10.3.5.1.2

EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:
Check if the given pair of linear equations has unique solution, infinitely many solutions, or no solution. In case there is a unique solution, find it by using cross-multiplication method.

$$2x + y = 5 \tag{0.1}$$
$$3x + 2y = 8 \tag{0.2}$$

Solution:

- A linear equation is said to be **consistent** if it has atleast one solution.
- A linear equation is said to be **inconsistent** if it has no solution.

Lines represented by the equation

$$a_1x + b_1y = c_1 \tag{0.3}$$
$$a_2x + b_2y = c_2 \tag{0.4}$$

are

- Intersecting, then
$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \tag{0.5}$$
- Coincident, then
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \tag{0.6}$$
- Parallel, then
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \tag{0.7}$$

For our Question,

$$\frac{a_1}{a_2} = 0.6667 \tag{0.8}$$
$$\frac{b_1}{b_2} = 0.5 \tag{0.9}$$
$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \tag{0.10}$$

The system has a unique solution
$\therefore$ It is consistent.

Cross multiplication method

The cross-multiplication method for solving a system of two linear equations:

is based on the Crammers rule:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{0.11}$$

$$D = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (2)(2) - (1)(3) = 4 - 3 = 1 \tag{0.12}$$

$$D_x = \begin{vmatrix} 5 & 1 \\ 8 & 2 \end{vmatrix} = (5)(2) - (1)(8) = 10 - 8 = 2 \tag{0.13}$$

$$D_y = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = (2)(8) - (5)(3) = 16 - 15 = 1 \tag{0.14}$$

$$x = \frac{D_x}{D} = \frac{2}{1} = 2, \quad y = \frac{D_y}{D} = \frac{1}{1} = 1 \tag{0.15}$$

$$\tag{0.16}$$

Matrix Method LU Decomposition Convert the given pair of linear equations into matrix form.

We get,

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \tag{0.17}$$

$$\mathbf{A}x = \mathbf{B} \tag{0.18}$$

To solve the above equation, we apply LU - factorization of matrix **A** We do so, because,

$$\mathbf{A} \mapsto LU \tag{0.19}$$

$$L \mapsto \text{(Lower triangular matrix)} \tag{0.20}$$

$$U \mapsto \text{(Upper triangular matrix)} \tag{0.21}$$

Let us consider

$$\mathbf{U}x = y \tag{0.22}$$

Then the equation (0.18) can be written as

$$\mathbf{L}y = \mathbf{B} \tag{0.23}$$

Now the above equation be easily solved using front substitution since **L** is lower triangular matrix. Thus obtaining a solution for y.

Now using back substitution in $y = \mathbf{U}x$ we can solve for the x since **U** is a lower triangular matrix.

LU factorizing $\mathbf{A}$ we get,

$$A = \begin{pmatrix} 1 & 0 \\ 1.5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0.5 \end{pmatrix} \tag{0.24}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1.5 & 1 \end{pmatrix} \tag{0.25}$$

$$U = \begin{pmatrix} 2 & 1 \\ 0 & 0.5 \end{pmatrix} \tag{0.26}$$

Factorization of LU:

Given a matrix $\mathbf{A}$ of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

1) Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.

2) For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k \tag{2.1}$$

3) For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k \tag{3.1}$$

The solution can be obtained in the following way:
Using forward substitution,

$$\begin{pmatrix} 1 & 0 \\ 1.5 & 1 \end{pmatrix} \mathbf{y} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \tag{3.2}$$

we get,

$$\mathbf{y} = \begin{pmatrix} 5 \\ 0.5 \end{pmatrix} \tag{3.3}$$

Now, solving for $\mathbf{x}$, via backward substitution

$$\begin{pmatrix} 2 & 1 \\ 0 & 0.5 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 0.5 \end{pmatrix} \tag{3.4}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.5}$$

Algorithms used for LU decomposition:

Doolittle Algorithm

Algorithm 1 Doolittle Algorithm for LU Decomposition

Require: A is an $n \times n$ matrix
Ensure: L is a lower triangular matrix with unit diagonal, U is an upper triangular matrix

Initialize L as an $n \times n$ identity matrix
Initialize U as an $n \times n$ zero matrix
for $k = 1$ to n **do**
  for $j = k$ to n **do**
    $U_{kj} \leftarrow A_{kj} - \sum_{m=1}^{k-1} L_{km} U_{mj}$
  end for
  for $i = k + 1$ to n **do**
    $L_{ik} \leftarrow \frac{A_{ik} - \sum_{m=1}^{k-1} L_{im} U_{mk}}{U_{kk}}$
  end for
end for
return L, U

Crout's Algorithm

Algorithm 2 Crout's Algorithm for LU Decomposition

Require: A is an $n \times n$ matrix
Ensure: L is a lower triangular matrix, U is an upper triangular matrix with unit diagonal

Initialize L as an $n \times n$ zero matrix
Initialize U as an $n \times n$ identity matrix
for $j = 1$ to n **do**
  for $i = j$ to n **do**
    $L_{ij} \leftarrow A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}$
  end for
  for $i = j + 1$ to n **do**
    $U_{ji} \leftarrow \frac{A_{ji} - \sum_{k=1}^{j-1} L_{jk} U_{ki}}{L_{jj}}$
  end for
end for
return L, U

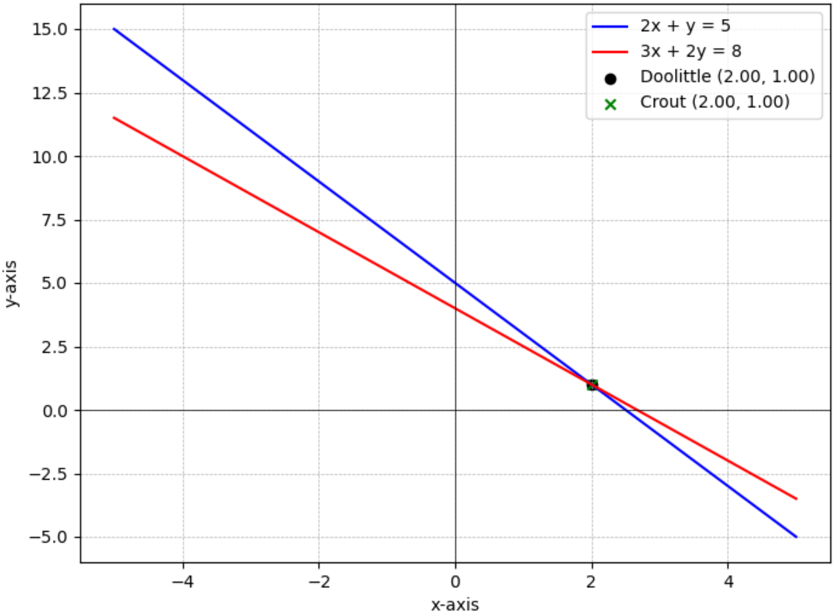

Fig. 3.1: Pair of Lines