Chapter 5

Simple Linear regression (solutions to exercises)

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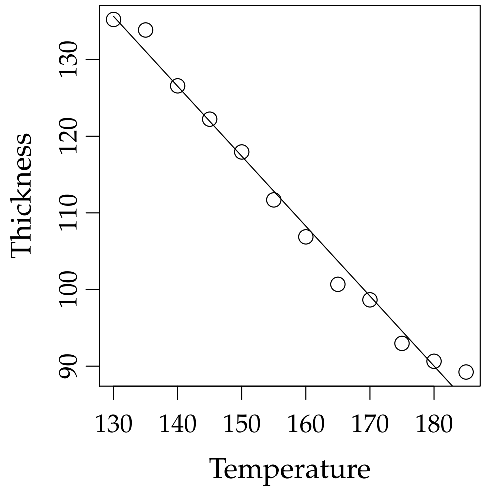
5.1 Plastic film folding machine

|||| Exercise 5.1 Plastic film folding machine

On a machine that folds plastic film the temperature may be varied in the range of 130-185 °C. For obtaining, if possible, a model for the influence of temperature on the folding thickness, $n = 12$ related set of values of temperature and the fold thickness were measured that is illustrated in the following figure:



a) Determine by looking at the figure, which of the following sets of estimates for the parameters in the usual regression model is correct:

1) $\hat{\beta}_0 = 0, \hat{\beta}_1 = -0.9, \hat{\sigma} = 36$
2) $\hat{\beta}_0 = 0, \hat{\beta}_1 = 0.9, \hat{\sigma} = 3.6$
3) $\hat{\beta}_0 = 252, \hat{\beta}_1 = -0.9, \hat{\sigma} = 3.6$
4) $\hat{\beta}_0 = -252, \hat{\beta}_1 = -0.9, \hat{\sigma} = 36$
5) $\hat{\beta}_0 = 252, \hat{\beta}_1 = -0.9, \hat{\sigma} = 36$

|||| Solution

First of all, the only possible intercept ($\hat{\beta}_0$) among the ones given in the answers is 252. And then the slope estimate of -0.9 in these two options looks reasonable. We just need to decide on whether the estimated standard deviation of the error $s_e = \hat{\sigma}$ is 3.6 or 36. From the figure it is clear that the points are NOT having an average vertical distance to the line in the size of 36, so 3.6 must be the correct number and hence the correct answer is:

3) $\hat{\beta}_0 = 252$, $\hat{\beta}_1 = -0.9$, $\hat{\sigma} = 3.6$

b) What is the only possible correct answer:

1) The proportion of explained variation is 50% and the correlation is 0.98
2) The proportion of explained variation is 0% and the correlation is -0.98
3) The proportion of explained variation is 96% and the correlation is -1
4) The proportion of explained variation is 96% and the correlation is 0.98
5) The proportion of explained variation is 96% and the correlation is -0.98

|||| Solution

The proportion of variation explained must be pretty high, so 0 can be ruled out. Answer 1 and 4 is also ruled out since the correlation clearly is negative. This also narrows the possibilities down to answer 3 and 5. And since the correlation is NOT exactly -1 (in which case the observations would be exactly on the line), the correct answer is:

5) The proportion of explained variation is 96% and the correlation is -0.98

5.2 Linear regression life time model

|||| Exercise 5.2 Linear regression life time model

A company manufactures an electronic device to be used in a very wide temperature range. The company knows that increased temperature shortens the life time of the device, and a study is therefore performed in which the life time is determined as a function of temperature. The following data is found:

Temperature in Celcius (t)	10	20	30	40	50	60	70	80	90
Life time in hours (y)	420	365	285	220	176	117	69	34	5

a) Calculate the 95% confidence interval for the slope in the usual linear regression model, which expresses the life time as a linear function of the temperature.

