1 Divergence

The following is a slight adaptation of the results of Wittwer, Pan Shi, and Hassani.

We consider the equation $u_{tt} = (u_x)^2$ on the real line. We start by writing the solution in the form

$$u(x,t) = f(x) + g(x)t + \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' (u_x(x,\tau'))^2 \ . \tag{1.1}$$

This corresponds to the initial conditions

$$u(x,0) = f(x) \ , \quad u_t(x,0) = g(x) \ .$$

We will consider the case that $f'(0) = g'(0) = 0$, and we ask how the solution behaves near $x = 0$. Depending on the curvatures $f''(0)$ and $g''(0)$, the second derivative $u_{xx}(x,t)$ will, or will not diverge at $x = 0$. Of course, if the functions f and g have vanishing derivatives at some other point(s) x_0, the same discussion will apply at those points, and there can be one of these points where $u_{xx}(x_0,t)$ diverges before the one at $x = 0$. In the following proposition, we will neglect this aspect.

Proposition 1.1. *Assume $f'(0) = g'(0) = 0$. Define*

$$c = \tfrac{1}{2} g''(0)^2 - \tfrac{2}{3} f''(0)^3 \ .$$

Then the following cases appear:
(i) If $g''(0) > 0$ then $u_{xx}(0,t)$ diverges in finite time t_* given by

$$t_* = \int_{f''(0)}^\infty \frac{\mathrm{d}b}{\sqrt{\frac{4}{3} b^3 + 2c}} \ . \tag{1.2}$$

(ii) If $g''(0) < 0$ then $u_{xx}(0,t)$ will converge to b_* in a finite time t_*, where

$$\tfrac{2}{3} b_*^3 = -c \ , \ \textit{and } t_* = \int_{b_*}^{f''(0)} \frac{\mathrm{d}b}{\sqrt{\frac{4}{3} b^3 + 2c}} \ . \tag{1.3}$$

At this point in time, we will have $u_t(0,t_*) = 0$ which corresponds to (iii) and the solution will diverge after another finite time (unless $u(0,t_*) = 0$.
(iii) If $g''(0) = 0$, and $f''(0) \neq 0$ then $u_{xx}(0,t)$ diverges in finite time t_* given again by (1.2).
(iv) If $g''(0) = 0$ and $f''(0) = 0$ then $u_{xx}(0,t)$ stays constant.

Remark 1.2. The divergence and limit times above are standard elliptic integrals. When $g''(0) = 0$, the finite divergence time t_* scales like $\mathcal{O}(|f''(0)|^{-1/2})$.

Remark 1.3. A case of a certain interest in cosmology appears when $f''(0) = 0$ and $g''(0) > 0$. In that case, $t_* \sim 2.547/(g''(0))^{1/6}$.

Proof. Define

$$a(t) = u_x(0,t) \ , \quad b(t) = u_{xx}(0,t) \ .$$

Then (1.1) leads to

$$a(t) = f'(0) + g'(0)t + 2\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' u_x(0,\tau') u_{xx}(0,\tau') \ ,$$

and therefore

$$\ddot{a}(t) = 2a(t)\, b(t)\ . \tag{1.4}$$

Similarly,

$$b(t) = f''(0) + g''(0)t + 2\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \left((u_{xx}(0,\tau'))^2 + u_x(0,\tau')u_{xxx}(0,\tau')\right)\ ,$$

From this, we deduce

$$\begin{aligned}\ddot{b}(t) &= 2(u_{xx}(0,t))^2 + 2u_x(0,t)u_{xxx}(0,t)\\ &= 2b(t)^2 + 2a(t)u_{xxx}(0,t)\ .\end{aligned} \tag{1.5}$$

Since we assume $f'(0) = g'(0) = 0$ we find from (1.4) that $a(t) = 0$ for all t for which $b(t)$ is finite. Therefore, (1.5) reduces to

$$\ddot{b}(t) = 2(b(t))^2\ . \tag{1.6}$$

We will discuss this equation. For computing the divergence time, it is useful to transform the equation as follows: Multiplying by $\dot{b}$ leads to

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{b}(t))^2 = \tfrac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}b(t)^3\ ,$$

or, for some c,

$$\tfrac{1}{2}(\dot{b}(t))^2 = \tfrac{2}{3}(b(t))^3 + c\ , \tag{1.7}$$

Note that looking at $t = 0$ we find

$$c = \tfrac{1}{2}\dot{b}(0)^2 - \tfrac{2}{3}b(0)^3 = \tfrac{1}{2}g''(0)^2 - \tfrac{2}{3}f''(0)^3\ , \tag{1.8}$$

which is the definition in the proposition.

We consider first the case where $g''(0) > 0$. Then $\dot{b}(0) > 0$ and from (1.7) we find that

$$\dot{b}(t) = \sqrt{\tfrac{4}{3}(b(t))^3 + 2c}\ , \tag{1.9}$$

which means b is increasing. Using standard techniques, we get

$$\mathrm{d}t = \frac{\mathrm{d}b}{\sqrt{\tfrac{4}{3}b^3 + 2c}}\ .$$

From (1.9) we deduce the divergence time t_*,

$$t_* = \int_{b(0)}^{\infty} \frac{\mathrm{d}b}{\sqrt{\tfrac{4}{3}b^3 + 2c}}\ . \tag{1.10}$$

This proves (1.2).

The case $g''(0) < 0$ is handled similarly, but now (1.9) is replaced by

$$\dot{b}(t) = -\sqrt{\tfrac{4}{3}(b(t))^3 + 2c}\ , \tag{1.11}$$

This means that b is decreasing until the square root in (1.11) vanishes. This defines b_*, and then (1.10) is replaced by

$$t_* = \int_{b_*}^{f''(0)} \frac{\mathrm{d}b}{\sqrt{\frac{4}{3}b^3 + 2c}} \,.$$

This leads to (1.3).

The assertions under (iii) are a simple variant of (i) and (ii). The difference is that because $g''(0) = 0$, we find now that $c = -\frac{2}{3}f''(0)^3$, and $c \neq 0$ by the assumption $f''(0) \neq 0$. The only remaining case is (iv), $g''(0) = f''(0) = 0$, which directly leads to $b(t) = b(0)$ by (1.6). $\square$