Title

Gradient Descent

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Question

Question:
A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Parameters and Values

Description	Parameter or Value
length of the wire	$28m$
Radius of Circle	x
Side of the square	s

Table: Parameters or Values

Theoretical Solution

length of the circle is its circumference
length of the circle $= 2\pi x$

$$s = \frac{28 - 2\pi x}{4} \tag{4.1}$$

The least value of $2\pi x$ can be 0 and the greatest value can be 28

$$x \in \left[0, \frac{14}{\pi}\right] \tag{4.2}$$

$$x_0 = 0 \tag{4.3}$$

Defining the convex function

Let $f(x)$ is function of sum of areas of circle and square

$$f(x) = s^2 + \pi x^2 \tag{4.4}$$

$$= \left(7 - \frac{\pi x}{2}\right)^2 + \pi x^2 \tag{4.5}$$

$$= 49 - 7\pi x + \left(\frac{\pi^2}{4} + \pi\right) x^2 \tag{4.6}$$

$$f(x) = ax^2 + bx + c \tag{4.7}$$

$$f'(x) = -7\pi + 2\left(\frac{\pi^2}{4} + \pi\right) x \tag{4.8}$$

$$f'(x) = 2ax + b \tag{4.9}$$

$$f''(x) = 2\left(\frac{\pi^2}{4} + \pi\right) \tag{4.10}$$

Critical points

$$f' = 0 \tag{4.11}$$

$$-7\pi + 2\left(\frac{\pi^2}{4} + \pi\right)x = 0 \tag{4.12}$$

$$x = \frac{14}{\pi + 4} \tag{4.13}$$

For

$$\text{Local Minimum } f'' > 0 \tag{4.14}$$

$$\text{Local Maximum } f'' < 0 \tag{4.15}$$

$$\text{Inflection Point } f'' = 0 \tag{4.16}$$

Hence (4.13) is critical point.

Numerical method results

$$\text{Length of circle} = \frac{28\pi}{\pi + 4} \tag{4.17}$$

$$\text{Length of square} = \frac{28}{\pi + 4} \tag{4.18}$$

Gradient Descent

Computational Solution: We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$x_{n+1} = x_n - \alpha f'(x_n) \tag{4.19}$$

$$x_{n+1} = x_n - \alpha (2ax_n + b) \tag{4.20}$$

$$x_{n+1} = x_n (1 - 2a\alpha) - \alpha b \tag{4.21}$$

Unilateral z-transform

Applying unilateral z-transform over the equation (4.21)

$$zX(z) - zx_0 = X(z)(1 - 2a\alpha) - \frac{\alpha b}{1 - z^{-1}} \tag{4.22}$$

$$X(z)\left[z - (1 - 2a\alpha)\right] = -\frac{\alpha b}{1 - z^{-1}} \tag{4.23}$$

$$X(z) = \frac{-\alpha b}{\left[z - (1 - 2a\alpha)\right]\left[1 - z^{-1}\right]} \tag{4.24}$$

$$X(z) = \frac{-b}{2a}\left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - (1 - 2a\alpha) z^{-1}}\right] \tag{4.25}$$

$$= \frac{-b}{2a}\sum_{n=-\infty}^{\infty}\left[1 - (1 - 2a\alpha)^n\right] z^{-n} u(n) \tag{4.26}$$

ROC

From the equation (4.26), ROC is

$$|z| > max\left\{1, |1-2a\alpha|\right\} \tag{4.27}$$

$$0 < |1-2a\alpha| < 1 \tag{4.28}$$

$$\alpha \in \left(0, \frac{1}{a}\right) \setminus \left\{\frac{1}{2a}\right\} \tag{4.29}$$

$$\alpha \in \left(0, \frac{1}{\frac{\pi^2}{4}+\pi}\right) \setminus \left\{\frac{1}{2\left(\frac{\pi^2}{4}+\pi\right)}\right\} \tag{4.30}$$

Theorem

An LTI system is stable if and only if the ROC of its system function contains the unit circle, $|z| = 1$

Finding x_{min}

If α satisfies the previous equation then

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{4.31}$$

$$\lim_{n\to\infty} \|\alpha\,(2ax_n + b)\| = 0 \tag{4.32}$$

$$\lim_{n\to\infty} \|2ax_n + b\| = 0 \tag{4.33}$$

$$\lim_{n\to\infty} 2ax_n = -b \tag{4.34}$$

$$\lim_{n\to\infty} x_n = \frac{-b}{2a} \tag{4.35}$$

$$x_{min} = \frac{-b}{2a} \tag{4.36}$$

$$x_{min} = \frac{14}{\pi + 4} \tag{4.37}$$

$$\tag{4.38}$$

Result

We take the initial guess $= 0$, step size $= 0.01$, tolerance $= 1e-6$
Using the gradient descent algorithm we get

$$x_{min} = 1.9603469573427723, f_{min} = 27.444858522066347 \tag{4.39}$$

Using Scipy:

$$\text{Minimum value of the function: } 27.444858522066315 \tag{4.40}$$
$$\text{Value of x at the minimum: } 1.96034704352491 \tag{4.41}$$

Alternate Computational Solution: CVXPY

We can also solve it using *cvxpy* module in python. On running the code we get,

$$f_{min} : 27.444858522066315, x_{min} : 1.9603470372904506 \quad (4.42)$$

CVXPY:

expressive syntax

supports various types of problems including convex and non - convex

Optimization variables are defined using the Variable class.

Constants that can be changed without redefining the problem.

Defines the goal of the optimization (minimization or maximization).

Logical conditions imposed on the variables. (constraints)

Combine the objective and constraints to form an optimization problem.

Plot

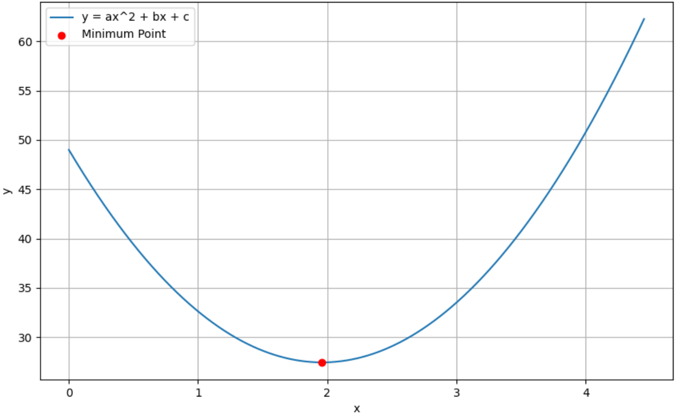

Figure: Minimum of the function