1 Context

In the effective field theory (EFT) of cosmology, it has been discovered in [xxx] that a field, called Π, obeys (among many other terms) a differential equation of the form

$$\partial_t^2\Pi = \alpha(\nabla_x\Pi)^2 + \beta\Delta\Pi \ .$$

The initial conditions for this equation are given by a source field Φ, which is generated by xxx. They are then of the form $\Pi(x, 0) = 0$, while $\partial_t\Pi(x, 0)$ is some non-zero function.

It was then discovered in [?] that such initial conditions can lead to a blowup in finite time, and that the relation between α and β (called the speed of sound) leads to a subtle interplay between this divergence, while for $\beta \neq 0$ the "sound wave" can lead to another type of divergence.

The aim of this paper is to discuss a 1-dimensional version of this problem, and to give some proofs and some general remarks about this phenomenon, which is important in cosmology because xxx.

Admittedly, while this is a modest start, it should be relatively easy to generalize this to a spherically symmetric situation.

2 The equations

We consider the equations

$$u_{tt} - \beta u_{xx} = \alpha(u_x)^2 \ , \quad \alpha \neq 0 \ , \tag{2.1}$$

for $t \geq 0$ and $x \in \mathbb{R}$. When $\beta \neq 0$ we define $u(x, t) = \frac{\beta}{\alpha}v(\sqrt{|\alpha|x/\beta}, |\alpha|t/\beta)$ and obtain

$$v_{tt} - v_{xx} = (v_x)^2 \ . \tag{2.2}$$

Thus, it suffices to consider (2.2).
When $\beta = 0$, we consider the equation

$$w_{tt} = (w_x)^2 \ . \tag{2.3}$$

Equations like (2.1)-(2.3) have been studied in great detail, but mostly in dimensions 2 and higher. Here, we are interested in the questions of divergence of such equations (in finite time). Such questions are of interest in cosmological models. In that case, it is probably more reasonable to keep (2.1), since the change of variables leading to (2.2) will of lead to a change of the initial conditions. But in cosmology, these initial conditions are fixed.

The behavior if the solutions can be quite different in the case $\beta = 0$ and $\beta \neq 0$. When $\beta = 0$, there is no wave equation and therefore we are confronted with what is essentially a set of local problems.

In the next section we discuss $\beta = 0$ and after that, we discuss $\beta \neq 0$.

3 The case $\beta = 0$

The following is a slight adaptation of the results of Wittwer, Pan Shi, and Hassani.

We consider the equation $u_{tt} = (u_x)^2$ on the real line. We start by writing the solution in the form

$$u(x, t) = f(x) + g(x)t + \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'(u_x(x, \tau'))^2 \ . \tag{3.1}$$

This corresponds to the initial conditions

$$u(x,0)=f(x)\ ,\quad u_t(x,0)=g(x)\ .$$

We will consider the case where $f'(0)=g'(0)=0$, and we ask how the solution behaves near $x=0$. Depending on the curvatures $f''(0)$ and $g''(0)$, the second derivative $u_{xx}(x,t)$ will, or will not diverge at $x=0$. Of course, if the functions f and g have vanishing derivatives at some other point(s) x_0, the same discussion will apply at those points, and there can be one of these points where $u_{xx}(x_0,t)$ diverges before the one at $x=0$. In the following proposition, we will neglect this aspect.

Proposition 3.1. *Assume $f'(0)=g'(0)=0$. Define*

$$c=\tfrac{1}{2}g''(0)^2-\tfrac{2}{3}f''(0)^3\ .$$

Then the following cases appear:
(i) If $g''(0)>0$ then $u_{xx}(0,t)$ diverges in finite time t_+ given by

$$t_+=\int_{f''(0)}^{\infty}\frac{\mathrm{d}b}{\sqrt{\frac{4}{3}b^3+2c}}\ . \tag{3.2}$$

(ii) If $g''(0)<0$ then $u_{xx}(0,t)$ will converge to b_* in a finite time t_-, where

$$\tfrac{2}{3}b_*^3=-c\ ,\ \textit{and}\ t_-=\int_{b_*}^{f''(0)}\frac{\mathrm{d}b}{\sqrt{\frac{4}{3}b^3+2c}}\ . \tag{3.3}$$

At this point in time, we will have $u_t(0,t_-)=0$ which corresponds to (iii) and the solution will diverge after another finite time t_+ (unless $u_{xx}(0,t_-)=0$).
(iii) If $g''(0)=0$, and $f''(0)\neq 0$ then $u_{xx}(0,t)$ diverges in finite time t_+ given again by (3.2).
(iv) If $g''(0)=0$ and $f''(0)=0$ then $u_{xx}(0,t)$ stays constant.

