We consider the equations

$$u_{tt} - \beta u_{xx} = \alpha (u_x)^2 \ , \tag{0.1}$$

for $t \geq 0$ and $x \in \mathbb{R}$. When $\beta \neq 0$ we can change variables to obtain a normalized equation

$$v_{tt} - v_{xx} = \frac{\alpha}{\beta}(v_x)^2 \ , \tag{0.2}$$

with $v(x,t) = u(x/\sqrt{\beta}, t)$. When $\beta = 0$, we consider the equation

$$w_{tt} = \alpha (w_x)^2 \ , \tag{0.3}$$

When $\beta \neq 0$ we are interested in the interplay of two types of divergence, *local* divergence, and *wave divergence*. In the local divergence, $u$ will diverge near $x = 1$ like a special solution of (0.3). In the wave divergence, there will be a divergence near the front of the advancing wave.

We illustrate the phenomenology with two numerical simulations:

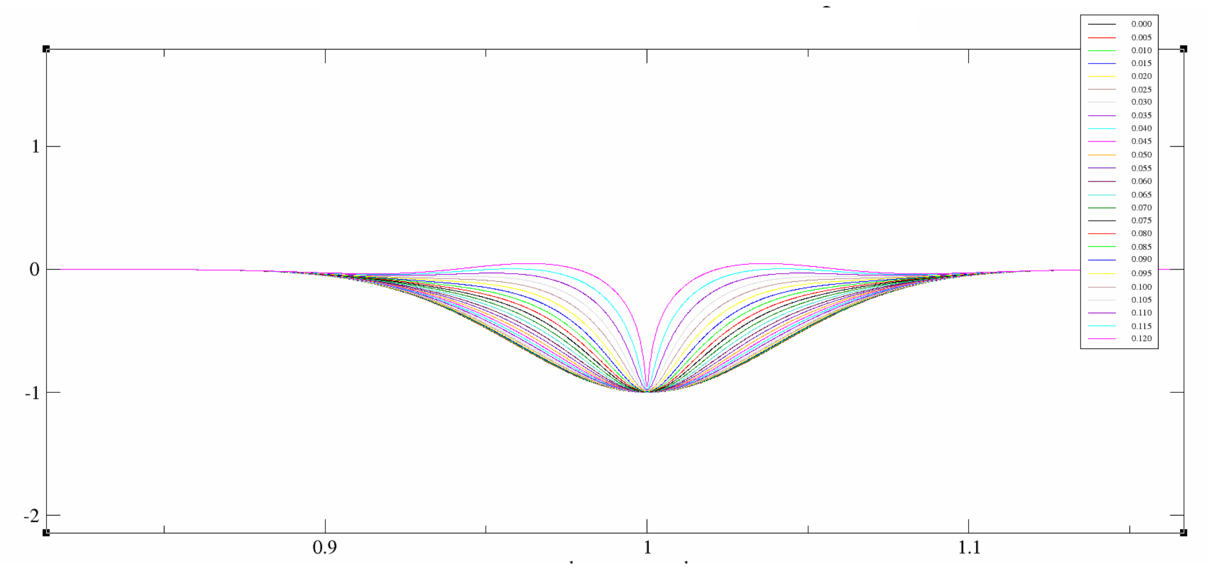


**Figure 1: Local divergence**:The parameters of (0.1) are $\alpha = 5$ and $\beta = 0.025$ with initial condition $u(x,0) = \exp(-30(x-1)^2)$, $u_t(x,0) = 0$. (Numerically we take periodic boundary conditions.) Note that the second derivative diverges in finite time. The simulation corresponds to $\alpha/\beta = 200$.

We expect that, for fixed $\beta$ a transition between local divergence and wave divergence will appear. The mathematical situation for the wave divergence can be explained by an adaptation of the work of Rammaha **?**. He shows that (0.2) will diverge in finite time under quite weak conditions: We adapt the proof to the 1d case in in Theorem 1.1, which will be given at the end.