|||| Solution

Either one could do all the regression computations to find the $\hat{\beta}_1 = -5.3133$ and then subsequently use the formula for the confidence interval for β_1 in Method 5.15

$$\hat{\beta}_1 \pm t_{1-\alpha/2} \cdot \hat{\sigma}_{\beta_1} = \hat{\beta}_1 \pm t_{\alpha/2} \cdot \hat{\sigma} \sqrt{\frac{1}{\sum_{i=1}^{n}(x_i - \bar{x})^2}},$$

or just run `lm` in `R` to find:

```
D <- data.frame(t=c(10,20,30,40,50,60,70,80,90),
                y=c(420,365,285,220,176,117,69,34,5))
fit <- lm(y ~ t, data=D)
summary(fit)


Call:
lm(formula = y ~ t, data = D)

Residuals:
   Min     1Q Median     3Q    Max 
-21.02 -12.62  -9.16  17.71  29.64 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  453.556     14.394    31.5  8.4e-09 ***
t             -5.313      0.256   -20.8  1.5e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 19.8 on 7 degrees of freedom
Multiple R-squared:  0.984, Adjusted R-squared:  0.982 
F-statistic:  432 on 1 and 7 DF,  p-value: 0.000000151
```

and use the knowledge of the information in the R-output that wht is know as the "standard error for the slope" can be directly read off as

$$\hat{\sigma}_{\beta_1} = \hat{\sigma}\sqrt{\frac{1}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = 0.2558,$$

and $t_{0.025}(7) = 2.364$ - in R:

```
qt(.975,7)

[1] 2.365
```

to get $-5.31 \pm 2.365 \cdot 0.2558$, or in R:

```
-5.31+c(-1,1)*qt(.975,7)*0.2558

[1] -5.915 -4.705
```

b) Can a relation between temperature and life time be documented on level

5%?

|||| Solution

Since the confidence interval does not include 0, it can be documented that there is a relationship between life time and temperature, also the p-value is $1.5 \cdot 10^{-7} < 0.05 = \alpha$, which also give strong evidence against the null-hypothesis.

5.3 Yield of chemical process

|||| Exercise 5.3 Yield of chemical process

The yield y of a chemical process is a random variable whose value is considered to be a linear function of the temperature x. The following data of corresponding values of x and y is found:

Temperature in °C (x)	0	25	50	75	100
Yield in grams (y)	14	38	54	76	95

The average and standard deviation of temperature and yield are

$$\bar{x} = 50,\ s_x = 39.52847,\ \bar{y} = 55.4,\ s_y = 31.66702,$$

In the exercise the usual linear regression model is used

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma_\varepsilon^2), \quad i = 1, \dots, 5$$

a) Can a significant relationship between yield and temperature be documented on the usual significance level $\alpha = 0.05$?

|||| Solution

It could most easily be solved by running the regression in R as:

```
D <- data.frame(x=c(0,25,50,75,100),
                y=c(14,38,54,76,95))
fit <- lm(y ~ x, data=D)
summary(fit)


Call:
lm(formula = y ~ x, data = D)

Residuals:
   1    2    3    4    5 
-1.4  2.6 -1.4  0.6 -0.4 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  15.4000     1.4967    10.3    0.002 ** 
x             0.8000     0.0244    32.7 0.000063 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.93 on 3 degrees of freedom
Multiple R-squared:  0.997, Adjusted R-squared:  0.996 
F-statistic: 1.07e+03 on 1 and 3 DF,  p-value: 0.0000627
```

Alternatively one could use hand calculations and use the formula in Theorem 5.12 for the t-test of the null hypothesis: $H_0 : \beta_1 = 0$.

The relevant test statistic and p-value can be read off in the R output as 32.7 and 0.000063. So the answer is:

Yes, as the relevant test statistic and p-value are resp. 32.7 and $0.00006 < 0.05 = \alpha$.

b) Give the 95% confidence interval of the expected yield at a temperature of $x_{\text{new}} = 80$ °C.

|||| Solution

We use the formula in Equation (5-59) for the confidence limit of the line (the expected value of Y_i for a value x_{new}):

$$\hat{\beta}_0 + \hat{\beta}_1 x_{\text{new}} \pm t_{1-\alpha/2}\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{\text{new}} - \bar{x})^2}{S_{xx}}},$$

and we have to compute $\hat{\beta}_0$, $\hat{\beta}_1$ and s_e either by hand OR in R as above:

$$\hat{\beta}_0 = 15.4,\ \hat{\beta}_1 = 0.8,\ \hat{\sigma} = 1.932.$$

So the confidence interval becomes

$$(15.4 + 0.8 \cdot 80) \pm 3.182 \cdot 1.932\sqrt{\frac{1}{5} + \frac{(80-50)^2}{6250}},$$

since

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}S_{xx} \Leftrightarrow$$
$$S_{xx} = (n-1)s_x^2 = 4 \cdot 39.528^2 = 6250.$$

