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EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:

In a bank, the principal continuously increases at a rate of $r\%$ per year. Find the value of r if Rs 100 doubles in 10 years ($\log_e 2 = 0.6931$).

Solution:

Variable	Description
P_0	initial principal amount
r	rate of increase per year
t	time in years
$C \& C_1$	arbitrary constants
P	principal at any time t

TABLE 0: Variables used

P is the principal at any time, according to the given question, the rate of change of principal can be written as follows.

$$\frac{dP}{dt} = \left(\frac{r}{100}\right) \times P \tag{0.1}$$

Separation of the variables in the equation (0.1)

$$\frac{dP}{p} = \left(\frac{r}{100}\right) \times dt \tag{0.2}$$

Integration on both sides (0.2)

$$\int \frac{dP}{p} = \int \left(\frac{r}{100}\right) dt \tag{0.3}$$

$$\log_e P = \frac{rt}{100} + C \tag{0.4}$$

$$P = e^{\frac{rt}{100} + C} \tag{0.5}$$

$$P = e^{\frac{rt}{100}} e^{C} \tag{0.6}$$

$$P = C_1 e^{\frac{rt}{100}} \tag{0.7}$$

At time $t = 0$, it is given that the principal is 100, that is, $P_0 = 100$.
Substitute in equation (0.7)

$$100 = C_1 \tag{0.8}$$

Principal can be written as

$$P = 100 \times e^{\frac{rt}{100}} \tag{0.9}$$

At $t = 10$, the principal doubles, that is, $P = 200$, using equation (0.9)

$$200 = 100 \times e^{\frac{r}{10}} \tag{0.10}$$
$$2 = e^{\frac{r}{10}} \tag{0.11}$$
$$\log_e 2 = \frac{r}{10} \tag{0.12}$$
$$r = 10 \times \log_e 2 \tag{0.13}$$
$$r = 6.931 \tag{0.14}$$

Verification of the solution computationally:
From the definition of $\frac{dy}{dx}$,

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} \tag{0.15}$$

rearrange the terms

$$y(x+h) = y(x) + h \times \frac{dy}{dx} \tag{0.16}$$

comparing the terms

$$P(t+h) = P(t) + h \times 0.06931 \times P \tag{0.17}$$

Let (t_0, P_0) be the initial conditions then

$$t_1 = t_0 + h \tag{0.18}$$
$$P_1 = P_0 + h \times 0.06931 \times P_0 \tag{0.19}$$

Generalizing this

$$t_{n+1} = t_n + h \tag{0.20}$$
$$P_{n+1} = P_n + h \times 0.06931 \times P_n \tag{0.21}$$

Where h is step and is small.

Laplace transforms:

- If $f(t)$ is a function, the Laplace transform of that function is

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{0.22}$$

- It is linear transformation, since integral is a linear operator
- **Laplace tranforms of some functions:**

$$f(t) = 0 \implies F(s) = 0 \tag{0.23}$$

$$f(t) = 1 \implies F(s) = \frac{1}{s} \text{ for } Re(s) > 0 \tag{0.24}$$

$$f(t) = t^n \implies F(s) = \frac{\Gamma(a+1)}{s^{n+1}} \text{ for } Re(s) > 0 \tag{0.25}$$

$$f(t) = e^{at} \implies F(s) = \frac{1}{s-a} \text{ for } Re(s) > a \tag{0.26}$$

$$f(t) = \sin at \implies F(s) = \frac{a}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.27}$$

$$f(t) = \cos at \implies F(s) = \frac{s}{s^2+a^2} \text{ for } Re(s) > 0 \tag{0.28}$$

$$f(t) = \sinh at \implies F(s) = \frac{a}{s^2-a^2} \text{ for } Re(s) > 0 \tag{0.29}$$

$$f(t) = \cosh at \implies F(s) = \frac{s}{s^2-a^2} \text{ for } Re(s) > 0 \tag{0.30}$$

$$f(t) = e^{\omega t} \sin at \implies F(s) = \frac{a}{(s-\omega)^2+a^2} \text{ for } Re(s) > \omega \tag{0.31}$$

$$f(t) = e^{\omega t} \cos at \implies F(s) = \frac{s-\omega}{(s-\omega)^2+a^2} \text{ for } Re(s) > \omega \tag{0.32}$$

- **Laplace transforms of derivatives:**

$$\mathcal{L}(f') = sF(s) - f(0) \tag{0.33}$$

$$\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0) \tag{0.34}$$

- **Laplace transform of unit step function $u(t)$:**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{0.35}$$

From (0.22)

$$\mathcal{L}(u(t)) = \int_0^\infty u(t) e^{-st} dt \tag{0.36}$$

For all non-negative values, $u(t) = 1$. Hence, the integral becomes,

$$F(s) = \int_0^\infty (1) e^{-st} dt \tag{0.37}$$

$$F(s) = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, \text{ for } Re(s) > 0 \tag{0.38}$$

- **Laplace transform of $e^{at}u(t)$:**
  From (0.22)

$$\mathcal{L}\left(e^{at}u(t)\right) = \int_0^{\infty} e^{at}u(t)e^{-st}dt \tag{0.39}$$
$$F(s) = \int_0^{\infty} e^{(a-s)t}dt \tag{0.40}$$
$$F(s) = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\infty} = \frac{1}{s-a}, \text{ for } Re(s) > a \tag{0.41}$$