Take as initial conditions smooth functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, both with compact support and assume the support is in $|x| < X$. Fix an $X_0 < X$ and define

$$\varepsilon = \tfrac{1}{2} \int_{x > X_0} (f(x) + (x - X_0) g(x)) \mathrm{d}x \ .$$

with

$$M(x) = \tfrac{1}{2} \left( f(x) + \int_x^X g(\xi) \mathrm{d}\xi \right) > 0$$

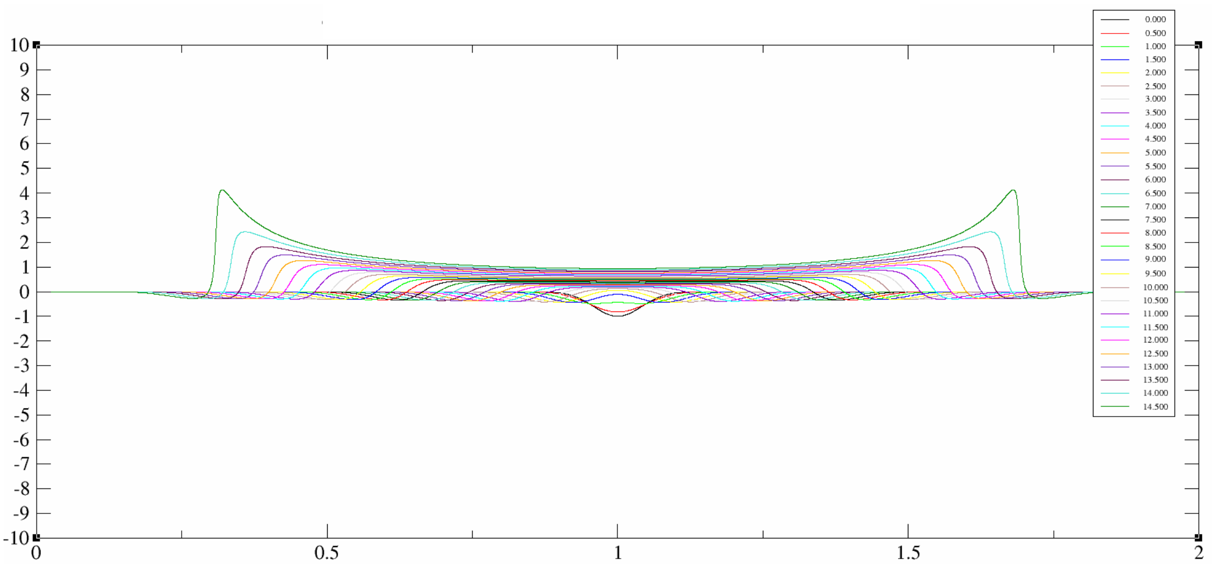


**Figure 2: Wave divergence**:The nature of divergence "at infinity." As time goes on, the support of the function spreads, and the function at the edge gets steeper, until the derivative will diverge at some time $T_*$. The simulation is for the variant equation (0.3), with $\alpha = 0.01$ and $\beta = 0.025$ and initial condition $u(x, 0) = \exp(-30(x-1)^2)$, $u_x(x, 0) = 0$. (Numerically we take periodic boundary conditions.) Now, $\alpha/\beta = 0.4$

for all $x \in (X_0, X)$. We show that the solution must diverge before a time $T_*$. We do not aim to get the best possible bound on this time.

Rammaha's result can be paraphrased as follows. The point of the proofs is that because of the wave character of (0.1), the solution will grow at the advancing front of the wave. The qualitative behavior is clearly seen in Fig. 2.

This discussion settles the question of what must happen at the advancing fronts. However, there can be a local divergence which can appear earlier in the "center" between the waves.

We will now argue that for $\beta = 0$ there are (admittedly unbounded) initial conditions for which local divergence will appear in arbitrarily short time. In particular, this time can be shorter than the wave divergence time.

This divergence is modeled most simply when $\beta = 0$. In this case, taking the (unbounded) initial condition $u(x, 0) = Ax^2$, Eq.(0.3) has the explicit solution

$$u(x,t) = \frac{3}{2\alpha(t-t_0)^2} x^2 ,$$

when

$$u(x,0) = \frac{3}{2\alpha t_0^2} x^2 ,$$
$$u_t(x,0) = \frac{3}{\alpha t_0^3} x^2 .$$

Clearly, $t_0$ is then just given by (This result is nice and ok, but it does not work when we want to have $u(x, 0) = 0$, which is the initial condition that Peter considered. There they showed that the curvature of the minimum blowsup. )JPE: need to discuss what to do here

JPE:probably omit what comes next

$$t_0 = 2\frac{u(x,0)}{u_t(x,0)} .$$

So we can make $t_0$ arbitrarily short as announced. Finally, in perturbation theory (up to order $x^4$ included) one can solve (0.2) explicitly (when $\alpha/\beta = 1$) in the form

$$u(x,t) = a(t)\left((x-t)^2 + (x+t)^2\right) + b(t)x + c(t) ,$$

with

$$\begin{aligned}
a(t) &= \frac{3}{4A(t-t_0)^2} , \\
b(t) &= C_1(t-t_0)^3 + C_2\frac{1}{(t-t_0)^2} , \\
c(t) &= 3\frac{\log(t_0-t)}{A} - 3\frac{t_0}{A(t-t_0)} - 3\frac{t_0^2}{2A(t-t_0)^2} + C_3 t + C_4 .
\end{aligned}$$

