Question(1.1):

Let **P** and **Q** be the user-feature and movie-feature matrix, respectively.

Let,

$$\boldsymbol{P} \times \boldsymbol{Q} = \boldsymbol{R'}$$

Where, **R’** is our approximation for the given user rating matrix, **R**.

We define error at, i-th row and j-th column as,

$$e_{ij}^2 = (\boldsymbol{R}_{ij} - \boldsymbol{R'}_{ij})^2 - (1)$$

$\boldsymbol{R'_{ij}}$ **=** Our prediction for the value at ith-row and j-th column.

$$\boldsymbol{R'}_{ij} = \sum_{k=0}^{k} \boldsymbol{P}_{ik} \times \boldsymbol{Q}_{kj}$$

$\boldsymbol{R'_{ij}}$ Could also be re-written as,

$$\boldsymbol{R'}_{ij} = \boldsymbol{P}_i \cdot \boldsymbol{Q}_j$$

Where, the dot (·) is dot product operator. Replacing this in eq(1),

$$e_{ij}^2 = \left(\boldsymbol{R}_{ij} - \boldsymbol{P}_i \cdot \boldsymbol{Q}_j\right)^2 - (2)$$

We now calculate, the calculate partial derivatives wrt each variable.

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{P_i}} = -2 * \left(\boldsymbol{R}_{ij} - \boldsymbol{P}_i \cdot \boldsymbol{Q}_j\right) * \left(\boldsymbol{Q}_j\right)$$

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{P_i}} = -2 * \left(e_{ij}\right) * (\boldsymbol{Q}_j)$$

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{Q_j}} = -2 * \left(e_{ij}\right) * (\boldsymbol{P_i})$$

Now, to update values, we move in the direction specified,

i.e.

new_value = current_value – (learning_rate * direction of maximum change)

$$\boldsymbol{P'}_i = \boldsymbol{P_i} + \alpha * 2 * \left(e_{ij}\right) * (\boldsymbol{Q}_j)$$

$$\boldsymbol{Q'}_j = \boldsymbol{Q_j} + \alpha * 2 * (e_{ij}) * (\boldsymbol{P}_i)$$

Question(1.2):

Now we introduce the bias vectors to our prediction. The updation formulae for Pi and Qj remain the same, while now we have updation terms for each bias vector.

$$\boldsymbol{R'}_{ij} = b\boldsymbol{U}_i + b\boldsymbol{I}_j + (\boldsymbol{P}_i \cdot \boldsymbol{Q}_j)$$

$$e_{ij}^2 = (\boldsymbol{R}_{ij} - (b\boldsymbol{U}_i + b\boldsymbol{I}_j + (\boldsymbol{P}_i \cdot \boldsymbol{Q}_j)))^2$$

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{bU_i}} = 2 * \left(\boldsymbol{R}_{ij} - b\boldsymbol{U}_i - b\boldsymbol{I}_j - (\boldsymbol{P}_i \cdot \boldsymbol{Q}_j)\right)$$

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{bU_i}} = 2 * (e_{ij}) * (-1)$$

$$\frac{\partial e_{ij}^2}{\partial \boldsymbol{bI_j}} = 2 * (e_{ij}) * (-1)$$

$$\boldsymbol{bU'}_i = \boldsymbol{bU_i} + \alpha * 2 * (e_{ij})$$

$$\boldsymbol{bI'}_j = \boldsymbol{bI_j} + \alpha * 2 * (e_{ij})$$