Solution of a differential Equation

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EE24BTECH11049

Question

Question:
In a bank, the principal continuously increases at a rate of $r\%$ per year. Find the value of r if Rs 100 doubles in 10 years ($\log_e 2 = 0.6931$).

Table

Solution:

Variable	Description
P_0	initial principal amount
r	rate of increase per year
t	time in years
$C \& C_1$	arbitrary constants
P	principal at any time t

Table: Variables used

Solution

P is the principal at any time, according to the given question, the rate of change of principal can be written as follows.

$$\frac{dP}{dt} = \left(\frac{r}{100}\right) \times P \tag{0.1}$$

Separation of the variables in the equation (0.1)

$$\frac{dP}{p} = \left(\frac{r}{100}\right) \times dt \tag{0.2}$$

Solution

Integration on both sides (0.2)

$$\int \frac{dP}{p} = \int \left(\frac{r}{100}\right) dt \tag{0.3}$$

$$\log_e P = \frac{rt}{100} + C \tag{0.4}$$

$$P = e^{\frac{rt}{100}+C} \tag{0.5}$$

$$P = e^{\frac{rt}{100}} e^{C} \tag{0.6}$$

$$P = C_1 e^{\frac{rt}{100}} \tag{0.7}$$

At time $t = 0$, it is given that the principal is 100, that is, $P_0 = 100$. Substitute in equation (0.7)

$$100 = C_1 \tag{0.8}$$

Solution

Principal can be written as

$$P = 100 \times e^{\frac{rt}{100}} \tag{0.9}$$

At $t = 10$, the principal doubles, that is, $P = 200$, using equation (0.9)

$$200 = 100 \times e^{\frac{r}{10}} \tag{0.10}$$

$$2 = e^{\frac{r}{10}} \tag{0.11}$$

$$\log_e 2 = \frac{r}{10} \tag{0.12}$$

$$r = 10 \times \log_e 2 \tag{0.13}$$

$$r = 6.931 \tag{0.14}$$

Differential Equation

The differential equation

$$\frac{dP}{dt} = 0.06931 \times P \tag{0.15}$$

Verification of the solution computationally

Euler's Method:

From the definition of $\frac{dy}{dx}$,

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} \tag{0.16}$$

$$\frac{dy}{dx} = \frac{y_{n+1} - y_n}{h} \tag{0.17}$$

As per the question

$$\frac{dP}{dt} = \frac{P_{n+1} - P_n}{h} \tag{0.18}$$

substitute (0.15)

$$\frac{P_{n+1} - P_n}{h} = 0.06931 \times P_n \tag{0.19}$$

Difference Equations

We represent the the differential equation in the following difference equations:

$$t_{n+1} = t_n + h \tag{0.20}$$

$$P_{n+1} - P_n = 0.06931 \times P_n \times h \tag{0.21}$$

$$P_{n+1} = P_n + h \times 0.06931 \times P_n \tag{0.22}$$

Where h is step and is small.

Laplace transforms

Laplacian Operator:
If $f(t)$ is a function, the Laplace transform of that function is

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{0.23}$$

It is linear transformation, since integral is a linear operator

Laplace tranforms of some functions:

$$f(t) = 0 \implies F(s) = 0 \tag{0.24}$$

$$f(t) = 1 \implies F(s) = \frac{1}{s} \text{ for } Re(s) > 0 \tag{0.25}$$

$$f(t) = t^n \implies F(s) = \frac{\Gamma(a+1)}{s^{n+1}} \text{ for } Re(s) > 0 \tag{0.26}$$

$$f(t) = e^{at} \implies F(s) = \frac{1}{s-a} \text{ for } Re(s) > a \tag{0.27}$$

$$f(t) = \sin at \implies F(s) = \frac{a}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.28}$$

$$f(t) = \cos at \implies F(s) = \frac{s}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.29}$$

Laplace transforms of derivatives

Laplace transforms of derivatives:

$$\mathcal{L}\left(f'\right) = sF(s) - f(0) \tag{0.30}$$

$$\mathcal{L}\left(f''\right) = s^2F(s) - sf(0) - f'(0) \tag{0.31}$$

Laplace transform of unit step function $u(t)$:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{0.32}$$

From (0.23)

$$\mathcal{L}\left(u(t)\right) = \int_0^{\infty} u(t) e^{-st} dt \tag{0.33}$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^{\infty} (1) e^{-st} dt \tag{0.34}$$

$$F(s) = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}, \text{ for } Re(s) > 0 \tag{0.35}$$