Therefore, to this order, one can also produce arbitrary rapid local divergence, just slightly shifted (by $t_0$) from the origin. (I don't understand the previous argument. The solution does not give us a reasonable behaviour. Because at $t = t_0$ it seems that the logarithm also has problem. )

# 1 Wave divergence

Consider the equation

$$u_{tt} - u_{xx} = u_x^2 .$$

Assume the initial conditions are

$$u(x,0) = f(x) , \quad u_t(x,0) = g(x) ,$$

with $f$, $g$ having support in $|x| < X$. Define, for $x > 0$,

$$M(x) = \tfrac{1}{2}f(x) + \tfrac{1}{2}\int_x^X g(\xi)\mathrm{d}\xi .$$

**Theorem 1.1.** *There is an $X_0 \in (0,X)$ for which $M(x) \geq 0$ for all $x \in (X_0,X)$, and also*

$$\int_{X_0}^X M(\xi)\mathrm{d}\xi \equiv \varepsilon > 0 .$$

*Then the local existence time is* **finite**.

*Proof.* It suffices to consider the situation where $u(x,0)$ has support in $|x| \leq X$, but we will consider only the side of positive $x$ in the sequel. Clearly, $u(x,t) = 0$ for $x \geq t + X$. One estimates now the function

$$H(t) = \int_{X_1}^t (t-\tau)\int_{\tau+X_0}^{\tau+X} u(\xi,\tau)\mathrm{d}\xi\,\mathrm{d}\tau .$$

Here, $X_1 = (X - X_0)/2$. From the definition, we get

$$H''(t) = \int_{t+X_0}^{t+X} u(\xi,t)\mathrm{d}\xi . \tag{1.1}$$

One has the explicit formula

$$u(x,t) = u_0(x,t) + \tfrac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} u_x(\xi,\tau)^2 \mathrm{d}\tau\, \mathrm{d}\xi\ , \tag{1.2}$$

with the "free evolution"

$$u_0(x,t) = \tfrac{1}{2}\left(f(x-t) + f(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} g(\xi)\mathrm{d}\xi\ .$$

When $x \ge t + X_0$ and $X \ge t \ge X_1$, then $x + t \ge X$ and therefore $f(x+t) = 0$ and therefore, in this region,

$$u_0(x,t) = \tfrac{1}{2} f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(\xi)\mathrm{d}\xi = M(x-t)\ ,$$

by the definition of $M$. We get from (1.2) and (1.1),

$$H''(t) = G_0(t) + G_1(t)\ ,$$

with

$$G_0(t) = \int_{t+X_0}^{t+X} u_0(x,t)\mathrm{d}x = \int_{t+X_0}^{t+X} M(x-t)\mathrm{d}x = \int_{X_0}^{X} M(x)\mathrm{d}x = \varepsilon\ ,$$

by the definition of $\varepsilon$ in Theorem 1.1. The nonlinearity leads to

$$G_1(t) = \int_{t+X_0}^{t+X} \mathrm{d}\xi \int_0^t \mathrm{d}\tau \int_{\xi-t+\tau}^{\xi+t-\tau} \mathrm{d}\eta\, u_x(\eta,\tau)^2\ .$$

In Lemma 1.2 below, we show

$$G_1(t) \ge \frac{1}{t+X_0}\int_{X_1}^t \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, (t-\tau)\,(\xi-\tau-X_0)\, u_x(\xi,\tau)^2\ . \tag{1.3}$$

We use now the Schwarz inequality in the form

$$\int \varphi\psi = \int \varphi^{1/2}(\varphi^{1/2}\psi) \le \left(\int \varphi\psi^2\right)^{1/2} \left(\int \varphi\right)^{1/2}\ ,$$

with $\varphi = (t-\tau)(\xi-\tau-X_0)$ and $\psi = u_x$. This leads to

$$G_1(t) \ge F^2(t)/J(t)\ ,$$

with

$$F(t) = \int_{X_1}^t \int_{\tau+X_0}^{\tau+X} (t-\tau)(\xi-\tau-X_0) u_x(\xi,\tau)\mathrm{d}\xi\, \mathrm{d}\tau$$

and

$$J(t) = \int_{X_1}^t \int_{\tau+X_0}^{\tau+X} (t-\tau)(\xi-\tau-X_0)\mathrm{d}\tau\, \mathrm{d}\xi = \frac{(X-X_0)^2(t-X_1)^2}{4}\ .$$

