

PREPARED FOR SUBMISSION TO JCAP

# $k$ -essence

**Abstract.**

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## 1 Introduction

## 2 Action

Starting from the covariant action of  $k$ -essence,

$$S_{\text{DE}} = \int d^4x \sqrt{-g} P(X, \phi) \quad (2.1)$$

Where the scalar field could be written as  $\phi(t, \vec{x}) = \bar{\phi}(t) + \delta\phi(t, \vec{x})$ . Now we choose the time coordinate such that  $\delta\phi = 0$  and thus  $t$  becomes a function of  $\phi$ . As a result, we have

$$\begin{aligned} X &= -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \rightarrow -g^{00} \dot{\bar{\phi}}(t)^2 \\ \phi &\rightarrow \bar{\phi}(t) \end{aligned} \quad (2.2)$$

So the action is written as,

$$S_{\text{DE}} = \int d^4x \sqrt{-g} P(-g^{00} \dot{\bar{\phi}}(t)^2, \bar{\phi}(t)) \quad (2.3)$$

where  $g^{00} = -1 + \delta g^{00}$ . So we can write the Lagrangian as,

$$\begin{aligned} P(\dot{\bar{\phi}}(t)^2(1 - \delta g^{00}), \bar{\phi}(t)) &= P(\bar{X}, \bar{\phi}(t)) - \bar{X} \frac{\partial P}{\partial \bar{X}} \Big|_{X=\bar{X}} \delta g^{00} + \bar{X}^2 \frac{\partial^2 P}{\partial \bar{X}^2} \Big|_{X=\bar{X}} (\delta g^{00})^2 \\ &\quad - \bar{X}^3 \frac{\partial^3 P}{\partial \bar{X}^3} \Big|_{X=\bar{X}} (\delta g^{00})^3 + \dots \end{aligned} \quad (2.4)$$

where we have defined  $\bar{X} = \dot{\bar{\phi}}(t)^2$ .

We want to make the scalar field perturbation explicit. We follow [1] and perform a diffeomorphism,

$$t \rightarrow \tilde{t}(\eta) = t(\eta) + \xi^0(t(\eta), \vec{x}), \quad \vec{x} \rightarrow \vec{\tilde{x}} = \vec{x}, \quad (2.5)$$

where

$$\pi(t(\eta), \vec{x}) \doteq \xi^0(t(\eta), \vec{x})/a(t(\eta)). \quad (2.6)$$

Where  $\eta$  is the conformal time,

$$\eta := \int \frac{dt}{a(t)} \quad (2.7)$$

$$g^{00} \rightarrow a^2(1 + \mathcal{H}\pi + \pi')^2 g^{00} + 2a^2(1 + \mathcal{H}\pi + \pi') \partial_\mu \pi g^{0\mu} + a(\eta)^2 g^{\mu\nu} \partial_\mu \pi \partial_\nu \pi \quad (2.8)$$

We have to pay attention to expressing the cosmic time quantities in terms of conformal time when changing coordinates. For instance,

$$\begin{aligned} f(t) &\rightarrow \tilde{f}(\tilde{t}) = f(t(\eta) + a(\eta)\pi(\eta, \vec{x})) \\ &= f(\eta) + f'(\eta)\pi(\eta, \vec{x}) + \frac{1}{2}\left(f''(\eta) - \mathcal{H}(\eta)f'(\eta)\right)\pi(\eta, \vec{x})^2 + \dots \end{aligned} \quad (2.9)$$