Laplace transform of $e^{at}u(t)$:

From (0.23)

$$\mathcal{L}\left(e^{at}u(t)\right) = \int_0^\infty e^{at}u(t)e^{-st}dt \tag{0.36}$$

$$F(s) = \int_0^\infty e^{(a-s)t}dt \tag{0.37}$$

$$F(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a}, \text{ for } Re(s) > a \tag{0.38}$$

Solution for the differential equation

Solution for the differential equation:
Let $f\left(t\right) = P\left(t\right)$
Apply Laplace transform to the equation (0.1)

$$\mathcal{L}\left(f'\right) = \mathcal{L}\left(0.06931 \times f\right) \tag{0.39}$$

$$\tag{0.40}$$

From equations (0.31) and (0.30)

$$sF\left(s\right) - f\left(0\right) = 0.06931F\left(s\right) \tag{0.41}$$

$$\left(s - 0.06931\right)F\left(s\right) = f\left(0\right) \tag{0.42}$$

$$F\left(s\right) = \frac{f\left(0\right)}{s - 0.06931} \tag{0.43}$$

Solution for the differential equation

Substitut the initial conditions $f\left(0\right)=100$

$$F\left(s\right)=\frac{100}{s-0.06931} \tag{0.44}$$

$$\mathcal{L}\left(f\left(t\right)\right)=\frac{100}{s-0.06931} \tag{0.45}$$

$$f\left(t\right)=\mathcal{L}^{-1}\left(\frac{100}{s-0.06931}\right) \tag{0.46}$$

from equation (0.38)

$$f\left(t\right)=100\times u\left(t\right)e^{0.06931t} \tag{0.47}$$

$$P\left(t\right)=100\times u\left(t\right)e^{0.06931t} \tag{0.48}$$

Z-transform

If $f\left(t\right)$ is a function, the Z-transform of that function is

$$X\left[z\right] = \mathcal{Z}\left(x\left[t\right]\right) = \sum_{t=-\infty}^{\infty} x\left[t\right]z^{-t} \tag{0.49}$$

Z-transform of some functions

$u(t)$:
From (0.49)

$$Y(z) = \sum_{t \to -\infty}^{\infty} u\left[t\right] z^{-t} \tag{0.50}$$

From (0.32), we simplify it as

$$Y(z) = \sum_{t=0}^{\infty} (1) z^{-t} \tag{0.51}$$

$$Y(z) = \frac{1}{1 - z^{-1}}, \text{ for } |z| > 1 \tag{0.52}$$

$a^t u(t)$:
From (0.49)

$$X[z] = \sum_{t \to -\infty}^{\infty} a^t u[t] z^{-t} \tag{0.53}$$

From (0.32), we simplify it as

$$X[z] = \sum_{t=0}^{\infty} a^t z^{-t} \tag{0.54}$$

$$X[z] = \sum_{t=0}^{\infty} \left(az^{-1}\right)^t \tag{0.55}$$

$$X[z] = \frac{1}{1 - az^{-1}}, \text{ for } |z| > |a| \tag{0.56}$$

Solution to Difference Equation using Z-transform

Some other useful results:

$$\mathcal{Z}\left[u_{n-1}\right] = z^{-1}\mathcal{Z}\left[u_n\right] \tag{0.57}$$

$$\mathcal{Z}\left[u_{n+1}\right] = z\left(\mathcal{Z}\left[u_n\right] - u_0\right) \tag{0.58}$$

Solution

$$P_{n+1} = P_n + h \times 0.06931 \times P_n \tag{0.59}$$
$$P_{n+1} = P_n \left(1 + 0.06931h\right) \tag{0.60}$$

Apply z-transform

$$\mathcal{Z}\left[P_{n+1}\right] = \mathcal{Z}\left[P_n \left(1 + 0.06931h\right)\right] \tag{0.61}$$
$$\mathcal{Z}\left[P_{n+1}\right] = \left(1 + 0.06931h\right) \mathcal{Z}\left[P_n\right] \tag{0.62}$$

