1 $\alpha = 0$

Consider, again

$$u_{tt} = u_{xx} \ , \tag{1.1}$$

with initial condition $u(x,0) = 0$ and $u_t(x,0) = g(x)$, with $g(x)$ having compact support, $g'(0) = 0$, $g''(0) > 0$. I typically think of $g(x) = -((1+\cos(x))/2)^n$, restricted to $[-\pi,\pi]$, which is differentiable when e.g. $n = 2$.

The solution of (1.1) is

$$u(x,t) = \tfrac{1}{2}\int_{x-t}^{x+t} g(\xi)\mathrm{d}\xi \ .$$

This implies that

$$u_{xx}(x,t) = \tfrac{1}{2}(g'(x+t) - g'(x-t)) \ .$$

Note now that if (for our example) $t > \pi/2$, the two terms above have distinct supports and for $x < 0$,

$$u_{xx}(x,t) = \tfrac{1}{2}g'(x+t) \ .$$

Since $g'(x)$ (inside its support) only vanishes for $x = 0$, we conclude that when x is not too far from t then

$$u_{xx}(x,t) = 0 \text{ if and only if } x \pm t = 0 \ . \tag{1.2}$$

Intuitively, this means that the root of u_{xx} moves away from 0 with exactly the speed of the wave.

2 $\alpha > 0$

In this case, the idea is that if α is small, then the roots of u_{xx} move *away* from the origin, while for α large enough, they move *towards* the origin, leading to local blowup.

Note that $u_{xx}(x_0,t) = 0$ means that $|u_x(x_0,t)|$ is extremal. Therefore, the extrema of $u_x(\cdot,t)$ (near 0), are at the positions where the quantity u_{xx} vanishes. One should use now the formula

$$u(x,t) = u_0(x,t) + \alpha\int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, (u_x(\xi,\tau))^2 \equiv u_0(x,t) + \alpha s(x,t) \ , \tag{2.1}$$

to estimate how the α-dependent term moves the roots of u_{xx}.

Consider the implicit equation

$$u_{xx}(z(t),t) = 0 \ .$$

By what we have said before, when $\alpha = 0$, then $z_0(t) = -t$ is a possible solution (when $t > \pi/2$). We now want to show that when $\alpha > 0$ then $z_\alpha(t) > z_0(t) = -t$. (Here, the index α is not a derivative, just a sign for the α-dependence.) Write

$$s(x,t) = \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi f(\xi,\tau) \ .$$

The derivatives of s are then

$$s_x = \int_0^t \mathrm{d}\tau \ (f(x+t-\tau,\tau) - f(x-t+\tau,\tau)) \ ,$$

$$
\begin{aligned}
s_{xx} &= \int_0^t \mathrm{d}\tau \ \left(f_x(x+t-\tau,\tau) - f_x(x-t+\tau,\tau)\right) \ , \\
s_{xxx} &= \int_0^t \mathrm{d}\tau \ \left(f_{xx}(x+t-\tau,\tau) - f_{xx}(x-t+\tau,\tau)\right) \ , \\
s_{xxt} &= \int_0^t \mathrm{d}\tau \ \left(f_{xx}(x+t-\tau,\tau) + f_{xx}(x-t+\tau,\tau)\right) \ .
\end{aligned}
$$

Since $u_{0,xx}(x,t) = \frac{1}{2}(g'(x+t) - g'(x-t))$, we also have

$$
\begin{aligned}
u_{0,xxx} &= \tfrac{1}{2}(g''(x+t) - g''(x-t)) \ , \\
u_{0,xxt} &= \tfrac{1}{2}(g''(x+t) + g''(x-t)) \ .
\end{aligned}
$$

Therefore,

$$
z'_\alpha(t) = -\frac{u_{xxt}(z(t),t)}{u_{xxx}(z(t),t)} = -\frac{u_{0,xxt} + \alpha s_{xxt}}{u_{0,xxx} + \alpha s_{xxx}} \ .
$$

We cannow restrict to negative x, and then we get

$$
z'_\alpha(t) = -\frac{\frac{1}{2}g''(x+t) + \alpha s_{xxt}}{\frac{1}{2}g''(x+t) + \alpha s_{xxx}} \ .
$$

Clearly, for $\alpha = 0$ we find $z'_0(t) = -1$.

