 934 MeV.

Question 12.4 Given that mass of proton is 1.007 u, mass of neutron is 1.009 u, and mass of uranium-235 nucleus is 234.992 u. Find the binding energy per nucleon of nuclide ${}^{235}_{92}$U.

Question 12.5 What is the binding energy for the hydrogen nucleus ${}^1_1$H?

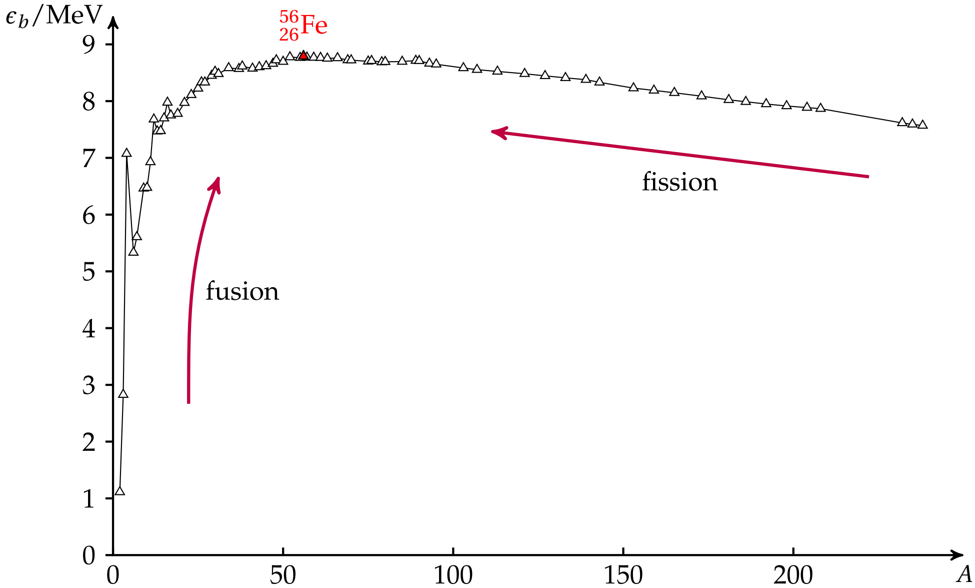
Question 12.6 Why is the rest mass of proton slightly larger than the unified atomic mass unit?

12.1.3 nuclear stability

binding energy per nucleon ϵ_b is closely related to nuclear stability

binding energy per nucleon gives average energy needed to remove a nucleon from nucleus

higher ϵ_b means more difficult to pull nucleons away, so nucleus has higher stability a graph of ϵ_b against nucleon number A can be plotted based on experimental data



binding energy per nucleon ϵ_b against nucleon number A

- ➢ $^{56}_{26}$Fe has greatest value of ϵ_b for all nucleus $\Rightarrow$ $^{56}_{26}$Fe is the most stable nuclide in nature
- ➢ any physical system would evolve such that it lowers total energy

  if there could be an increase in total binding energy, nuclear reactions could occur

12.1.4 fusion

for nuclei smaller than $^{56}_{26}$Fe, they could combine together and release energy

> **fusion** is the process where two light nuclei join together to form a heavier nucleus

- ➢ fusion occurs only at very *high temperatures*

  nuclei are all positively charged, for two nuclei to come close enough to fuse together, high initial K.E. is needed for them to overcome their *electrostatic repusion*
- ➢ vast amount of energy of the sun and other *stars* are created through fusion

  extreme high temperature and high density at core of stars make fusion reactions possible

[82]

12.1.5 fission

for a nucleus larger than $^{56}_{26}\text{Fe}$, it can break into two or more parts and release energy

> **fission** is the process where a massive nucleus splits into two smaller nuclei of about the same size (and is usually associated with release of several neutrons)

- note that fission requires the two product nuclei are of simliar size

  α-decay (emission of helium necleus from an unstable nucleus) is not a fission process
- fission reactions can occur *spontaneously*, i.e. nuclei can undergo fission by themselves

  fission can also be *induced* by bombarding the nucleus with an incident neutron

  induced fission is used by humans to generate nuclear power or to build nuclear weapons

nuclear reactors (⋆)

nuclear reactors make use of energy released from induced fission reactions

thermal energy produced from fission is further converted to electrical or mechanical forms

uranium-235 ($^{235}_{92}\text{U}$) is the most widely used fuel for fuel nuclear reactors

one of the many fission reactions of uranium-235 is: $^{235}_{92}\text{U} + ^{1}_{0}\text{n} \longrightarrow ^{141}_{56}\text{Ba} + ^{92}_{36}\text{Kr} + 3^{1}_{0}\text{n}$