Remark 3.2. The divergence and limit times above are standard elliptic integrals. When $g''(0)=0$, the finite divergence time t_+ scales like $\mathcal{O}(|f''(0)|^{-1/2})$.

Remark 3.3. Assume that $u(x,0)=0$ and $u_t(x,0)$ is a smooth, bounded, functions with well-separated extrema. Such initial conditions are typical for questions in cosmology. In this case, the blowup will happen first in that point x_0 for which $C\equiv u_{txx}(x_0,0)$ is maximal. In that case, $t_+\sim 2.547/C^{1/6}$.

Remark 3.4. Note that t_++t_- is the total time for an initial condition $g''(0)<0$ to diverge.

Proof. Define

$$a(t)=u_x(0,t)\ ,\quad b(t)=u_{xx}(0,t)\ .$$

Then (3.1) leads to

$$a(t)=f'(0)+g'(0)t+2\int_0^t\mathrm{d}\tau\int_0^\tau\mathrm{d}\tau' u_x(0,\tau')u_{xx}(0,\tau')\ ,$$

and therefore

$$\ddot{a}(t)=2a(t)\,b(t)\ . \tag{3.4}$$

Similarly,

$$b(t) = f''(0) + g''(0)t + 2\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \left((u_{xx}(0,\tau'))^2 + u_x(0,\tau')u_{xxx}(0,\tau') \right) ,$$

From this, we deduce

$$\begin{aligned} \ddot{b}(t) &= 2(u_{xx}(0,t))^2 + 2u_x(0,t)u_{xxx}(0,t) \\ &= 2b(t)^2 + 2a(t)u_{xxx}(0,t) . \end{aligned} \tag{3.5}$$

Since we assume $f'(0) = g'(0) = 0$ we find from (3.4) that $a(t) = 0$ for all t for which $b(t)$ is finite. Therefore, (3.5) reduces to

$$\ddot{b}(t) = 2(b(t))^2 . \tag{3.6}$$

We will discuss this equation. For computing the divergence time, it is useful to transform the equation as follows: Multiplying by $\dot{b}$ leads to

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{b}(t))^2 = \tfrac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}b(t)^3 ,$$

or, for some c,

$$\tfrac{1}{2}(\dot{b}(t))^2 = \tfrac{2}{3}(b(t))^3 + c , \tag{3.7}$$

Note that looking at $t = 0$ we find

$$c = \tfrac{1}{2}\dot{b}(0)^2 - \tfrac{2}{3}b(0)^3 = \tfrac{1}{2}g''(0)^2 - \tfrac{2}{3}f''(0)^3 , \tag{3.8}$$

which is the definition in the proposition.

We consider first the case where $g''(0) > 0$. Then $\dot{b}(0) > 0$ and from (3.7) we find that

$$\dot{b}(t) = \sqrt{\tfrac{4}{3}(b(t))^3 + 2c} , \tag{3.9}$$

which means b is increasing. Using standard techniques, we get

$$\mathrm{d}t = \frac{\mathrm{d}b}{\sqrt{\tfrac{4}{3}b^3 + 2c}} .$$

From (3.9) we deduce the divergence time t_+,

$$t_+ = \int_{b(0)}^{\infty} \frac{\mathrm{d}b}{\sqrt{\tfrac{4}{3}b^3 + 2c}} . \tag{3.10}$$

This proves (3.2).

The case $g''(0) < 0$ is handled similarly, but now (3.9) is replaced by

$$\dot{b}(t) = -\sqrt{\tfrac{4}{3}b(t)^3 + 2c} , \tag{3.11}$$

This means that b is decreasing until the square root in (3.11) vanishes. This defines b_*, and then (3.10) is replaced by

$$t_- = \int_{b_*}^{f''(0)} \frac{\mathrm{d}b}{\sqrt{\tfrac{4}{3}b^3 + 2c}} .$$

This leads to (3.3).