Thus the answer is

$$79.40 \pm 3.61 = [75.79,\ 83.01].$$

In R this could be by:

```
predict(fit, newdata=data.frame(x=80), interval="confidence",
        level=0.95)

   fit   lwr   upr
1 79.4 75.79 83.01
```

c) What is the upper quartile of the residuals?

|||| Solution

The five residuals become: -1.4, 2.6, -1.4, 0.6 og -0.4.

We use the basic definition of finding a quantile (from Definition 1.7) and the upper quartile is $q_{0.75}$ (see Definition 1.8). We set $n = 5$, $p = 0.75$, so

$$np = 3.75$$

So the upper quartile is the 4th observation in the ordered sequence:

$$-1.4, -1.4, -0.4, 0.6, 2.6.$$

This is also found in the `summary()` output above under

```
Residuals:
   1    2    3    4    5
-1.4  2.6 -1.4  0.6 -0.4
```

So the answer is: 0.6.

5.4 Plastic material

Exercise 5.4 Plastic material

In the manufacturing of a plastic material, it is believed that the cooling time has an influence on the impact strength. Therefore a study is carried out in which plastic material impact strength is determined for 4 different cooling times. The results of this experiment are shown in the following table:

Cooling times in seconds (x)	15	25	35	40
Impact strength in kJ/m^2 (y)	42.1	36.0	31.8	28.7

The following statistics may be used:

$$\bar{x} = 28.75, \ \bar{y} = 34.65, \ S_{xx} = 368.75.$$

a) What is the 95% confidence interval for the slope of the regression model, expressing the impact strength as a linear function of the cooling time?

|||| Solution

The easiest way to get to the confidence interval is to use the standard error for the slope ($\hat{\sigma}_{\beta_1}$ or denoted with SE_{β_1}) given in the R output:

```
x <- c(15,25,35,40)
y <- c(42.1,36.0,31.8,28.7)
summary(lm(y ~ x))


Call:
lm(formula = y ~ x)

Residuals:
      1       2       3       4 
 0.2814 -0.6051  0.4085 -0.0847 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   49.639      0.878    56.5  0.00031 ***
x             -0.521      0.029   -18.0  0.00308 ** 
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.556 on 2 degrees of freedom
Multiple R-squared:  0.994,	Adjusted R-squared:  0.991 
F-statistic:  324 on 1 and 2 DF,  p-value: 0.00308
```

the standard error for the slope is $\hat{\sigma}_{\beta_1} = 0.029$ (also known as the sampling distribution standard deviation for $\hat{\beta}_1$). Finding the relevant t-quantile (with $\nu = 2$ degrees of freedom (either of):

```
c(qt(0.025, df=2), qt(0.975, df=2))

[1] -4.303  4.303
```

$|t_{0.025}| = 4.303$, which using Theorem 5.15 gives

$$-0.521 \pm 4.303 \cdot 0.029,$$

giving

$$-0.521 \pm 0.125,$$

or, that we say with high confidence that the true parameter value is in the interval, i.e.

$$-0.646 \leq \beta_1 \leq -0.396.$$

b) Can you conclude that there is a relation between the impact strength and the cooling time at significance level $\alpha = 5\%$?

|||| Solution

The relevant p-value can be read off directly from the `summary` output: 0.00308, and we can conclude: *Yes, as the relevant p-value is 0.00308, which is smaller than 0.05.*

c) For a similar plastic material the tabulated value for the linear relation between temperature and impact strength (i.e the slope) is -0.30. If the following hypothesis is tested (at level $\alpha = 0.05$)

$$H_0 : \beta_1 = -0.30$$
$$H_1 : \beta_1 \neq -0.30$$

with the usual t-test statistic for such a test, what is the range (for t) within which the hypothesis is accepted?

