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k-essence

Abstract.

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Starting from the covariant action of k-essence,

$$S_{\rm DE} = \int d^4x\sqrt{-g}P(X,\phi) \tag{2.1}$$

Where the scalar field could be written as $\phi(t,\vec{x}) = \bar{\phi}(t) + \delta\phi(t,\vec{x})$. Now we choose the time coordinate such that $\delta\phi = 0$ and thus t becomes a function of ϕ. As a result, we have

$$\begin{aligned} &X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to -g^{00}\dot{\bar{\phi}}(t)^2 \\ &\phi \to \bar{\phi}(t) \end{aligned} \tag{2.2}$$

So the action is written as,

$$S_{\rm DE} = \int d^4x\sqrt{-g}P(-g^{00}\dot{\bar{\phi}}(t)^2, \bar{\phi}(t)) \tag{2.3}$$

where $g^{00} = -1 + \delta g^{00}$. So we can write the Lagrangian as,

$$\begin{aligned} P\Big(\dot{\bar{\phi}}(t)^2(1-\delta g^{00}), \bar{\phi}(t)\Big) = P(\bar{X},\bar{\phi}(t)) - \bar{X}\frac{\partial P}{\partial \bar{X}}\Big|_{X=\bar{X}}\delta g^{00} + \frac{\bar{X}^2}{2!}\frac{\partial^2 P}{\partial \bar{X}^2}\Big|_{X=\bar{X}}(\delta g^{00})^2 \\ - \frac{\bar{X}^3}{3!}\frac{\partial^3 P}{\partial \bar{X}^3}\Big|_{X=\bar{X}}(\delta g^{00})^3 + \cdots \end{aligned} \tag{2.4}$$

where we have defined $\bar{X} = \dot{\bar{\phi}}(t)^2$. As a result the action could be written in the following form:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}\left(\delta g^{00}\right)^2 + \frac{M_3^4(t)}{3!}\left(\delta g^{00}\right)^3 + \cdots\right]. \tag{2.5}$$

2.1 Stueckelberg trick

We want to make the scalar field perturbation explicit. We follow [1] and perform a diffeomorphism,

$$t \to \tilde{t}(\eta) = t(\eta) + \xi^0(t(\eta),\vec{x}) , \qquad \vec{x} \to \tilde{\vec{x}} = \vec{x} , \tag{2.6}$$

where

$$\pi(t(\eta),\vec{x}) \doteq \xi^0(t(\eta),\vec{x})/a(t(\eta)) . \tag{2.7}$$

Where η is the conformal time,

$$\eta := \int \frac{dt}{a(t)} \tag{2.8}$$

$$g^{00} \to a^2(1+\mathcal{H}\pi+\pi')^2 g^{00} + 2a^2(1+\mathcal{H}\pi+\pi')\partial_\mu\pi g^{0\mu} + a(\eta)^2 g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi \tag{2.9}$$

We have to pay attention to expressing the cosmic time quantities in terms of conformal time when changing coordinates. For instance,

$$\begin{aligned} f(t) \to \tilde{f}(\tilde{t}) &= f(t(\eta) + a(\eta)\pi(\eta,\vec{x})) \\ &= f(\eta) + f'(\eta)\pi(\eta,\vec{x}) + \frac{1}{2}\Big(f''(\eta) - \mathcal{H}(\eta)f'(\eta)\Big)\pi(\eta,\vec{x})^2 + ... \end{aligned} \tag{2.10}$$

On the other hand, since the geometric quantities g^{00} and δK are constructed based on the metric in conformal time (based on our notation), so in order to perform the transformation on the vector A^μ we have

$$A^\mu \to \tilde{A}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} A^\nu \tag{2.11}$$

where results in,

$$\begin{aligned} A^0 \to \tilde{A}^0 &= \frac{\partial \tilde{x}^0}{\partial x^0}A^0 + \frac{\partial \tilde{x}^0}{\partial x^i}A^i = \frac{\partial(t(\eta) + a(\eta)\pi(\eta,\vec{x}))}{\partial \eta}A^0 + \frac{\partial(t(\eta) + a(\eta)\pi(\eta,\vec{x}))}{\partial x^i}A^i \\ &= a(\eta)\big(1+\mathcal{H}\pi(\eta,\vec{x}) + \pi'(\eta,\vec{x})\Big)A^0 + a(\eta)\partial_i\pi(\eta,\vec{x})A^i \end{aligned} \tag{2.12}$$