The assertions under (iii) are a simple variant of (i) and (ii). The difference is that because $g''(0) = 0$, we find now that $c = -\frac{2}{3}f''(0)^3$, and $c \neq 0$ by the assumption $f''(0) \neq 0$. The only remaining case is (iv), $g''(0) = f''(0) = 0$, which directly leads to $b(t) = b(0)$ by (3.6). □

3.1 Some illustrations for the case $\beta = 0$

The two figures Fig. 1 and 2 illustrate the typical behavior for the case when of an initial condition with positive curvature at 0, corresponding to the case (i) of the proposition. $u(x, 0) = 0$ and

$$u_t(x, 0) = \begin{cases} -((1 + \cos(x))/2)^2, & \text{for } x \in [-\pi, \pi] , \\ 0 , & \text{otherwise.} \end{cases}$$

The x-scale of the two figures is arbitrary.

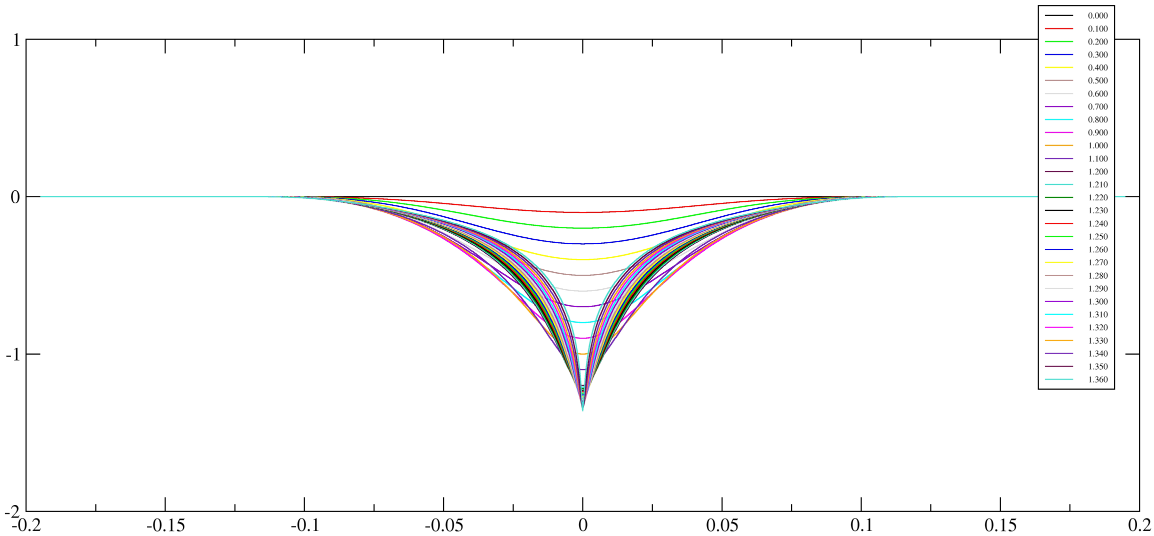

Figure 1: Local divergence: Shown is $u(x, t)$ as a function of time, up to divergence.