Consider now “our” case when $f = (u_x)^2$. Then, $f_x = 2u_x u_{xx}$ and

$$
f_{xx} = 2(u_{xx})^2 + 2u_{xxx}u_x \ .
$$

3 Perturbation theory

Let us replace (2.1) by the first order approximation:

$$
u_0(x,t) + \alpha \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} (u_{0,x}(\xi,\tau))^2 \ .
$$

Then the implicit equation for z is

$$
u_{0xx}(z(t),t) + \alpha\partial_x^2 \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} (u_{0x}(\xi,\tau))^2 \bigg|_{x=z(t)} = 0 \ .
$$

Let $X = \int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} (u_{0x}(\xi,\tau))^2$. Looking only on the negative (x) side, we have $u_{0x}(x,t) = \frac{1}{2}g(x+t)$ and therefore this leads (near $x+t=0$), to

$$
X = \frac{1}{4}\int_0^t \mathrm{d}\tau \int_{x-t+\tau}^{x+t-\tau} \mathrm{d}\xi \, g(\xi+\tau)^2 \ .
$$

The derivatives are

$$
\partial_x X = \frac{1}{4}\int_0^t \mathrm{d}\tau g(x+t-\tau+\tau)^2 - g(x-t+\tau+\tau)^2
$$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^t \mathrm{d}\tau g(x+t)^2 - g(x-t+2\tau)^2 \; , \\
\partial_x^2 X &= \frac{1}{4}\int_0^t \mathrm{d}\tau\, \partial_x(g(x+t)^2) - \partial_x(g(x-t+2\tau)^2) \\
&= -\frac{1}{4}\left(g(x+t)^2 - g(x-t)^2\right) \; .
\end{aligned}
$$

Therefore, we find, in perturbation theory, the equation for $z(t)$ as

$$
\frac{1}{2}g'(z(t)+t) - \frac{\alpha}{4}(g(z(t)+t))^2 = 0 \; . \tag{3.1}
$$

Since $g(0) = -1$, $g'(0) = 0$, and $g''(0) > 0$, we can write $g(x) = -1 + Ax^2 + \mathcal{O}(x^3)$, with $A > 0$. Then (3.1) with $z(t) + t = \varepsilon$ leads to:

$$
g'(\varepsilon)/2 = \alpha g(\varepsilon)^2/4 \; ,
$$

and therefore

$$
z(t) = -t + \varepsilon = -t + \frac{\alpha}{4A} = -t + \frac{\alpha g(0)^2}{4g''(0)}
$$

Note that under our assumptions on g the shift $\alpha g(0)^2/(4g''(0))$ is *positive*, *i.e.,* the extrema move towards the center.

4 Disorderd thoughts

Here I sketch (badly) what I have in mind:

At any time t, I want to consider the current $u(x,t)$ as a new initial condition. But, we are not interested in the full evolution of the solution, but only in the motion of $z(t)$ which is defined by $u_{xx}(z(t),t) = 0$. So I consider $f(x) = u(x,t)$ and $g(x) = u_t(x,t)$ as my initial conditions at time t. The "free" evolution from this initial condition at time $t + \varepsilon$ should be (on one side only)

$$
\tfrac{1}{2}f(x+\varepsilon) + \tfrac{1}{2}\int_{x-\varepsilon}^{x+\varepsilon} g(\xi)\mathrm{d}\xi + \frac{\alpha}{2}\int_0^{\varepsilon}\mathrm{d}\tau \int_{x-\varepsilon+\tau}^{x+\varepsilon-\tau} \mathrm{d}\xi (u_x(\xi, t+\tau))^2 \; . \tag{4.1}
$$

I have a feeling that the "correct" speed in this case should be $z'(t)$ and not 1 so that perhaps one should have taken for example

$$
\tfrac{1}{2}f(x+z'(t)\varepsilon) + \tfrac{1}{2}\int_{x-z'(t)\varepsilon}^{x+z'(t)\varepsilon} g(\xi)\mathrm{d}\xi + \frac{\alpha}{2}\int_0^{z'(t)\varepsilon}\mathrm{d}\tau \int_{x-z'(t)\varepsilon+\tau}^{x+z'(t)\varepsilon-\tau} \mathrm{d}\xi (u_x(\xi, t+\tau))^2 \; . \tag{4.2}
$$

I have somehow the idea that the form (4.2) should somehow make better cancellations, but I am stuck here.

Anyway, the idea is now that (4.1) or (4.2), *near the root of* u_{xx} can be treated again with the implicit function idea as in (3.1).