In order to transform the vectors and convectors it is convenient to define the following matrices [2],

$$R^\mu_\nu \doteq \frac{\partial \tilde{x}^\mu}{\partial x^\nu}, \quad (R^{-1})^\mu_\nu \doteq \frac{\partial \tilde{x}_\mu}{\partial x_\nu} \tag{2.13}$$

where R and R^{-1} read as,

$$R^\mu_\nu = \begin{pmatrix} a(1+\mathcal{H}\pi+\pi') & a\partial_i\pi \\ 0 & \delta^i_j \end{pmatrix}_{\mu\nu} \tag{2.14}$$

$$(R^{-1})^\mu_\nu = \begin{pmatrix} \frac{1}{a(1+\mathcal{H}\pi+\pi')} & -\frac{\partial_i\pi}{1+\mathcal{H}\pi+\pi'} \\ 0 & \delta^j_i \end{pmatrix}_{\mu\nu} \tag{2.15}$$

Since the scalar $A^\mu A_\mu$ should be invariant under the coordinate transformation,

$$A^\mu A_\mu \to (R^{-1})^\nu_\mu R^\mu_\rho A^\rho A_\nu \tag{2.16}$$

As expected we can easily verify that,

$$(R^{-1})^\nu_\mu R^\mu_\rho = I \tag{2.17}$$

As a result in order to perform gauge transformation, for any upper and lower-index we need to apply R and R^{-1} respectively. So we have the following relations,

$$g^{\mu\nu} \to R^\mu_\rho R^\nu_\sigma g^{\rho\sigma} \tag{2.18}$$

$$g_{\mu\nu} \to (R^{-1})^{\rho}_{\mu}(R^{-1})^{\sigma}_{\nu}g^{\rho\sigma} \tag{2.19}$$

$$g^{00} \to a^2(1+\mathcal{H}\pi+\pi')^2 g^{00} + 2a^2(1+\mathcal{H}\pi+\pi')\partial_\mu\pi g^{0\mu} + a(\eta)^2 g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi \tag{2.20}$$

$$g^{0i} \to a(1+\mathcal{H}\pi+\pi')g^{0i} + a(\eta)\partial_j\pi g^{ij} \tag{2.21}$$

$$g^{ij} \to g^{ij} \tag{2.22}$$

2.2 Action

The cpvariant action after the Stueckelberg trick reads,

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big\{ &\frac{M_{\rm Pl}^2}{2}R - \Lambda\left[t(\eta)+a(\eta)\pi\right] - c\left[t(\eta)+a(\eta)\pi\right]a^2(\eta)\big[(1+\mathcal{H}\pi+\pi')^2 g^{00} \\ &+ 2(1+\mathcal{H}\pi+\pi')g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi\big] + \frac{M_2^4\left[t(\eta)+a(\eta)\pi\right]}{2} \\ &\times a^4(\eta)\big[(1+\mathcal{H}\pi+\pi')^2 g^{00} + 2(1+\mathcal{H}\pi+\pi')g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \bar{g}^{00}\big]^2 \\ + \frac{M_3^4\left[t(\eta)+a(\eta)\pi\right]}{3!} &\times a^6(\eta)\big[(1+\mathcal{H}\pi+\pi')^2 g^{00} + 2(1+\mathcal{H}\pi+\pi')g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \bar{g}^{00}\big]^3\Big\}\,, \end{aligned} \tag{2.23}$$

Here, we only focus on the last term in the EFT expansion, i.e.,

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big\{ &\frac{M_3^4\left[t(\eta)+a(\eta)\pi\right]}{3!} \\ &\times a^6(\eta)\big[(1+\mathcal{H}\pi+\pi')^2 g^{00} + 2(1+\mathcal{H}\pi+\pi')g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \bar{g}^{00}\big]^3\Big\}\,, \end{aligned} \tag{2.24}$$

References

[1] F. Hassani, J. Adamek, M. Kunz, and F. Vernizzi, *k-evolution: a relativistic n-body code for clustering dark energy*, *Journal of Cosmology and Astroparticle Physics* **2019** (Dec, 2019) 011?011.

[2] M. Lewandowski, A. Maleknejad, and L. Senatore, *An effective description of dark matter and dark energy in the mildly non-linear regime*, *JCAP* **05** (2017) 038, [arXiv:1611.07966].