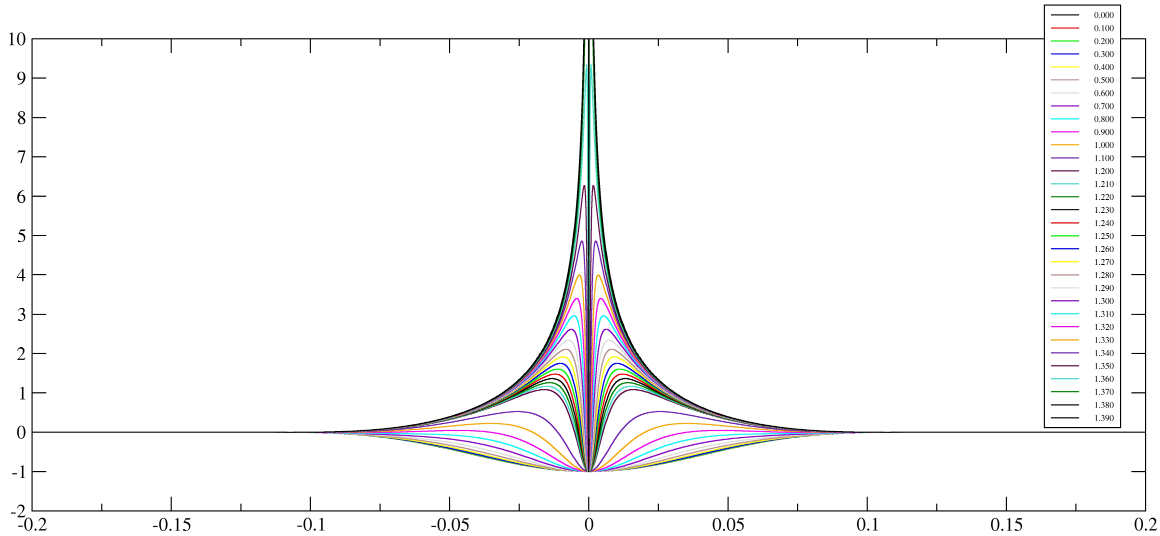

Figure 2: Local divergence: Shown is $u_t(x,t)$ as a function of time, up to divergence.

4 The case $\beta \neq 0$

When $\beta \neq 0$ we are interested in the interplay of two types of divergence, *local* divergence, and *wave divergence*. In the local divergence, u will diverge near $x = 1$ like a special solution of (2.3). In the wave divergence, there will be a divergence near the front of the advancing wave.

We illustrate the phenomenology with two numerical simulations, Fig. 3 and Fig. 4.

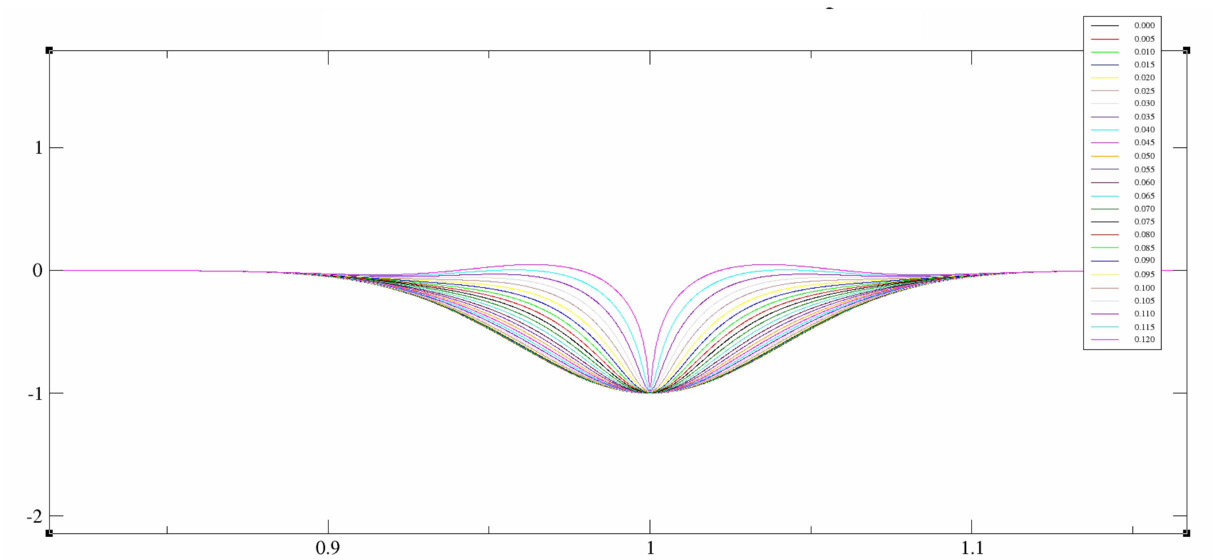

Figure 3: Local divergence:The parameters of (2.1) are $\alpha = 5$ and $\beta = 0.025$ with initial condition $u(x,0) = -\exp(-30(x-1)^2)$, $u_t(x,0) = 0$. (Numerically we take periodic boundary conditions.) Note that the second derivative diverges in finite time. The simulation corresponds to $\alpha/\beta = 200$.