Let,

$$\mathcal{Z}\left[P_n\right] = P\left[z\right] \tag{0.63}$$

Then,

$$\mathcal{Z}\left[P_{n+1}\right] = zP\left[z\right] - zP_0 \tag{0.64}$$

Now,

$$zP[z] - zP_0 = P[z](1 + 0.06931h) \tag{0.65}$$

$$P[z]\,[z - (1 + 0.06931h)] = zP_0 \tag{0.66}$$

$$P[z] = P_0\left[\frac{z}{z - (1 + 0.06931h)}\right] \tag{0.67}$$

By inversing, we get

$$P_n = P_0 \times (1 + 0.06931h)^n \tag{0.68}$$

We know that,

$$1 + 0.06931h \approx e^{0.06931h} \tag{0.69}$$

then,

$$P_n = P_0\left(e^{0.06931h}\right)^n \tag{0.70}$$

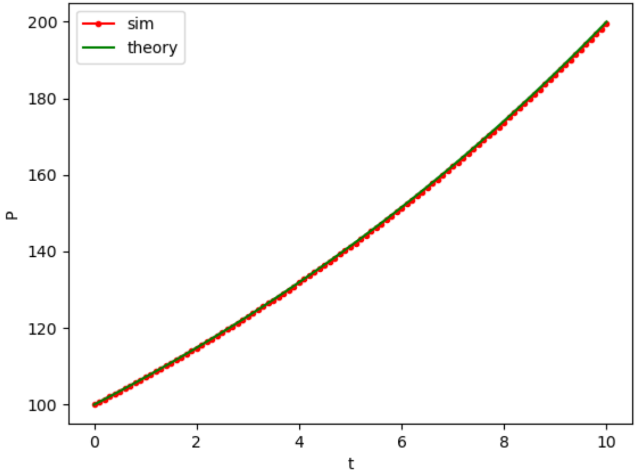
$$P_n = P_o e^{0.06931nh} \tag{0.71}$$

As h is the small division of time and n are the total no.of divisions, then nh turns to be t at that point, So

$$P(t) = P_0 e^{0.06931t} \tag{0.72}$$

We iterate this by taking the initial conditions from $t = 0$ to $t = 10$
By plotting all the points (t, P) we get the graph of function P varying with t.
The comparison between theoretical and simulation curves is shown in the figure, we can clearly see that both the curves are coincides which verifies our solution.
For the following approximate graph, I chose $h = 0.1$

A Plot of the Given Question



C Code: Generate the points I

```
#include <stdio.h>
#include <stdlib.h>

typedef struct
{
    double* t;
    double* P;
} points;

void generate_points(points* point, double P_0, double t_0, double t_end, double h)
{
    int size = (int)((t_end - t_0) / h) + 1;
    point->t = malloc(size * sizeof(double));
    point->P = malloc(size * sizeof(double));
    if (point->t == NULL || point->P == NULL) {
        printf("Memory allocation failed.\n");
        exit(1);
    }
    point->t[0] = t_0;
    point->P[0] = P_0;

    for (int i = 1; i < size; i++)
    {
        point->t[i] = point->t[i - 1] + h;
        double slope = point->P[i - 1] * 6.931 / 100;
        point->P[i] = point->P[i - 1] + h * slope;
    }
}
```

C Code: Generate the points II

```c
void free_points(points* point)
{
    free(point->t);
    free(point->P);
}
```

Python: To plot the points I

```python
import numpy as np
import ctypes
import matplotlib.pyplot as plt

gen = ctypes.CDLL('./points.so')

class Points(ctypes.Structure):
    _fields_ = [("t", ctypes.POINTER(ctypes.c_double)),
                ("P", ctypes.POINTER(ctypes.c_double))]

gen.generate_points.argtypes = [ctypes.POINTER(Points), ctypes.c_double, ctypes.c_double, ctypes.c_double,
      ctypes.c_double]
gen.generate_points.restype = None

gen.free_points.argtypes = [ctypes.POINTER(Points)]
gen.free_points.restype = None

P_0 = 100.0
t_0 = 0.0
t_end = 10.0
h = 0.1

points = Points()

gen.generate_points(ctypes.byref(points), P_0, t_0, t_end, h)

size = int((t_end - t_0) / h) + 1
```

Python: To plot the points II

```
t_sim = np.ctypeslib.as_array(points.t, shape=(size,))
P_sim = np.ctypeslib.as_array(points.P, shape=(size,))

plt.figure()
plt.plot(t_sim, P_sim, c='r', marker='.', label="sim")

t_theory = np.linspace(t_0, t_end, 1000)
P_theory = P_0 * np.exp(t_theory * 6.931 / 100)
plt.plot(t_theory, P_theory, c='g', label="theory")

plt.xlabel("t")
plt.ylabel("P")
plt.legend()
plt.savefig('../figs/fig.png')
plt.show()

gen.free_points(ctypes.byref(points))
```