

1 Divergence

The following is a slight adaptation of the results of Wittwer, Pan Shi, and Hassani.

We consider the equation $u_{tt} = (u_x)^2$ on the real line. We start by writing the solution in the form

$$u(x, t) = f(x) + g(x)t + \int_0^t d\tau \int_0^\tau d\tau' (u_x(x, \tau'))^2 . \quad (1.1)$$

This corresponds to the initial conditions

$$u(x, 0) = f(x) , \quad u_t(x, 0) = g(x) .$$

We will consider the case that $f'(0) = g'(0) = 0$, and we ask how the solution behaves near $x = 0$. Depending on the curvatures $f''(0)$ and $g''(0)$, the second derivative $u_{xx}(x, t)$ will, or will not diverge at $x = 0$. Of course, if the functions f and g have vanishing derivatives at some other point(s) x_0 , the same discussion will apply at those points, and there can be one of these points where $u_{xx}(x_0, t)$ diverges before the one at $x = 0$. In the following proposition, we will neglect this aspect.

Proposition 1.1. *Assume $f'(0) = g'(0) = 0$. Define*

$$c = \frac{1}{2}g''(0)^2 - \frac{2}{3}f''(0)^3 .$$

Then the following cases appear:

(i) *If $g''(0) > 0$ then $u_{xx}(0, t)$ diverges in finite time t_* given by*

$$t_* = \int_{f''(0)}^\infty \frac{db}{\sqrt{\frac{4}{3}b^3 + 2c}} . \quad (1.2)$$

(ii) *If $g''(0) < 0$ then $u_{xx}(0, t)$ will converge to b_* in a finite time t_* , where*

$$\frac{2}{3}b_*^3 = -c , \text{ and } t_* = \int_{b_*}^{f''(0)} \frac{db}{\sqrt{\frac{4}{3}b^3 + 2c}} . \quad (1.3)$$

(iii) *If $g''(0) = 0$, and $f''(0) \neq 0$ then $u_{xx}(0, t)$ diverges in finite time t_* given again by (1.2).*

(iv) *If $g''(0) = 0$ and $f''(0) = 0$ then $u_{xx}(0, t)$ stays constant.*

Remark 1.2. The divergence and limit times above are standard elliptic integrals. When $g''(0) = 0$, the finite divergence time t_* scales like $\mathcal{O}(|f''(0)|^{-1/2})$.

Remark 1.3. A case of a certain interest in cosmology appears when $f''(0) = 0$ and $g''(0) > 0$. In that case, $t_* \sim 2.547/(g''(0))^{1/6}$.

Proof. Define

$$a(t) = u_x(0, t) , \quad b(t) = u_{xx}(0, t) .$$

Then (1.1) leads to

$$a(t) = f'(0) + g'(0)t + 2 \int_0^t d\tau \int_0^\tau d\tau' u_x(0, \tau') u_{xx}(0, \tau') ,$$

and therefore

$$\ddot{a}(t) = 2a(t)b(t) . \quad (1.4)$$

Similarly,

$$b(t) = f''(0) + g''(0)t + 2 \int_0^t d\tau \int_0^\tau d\tau' \left((u_{xx}(0, \tau'))^2 + u_x(0, \tau')u_{xxx}(0, \tau') \right) ,$$

From this, we deduce

$$\begin{aligned} \ddot{b}(t) &= 2(u_{xx}(0, t))^2 + 2u_x(0, t)u_{xxx}(0, t) \\ &= 2b(t)^2 + 2a(t)u_{xxx}(0, t) . \end{aligned} \quad (1.5)$$

Since we assume $f'(0) = g'(0) = 0$ we find from (1.4) that $a(t) = 0$ for all t for which $b(t)$ is finite. Therefore, (1.5) reduces to

$$\ddot{b}(t) = 2(b(t))^2 . \quad (1.6)$$

We will discuss this equation. For computing the divergence time, it is useful to transform the equation as follows: Multiplying by \dot{b} leads to

$$\frac{1}{2} \frac{d}{dt} (\dot{b}(t))^2 = \frac{2}{3} \frac{d}{dt} b(t)^3 ,$$

or, for some c ,

$$\frac{1}{2} (\dot{b}(t))^2 = \frac{2}{3} (b(t))^3 + c , \quad (1.7)$$

Note that looking at $t = 0$ we find

$$c = \frac{1}{2} \dot{b}(0)^2 - \frac{2}{3} b(0)^3 = \frac{1}{2} g''(0)^2 - \frac{2}{3} f''(0)^3 , \quad (1.8)$$

which is the definition in the proposition.

We consider first the case where $g''(0) > 0$. Then $\dot{b}(0) > 0$ and from (1.7) we find that

$$\dot{b}(t) = \sqrt{\frac{4}{3} (b(t))^3 + 2c} , \quad (1.9)$$

which means b is increasing. Using standard techniques, we get

$$dt = \frac{db}{\sqrt{\frac{4}{3} b^3 + 2c}} .$$

From (1.9) we deduce the divergence time t_* ,

$$t_* = \int_{b(0)}^{\infty} \frac{db}{\sqrt{\frac{4}{3} b^3 + 2c}} . \quad (1.10)$$

This proves (1.2).

The case $g''(0) < 0$ is handled similarly, but now (1.9) is replaced by

$$\dot{b}(t) = -\sqrt{\frac{4}{3} (b(t))^3 + 2c} , \quad (1.11)$$

This means that b is decreasing until the square root in (1.11) vanishes. This defines b_* , and then (1.10) is replaced by

$$t_* = \int_{b_*}^{f''(0)} \frac{db}{\sqrt{\frac{4}{3}b^3 + 2c}} .$$

This leads to (1.3).

The assertions under (iii) are a simple variant of (i) and (ii). The difference is that because $g''(0) = 0$, we find now that $c = -\frac{2}{3}f''(0)^3$, and $c \neq 0$ by the assumption $f''(0) \neq 0$. The only remaining case is (iv), $g''(0) = f''(0) = 0$, which directly leads to $b(t) = b(0)$ by (1.6). \square