We expect that, for fixed α/β a transition between local divergence and wave divergence will appear, when the initial condition is fixed. The mathematical situation for the wave divergence can be explained by an adaptation of the work of Rammaha [rammaha]. He shows that (2.2)

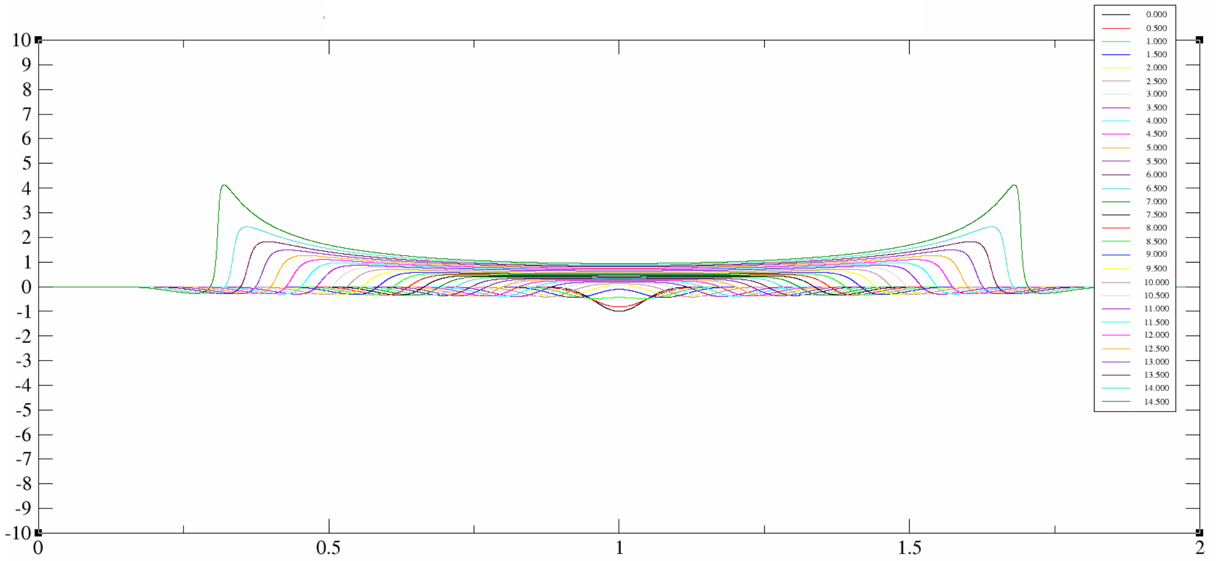

Figure 4: Wave divergence:The nature of divergence "at infinity." As time goes on, the support of the function spreads, and the function at the edge gets steeper, until the derivative will diverge at some time T_*. The simulation is for the variant equation (2.3), with $\alpha = 0.01$ and $\beta = 0.025$ and initial condition $u(x, 0) = -\exp(-30(x-1)^2)$, $u_x(x, 0) = 0$. (Numerically we take periodic boundary conditions.) Now, $\alpha/\beta = 0.4$

will diverge in finite time under quite weak conditions: We adapt the proof to the 1d case in Theorem 5.1, which will be given at the end.

Take as initial conditions smooth functions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, both with compact support and assume the support is in $|x| < X$. Fix an $X_0 < X$ and define

$$\varepsilon = \tfrac{1}{2} \int_{x>X_0} (f(x) + (x - X_0)g(x))\mathrm{d}x \ .$$

with

$$M(x) = \tfrac{1}{2} \left(f(x) + \int_x^X g(\xi)\mathrm{d}\xi \right) > 0$$

for all $x \in (X_0, X)$. We show that the solution must diverge before a time T_*. We do not aim to get the best possible bound on this time.

Rammaha's result can be paraphrased as follows. The point of the proofs is that because of the wave character of (2.1), the solution will grow at the advancing front of the wave. The qualitative behavior is clearly seen in Fig. 4.

This discussion settles the question of what must happen at the advancing fronts. However, there can be a local divergence which can appear earlier in the "center" between the waves.

We have considered the shape of the exploding wave front. While we have no theoretical insights, perhaps the following remarks can help future research: It seems that the exploding front has a universal shape, at least near the maximum of the exploding profile. However, extensive numerical tests show that while these profiles rescale properly in each instance of the equation, the profiles for different α/β and fixed initial condition seem not to be the same. We tested the "usual" rescaling ideas, but they manifestly do not work.