- each reaction releases more than one neutrons

  they can trigger further fission reactions, making *chain reaction* possible

  hence huge amount of energy can be released in a very short time

[82]Even for the simplest fusion reaction, fusing hydrogen into helium, at least 1.5×10^{7} K is required. This is basically how the sun shines and nurtures lives on the earth, and also many other stars in the universe power energies. For a sufficiently massive star, when it burns out of its fuel, its core collapses and becomes hot enough to start to fuse heavier elements. If a star is not able to fuse again, nuclear reaction ceases, the star collapses and becomes a *white dwarf*. For a giant star, nuclear reactions can go all the way up to $^{56}_{26}\text{Fe}$. At this point, no more energy can be produced through fusion. The star collapses extremely rapidly and then explodes, creating a *supernova*, during which all elements heavier than $^{56}_{26}\text{Fe}$ are formed. (⋆)

➢ rate of fission should be controlled, a runaway reaction could lead to disastrous explosion

control rods (e.g., boron) are used to absorb neutrons: ${}^{10}_{5}\mathrm{B} + {}^{1}_{0}\mathrm{n} \longrightarrow {}^{7}_{3}\mathrm{Li} + {}^{4}_{2}\mathrm{He}$

control rods are adjusted so that one neutron per reaction goes on to produce further fission

in emergency, release of boron rods shuts down reactor

➢ reaction is expected to continue at steady rate

need sufficient amount of neutrons to maintain the chain reaction

this requires a *critical mass* for the amount of ${}^{235}_{92}\mathrm{U}$ fuel

➢ only low energy neutrons (*thermal neutrons*) can be *captured* by ${}^{235}_{92}\mathrm{U}$

but neutrons released through fission are very energetic

a *moderator* (e.g., water) is needed to slow down neutrons

➢ heat produced in fission is removed by *coolant* (e.g., water)

this heat can be used to power generators to produce electricity

➢ reactor is surrounded by a *shield* (e.g., a thick concrete) to prevent radiation from escaping

12.1.6 energy release from nuclear reactions

in this section, we consider only those nuclear reactions that release energy

let's write a generic nuclear reaction as: original particles $\longrightarrow$ product particles $+\,Q$

two approaches to compute Q, the amount of energy release, will be introduced

method 1: use change in *mass* during the reaction

there must be a decrease in total mass to be converted into energy release

using mass-energy relation: $Q = \Delta mc^2 = (m_{\mathrm{org}} - m_{\mathrm{prod}})c^2$

method 2: use change of *total binding energy*

energy is released means product particles are more stable

so total binding energy of product particles is higher than that of original particles

energy released during reaction: $Q = E_{B,\mathrm{prod}} - E_{B,\mathrm{org}}$

Example 12.3 The sun is a huge nuclear reactor. It fuses hydrogen nuclei into helium nuclei to produce large amounts of energy. The reaction can be expressed by the formula:

$$4\,{}^{1}_{1}\mathrm{H} \longrightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{0}_{+1}\mathrm{e} \quad ({}^{0}_{+1}\mathrm{e} \text{ is a } \textit{positron}\text{, the anti-particle of electron})$$

(a) Evaluate the energy released in one reaction. (b) Given that the power output of the sun is roughly 4.0×10^{26} W, estimate how much hydrogen the sun burns every second. (Data for this question: $m_{\mathrm{p}} = 1.672622 \times 10^{-27}$ kg, $m_{\mathrm{He}} = 6.644657 \times 10^{-27}$ kg, $m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg)

