Trapezoidal

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January 16, 2025

Question

Find the area enclosed between the ellipse $\frac{x^2}{4} + \frac{y^2}{36} = 1$ and the line $\frac{x}{2} + \frac{y}{6} = 1$.

Equations

Given	Formula
$\frac{x^2}{4} + \frac{y^2}{36} = 1$	$\mathbf{x}^\mathsf{T}\mathbf{V}\mathbf{x} + 2\mathbf{u}^\mathsf{T}\mathbf{x} + f$
$\frac{x}{2} + \frac{y}{6} = 1$	$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}$

Table: Equations

Matrix Parameters

Substituting the given values of we have

Conic:

$$\mathbf{V} = \begin{pmatrix} 36 & 0 \\ 0 & 4 \end{pmatrix} \tag{3.1}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2}$$

$$f = -144 \tag{3.3}$$

Line:

$$\mathbf{h} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{3.4}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{3.5}$$

Intersection of line with the conic

If a line intersects a conic, the κ value of the intersection points is given by

$$\kappa_i = \frac{-\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - g\left(h\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{3.6}$$

Substituting the given values, we get κ of the points of intersections as

$$\kappa_i = 0, 2 \tag{3.7}$$

Hence the points of intersection are $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Area of the required region

Now $\frac{x^2}{4} + \frac{y^2}{36} = 1$ gives $y = \pm 3\sqrt{4 - x^2}$. But the common area lies in the first quadrant because the the points of intersection are on positive x and y axes.

The area bounded by the curve and the line is

Numerical solution:

$$= \int_0^2 3\sqrt{4 - x^2} - (6 - 3x)\, dx \tag{3.8}$$

$$= 3\left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2 \tag{3.9}$$

$$= 3\left[0 + 2\sin^{-1}(1)\right] - [12 - 6] \tag{3.10}$$

$$= 3\pi - 6 \approx 3.42 \tag{3.11}$$

Trapezoidal Rule I

Split the interval $[0, 2]$ into N parts

$$h = \frac{2 - 0}{N} \tag{3.12}$$

Consider the points

$$x_0 = 0 \tag{3.13}$$

$$x_N = 2 \tag{3.14}$$

$$x_{i+1} = x_i + h \tag{3.15}$$

Trapezoidal rule

Summing the areas of the trapezoids formed, we approximate the area between the line and curve Let

$$A = \int_0^2 \left(3\sqrt{4 - x^2} - (6 - 3x)\right)\, dx \tag{3.16}$$

Trapezoidal Rule II

It can be approximated as

$$f(x) = 3\sqrt{4-x^2} - 6 + 3x \tag{3.17}$$

$$A \approx \frac{h}{2}\sum_{i=1}^{N}\left(f\left(x_{i-1}\right) + f\left(x_i\right)\right) \tag{3.18}$$

$$j_{i+1} = j_i + \frac{h}{2}\left(f\left(x_i\right) + f\left(x_{i+1}\right)\right) \tag{3.19}$$

$$j_{i+1} = j_i + \frac{h}{2}\left(3\sqrt{4-x_i^2} - 6 + 3x_i + 3\sqrt{4-x_{i+1}^2} - 6 + 3x_{i+1}\right) \tag{3.20}$$

Results

Result:
Theoretical Area: 3.4247779607693793
Computed Area: 3.4247767135897336

Area between line and ellipse

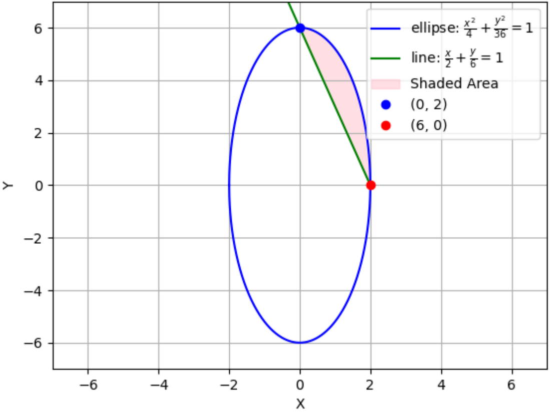

Figure: Area between line and ellipse