5 Wave divergence

Consider the equation

$$u_{tt} - u_{xx} = u_x^2 \ .$$

Assume the initial conditions are

$$u(x,0) = f(x) \ , \quad u_t(x,0) = g(x) \ ,$$

with f, g having support in $|x| < X$. Define, for $x > 0$,

$$M(x) = \tfrac{1}{2} f(x) + \tfrac{1}{2} \int_x^X g(\xi) \mathrm{d}\xi \ .$$

Theorem 5.1. *Assume there is an $X_0 \in (0, X)$ for which $M(x) \geq 0$ for all $x \in (X_0, X)$, and also*

$$\int_{X_0}^X M(\xi) \mathrm{d}\xi \equiv \varepsilon > 0 \ .$$

Then the local existence time is **finite***.*

Proof. It suffices to consider the situation where $u(x,0)$ and $u_t(x,0)$ have support in $|x| \leq X$, but we will consider only the side of positive x in the sequel. Clearly, $u(x,t) = 0$ for $x \geq t+X$. One estimates now the function

$$H(t) = \int_{X_1}^t (t-\tau) \int_{\tau+X_0}^{\tau+X} u(\xi,\tau) \mathrm{d}\xi \, \mathrm{d}\tau \ .$$

Here, $X_1 = (X - X_0)/2$. From the definition, we get

$$H''(t) = \int_{t+X_0}^{t+X} u(\xi,t) \mathrm{d}\xi \ . \tag{5.1}$$

One has the explicit formula

$$u(x,t) = u_0(x,t) + \tfrac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} u_x(\xi,\tau)^2 \mathrm{d}\tau \, \mathrm{d}\xi \ , \tag{5.2}$$

with the "free evolution"

$$u_0(x,t) = \tfrac{1}{2} \left(f(x-t) + f(x+t) \right) + \tfrac{1}{2} \int_{x-t}^{x+t} g(\xi) \mathrm{d}\xi \ .$$

When $x \geq t + X_0$ and $X \geq t \geq X_1$, then $x + t \geq X$ and therefore $f(x+t) = 0$ and therefore, in this region,

$$u_0(x,t) = \tfrac{1}{2} f(x-t) + \tfrac{1}{2} \int_{x-t}^{x+t} g(\xi) \mathrm{d}\xi \ .$$

We get from (5.2) and (5.1),

$$H''(t) = G_0(t) + G_1(t) \ ,$$

with

$$G_0(t) = \int_{t+X_0}^{t+X} u_0(x,t) \mathrm{d}x = \int_{t+X_0}^{t+X} M(x-t) \mathrm{d}x = \int_{X_0}^X M(x) \mathrm{d}x = \varepsilon \ ,$$

by the definition of ε in Theorem 5.1. The nonlinearity leads to

$$G_1(t) = \int_{t+X_0}^{t+X} \mathrm{d}\xi \int_0^t \mathrm{d}\tau \int_{\xi-t+\tau}^{\xi+t-\tau} \mathrm{d}\eta \, u_x(\eta,\tau)^2 \ .$$

In Lemma 5.2 below, we show

$$G_1(t) \geq \frac{1}{t+X_0} \int_{X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi \, (t-\tau)\,(\xi-\tau-X_0)\, u_x(\xi,\tau)^2 \,. \tag{5.3}$$

We use now the Schwarz inequality in the form

$$\int \varphi\psi = \int \varphi^{1/2}(\varphi^{1/2}\psi) \leq \Big(\int \varphi\psi^2\Big)^{1/2} (\int \varphi)^{1/2} \,,$$

with $\varphi = (t-\tau)(\xi-\tau-X_0)$ and $\psi = u_x$. This leads to

$$G_1(t) \geq F^2(t)/J(t) \,,$$

with

$$F(t) = \int_{X_1}^{t} \int_{\tau+X_0}^{\tau+X} (t-\tau)(\xi-\tau-X_0) u_x(\xi,\tau) \mathrm{d}\xi \, \mathrm{d}\tau$$

and

$$J(t) = \int_{X_1}^{t} \int_{\tau+X_0}^{\tau+X} (t-\tau)(\xi-\tau-X_0) \mathrm{d}\tau \, \mathrm{d}\xi = \frac{(X-X_0)^2(t-X_1)^2}{4} \,.$$