✍ energy released in one reaction:

$$E_0 = \Delta mc^2 = (4m_{\mathrm{H}} - m_{\mathrm{He}} - 2m_{\mathrm{e}})c^2 \approx 4.40 \times 10^{-29} \times (3.0 \times 10^8)^2 \approx 3.96 \times 10^{-12} \text{ J}$$

in interval Δt, sun outputs a total energy of $P\Delta t$ via a number of ΔN fusion reactions

we write $P\Delta t = \Delta N E_0$, so number of fusion reactions every second is:

$$\frac{\Delta N}{\Delta t} = \frac{P}{E_0} = \frac{4.0 \times 10^{26}}{3.96 \times 10^{-12}} \approx 1.0 \times 10^{38} \text{ s}^{-1}$$

each reaction takes four hydrogen nuclei, so rate of hydrogen consumption:

$$\frac{\Delta M_{\mathrm{H}}}{\Delta t} = \frac{4\Delta N}{\Delta t} \times m_{\mathrm{H}} \approx 4 \times 1.0 \times 10^{38} \times 1.67 \times 10^{-27} \approx 6.7 \times 10^{11} \text{ kg s}^{-1}$$

the sun burns over 600 billion kilograms of hydrogen every second, just think about it! □

Example 12.4 Uranium-235 nuclei ($^{235}_{92}\mathrm{U}$) bombarded with slow neutrons can undergo nuclear reaction:

$$^{1}_{0}\mathrm{n} + ^{235}_{92}\mathrm{U} \longrightarrow ^{141}_{56}\mathrm{Ba} + ^{92}_{36}\mathrm{Kr} + x^{1}_{0}\mathrm{n}$$

	binding energy per nucleon
$^{235}_{92}\mathrm{U}$	7.591 MeV
$^{141}_{56}\mathrm{Ba}$	8.326 MeV
$^{92}_{36}\mathrm{Kr}$	8.513 MeV

(a) Determine the number of x. (b) Using the data in the table, find the energy released for one reaction. (c) What is the change in mass, if any, before and after the reaction?

✍ from conservation of mass number: $1 + 235 = 141 + 92 + x \Rightarrow x = 3$

energy release during reaction equals change in total binding energy:

$$Q = E_{B,\mathrm{Ba}} + E_{B,\mathrm{Kr}} - E_{B,\mathrm{U}} = 141 \times 8.326 + 92 \times 8.513 - 235 \times 7.591 = 173.277 \text{ MeV}$$

reduction in total mass: $\Delta m = \dfrac{Q}{c^2} = \dfrac{173.277 \times 1.60 \times 10^{-13}}{(3.0 \times 10^8)^2} \approx 3.08^{-28}$ kg □

Question 12.7 Uranium-235 nuclei can undergo another fission process when being bombarded with neutrons. This is represented by:

$$^{1}_{0}\mathrm{n} + ^{235}_{92}\mathrm{U} \longrightarrow ^{144}_{55}\mathrm{Cs} + ^{90}_{37}\mathrm{Rb} + 2^{1}_{0}\mathrm{n}$$

	binding energy per nucleon
$^{235}_{92}\mathrm{U}$	7.591 MeV
$^{144}_{55}\mathrm{Cs}$	8.212 MeV
$^{92}_{36}\mathrm{Kr}$	8.631 MeV

Calculate the energy released in this reaction.

Question 12.8 The *proton–proton chain reaction* is a set of nuclear reactions through which stars fuse hydrogen into helium. One intermediate reaction may be summarised as:

$$^1_1\mathrm{H} + ^2_1\mathrm{D} \longrightarrow ^3_2\mathrm{He} + \text{energy}$$

	mass/u
$^1_1$H	1.00728
$^2_1$D	2.01356
$^3_2$He	3.01605

Calculate the energy released in this reaction.

Question 12.9 Another fusion reaction in the proton-proton chain is represented by:

$$^3_2\mathrm{He} + ^3_2\mathrm{He} \longrightarrow ^4_2\mathrm{He} + 2^1_1\mathrm{p} + \text{energy}$$

where energy released through this reaction is 12.86 MeV. Given that binding energy per nucleon of $^4_2$He is 7.074 MeV, calculate the binding energy per nucleon for $^3_2$He.

Question 12.10 If the total mass of the product particles is greater than that of the original particles in a nuclear reaction, what can you say about the energy of the original particles?

