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EE24BTECH11049 - Patnam Shariq Faraz Muhammed

Question:

In a bank, the principal continuously increases at a rate of $r\%$ per year. Find the value of r if Rs 100 doubles in 10 years ($\log_e 2 = 0.6931$).

Solution:

Variable	Description
P_0	initial principal amount
r	rate of increase per year
t	time in years
$C \& C_1$	arbitrary constants
P	principal at any time t

TABLE 0: Variables used

P is the principal at any time, according to the given question, the rate of change of principal can be written as follows.

$$\frac{dP}{dt} = \left(\frac{r}{100}\right) \times P \tag{0.1}$$

Separation of the variables in the equation (0.1)

$$\frac{dP}{p} = \left(\frac{r}{100}\right) \times dt \tag{0.2}$$

Integration on both sides (0.2)

$$\int \frac{dP}{p} = \int \left(\frac{r}{100}\right) dt \tag{0.3}$$

$$\log_e P = \frac{rt}{100} + C \tag{0.4}$$

$$P = e^{\frac{rt}{100} + C} \tag{0.5}$$

$$P = e^{\frac{rt}{100}} e^{C} \tag{0.6}$$

$$P = C_1 e^{\frac{rt}{100}} \tag{0.7}$$

At time $t = 0$, it is given that the principal is 100, that is, $P_0 = 100$.
Substitute in equation (0.7)

$$100 = C_1 \tag{0.8}$$

Principal can be written as

$$P = 100 \times e^{\frac{rt}{100}} \tag{0.9}$$

At $t = 10$, the principal doubles, that is, $P = 200$, using equation (0.9)

$$200 = 100 \times e^{\frac{r}{10}} \tag{0.10}$$

$$2 = e^{\frac{r}{10}} \tag{0.11}$$

$$\log_e 2 = \frac{r}{10} \tag{0.12}$$

$$r = 10 \times \log_e 2 \tag{0.13}$$

$$r = 6.931 \tag{0.14}$$

Verification of the solution computationally:

From the definition of $\frac{dy}{dx}$,

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} \tag{0.15}$$

rearrange the terms

$$y(x+h) = y(x) + h \times \frac{dy}{dx} \tag{0.16}$$

comparing the terms

$$P(t+h) = P(t) + h \times 0.06931 \times P \tag{0.17}$$

Let (t_0, P_0) be the initial conditions then

$$t_1 = t_0 + h \tag{0.18}$$

$$P_1 = P_0 + h \times 0.06931 \times P_0 \tag{0.19}$$

Generalizing this

$$t_{n+1} = t_n + h \tag{0.20}$$

$$P_{n+1} = P_n + h \times 0.06931 \times P_n \tag{0.21}$$

Where h is step and is small. We iterate this by taking the initial conditions from $t = 0$ to $t = 10$

By plotting all the points (t, P) we get the graph of function P varying with t.

The comparison between theoretical and simulation curves is shown in the figure, we can

clearly see that both the curves are coincides which verifies our solution.
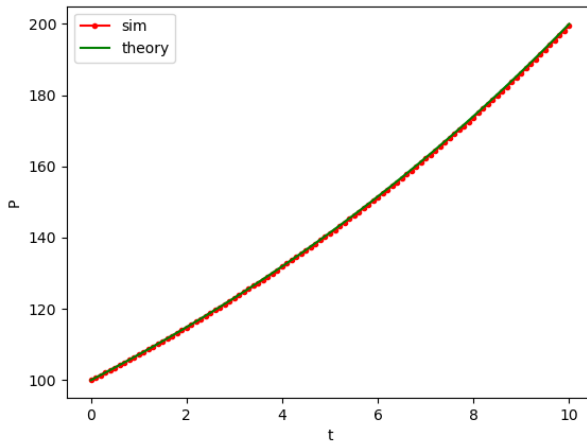For the following approximate graph, I chose $h = 0.1$

Fig. 0.1: Approximate solution of the DE