9-9.5-4

EE24BTECH11049
Patnam Shariq Faraz Muhammed

QUESTION:
Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$, using integration.
SOLUTION:

Variable	Description
$y^2 = 4x$	Parabola
$4x^2 + 4y^2 = 9$	Circle

The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$
$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top$$
$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$
$$f = 0$$

The given circle can be expressed as conics with parameters

$$\mathbf{V} = \frac{1}{4} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$
$$\mathbf{u} = 0$$
$$f = -\frac{81}{16}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top (\mathbf{V_1} + \mu \mathbf{V_2}) \mathbf{x} + 2 (\mathbf{u_1} + \mu \mathbf{u_2})^\top \mathbf{x} + (f_1 + \mu f_2) = 0$$
$$\mu = -\frac{4}{9}$$

On solving we get the points of intersection to be $\begin{pmatrix}\frac{1}{2}\\ \sqrt{2}\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\ -\sqrt{2}\end{pmatrix}$

The desired area of region is given as

$$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx\right]$$

$$= 2\left[\frac{4}{3}\sqrt{x^3}\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

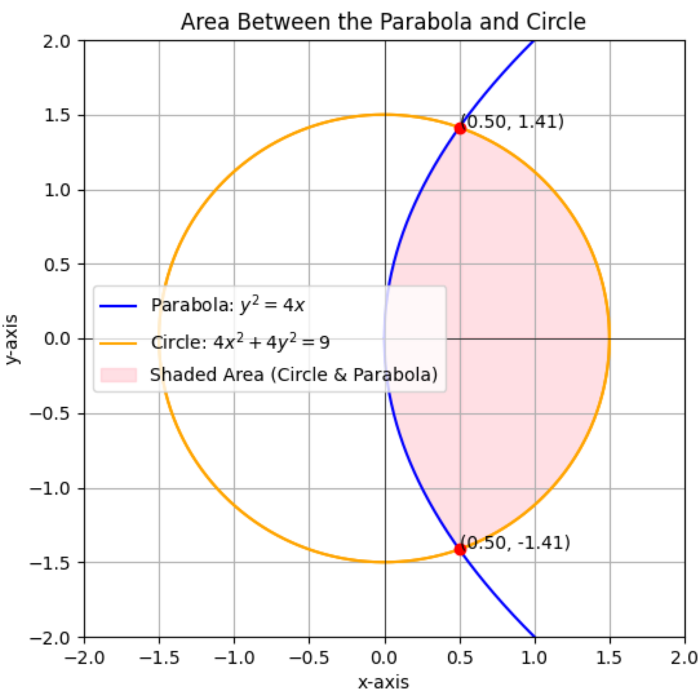

Fig. 0.1: A plot of the given question.