12.2 radioactive decay

12.2.1 radioactive decay

the process of *random* and *spontaneous* emission of α-, β-, and γ-radiation from unstable nuclei is called **radioactive decay**

- **random** means decay events are not predictable

  we cannot tell precisely when a particular nucleus is about to decay

  we can only tell the *probability* of decay within a certain time

- **spontaneous** means rate of decay does not depend on external conditions

  temperature, pressure, or has no effect on rate of radioactive decays

suppose initially a sample contains N radioactive nuclei that are about to decay

after a time interval Δt, ΔN nuclei undergo decay processes

probability of decay for nuclei in the sample during this time interval is $\dfrac{\Delta N}{N}$

divide by Δt, probability of decay for each nucleus per unit time can be found since nuclear decay is spontaneous, this number is a constant, called the decay constant

decay constant is the probability for one nucleus to decay per unit time: $\boxed{\lambda = \dfrac{\Delta N}{N \Delta t}}$

➢ to describe the rate of decay, we introduce the notion of **activity**

activity is defined as the number of nuclei that undergo decay per unit time: $\boxed{A = \dfrac{\Delta N}{\Delta t}}$

this can be given in a differential form: $A = -\dfrac{\mathrm{d}N}{\mathrm{d}t}$

a minus sign is included because number of nuclei N decreases with time

➢ compare with the expression for decay constant, we find: $\boxed{A = \lambda N}$

this equation shows that if there are more nuclei present, the sample has greater activity

➢ units for decay constant and activity: $[\lambda] = [A] = \mathrm{s}^{-1}$

for activity, one decay event per unit time is defined as one *becquerel*: $1\ \mathrm{Bq} = 1\ \mathrm{s}^{-1}$

12.2.2 decay equations

variation of number of undecayed nuclei with time can be derived from the equation: $A = \lambda N$

recall that $A = -\dfrac{\mathrm{d}N}{\mathrm{d}t}$, so this equation is a differential equation in disguise: $-\dfrac{\mathrm{d}N}{\mathrm{d}t} = \lambda N$

this can be solved by separating variables and integrating

$$-\lambda \mathrm{d}t = \frac{\mathrm{d}N}{N} \Rightarrow -\lambda \int_0^t \mathrm{d}t = \int_{N_0}^{N} \frac{\mathrm{d}N}{N} \Rightarrow -\lambda t \Big|_0^t = \ln N \Big|_{N_0}^{N} \Rightarrow -\lambda t = \ln \frac{N}{N_0} \Rightarrow N = N_0 \mathrm{e}^{-\lambda t}$$

this shows that decay events obey *exponential decay laws*

➢ number of undecayed nuclei at time t is given by: $\boxed{N(t) = N_0 \mathrm{e}^{-\lambda t}}$

N_0 is initial number of nuclei in the sample at $t = 0$

➢ since activity $A = \lambda N$, so activity varies with time as: $\boxed{A(t) = A_0 \mathrm{e}^{-\lambda t}}$

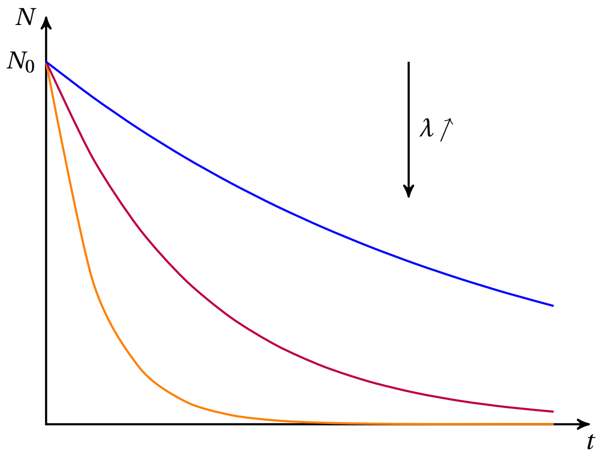
➢ $\lambda \nearrow \Rightarrow$ higher probability of decay, or greater rate of decay

Example 12.5 A sample containing 6.0×10^7 iodine-131 nuclei has an activity of 60 Bq. (a) What is the decay constant of iodine-131? (b) How many nuclei remain undecayed after 20 days?

✍ decay constant: $\lambda = \dfrac{A}{N} = \dfrac{60}{6.0 \times 10^7} \approx 1.0 \times 10^{-6}\ \mathrm{s}^{-1}$

number of remaining nuclei: $N = N_0 \mathrm{e}^{-\lambda t} = 6.0 \times 10^7 \times \mathrm{e}^{-1.0 \times 10^{-6} \times 20 \times 3600} \approx 1.1 \times 10^7$ □

Example 12.6 ${}^{24}_{11}\mathrm{Na}$, an isotope of sodium, has a decay constant of $1.28 \times 10^{-5}\ \mathrm{s}^{-1}$. Suppose a



exponential decay law for number of nuclei: $N(t) = N_0 e^{-\lambda t}$

sample initially contains a mass of 9.0 μg of $^{24}_{11}$Na. Find (a) initial number of $^{24}_{11}$Na nuclei, (b) initial activity of this sample, (c) the activity after 48 hours.