If we integrate the expression for $F$ by parts (in $\xi$), we get

$$F(t) = -\int_{X_1}^t \int_{\tau+X_0}^{\tau+X} (t-\tau) u(\xi,\tau)\mathrm{d}\xi\, \mathrm{d}\tau = -H(t)\ .$$

Therefore, we find finally

$$H''(t) \geq G_0(t) + \frac{H(t)^2}{J(t)} \ . \tag{1.4}$$

Fix now $T$ and we will show that the solution cannot exist for $T > T_*$, where $T_*$ will be computed in the proof: We use here Lemma 1 from **?** adapted to the 1d case. The ingredients are that

$$H''(t) \geq G_0(t) = \varepsilon > 0 \ , \tag{1.5}$$

for all $t \geq 0$ and

$$H''(t) \geq G_1(t) \geq 4\frac{H(t)^2}{(X-X_0)^2(t-X_1)^2} \ , \tag{1.6}$$

for $t > X_1$ (as long as the solution exists). Furthermore, $H(0) = H'(0) = 0$.

Fix now $T_1 = 2(X_1 + 1)$. Then, for $t > T_1$, we have $t - X_1 > \frac{1}{2}(t+1)$, and we replace from now on (1.6) by the simpler

$$H''(t) \geq K_1\frac{H(t)^2}{(t+1)^2} \ , \ \text{for } t > T_1 \ , \tag{1.7}$$

with $K_1 = 16/(X-X_0)^2$.

The idea is now to deduce from (1.5) and (1.7) an inequality of the form

$$H'(t) \geq CH^{1+\delta}(t) \text{ for } t > T_* \text{ with } \delta > 0 \ . \tag{1.8}$$

This implies divergence in finite time, when $H(T_0) > 0$. Indeed, if $H(T_0) = c^{-1/\delta} > 0$, then

$$H(t) = \frac{1}{(c - C\delta(t-T_0))^{1/\delta}} \ . \tag{1.9}$$

We now begin the proof proper. If $B > 0$, we will use repeatedly the inequality

$$\frac{x}{x+B} \geq \tfrac{1}{2} \ , \ \text{for all } x \geq B \ . \tag{1.10}$$

From (1.5) we find

$$H(t) \geq K_2\varepsilon t^2 \ , \ \text{for all } t > 0 \ , \tag{1.11}$$

with $K_2 = \frac{1}{2}$.

Subsituting (1.11) into (1.7), we get

$$H''(t) \geq K_1K_2\varepsilon H(t)\frac{t^2}{(t+1)^2} \geq K_3\varepsilon H(t) \ , \ \text{when } t > T_2 \ , \tag{1.12}$$

for some large enough $T_2 = \text{const.}T_1$, not depending on $\varepsilon$. Since $H'(t) > 0$, we can multiply (1.12) by $H'$ and write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}(H'(t)^2) \geq K_3\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}(H(t)^2) \text{ when } t > T_2 \ ,$$

We integrate from $T_2$ to $t$ and obtain

$$H'(t)^2 \geq K_3\varepsilon\left(H(t)^2 + H'(T_2)^2 - H(T_2)^2\right) = K_3\varepsilon H(t)^2 + K_4\varepsilon \text{ when } t > T_2 \ ,$$

for some $K_4$. From (1.5), we conclude that for large enough $T_3$, one has

$$K_3\varepsilon H(t)^2 + K_4\varepsilon \geq K_5\varepsilon H(t)^2 \ , \ \text{when } t > T_3 \ .$$

Combining the last two equations we find

$$H'(t) \geq K_5^{1/2} \varepsilon^{1/2} H(t), \quad \text{when } t > T_3 \, .$$

Integrating from $T_3$ to $t$ this leads to

$$H(t) \geq H(T_3) \exp\left(K_6 \varepsilon^{1/2}(t - T_3)\right) \geq H(T_3) \exp\left(\tfrac{1}{2} K_6 \varepsilon^{1/2} t\right) \, , \text{ when } t > 2T_3, \qquad (1.13)$$

with $K_6 = K_5^{1/2}$. Substituting again into (1.7), we get

$$H''(t) \geq K_7 H(t)^{1+\delta} \, , \text{ for any } \delta > 0 \, ,$$

since the exponential in (1.13) (to the power $\delta > 0$) will dominate the factor $(t+1)^{-2}$ of (1.7), if only $t$ is sufficiently large. (Note that $K_7$ and this new minimal time $T_4$ will depend on $\delta$.) We now multiply the last equation by $H'$ and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(H'(t)^2) \geq \frac{2K_7}{2+\delta} \frac{\mathrm{d}}{\mathrm{d}t} \left(H(t)^{2+\delta}\right) \, , \text{ for } t > T_4 \, .$$