On the other hand, since the geometric quantities  $g^{00}$  and  $\delta K$  are constructed based on the metric in conformal time (based on our notation), so in order to perform the transformation on the vector  $A^\mu$  we have

$$A^\mu \rightarrow \tilde{A}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} A^\nu \quad (2.10)$$

where results in,

$$\begin{aligned} A^0 \rightarrow \tilde{A}^0 &= \frac{\partial \tilde{x}^0}{\partial x^0} A^0 + \frac{\partial \tilde{x}^0}{\partial x^i} A^i = \frac{\partial(t(\eta) + a(\eta)\pi(\eta, \vec{x}))}{\partial \eta} A^0 + \frac{\partial(t(\eta) + a(\eta)\pi(\eta, \vec{x}))}{\partial x^i} A^i \\ &= a(\eta)(1 + \mathcal{H}\pi(\eta, \vec{x}) + \pi'(\eta, \vec{x}))A^0 + a(\eta)\partial_i \pi(\eta, \vec{x})A^i \end{aligned} \quad (2.11)$$

In order to transform the vectors and convectors it is convenient to define the following matrices [2],

$$R_\nu^\mu \doteq \frac{\partial \tilde{x}^\mu}{\partial x^\nu}, \quad (R^{-1})_\nu^\mu \doteq \frac{\partial \tilde{x}_\mu}{\partial x_\nu} \quad (2.12)$$

where  $R$  and  $R^{-1}$  read as,

$$R_\nu^\mu = \begin{pmatrix} a(1 + \mathcal{H}\pi + \pi') & a\partial_i \pi \\ 0 & \delta_j^i \end{pmatrix}_{\mu\nu} \quad (2.13)$$

$$(R^{-1})_\nu^\mu = \begin{pmatrix} \frac{1}{a(1 + \mathcal{H}\pi + \pi')} & -\frac{\partial_i \pi}{1 + \mathcal{H}\pi + \pi'} \\ 0 & \delta_i^j \end{pmatrix}_{\mu\nu} \quad (2.14)$$

Since the scalar  $A^\mu A_\mu$  should be invariant under the coordinate transformation,

$$A^\mu A_\mu \rightarrow (R^{-1})_\mu^\nu R_\rho^\mu A^\rho A_\nu \quad (2.15)$$

As expected we can easily verify that,

$$(R^{-1})_\mu^\nu R_\rho^\mu = I \quad (2.16)$$

As a result in order to perform gauge transformation, for any upper and lower-index we need to apply  $R$  and  $R^{-1}$  respectively. So we have the following relations,

$$g^{\mu\nu} \rightarrow R_\rho^\mu R_\sigma^\nu g^{\rho\sigma} \quad (2.17)$$

$$g_{\mu\nu} \rightarrow (R^{-1})_\mu^\rho (R^{-1})_\nu^\sigma g_{\rho\sigma} \quad (2.18)$$

$$\partial_0 \rightarrow (R^{-1})_0^\rho \partial_\rho = \frac{1}{a(1 + \mathcal{H}\pi + \pi')} \partial_0, \quad (2.19)$$

$$\partial_i \rightarrow (R^{-1})_i^\rho \partial_\rho = -\frac{\partial_i \pi}{1 + \mathcal{H}\pi + \pi'} \partial_0 + \partial_i \quad (2.20)$$

$$g^{00} \rightarrow a^2(1 + \mathcal{H}\pi + \pi')^2 g^{00} + 2a^2(1 + \mathcal{H}\pi + \pi') \partial_\mu \pi g^{0\mu} + a(\eta)^2 g^{\mu\nu} \partial_\mu \pi \partial_\nu \pi \quad (2.21)$$

$$g^{0i} \rightarrow a(1 + \mathcal{H}\pi + \pi') g^{0i} + a(\eta) \partial_j \pi g^{ij} \quad (2.22)$$

$$g^{ij} \rightarrow g^{ij} \quad (2.23)$$

## References

- [1] F. Hassani, J. Adamek, M. Kunz, and F. Vernizzi, *k-evolution: a relativistic n-body code for clustering dark energy*, *Journal of Cosmology and Astroparticle Physics* **2019** (Dec, 2019) 011?011.
- [2] M. Lewandowski, A. Maleknejad, and L. Senatore, *An effective description of dark matter and dark energy in the mildly non-linear regime*, *JCAP* **05** (2017) 038, [[arXiv:1611.07966](https://arxiv.org/abs/1611.07966)].