

9-9.5-4

EE24BTECH11049

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QUESTION:

Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$, using integration.

SOLUTION:

Variable	Description
$y^2 = 4x$	Parabola
$4x^2 + 4y^2 = 9$	Circle

The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$f = 0$$

The given circle can be expressed as conics with parameters

$$\mathbf{V} = \frac{1}{4} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\mathbf{u} = 0$$

$$f = -\frac{81}{16}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0$$

$$\mu = -\frac{4}{9}$$

On solving we get the points of intersection to be $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

The desired area of region is given as

$$\begin{aligned}
 &= 2 \left[\int_0^{\frac{1}{\sqrt{2}}} 2\sqrt{x} dx + \int_{\frac{1}{\sqrt{2}}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2} dx \right] \\
 &= 2 \left[\frac{4}{3} \sqrt{x^3} \right]_0^{\frac{1}{\sqrt{2}}} + 2 \left[\frac{x}{2} \sqrt{\frac{9}{4} - x^2} + \frac{9}{8} \sin^{-1} \left(\frac{2x}{3} \right) \right]_{\frac{1}{\sqrt{2}}}^{\frac{3}{2}} \\
 &= \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8} \sin^{-1} \left(\frac{1}{3} \right)
 \end{aligned}$$

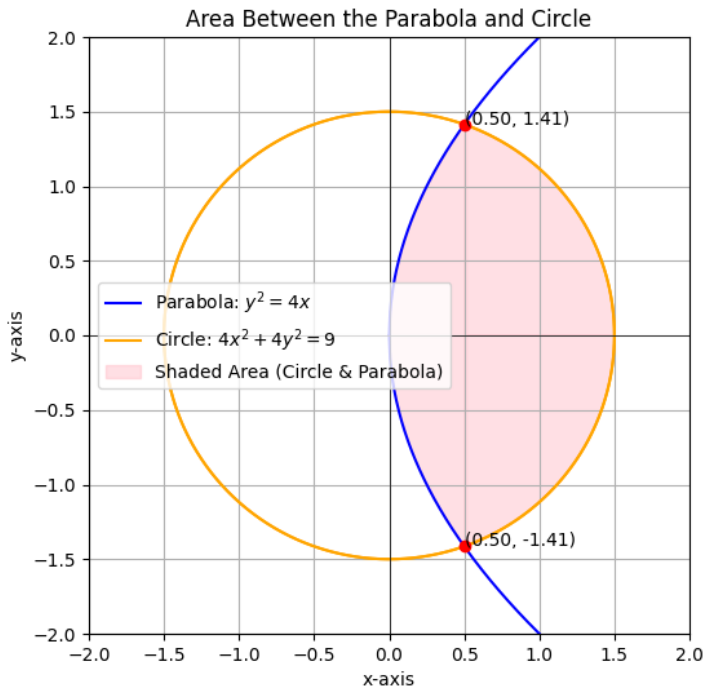


Fig. 0.1: A plot of the given question.