- **Solution for the differential equation:**
  Let $f(t) = P(t)$
  Apply Laplace transform to the equation (0.1)

$$\mathcal{L}(f') = \mathcal{L}(0.06931 \times f) \tag{0.42}$$
$$\tag{0.43}$$

From equations (0.34) and (0.33)

$$sF(s) - f(0) = 0.06931F(s) \tag{0.44}$$
$$(s - 0.06931)F(s) = f(0) \tag{0.45}$$
$$F(s) = \frac{f(0)}{s - 0.06931} \tag{0.46}$$

Substitute the initial conditions $f(0) = 100$

$$F(s) = \frac{100}{s - 0.06931} \tag{0.47}$$
$$\mathcal{L}(f(t)) = \frac{100}{s - 0.06931} \tag{0.48}$$
$$f(t) = \mathcal{L}^{-1}\left(\frac{100}{s - 0.06931}\right) \tag{0.49}$$
$$\tag{0.50}$$

from equation (0.41)

$$f(t) = 100 \times u(t)e^{0.06931t} \tag{0.51}$$
$$P(t) = 100 \times u(t)e^{0.06931t} \tag{0.52}$$

Z-transform:

- If $f(t)$ is a function, the Z-transform of that function is

$$X[z] = \mathcal{Z}(x[t]) = \sum_{t=-\infty}^{\infty} x[t]z^{-t} \tag{0.53}$$

Z-transform of some functions

- $u(t)$:

From (0.53)

$$Y(z) = \sum_{t \to -\infty}^{\infty} u[t] z^{-t} \tag{0.54}$$

From (0.35), we simplify it as

$$Y(z) = \sum_{t=0}^{\infty} (1) z^{-t} \tag{0.55}$$

$$Y(z) = \frac{1}{1 - z^{-1}}, \text{ for } |z| > 1 \tag{0.56}$$

- $a^t u(t)$:
  From (0.53)

$$X[z] = \sum_{t \to -\infty}^{\infty} a^t u[t] z^{-t} \tag{0.57}$$

From (0.35), we simplify it as

$$X[z] = \sum_{t=0}^{\infty} a^t z^{-t} \tag{0.58}$$

$$X[z] = \sum_{t=0}^{\infty} \left(a z^{-1}\right)^t \tag{0.59}$$

$$X[z] = \frac{1}{1 - a z^{-1}}, \text{ for } |z| > |a| \tag{0.60}$$

- **Some other useful results** :

$$\mathcal{Z}[u_{n-1}] = z^{-1} \mathcal{Z}[u_n] \tag{0.61}$$

$$\mathcal{Z}[u_{n+1}] = z(\mathcal{Z}[u_n] - u_0) \tag{0.62}$$

- **Solution:** from (0.19)

$$P_{n+1} = P_n + h \times 0.06931 \times P_n \tag{0.63}$$

$$P_{n+1} = P_n (1 + 0.06931h) \tag{0.64}$$

Apply z-transform

$$\mathcal{Z}[P_{n+1}] = \mathcal{Z}[P_n (1 + 0.06931h)] \tag{0.65}$$

$$\mathcal{Z}[P_{n+1}] = (1 + 0.06931h) \mathcal{Z}[P_n] \tag{0.66}$$

Let,

$$\mathcal{Z}[P_n] = P[z] \tag{0.67}$$

Then,

$$\mathcal{Z}\left[P_{n+1}\right] = zP\left[z\right] - zP_0 \tag{0.68}$$

Now,

$$zP\left[z\right] - zP_0 = P\left[z\right]\left(1 + 0.06931h\right) \tag{0.69}$$

$$P\left[z\right]\ \left[z - \left(1 + 0.06931h\right)\right] = zP_0 \tag{0.70}$$

$$P\left[z\right] = P_0\left[\frac{z}{z - \left(1 + 0.06931h\right)}\right] \tag{0.71}$$

By inversing, we get

$$P_n = P_0 \times \left(1 + 0.06931h\right)^n \tag{0.72}$$

We know that,

$$1 + 0.06931h \approx e^{0.06931h} \tag{0.73}$$

then,

$$P_n = P_0\left(e^{0.06931h}\right)^n \tag{0.74}$$

$$P_n = P_o e^{0.06931nh} \tag{0.75}$$

As h is the small division of time and n are the total no.of divisions, nh turns to be t at that point,Then

$$P\left(t\right) = P_0 e^{0.06931t} \tag{0.76}$$

We iterate this by taking the initial conditions from $t = 0$ to $t = 10$
By plotting all the points (t, P) we get the graph of function P varying with t.
The comparison between theoretical and simulation curves is shown in the figure, we can clearly see that both the curves are coincides which verifies our solution.
For the following approximate graph, I chose $h = 0.1$

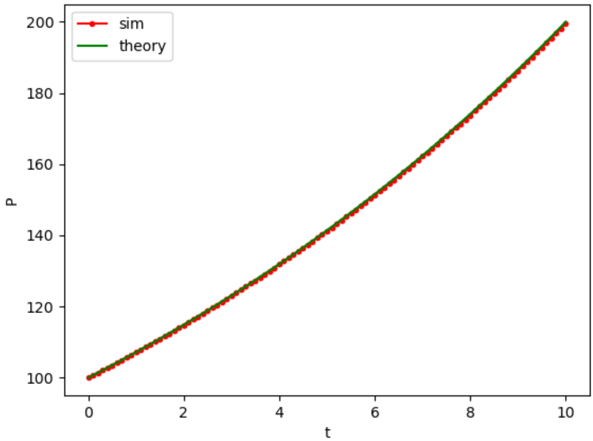

Fig. 0.1: Approximate solution of the DE