8.ex.8

EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:
A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
Solution:

Description	Parameter or Value
length of the wire	$28m$
Radius of Circle	x
Side of the square	s

TABLE 0: Parameters or Values

- **Theoretical Solution:**
  length of the circle is its circumference
  length of the circle = $2\pi x$

$$s = 28 - 2\pi x \tag{0.1}$$

The least value of $2\pi x$ can be 0 and the greatest value can be 28

$$x \in \left[0, \frac{14}{\pi}\right] \tag{0.2}$$
$$x_0 = 0 \tag{0.3}$$

Let $f(x)$ is function of sum of areas of circle and square

$$f(x) = s^2 + \pi x^2 \tag{0.4}$$
$$= (28 - 2\pi x)^2 + \pi x^2 \tag{0.5}$$
$$= 784 - 112\pi x + \left(4\pi^2 + \pi\right) x^2 \tag{0.6}$$
$$f(x) = ax^2 + bx + c \tag{0.7}$$
$$f'(x) = -112\pi + 2\left(4\pi^2 + \pi\right) x \tag{0.8}$$
$$f'(x) = 2ax + b \tag{0.9}$$
$$f''(x) = 2\left(4\pi^2 + \pi\right) \tag{0.10}$$
$$a = 4\pi^2 + \pi \tag{0.11}$$
$$b = -112\pi \tag{0.12}$$
$$c = 784 \tag{0.13}$$

Critical points,

$$f' = 0 \tag{0.14}$$

$$-112\pi + 2\left(4\pi^2 + \pi\right)x = 0 \tag{0.15}$$

$$x = \frac{56}{4\pi + 1} \tag{0.16}$$

For

$$\text{Local Minimum } f'' > 0 \tag{0.17}$$

$$\text{Local Maximum } f'' < 0 \tag{0.18}$$

$$\text{Inflection Point } f'' = 0 \tag{0.19}$$

Hence (0.16) is critical point.

$$\text{Length of circle } = \frac{112\pi}{4\pi + 1} \tag{0.20}$$

$$\text{Length of square} = \frac{28}{4\pi + 1} \tag{0.21}$$

- **Computational Solution:** We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$x_{n+1}x_n - \alpha f'(x_n) \tag{0.22}$$

$$x_{n+1} = x_n - \alpha(2ax_n + b) \tag{0.23}$$

$$x_{n+1} = x_n(1 - 2a\alpha) + \alpha b \tag{0.24}$$

Applying unilateral z-transform over the equation (0.24)

$$zX(z) - zx_0 = X(z)(1 - 2a\alpha) + \frac{\alpha b}{1 - z^{-1}} \tag{0.25}$$

$$X(z)\ [z - (1 - 2a\alpha)] = \frac{\alpha b}{1 - z^{-1}} \tag{0.26}$$

$$X(z) = \frac{\alpha b}{[z - (1 - 2a\alpha)]\ [1 - z^{-1}]} \tag{0.27}$$

$$X(z) = \frac{b}{2a}\left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - (1 - 2a\alpha)z^{-1}}\right] \tag{0.28}$$

$$= \frac{b}{2a}\sum_{n=-\infty}^{\infty}[1 - (1 - 2a\alpha)^n]\, z^{-n}u(n) \tag{0.29}$$

From the equation (0.29), ROC is

$$|z| > max\{1, |1 - 2a\alpha|\} \tag{0.30}$$

$$0 < |1 - 2a\alpha| < 1 \tag{0.31}$$

$$\tag{0.32}$$

$$\alpha \in \left(0, \frac{1}{a}\right) \setminus \left\{\frac{1}{2a}\right\} \tag{0.33}$$

$$\alpha \in \left(0, \frac{1}{4\pi^2 + \pi}\right) \setminus \left\{\frac{1}{2\left(4\pi^2 + \pi\right)}\right\} \tag{0.34}$$

If α satisfies the previous equation then

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{0.35}$$

$$\lim_{n\to\infty} \|\alpha\left(2ax_n + b\right)\| = 0 \tag{0.36}$$

$$\lim_{n\to\infty} \|2ax_n + b\| = 0 \tag{0.37}$$

$$\lim_{n\to\infty} 2ax_n = -b \tag{0.38}$$

$$\lim_{n\to\infty} x_n = \frac{-b}{2a} \tag{0.39}$$

$$x_{min} = \frac{-b}{2a} \tag{0.40}$$

$$x_{min} = \frac{56}{1 + 4\pi} \tag{0.41}$$

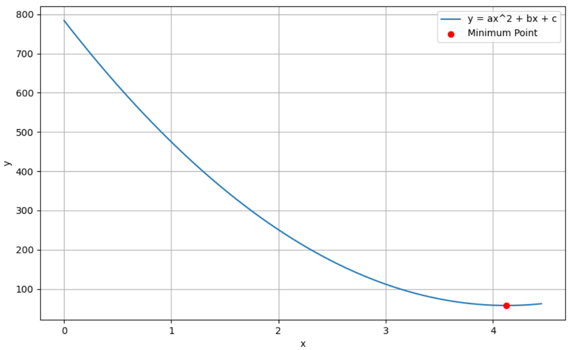

Fig. 0.1: Minimum of the function