6.5.22

EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:
A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
Solution:

Description	**Parameter or Value**
length of the wire	$28m$
Radius of Circle	x
Side of the square	s

TABLE 0: Parameters or Values

- **Theoretical Solution:**
  length of the circle is its circumference
  length of the circle = $2\pi x$

$$s = \frac{28 - 2\pi x}{4} \tag{0.1}$$

  The least value of $2\pi x$ can be 0 and the greatest value can be 28

$$x \in \left[0, \frac{14}{\pi}\right] \tag{0.2}$$

$$x_0 = 0 \tag{0.3}$$

Let $f(x)$ is function of sum of areas of circle and square

$$
\begin{aligned}
f(x) &= s^2 + \pi x^2 && (0.4)\\
&= \left(7 - \frac{\pi x}{2}\right)^2 + \pi x^2 && (0.5)\\
&= 49 - 7\pi x + \left(\frac{\pi^2}{4} + \pi\right) x^2 && (0.6)\\
f(x) &= ax^2 + bx + c && (0.7)\\
f'(x) &= -7\pi + 2\left(\frac{\pi^2}{4} + \pi\right) x && (0.8)\\
f'(x) &= 2ax + b && (0.9)\\
f''(x) &= 2\left(\frac{\pi^2}{4} + \pi\right) && (0.10)\\
a &= \frac{\pi^2}{4} + \pi && (0.11)\\
b &= -7\pi && (0.12)\\
c &= 49 && (0.13)
\end{aligned}
$$

Critical points,

$$
\begin{aligned}
f' &= 0 && (0.14)\\
-7\pi + 2\left(\frac{\pi^2}{4} + \pi\right) x &= 0 && (0.15)\\
x &= \frac{14}{\pi + 4} && (0.16)
\end{aligned}
$$

For

$$
\begin{aligned}
\text{Local Minimum } f'' &> 0 && (0.17)\\
\text{Local Maximum } f'' &< 0 && (0.18)\\
\text{Inflection Point } f'' &= 0 && (0.19)
\end{aligned}
$$

Hence (0.16) is critical point.

$$
\begin{aligned}
\text{Length of circle } &= \frac{28\pi}{\pi + 4} && (0.20)\\
\text{Length of square} &= \frac{1}{\pi + 4} && (0.21)
\end{aligned}
$$

- **Computational Solution:** We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$
\begin{aligned}
x_{n+1} &= x_n - \alpha f'(x_n) && (0.22)\\
x_{n+1} &= x_n - \alpha(2ax_n + b) && (0.23)\\
x_{n+1} &= x_n(1 - 2a\alpha) - \alpha b && (0.24)
\end{aligned}
$$

Applying unilateral z-transform over the equation (0.24)

$$zX(z) - zx_0 = X(z)(1-2a\alpha) - \frac{\alpha b}{1-z^{-1}} \tag{0.25}$$

$$X(z)\ [z-(1-2a\alpha)] = -\frac{\alpha b}{1-z^{-1}} \tag{0.26}$$

$$X(z) = \frac{-\alpha b}{[z-(1-2a\alpha)]\ [1-z^{-1}]} \tag{0.27}$$

$$X(z) = \frac{-b}{2a}\left[\frac{1}{1-z^{-1}} - \frac{1}{1-(1-2a\alpha)z^{-1}}\right] \tag{0.28}$$

$$= \frac{-b}{2a}\sum_{n=-\infty}^{\infty}\left[1-(1-2a\alpha)^n\right]z^{-n}u(n) \tag{0.29}$$

From the equation (0.29), ROC is

$$|z| > max\{1, |1-2a\alpha|\} \tag{0.30}$$

$$0 < |1-2a\alpha| < 1 \tag{0.31}$$

$$\alpha \in \left(0, \frac{1}{a}\right) \setminus \left\{\frac{1}{2a}\right\} \tag{0.32}$$

$$\alpha \in \left(0, \frac{1}{\frac{\pi^2}{4}+\pi}\right) \setminus \left\{\frac{1}{2\left(\frac{\pi^2}{4}+\pi\right)}\right\} \tag{0.33}$$

If α satisfies the previous equation then

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{0.34}$$

$$\lim_{n\to\infty} \|\alpha(2ax_n + b)\| = 0 \tag{0.35}$$

$$\lim_{n\to\infty} \|2ax_n + b\| = 0 \tag{0.36}$$

$$\lim_{n\to\infty} 2ax_n = -b \tag{0.37}$$

$$\lim_{n\to\infty} x_n = \frac{-b}{2a} \tag{0.38}$$

$$x_{min} = \frac{-b}{2a} \tag{0.39}$$

$$x_{min} = \frac{14}{\pi+4} \tag{0.40}$$

We take the initial guess = 0, step size = 0.01, tolerance = $1e-6$
Using the gradient descent algorithm we get

$$x_{min} = 1.9603469573427723, f_{min} = 27.444858522066347 \tag{0.41}$$

Alternate Computational Solution:

We can also solve it using *cvxpy* module in python. On running the code we get,

$$f_{min} : 27.444858522066315 \tag{0.42}$$

$$x_{min} : 1.9603470372904506 \tag{0.43}$$

Using Scipy:

$$\text{Minimum value of the function: } 27.444858522066315 \tag{0.44}$$

$$\text{Value of x at the minimum: } 1.96034704352491 \tag{0.45}$$

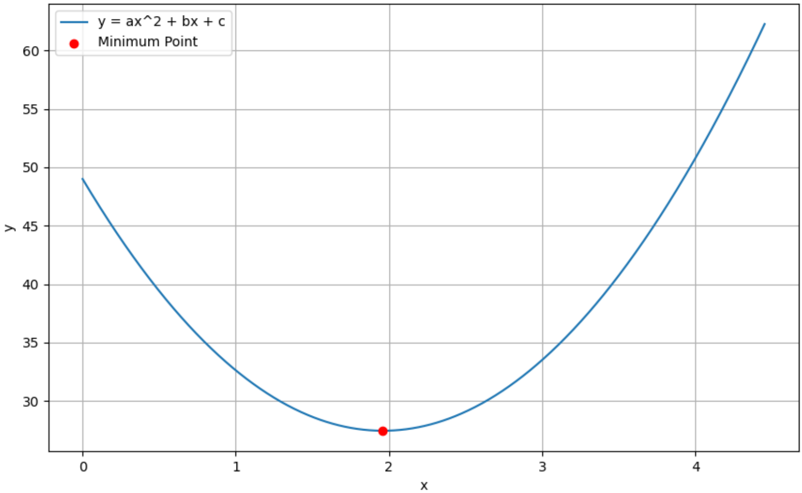

Fig. 0.1: Minimum of the function