Numerical Solutions to Equations

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Outline

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Problem

Problem Statement

Find the values of k for which the quadratic equation $2x^2 + kx + 5 = 0$. So they've two equal roots.

Theoritical Solution

Value Of k

If the roots are equal then the value of Discriminant is to 0

$$b^2 - 4ac = 0 \tag{3.1}$$
$$k^2 - 4 \times 2 \times 5 = 0 \tag{3.2}$$
$$k = \pm 2\sqrt{10} \tag{3.3}$$

We get two equations since there are two values for k.

$$2x^2 + 2\sqrt{10}x + 5 = 0 \tag{3.4}$$
$$2x^2 - 2\sqrt{10}x + 5 = 0 \tag{3.5}$$

Matrix Method

Characteristic Polynomial

For a Matrix A of order n, the characteristic equation is given by,

$$det\,(A - \lambda I) = a_n\lambda^n + a_{n-1}\lambda^{n-1} \cdots + a_0 = 0 \tag{4.1}$$

We know that the roots of the characteristic polynomial are the eigenvalues of the matrix A.

We need to construct a matrix such we can find the eigen value using any iterative algorithm.

Companion Matrix

Matrix A is said to be the companion of a polynomial $f(x)$ if

$$det(A - \lambda I) = 0 \implies f(x) = 0 \tag{4.2}$$

For,

$$f(x) = c_0 + c_1 x + \cdots + x^n \tag{4.3}$$

The companion matrix is,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{4.4}$$

My Companion Matrix

For $x^2 + \sqrt{10}x + 2.5 = 0$
Companion matrix

$$C_1 = \begin{pmatrix} 0 & -2.5 \\ 1 & -\sqrt{10} \end{pmatrix} \tag{4.5}$$

For $x^2 - \sqrt{10}x + 2.5 = 0$
Companion matrix

$$C_2 = \begin{pmatrix} 0 & -2.5 \\ 1 & \sqrt{10} \end{pmatrix} \tag{4.6}$$

Matrix Method

1. Construct the companion matrix of the obtained polynomials.
2. Use power iteration algorithm to find the largest eigen value of the companion matrix, called as λ_{max}.
3. This eigen value obtained is the root of the polynomial.
4. Now divide the polynomial with $x - \lambda_{\text{max}}$ to remove the root.
5. Repeat the above steps iteratively on the reduced polynomial until a degree-1 equation is obtained, which gives the final root directly.

Power Iteration

The power iteration algorithm computes the largest eigenvalue λ_{max} of this matrix by iteratively applying the following steps:

$$\tilde{\mathbf{x}}_n = C\mathbf{x}_{n-1}, \tag{4.7}$$

$$\mathbf{x}_n = \frac{\tilde{\mathbf{x}}_n}{\|\tilde{\mathbf{x}}_n\|}, \tag{4.8}$$

The iteration stops when:

$$|\lambda_{\text{max}}^{(n)} - \lambda_{\text{max}}^{(n-1)}| < \epsilon, \tag{4.9}$$

where ϵ is the desired precision, and $\lambda_n = \frac{\mathbf{x}_n^\top C\mathbf{x}_n}{\mathbf{x}_n^\top \mathbf{x}_n}$.
(Rayleigh Quotient)

Power Iteration

Once $\lambda_{\max}$ is found, synthetic division is performed to reduce the polynomial:

$$P(x) = P(x) \div (x - \lambda_{\max}), \tag{4.10}$$

$$P(x) = c_{1,0} + c_{1,1}x + c_{1,2}x^2 + \cdots + c_{1,(n-1)}x^{n-1}. \tag{4.11}$$

Now, the new companion matrix will be evaluated.
This process is repeated iteratively until the polynomial is reduced to a degree-1 equation:

$$P_1(x) = c_{n-1,0} + c_{n-1,1}x. \tag{4.12}$$

Thus, the update equation would be:

$$P_{n-1}(x) = P_n(x) \div (x - \lambda_n), \tag{4.13}$$

where λ_n can be found using the power iteration of the companion matrix of $P_n(x)$.

Why does it converge?

1. Power iteration works when the matrix A has a dominant eigen value $\lambda_{\max}$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{4.14}$$

2. Corresponding eigen vector is v_1

Eigen basis:

3. Assume $v_1, v_2, \ldots, v_n$ be the eigen values of matrix A.

4. Then the initial vector x_0 can be expressed as linear combination of eigen values

$$x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \tag{4.15}$$

Result

Quad1:

Root 1: -1.582720

Root 2: -1.579558

Quad2:

Root 1: 1.582720

Root 2: 1.579558

Why does it converge?

After k iterations, vector x_k becomes

$$x_k = \frac{c_1\lambda_1^k v_1 + c_2\lambda_2^k v_2 + \cdots + c_n\lambda_n^k v_n}{\|c_1\lambda_1^k v_1 + c_2\lambda_2^k v_2 + \cdots + c_n\lambda_n^k v_n\|} \tag{4.16}$$

since λ_1 is the dominant eigen value as k increases (more precisely $k \to \infty$) the term $c_1\lambda_1^k v_1$ dominates and all the other terms become negligible
The normalization step $\mathbf{x}_k = \frac{\tilde{\mathbf{x}}_k}{\|\tilde{\mathbf{x}}_k\|}$ensures that the vector x_k does not grow indefinitely and stabilizes the iteration.

Rate of Convergence

The rate of convergence of power iteration depends on the ratio of the second-largest eigenvalue to the largest eigenvalue:

$$r = \left|\frac{\lambda_2}{\lambda_1}\right| \tag{4.17}$$

1. If r is small, the algorithm converges quickly.
2. If r is close to 1, the algorithm converges slowly.