

6.5.22

EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Solution:

Description	Parameter or Value
length of the wire	$28m$
Radius of Circle	x
Side of the square	s

TABLE 0: Parameters or Values

• Theoretical Solution:

length of the circle is its circumference

length of the circle = $2\pi x$

$$s = \frac{28 - 2\pi x}{4} \quad (0.1)$$

The least value of $2\pi x$ can be 0 and the greatest value can be 28

$$x \in \left[0, \frac{14}{\pi} \right] \quad (0.2)$$

$$x_0 = 0 \quad (0.3)$$

Let $f(x)$ is function of sum of areas of circle and square

$$f(x) = s^2 + \pi x^2 \quad (0.4)$$

$$= \left(7 - \frac{\pi x}{2}\right)^2 + \pi x^2 \quad (0.5)$$

$$= 49 - 7\pi x + \left(\frac{\pi^2}{4} + \pi\right)x^2 \quad (0.6)$$

$$f(x) = ax^2 + bx + c \quad (0.7)$$

$$f'(x) = -7\pi + 2\left(\frac{\pi^2}{4} + \pi\right)x \quad (0.8)$$

$$f'(x) = 2ax + b \quad (0.9)$$

$$f''(x) = 2\left(\frac{\pi^2}{4} + \pi\right) \quad (0.10)$$

$$a = \frac{\pi^2}{4} + \pi \quad (0.11)$$

$$b = -7\pi \quad (0.12)$$

$$c = 49 \quad (0.13)$$

Critical points,

$$f' = 0 \quad (0.14)$$

$$-7\pi + 2\left(\frac{\pi^2}{4} + \pi\right)x = 0 \quad (0.15)$$

$$x = \frac{14}{\pi + 4} \quad (0.16)$$

For

$$\text{Local Minimum } f'' > 0 \quad (0.17)$$

$$\text{Local Maximum } f'' < 0 \quad (0.18)$$

$$\text{Inflection Point } f'' = 0 \quad (0.19)$$

Hence (0.16) is critical point.

$$\text{Length of circle} = \frac{28\pi}{\pi + 4} \quad (0.20)$$

$$\text{Length of square} = \frac{28}{\pi + 4} \quad (0.21)$$

- **Computational Solution:** We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$x_{n+1} = x_n - \alpha f'(x_n) \quad (0.22)$$

$$x_{n+1} = x_n - \alpha(2ax_n + b) \quad (0.23)$$

$$x_{n+1} = x_n(1 - 2a\alpha) - \alpha b \quad (0.24)$$

Applying unilateral z-transform over the equation (0.24)

$$zX(z) - zx_0 = X(z)(1 - 2a\alpha) - \frac{ab}{1 - z^{-1}} \quad (0.25)$$

$$X(z)[z - (1 - 2a\alpha)] = -\frac{ab}{1 - z^{-1}} \quad (0.26)$$

$$X(z) = \frac{-ab}{[z - (1 - 2a\alpha)][1 - z^{-1}]} \quad (0.27)$$

$$X(z) = \frac{-b}{2a} \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - (1 - 2a\alpha)z^{-1}} \right] \quad (0.28)$$

$$= \frac{-b}{2a} \sum_{n=-\infty}^{\infty} [1 - (1 - 2a\alpha)^n] z^{-n} u(n) \quad (0.29)$$

From the equation (0.29), ROC is

$$|z| > \max\{1, |1 - 2a\alpha|\} \quad (0.30)$$

$$0 < |1 - 2a\alpha| < 1 \quad (0.31)$$

$$\alpha \in \left(0, \frac{1}{a}\right) \setminus \left\{\frac{1}{2a}\right\} \quad (0.32)$$

$$\alpha \in \left(0, \frac{1}{\frac{\pi^2}{4} + \pi}\right) \setminus \left\{\frac{1}{2\left(\frac{\pi^2}{4} + \pi\right)}\right\} \quad (0.33)$$

If α satisfies the previous equation then

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \quad (0.34)$$

$$\lim_{n \rightarrow \infty} \|\alpha(2ax_n + b)\| = 0 \quad (0.35)$$

$$\lim_{n \rightarrow \infty} \|2ax_n + b\| = 0 \quad (0.36)$$

$$\lim_{n \rightarrow \infty} 2ax_n = -b \quad (0.37)$$

$$\lim_{n \rightarrow \infty} x_n = \frac{-b}{2a} \quad (0.38)$$

$$x_{min} = \frac{-b}{2a} \quad (0.39)$$

$$x_{min} = \frac{14}{\pi + 4} \quad (0.40)$$

We take the initial guess = 0, step size = 0.01, tolerance = $1e-6$

Using the gradient descent algorithm we get

$$x_{min} = 1.9603469573427723, f_{min} = 27.444858522066347 \quad (0.41)$$

Alternate Computational Solution:

We can also solve it using *cvxpy* module in python. On running the code we get,

$$f_{min} : 27.444858522066315 \quad (0.42)$$

$$x_{min} : 1.9603470372904506 \quad (0.43)$$

CVXPY:

- expressive syntax
- supports various types of problems including convex and non - convex
- Optimization variables are defined using the Variable class.
- Constants that can be changed without redefining the problem.
- Defines the goal of the optimization (minimization or maximization).
- Logical conditions imposed on the variables. (constraints)
- Combine the objective and constraints to form an optimization problem.

Using Scipy:

$$\text{Minimum value of the function: } 27.444858522066315 \quad (0.44)$$

$$\text{Value of } x \text{ at the minimum: } 1.96034704352491 \quad (0.45)$$

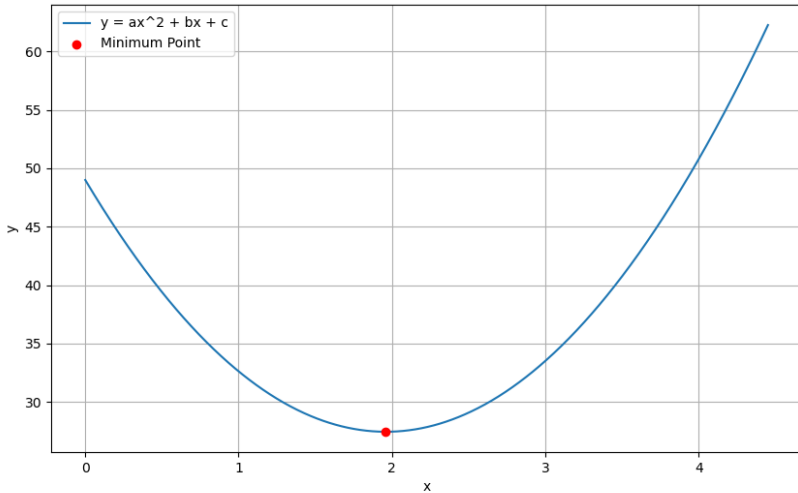


Fig. 0.1: Minimum of the function