|||| Solution

The so-called critical values for the t-statistic with $\nu = 2$ degrees of freedom is found as (or at least the negative one of the two): $t_{0.025} = -4.303$ - in R: `qt(0.975,2)`). So the answer becomes:

$$[-4.303, 4.303].$$

5.5 Water polution

Exercise 5.5 Water polution

In a study of pollution in a water stream, the concentration of pollution is measured at 5 different locations. The locations are at different distances to the pollution source. In the table below, these distances and the average pollution are given:

Distance to the pollution source (in km)	2	4	6	8	10
Average concentration	11.5	10.2	10.3	9.68	9.32

a) What are the parameter estimates for the three unknown parameters in the usual linear regression model: 1) The intercept (β_0), 2) the slope (β_1) and 3) error standard deviation (σ)?

|||| Solution

The question is solved by considering the following R-output:

```
D <- data.frame(concentration=c(11.5, 10.2, 10.3, 9.68, 9.32),
                distance=c(2, 4, 6, 8, 10))
fit <- lm(concentration ~ distance, data=D)
summary(fit)


Call:
lm(formula = concentration ~ distance, data = D)

Residuals:
     1      2      3      4      5
 0.324 -0.488  0.100 -0.032  0.096

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   11.664      0.365   31.96 0.000067 ***
distance      -0.244      0.055   -4.43    0.021 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.348 on 3 degrees of freedom
Multiple R-squared:  0.868, Adjusted R-squared:  0.823
F-statistic: 19.7 on 1 and 3 DF,  p-value: 0.0213
```

Given the knowledge of the R-output structure, the three values can be read off directly from the output.
So the correct answer is: $\hat{\beta}_0 = 11.7$, $\hat{\beta}_1 = -0.244$ and $SE_{\hat{\sigma}} = \hat{\sigma} = 0.348$.

b) How large a part of the variation in concentration can be explained by the distance?

|||| Solution

The amount of variation in the model output (Y) explained by the variable input (x) can be found from the squared correlation, that can be read off directly from the output as "`Multiple R-squared`". So the correct answer is: $R^2 = 86.8\%$ (it is actually an estimate of the variation in concentration which can be explained by distance, since it is what we found with the particular data at hand. If the sample was taken again, then this value would vary. We should actually calculate a confidence interval for R^2 to understand how accurate this estimate is!).

c) What is a 95%-confidence interval for the expected pollution concentration 7 km from the pollution source?

|||| Solution

The wanted number is estimated by the point on the line (using $x_{\text{new}} = 7$)

$$-0.244 \cdot 7 + 11.664 = 9.96,$$

and the confidence interval is given by

$$9.96 \pm t_{0.025}(3) \cdot \hat{\sigma} \sqrt{\frac{1}{5} + \frac{(7-6)^2}{S_{xx}}},$$

where $S_{xx} = 4^2 + 2^2 + 0^2 + 2^2 + 4^2 = 40$ and $t_{0.025}(3) = 3.182$ (in R: `qt(0.975,3)`) we have that

$$3.182 \cdot 0.348 \sqrt{\frac{1}{5} + \frac{1}{40}} = 0.525,$$

where s_x is:

```
sd(D$distance)

[1] 3.162
```

and thus

$$S_{xx} = (n-1) \cdot s_x^2 = 4 \cdot 3.162^2 = 40.$$

This could also have been found by

```
predict(fit, newdata=data.frame(distance=7), interval="confidence",
        level=0.95)

    fit   lwr   upr
1 9.956 9.431 10.48
```

So the correct answer is:

$$9.96 \pm 0.525 = [9.43, 10.5].$$

5.6 Membrane pressure drop

|||| Exercise 5.6 Membrane pressure drop

When purifying drinking water you can use a so-called membrane filtration. In an experiment one wishes to examine the relationship between the pressure drop across a membrane and the flux (flow per area) through the membrane. We observe the following 10 related values of pressure (x) and flux (y):

	1	2	3	4	5	6	7	8	9	10
Pressure (x)	1.02	2.08	2.89	4.01	5.32	5.83	7.26	7.96	9.11	9.99
Flux (y)	1.15	0.85	1.56	1.72	4.32	5.07	5.00	5.31	6.17	7.04