✍ initial number of nuclei: $N_0 = \dfrac{9.0\ \mu g}{24u} = \dfrac{9.0 \times 10^{-9}}{24 \times 1.66 \times 10^{-27}} \approx 2.26 \times 10^{17}$

initial acitivy: $A_0 = \lambda N_0 = 1.28 \times 10^{-5} \times 2.26 \times 10^{17} \approx 2.89 \times 10^{12}$ Bq

activity after 20 hours: $A = A_0 e^{-\lambda t} = 2.89 \times 10^{12} \times e^{-1.28 \times 10^{-5} \times 48 \times 3600} \approx 3.17 \times 10^{11}$ Bq □

Question 12.11 A sample of polonium-205 has an activity of 6.7×10^{15} Bq. The decay constant of polonium-205 is known to be $1.16 \times 10^{-4}\ \mathrm{s}^{-1}$. (a) Find the number of nuclei needed to give this activity. (b) Find the mass of polonium-205 in this sample. (c) Calculate the time needed for the activity reduce to 1% of its initial value.

Question 12.12 Technetium-99m is widely used as a radioactive tracer for medical diagnostic procedures. If some of this isotope, with an activity of 900 MBq was injected into a patient. The activity is found to reduce to 56.4 MBq after 24 hours. (a) What is the decay constant of technetium-99m? (b) How many technetium-99m nuclei are stil left in the patient's body?

12.2.3 half-life

it is more convenient to define a time quantity to describe how fast radioactive nuclei decay

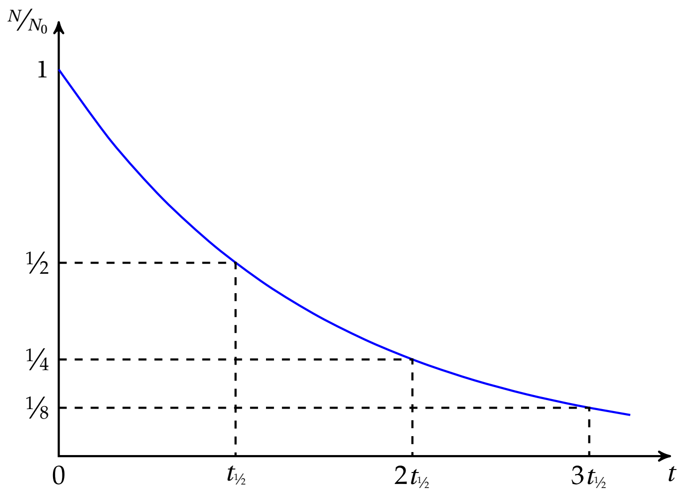
it is useful to define half-life of a radioactive sample

> **half-life** ($t_{½}$) is the mean time taken for number of radioactive nuclei in the sample, or activity of the sample, to reduce to half of its initial value

- half-life $t_{½}$ is closely related to decay constant λ

  by definition, at $t = t_{½}$, $N = \dfrac{1}{2}N_0$, $A = \dfrac{1}{2}A_0$ $\Rightarrow$ $e^{-\lambda t_{½}} = \dfrac{1}{2}$ $\Rightarrow$ $e^{\lambda t_{½}} = 2$ $\Rightarrow$ $\boxed{\lambda t_{½} = \ln 2}$
- λ is a constant, so $t_{½}$ is also a constant over the lifetime of nuclear decay

  half-life of a sample of isotope does not depend on initial number of nuclei or initial activity



half-life of radioactive decay

Example 12.7 Radium-224 has a half-life of 3.63 days. (a) What is the activity from 6.0 mg of pure radium-224? (b) How many radium-224 nuclei have undergone decay after 10 days?