If we integrate the expression for F by parts (in ξ), we get

$$F(t) = -\int_{X_1}^{t} \int_{\tau+X_0}^{\tau+X} (t-\tau) u(\xi,\tau) \mathrm{d}\xi \, \mathrm{d}\tau = -H(t) \,.$$

Therefore, we find finally

$$H''(t) \geq G_0(t) + \frac{H(t)^2}{J(t)} \,. \tag{5.4}$$

Fix now T and we will show that the solution cannot exist for $T > T_*$, where T_* will be computed in the proof: We use here Lemma 1 from [rammaha2] adapted to the 1d case. The ingredients are that

$$H''(t) \geq G_0(t) = \varepsilon > 0 \,, \tag{5.5}$$

for all $t \geq 0$ and

$$H''(t) \geq G_1(t) \geq 4 \frac{H(t)^2}{(X-X_0)^2(t-X_1)^2} \,, \tag{5.6}$$

for $t > X_1$ (as long as the solution exists). Furthermore, $H(X_1) = H'(X_1) = 0$.

Fix now $T_1 = 2(X_1+1)$. Then, for $t > T_1$, we have $t - X_1 > \frac{1}{2}(t+1)$, and we replace from now on (5.6) by the simpler

$$H''(t) \geq K_1 \frac{H(t)^2}{(t+1)^2} \,, \text{ for } t > T_1 \,, \tag{5.7}$$

with $K_1 = 16/(X-X_0)^2$.

The idea is now to deduce from (5.5) and (5.7) an inequality of the form

$$H'(t) \geq C H^{1+\delta}(t) \text{ for } t > T_* \text{ with } \delta > 0 \,. \tag{5.8}$$

This implies divergence in finite time, when $H(T_0) > 0$. Indeed, if $H(T_0) = c^{-1/\delta} > 0$, then

$$H(t) = \frac{1}{(c - C\delta(t-T_0))^{1/\delta}} \,. \tag{5.9}$$

One can reformulate this as follows: If $H(T_0) = A$ and $A \leq 1/e$, the optimizing δ in (5.9) is ≤ 1 and therefore we find that the divergence time is proportional to $-\log(A)$. Note that, if, for example, the leading edge of the support (at $x = 0$) is like $|x|^2$ for $x < 0$, then this will lead to earlier divergence compared to $|x|^3$.

We now begin the proof proper. If $B > 0$, we will use repeatedly the inequality

$$\frac{x}{x+B} \geq \tfrac{1}{2} \text{ , for all } x \geq B \text{ .} \tag{5.10}$$

From (5.5) we find

$$H(t) \geq K_2 \varepsilon t^2 \text{ , for all } t > 0 \text{ ,} \tag{5.11}$$

with $K_2 = \frac{1}{2}$.

Substituting (5.11) into (5.7), we get

$$H''(t) \geq K_1 K_2 \varepsilon H(t) \frac{t^2}{(t+1)^2} \geq K_3 \varepsilon H(t) \text{ , when } t > T_2 \text{ ,} \tag{5.12}$$

for some large enough $T_2 = \text{const.}T_1$, not depending on ε. Since $H'(t) > 0$, we can multiply (5.12) by H' and write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}(H'(t)^2) \geq K_3 \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(H(t)^2) \text{ when } t > T_2 \text{ .}$$

We integrate from T_2 to t and obtain

$$H'(t)^2 \geq K_3 \varepsilon \left(H(t)^2 + H'(T_2)^2 - H(T_2)^2 \right) = K_3 \varepsilon H(t)^2 + K_4 \varepsilon \text{ when } t > T_2 \text{ ,}$$

for some K_4. From (5.5), we conclude that for large enough T_3, one has

$$K_3 \varepsilon H(t)^2 + K_4 \varepsilon \geq K_5 \varepsilon H(t)^2 \text{ , when } t > T_3 \text{ .}$$