Copy this into R to avoid typing in the data:

```
D <- data.frame(
  pressure=c(1.02,2.08,2.89,4.01,5.32,5.83,7.26,7.96,9.11,9.99),
  flux=c(1.15,0.85,1.56,1.72,4.32,5.07,5.00,5.31,6.17,7.04)
)
```

a) What is the empirical correlation between pressure and flux estimated to? Give also an interpretation of the correlation.

|||| Solution

The questions are most easily solved by using `lm` in R:

```
D <- data.frame(
  pressure=c(1.02,2.08,2.89,4.01,5.32,5.83,7.26,7.96,9.11,9.99),
  flux=c(1.15,0.85,1.56,1.72,4.32,5.07,5.00,5.31,6.17,7.04)
)
fit <- lm(flux ~ pressure, data=D)
summary(fit)


Call:
lm(formula = flux ~ pressure, data = D)

Residuals:
   Min     1Q Median     3Q    Max 
-0.989 -0.318 -0.140  0.454  1.046 

Coefficients:
            Estimate Std. Error t value  Pr(>|t|)    
(Intercept)  -0.1886     0.4417   -0.43      0.68    
pressure      0.7225     0.0706   10.23 0.0000072 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.645 on 8 degrees of freedom
Multiple R-squared:  0.929,	Adjusted R-squared:  0.92 
F-statistic:  105 on 1 and 8 DF,  p-value: 0.00000718
```

The found coefficient of determination (see Theorem 5.25) can be read off the R output to be 0.929. The sign of the correlation is the same as the sign of the slope, which can be read off to be positive ($\hat{\beta}_1 = 0.7225$), so the correlation is

$$\hat{\rho} = r = \sqrt{0.929} = 0.964.$$

So the empirical correlation is 0.964, and thus flux is found to increase with increasing pressure.

b) What is a 90% confidence interval for the slope β_1 in the usual regression model?

|||| Solution

We use the formula for the slope (β_1, see Method 5.15) confidence interval, and can actually just realize that the correct t-quantile to use is the $t_{1-0.05}(8) = 1.860$ (in R: `qt(0.95,8)`), and the other values we read of the `summary` output.
So the confidence interval is: $0.7225 \pm 1.860 \cdot 0.0706$.

c) How large a part of the flux-variation ($\sum_{i=1}^{10}(y_i - \bar{y})^2$) is not explained by pressure differences?

|||| Solution

The squared correlation, $r^2 = 0.929$ express the explained variation, this means that $1 - 0.929 = 0.071$ express the unexplained variation by the model.

d) Can you at significance level $\alpha = 0.05$ reject the hypothesis that the line passes through $(0,0)$?

|||| Solution

The hypothesis is the same as:

$$H_0 : \beta_0 = 0$$

which is the hypothesis results provided in the output in the `"intercept"` row of `summary`, so: *No, since the relevant p-value is 0.68, which is larger than* α.

e) A confidence interval for the line at three different pressure levels: $x_{\text{new}}^{\text{A}} = 3.5$, $x_{\text{new}}^{\text{B}} = 5.0$ and $x_{\text{new}}^{\text{C}} = 9.5$ will look as follows:

$$\hat{\beta}_0 + \hat{\beta}_1 \cdot x_{\text{new}}^{U} \pm C_U$$

where U then is either A, B or C. Write the constants C_U in increasing order.

|||| Solution

The formula for the Confidence limits of $\alpha + \beta x_{\text{new}}$ includes the following term:

$$\frac{(x_{\text{new}} - \bar{x})^2}{S_{xx}}$$

and this is the ONLY term in C_U that makes C_U different between the three Us. And since $\bar{x} = 5.547$ it is clear that

$$(5.0 - 5.547)^2 < (3.5 - 5.547)^2 < (9.5 - 5.547)^2$$

and hence

$$(x^{\text{B}}_{\text{new}} - 5.547)^2 < (x^{\text{A}}_{\text{new}} - 5.547)^2 < (x^{\text{C}}_{\text{new}} - 5.547)^2$$

So $C_{\text{B}} < C_{\text{A}} < C_{\text{C}}$

5.7 Membrane pressure drop (matrix form)

|||| Exercise 5.7 Membrane pressure drop (matrix form)

This exercise uses the data presented in Exercise 6 above.

a) Find parameters values, standard errors, t-test statistics, and p-values for the standard hypotheses tests.