✍ decay constant: $\lambda = \dfrac{\ln 2}{t_{½}} = \dfrac{\ln 2}{3.63 \times 24 \times 3600} \approx 2.21 \times 10^{-6}\ \mathrm{s}^{-1}$

initial number of Ra-224 nuclei: $N_0 = \dfrac{6.0\ \mathrm{mg}}{224\ \mathrm{u}} = \dfrac{6.0 \times 10^{-6}}{224 \times 1.66 \times 10^{-27}} \approx 1.61 \times 10^{19}$

initial acitivy: $A_0 = \lambda N_0 = 2.21 \times 10^{-6} \times 1.61 \times 10^{19} \approx 3.57 \times 10^{13}$ Bq

number of undecayed nuclei: $N = N_0 e^{-\lambda t} = 1.61 \times 10^{19} \times e^{-2.21 \times 10^{-6} \times 10 \times 24 \times 3600} \approx 2.39 \times 10^{18}$

number of nuclei that have decayed: $\Delta N = N_0 - N = 1.61 \times 10^{19} - 2.39 \times 10^{18} \approx 1.37 \times 10^{19}$ □

Example 12.8 Living trees contain a certain percentage of ${}^{14}_{6}C$, an radioactive isotope of carbon that has a half-life of 5570 years. A sample of dead wood is found to have an activity of 0.42 Bq, while an equal mass of living wood has an activity of 1.60 Bq. Find the age of the dead wood.

✍ $$A = A_0 e^{-\lambda t} = A_0 e^{-\frac{\ln 2}{t_{1/2}} t} \Rightarrow 0.42 = 1.60 e^{-\frac{\ln 2}{5570} t} \Rightarrow -\frac{\ln 2}{5570} t = \ln\left(\frac{0.42}{1.60}\right) \Rightarrow t \approx 10700 \text{ years}$$ □

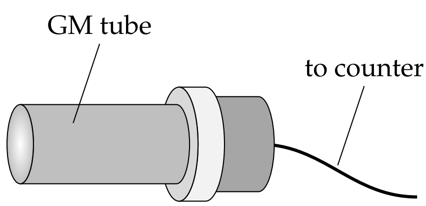
Question 12.13 The number of uranium-238 nuclei in a rock sample is believed to have decreased from 3.50×10^{17} to 3.27×10^{17} in 480 million years. Estimate the half-life of uranium-238.

Question 12.14 Plutonium-238 is a powerful alpha emitter with a half-life of 87.7 years. One decay of plutonium-238 releases an energy of about $9.0 \times 10{-}13$ J. A nuclear battery containing a sealed plutonium source is implanted into patient's body to power heart pacemakers. The battery has an initial activity of 6.0×10^{10} Bq. (a) Calculate the initial power released by the source. (b) Find the mass of plutonium required to produce this power. (c) It is required that power output to the pacemaker is at least 60% of the initial power. Calculate the time, in years, for which the battery provides sufficient power.

Question 12.15 Show that the variation of the number of undecayed nuclei with time t can be given by: $N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$, where N_0 is the initial number of nuclei.

12.2.4 measurement of radioactive decay

a Geiger–Müller tube, or a **GM tube**, is a device used to detect ionizing radiation



GM tube measures the number of α-particles, β-particles and γ-photons arriving per unit time

the number of decays recorded per unit time by GM tube is called the **count rate** R

since GM tube only picks up emissions to one particular direction, but a radioactive source emits radiation in *all* directions, so count rate is a fraction of the activity of the sample

activity obeys exponential decay, so we would expect count rate to satisfy: $\boxed{R = R_0 e^{-\lambda t}}$

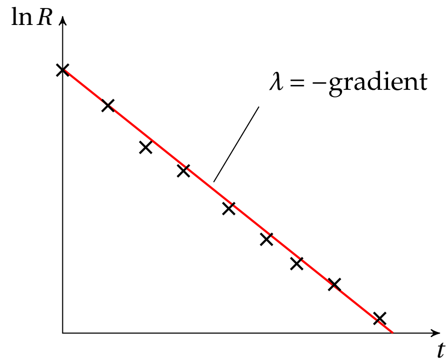
verifying the exponential decay law

to verify the exponential decay law, we first rearrang the equation as: $\ln R = -\lambda t + \ln R_0$

if a set of measurements of R at time t are obtained, a graph of $\ln R$ against t can be plotted

if trend curve shows a straight line, the relation $R = R_0 e^{-\lambda t}$ is then verified

decay constant λ is given by negative gradient of the best-fit line



$\ln R = -\lambda t + \ln R_0$

issues with count rate

for now, we take for granted that the count rate measured is 100% accurate

but in practice, there might be a number of factors leading to measurement errors