Combining the last two equations we find

$$H'(t) \geq K_5^{1/2} \varepsilon^{1/2} H(t), \text{ when } t > T_3 \text{ .}$$

Integrating from T_3 to t this leads to

$$H(t) \geq H(T_3) \exp\left(K_6 \varepsilon^{1/2} (t - T_3) \right) \geq H(T_3) \exp\left(\tfrac{1}{2} K_6 \varepsilon^{1/2} t \right) \text{ , when } t > 2T_3, \tag{5.13}$$

with $K_6 = K_5^{1/2}$. Substituting again into (5.7), we get

$$H''(t) \geq K_7 H(t)^{1+\delta} \text{ , for any } \delta > 0 \text{ ,}$$

since the exponential in (5.13) (to the power $\delta > 0$) will dominate the factor $(t+1)^{-2}$ of (5.7), if only t is sufficiently large. (Note that K_7 and this new minimal time T_4 will depend on δ.) We now multiply the last equation by H' and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(H'(t)^2) \geq \frac{2K_7}{2+\delta} \frac{\mathrm{d}}{\mathrm{d}t} \left(H(t)^{2+\delta} \right) \text{ , for } t > T_4 \text{ .}$$

Integrating from T_4 to t we find

$$H'(t)^2 \geq \frac{2K_1}{2+\delta} \left(H(t)^{2+\delta} - H(T_4)^{2+\delta} \right) + H'(T_4)^2 \text{ .}$$

Taking square roots on both sides and choosing T_* sufficiently larger than T_4, we finally arrive at (5.8) from which we see that there is divergence in finite time, as in (5.9). $\square$

We still need to show the inequality (5.3).

Lemma 5.2. *Let*

$$G_1(t) = \int_{t+X_0}^{t+X} \mathrm{d}x \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \,.$$

Let $X > X_0 > 0$, and assume $u_x(x,t) = 0$ for all $|x| \geq t + X$, $0 \leq t \leq T$. Then one has for all $T \geq t \geq X_1 = (X - X_0)/2$, the inequality

$$G_1(t) \geq \frac{1}{t+X_0} \int_{X_1}^t \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, (t-\tau)\,(\xi-\tau-X_0)\, u_x(\xi,\tau)^2 \,. \tag{5.14}$$

Proof.

$$\begin{aligned} G_1(t) &= \int_{t+X_0}^{t+X} \mathrm{d}x \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \\ &= \int_0^t \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \,. \end{aligned}$$

Note that for $t \geq X_1$, the function $u_x(\xi,\tau)$ vanishes for $\xi \geq \tau + X$. Therefore, since also $2t - \tau \geq \tau$, we find

$$\begin{aligned} &\int_0^{t-X_1} \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \\ &= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{2t+X-\tau} \mathrm{d}\xi \int_{t+X_0}^{\xi+t-\tau} \mathrm{d}x\, u_x(\xi,\tau)^2 \\ &= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1\,\mathrm{d}x \,. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_{t-X_1}^t \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \\ &= \int_{t-X_1}^t \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1\mathrm{d}x \\ &\quad + \int_{t-X_1}^t \mathrm{d}\tau \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{\xi-t+\tau}^{\xi+t-\tau} 1\mathrm{d}x \,. \end{aligned}$$

Therefore, when $t \geq X_1$ the function G_1 can be decomposed as

$$\begin{aligned} G_1(t) &= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1\mathrm{d}x \\ &\quad + \int_{t-X_1}^t \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1\mathrm{d}x \\ &\quad + \int_{t-X_1}^t \mathrm{d}\tau \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{\xi-t+\tau}^{\xi+t-\tau} 1\mathrm{d}x \,. \end{aligned}$$

The integrals over $\mathrm{d}x$ lead to

$$
\begin{aligned}
G_1(t) = &\int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi \; (\xi - \tau - X_0)\, u_x(\xi,\tau)^2 \\
&+ \int_{t-X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi \, (\xi - \tau - X_0)\, u_x(\xi,\tau)^2 \\
&+ \int_{t-X_1}^{t} \mathrm{d}\tau \, 2(t-\tau) \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi \; u_x(\xi,\tau)^2 \; .
\end{aligned}
$$

Since

$$
t - \tau \leq t + X_0 \; , \text{ and } \; \xi - \tau - X_0 \leq t + X_0 \; ,
$$

and both are positive, we find finally that

$$
G_1(t) \geq \frac{1}{t+X_0} \int_{X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi \, (t-\tau)\, (\xi - \tau - X_0)\, u_x(\xi,\tau)^2 \; .
$$

$\square$