Copy this into R to avoid typing in the data:

```
D <- data.frame(
  pressure=c(1.02,2.08,2.89,4.01,5.32,5.83,7.26,7.96,9.11,9.99),
  flux=c(1.15,0.85,1.56,1.72,4.32,5.07,5.00,5.31,6.17,7.04)
)
```

|||| Solution

```
D <- data.frame(
  pressure=c(1.02,2.08,2.89,4.01,5.32,5.83,7.26,7.96,9.11,9.99),
  flux=c(1.15,0.85,1.56,1.72,4.32,5.07,5.00,5.31,6.17,7.04)
)
fit <- lm(flux ~ pressure, data=D)
summary(fit)


Call:
lm(formula = flux ~ pressure, data = D)

Residuals:
   Min     1Q Median     3Q    Max
-0.989 -0.318 -0.140  0.454  1.046

Coefficients:
            Estimate Std. Error t value  Pr(>|t|)
(Intercept)  -0.1886     0.4417   -0.43      0.68
pressure      0.7225     0.0706   10.23 0.0000072 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.645 on 8 degrees of freedom
Multiple R-squared:  0.929, Adjusted R-squared:  0.92
F-statistic:  105 on 1 and 8 DF,  p-value: 0.00000718
```

The parameter estimates are given in the first column, the standard errors in the second column, the t-test statistics are given in the third column and the p-values of the standard hypothesis are given in the last column.

b) Reproduce the above numbers by matrix vector calculations. You will need some matrix notation in R:
   - Matrix multiplication (XY): `X%*%Y`
   - Matrix transpose (X^T): `t(X)`
   - Matrix inverse (X^{-1}): `solve(X)`
   - Make a matrix from vectors ($X = [x_1^T; x_2^T]$): `cbind(x1,x2)`

   See also Example 5.24.

|||| Solution

```
X <- cbind(1, D$pressure)
y <- D$flux
n <- length(y)
beta <- solve(t(X) %*%X ) %*% t(X) %*% y
beta

        [,1]
[1,] -0.1886
[2,]  0.7225


e <- y - X %*% beta
s <- sqrt(sum(e^2)/(n-2))
Vbeta <- s^2 * solve(t(X) %*%X )
se.beta <- sqrt(diag(Vbeta))
t.obs <- beta / se.beta
p.value <- 2 * (1 - pt(abs(t.obs), df = n-2))

## Collection in a table
analasis.table <- cbind(beta, se.beta, t.obs, p.value)
analasis.table

             se.beta
[1,] -0.1886 0.44171 -0.4269 0.680696710
[2,]  0.7225 0.07064 10.2269 0.000007177


## Put some names on our table
colnames(analasis.table) <- c("Estimates","Std.Error","t.obs","p.value")
rownames(analasis.table) <- c("beta1","beta2")
analasis.table

      Estimates Std.Error   t.obs     p.value
beta1   -0.1886   0.44171 -0.4269 0.680696710
beta2    0.7225   0.07064 10.2269 0.000007177


## Done!!
```

5.8 Independence and correlation

Exercise 5.8 Independence and correlation

Consider the layout of independent variable in Example 5.11,

a) Show that $S_{xx} = \frac{n\cdot(n+1)}{12\cdot(n-1)}$.

Hint: you can use the following relations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