- there is always *background radiation* from earth minerals, cosmic rays, food and water, etc.

  background reading must be subtracted to give a corrected count rate
- α- and β-particles emitted could be absorbed by sample itself
- product nuclei could also be radioactive, contributing to additional counts
- GM tube has *dead time*, or *resolving time*

  when a count is recorded, it takes GM tube a certain time to reset for the next count

Example 12.9 A GM tube is placed close to a source of radioactive isotope. The variation with time t of the measured count rate R is shown. Determine the half-life of this isotope.

✍ there is evidence of *background radiation* as the count rate decreases and tends to a non-zero value of 20 s^{-1}, so we need subtract this number to obtain the true count rate of sample

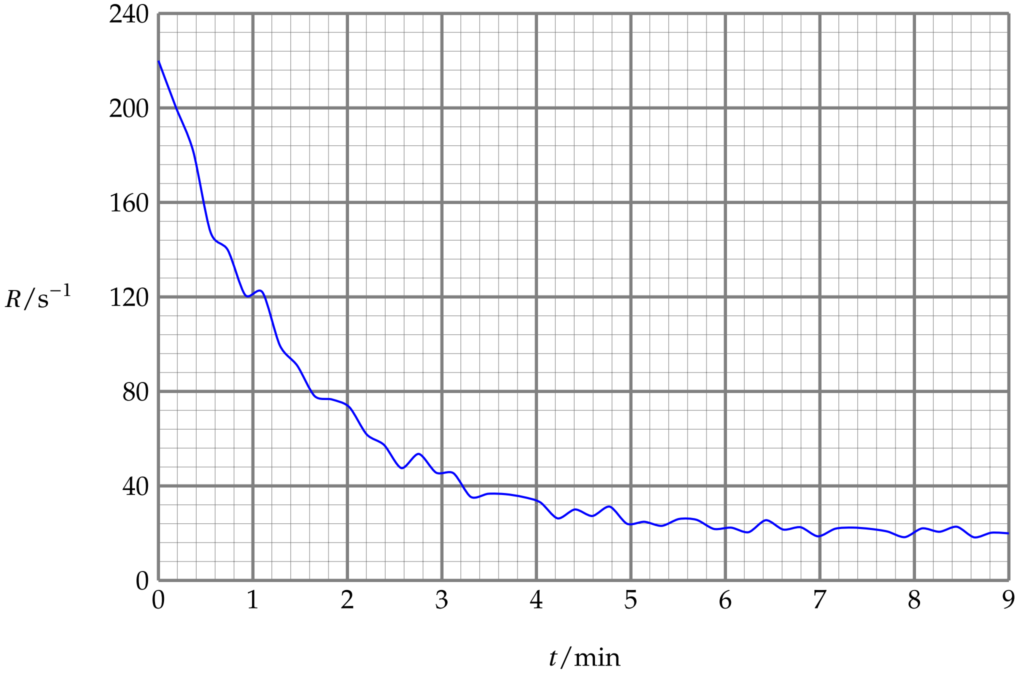
there are also *fluctuations* in count rate due to *random* nature of decay processes, so we need to make several calculations and take average to obtain an accurate value for half-life

let's call true count rate R_T, and call measured count rate R

as $R_T = 200\ s^{-1} \to 100\ s^{-1}$, $R = 220\ s^{-1} \to 120\ s^{-1}$, so $t_{1/2} \approx 1.0 - 0.0 \approx 1.0$ min

as $R_T = 160\ s^{-1} \to 80\ s^{-1}$, $R = 180\ s^{-1} \to 100\ s^{-1}$, so $t_{1/2} \approx 1.3 - 0.4 \approx 0.9$ min

as $R_T = 120\ s^{-1} \to 60\ s^{-1}$, $R = 140\ s^{-1} \to 80\ s^{-1}$, so $t_{1/2} \approx 1.6 - 0.7 \approx 0.9$ min



as $R_T = 80\ s^{-1} \to 40\ s^{-1}$, $R = 100\ s^{-1} \to 60\ s^{-1}$, so $t_{½} \approx 2.3 - 1.3 \approx 1.0$ min

take average for these results, we find $t_{½} \approx 0.95$ min $\square$

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