Integrating from $T_4$ to $t$ we find

$$H'(t)^2 \geq \frac{2K_1}{2+\delta} \left(H(t)^{2+\delta} - H(T_4)^{2+\delta}\right) + H'(T_4)^2 \, .$$

Taking square roots on both sides and choosing $T_*$ sufficiently larger than $T_4$, we finally arrive at (1.8) from which we see that there is divergence in finite time, as in (1.9). $\square$

We still need to show the inequality (1.3).

**Lemma 1.2.** *Let*

$$G_1(t) = \int_{t+X_0}^{t+X} \mathrm{d}x \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, u_x(\xi, \tau)^2 \, .$$

*Let $X > X_0 > 0$, and assume $u_x(x.t) = 0$ for all $|x| \geq t + X$, $0 \leq t \leq T$. Then one has for all $T \geq t \geq X_1 = (X - X_0)/2$, the inequality*

$$G_1(t) \geq \frac{1}{t + X_0} \int_{X_1}^t \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi \, (t-\tau) \, (\xi - \tau - X_0) \, u_x(\xi, \tau)^2 \, . \qquad (1.14)$$

*Proof.*

$$\begin{aligned} G_1(t) &= \int_{t+X_0}^{t+X} \mathrm{d}x \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, u_x(\xi, \tau)^2 \\ &= \int_0^t \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, u_x(\xi, \tau)^2 \, . \end{aligned}$$

Note that for $t \geq X_1$, the function $u_x(\xi, \tau)$ vanishes for $\xi \geq \tau + X$. Therefore, since also $2t - \tau \geq \tau$, we find

$$\int_0^{t-X_1} \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, u_x(\xi, \tau)^2$$

$$
\begin{aligned}
&= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{2t+X-\tau} \mathrm{d}\xi \int_{t+X_0}^{\xi+t-\tau} \mathrm{d}x\, u_x(\xi,\tau)^2 \\
&= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1\, \mathrm{d}x\ .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{t-X_1}^{t} \mathrm{d}\tau \int_{t+X_0}^{t+X} \mathrm{d}x \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \\
&= \int_{t-X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1 \mathrm{d}x \\
&\quad + \int_{t-X_1}^{t} \mathrm{d}\tau \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{\xi-t+\tau}^{\xi+t-\tau} 1 \mathrm{d}x\ .
\end{aligned}
$$

Therefore, when $t \geq X_1$ the function $G_1$ can be decomposed as

$$
\begin{aligned}
G_1(t) &= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1 \mathrm{d}x \\
&\quad + \int_{t-X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{t+X_0}^{\xi+t-\tau} 1 \mathrm{d}x \\
&\quad + \int_{t-X_1}^{t} \mathrm{d}\tau \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi\, u_x(\xi,\tau)^2 \int_{\xi-t+\tau}^{\xi+t-\tau} 1 \mathrm{d}x\ .
\end{aligned}
$$

The integrals over $\mathrm{d}x$ lead to

$$
\begin{aligned}
G_1(t) &= \int_0^{t-X_1} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\ (\xi-\tau-X_0)\, u_x(\xi,\tau)^2 \\
&\quad + \int_{t-X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{2t-\tau+X_0} \mathrm{d}\xi\, (\xi-\tau-X_0)\, u_x(\xi,\tau)^2 \\
&\quad + \int_{t-X_1}^{t} \mathrm{d}\tau\, 2(t-\tau) \int_{2t-\tau+X_0}^{\tau+X} \mathrm{d}\xi\ u_x(\xi,\tau)^2\ .
\end{aligned}
$$

Since

$$
t-\tau \leq t+X_0\ , \text{ and } \ \xi-\tau-X_0 \leq t+X_0\ ,
$$

and both are positive, we find finally that

$$
G_1(t) \geq \frac{1}{t+X_0} \int_{X_1}^{t} \mathrm{d}\tau \int_{\tau+X_0}^{\tau+X} \mathrm{d}\xi\, (t-\tau)\, (\xi-\tau-X_0)\, u_x(\xi,\tau)^2\ .
$$

□