$\bar{x}$ becomes

$$\begin{aligned}\bar{x} &= \frac{1}{n}\sum_{i=1}^{n}\frac{i-1}{n-1} = \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)\\ &= \frac{1}{n(n-1)}\left(\frac{n(n+1)}{2} - n\right) = \frac{1}{2},\end{aligned}$$

and S_{xx} becomes

$$\begin{aligned}S_{xx} &= \sum_{i=1}^{n}\left(\frac{i-1}{n-1} - \frac{1}{2}\right)^2\\ &= -\frac{n}{4} + \frac{1}{(n-1)^2}\sum_{i=1}^{n}(i^2+1-2i)\\ &= -\frac{n}{4} + \frac{1}{(n-1)^2}\left(\frac{n(n+1)(2n+1)-6n^2}{6}\right)\\ &= \frac{n}{(n-1)^2}\left(\frac{4n^2+6n+2-12n-3(n-1)^2}{12}\right)\\ &= \frac{n}{(n-1)^2}\left(\frac{n^2-1}{12}\right) = \frac{n(n+1)}{12(n-1)}.\end{aligned}$$

b) Show that the asymptotic correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$ is

$$\lim_{n\to\infty} \rho_n(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\sqrt{3}}{2}.$$

|||| Solution

The correlation between $\hat{\beta}_0$ and $\hat{\beta}_0$ is

$$\begin{aligned}
\rho_n(\hat{\beta}_0, \hat{\beta}_1) &= \frac{\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1)}{\sqrt{\mathrm{V}(\hat{\beta}_0)\,\mathrm{V}(\hat{\beta}_1)}} \\
&= -\frac{\sigma^2 \bar{x}/S_{xx}}{\sqrt{\sigma^4\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\frac{1}{S_{xx}}}} \\
&= -\frac{\bar{x}/S_{xx}}{\frac{1}{S_{xx}}\sqrt{\left(\frac{S_{xx}}{n} + \bar{x}^2\right)}} \\
&= -\frac{\bar{x}}{\sqrt{\frac{S_{xx}}{n} + \bar{x}^2}}.
\end{aligned}$$

Notice that the correlation is not a function of the variance (σ^2), but only a function of the independent variables. Now insert the values of $\bar{x}$ and S_{xx}

$$\begin{aligned}
\rho_n(\hat{\beta}_0, \hat{\beta}_1) &= -\frac{1}{2\sqrt{\frac{n+1}{12(n-1)} + \frac{1}{4}}} = -\frac{1}{2\sqrt{\frac{n+1+3(n-1)}{12(n-1)}}} \\
&= -\frac{1}{2\sqrt{\frac{2n-1}{6(n-1)}}} = -\frac{\sqrt{6(n-1)}}{2\sqrt{2n-1}} \\
&= -\frac{1}{2}\sqrt{\frac{6(n-1)}{2(n-1/2)}} = -\frac{\sqrt{3}}{2}\sqrt{\frac{n-1}{n-1/2}}
\end{aligned}$$

.

which converges to $-\frac{\sqrt{3}}{2}$ for $n \to \infty$.

Consider a layout of the independent variable where $n = 2k$ and $x_i = 0$ for $i \leq k$ and $x_i = 1$ for $k < i \leq n$.

c) Find S_{xx} for the new layout of x.

|||| Solution

$$\bar{x} = \frac{1}{2},$$

and

$$\begin{aligned} S_{xx}^{\text{new}} &= \sum_{i=1}^{k} \left(0 - \frac{1}{2}\right)^2 + \sum_{i=k+1}^{2k} \left(1 - \frac{1}{2}\right)^2 \\ &= \frac{k}{4} + \frac{k}{4} = \frac{k}{2} = \frac{n}{4}. \end{aligned}$$

d) Compare S_{xx} for the two layouts of x.

|||| Solution

$$\frac{S_{xx}}{S_{xx}^{\text{new}}} = \frac{n(n+1)}{12(n-1)} \frac{4}{n} = \frac{(n+1)}{3(n-1)} < 1; \quad for \quad n > 2$$

which imply that $S_{xx}^{\text{new}} > S_{xx}$ for all $n > 2$.

e) What is the consequence for the parameter variance in the two layouts?

|||| Solution

The larger S_{xx} for the new layout imply that the parameter variance is smaller for the new layout (given that data comes from the same model).

f) Discuss pro's and cons for the two layouts.

|||| Solution

The smaller parameter variance for the new layout would suggest that we should use this layout. However, we would not be able to check that data is in fact generated by a linear model. Consider e.g. data generated by the model

$$y_i = \beta_0 + \beta_1 x_i^2 + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2),$$

if we only look at $x_i = 0$ or $x_i = 1$ we will not be able to detect that the relationship is in fact non-linear.