Notes on k-essence models with $c_s^2 \ll 1$

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Let us start with the k-essence simple action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\rm Pl}^2 R + K(\phi, X) + \mathcal{L}_m \right] , \tag{1}$$

where K is a free function of the scalar field ϕ and its kinetic term X, and $\mathcal{L}_m$ is lagrangian describing the matter sector. The typical k-essence model has a luminal sound speed

$$c_s^2 \equiv \frac{\delta p_{DE}}{\delta \rho_{DE}} \simeq 1 , \tag{2}$$

and a general time dependent equation of state

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} . \tag{3}$$

Usually $w_{DE} \simeq -1$ at late times, but just because the models studied in the literature are tuned to drive the cosmic acceleration. However, Eq. (1) has been used to build models that were mimicking dark matter – the so called unified dark matter models – i.e.

$$c_s^2 \ll 1 \tag{4}$$

$$w_{DE} \simeq 0 , \tag{5}$$

during matter domination. Then, it seems natural to try to resurrect some of these models and adapt it to our needs, i.e.

$$c_s^2 \ll 1 \tag{6}$$

$$w_{DE} \simeq -1 . \tag{7}$$

We focus on models with this form

$$K(\phi, X) = -g_0 + g_2 \left(X - \hat{X} \right)^2 + g_4 \left(X - \hat{X} \right)^4 , \tag{8}$$

where g_0, g_2, g_4 and $\hat{X}$ are free parameters of the theory (g_0 will be tuned to satisfy flatness).

1 Covariant equations

Varying the action Eq. (1) w.r.t. the metric $g_{\mu\nu}$ gives

$$M_{\rm Pl}^2 G_{\mu\nu} = T_{\mu\nu} + K g_{\mu\nu} + K_X \phi_{;\mu}\phi_{;\nu}\,, \tag{9}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter. The scalar field equation can be obtained by varying Eq. (1) w.r.t. ϕ

$$K_X \Box\phi - K_{XX}\phi_{;\alpha}\phi_{;\beta}\phi^{;\alpha\beta} = 0\,. \tag{10}$$

The scalar field equation rewritten with partial derivatives is

$$\begin{aligned}\left(g^{\alpha\beta}K_X - \partial^\alpha\phi\partial^\beta\phi K_{XX}\right)\partial_\beta\partial_\alpha\phi = &-K_\phi - \partial_\alpha\phi\partial^\alpha\phi K_{\phi X}\\ &+ g^{\beta\gamma}\left(\partial_\gamma g_{\alpha\beta} - \frac{1}{2}\partial_\alpha g_{\beta\gamma}\right)\partial^\alpha\phi K_X - \frac{1}{2}\partial^\alpha\phi\partial^\beta\phi\partial_\gamma g_{\alpha\beta}\partial^\gamma\phi K_{XX}\,.\end{aligned} \tag{11}$$

This, rewritten with in mind our theory, Eq. (8), becomes

- this is the raw output of Mathematica

$$\begin{aligned}&g_2\hat{X}g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\beta g_{\gamma\eta} + 2g_4\hat{X}^3 g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\beta g_{\gamma\eta} - g_2 g^{\alpha\beta}g^{\gamma\eta}X\partial_\alpha\phi\partial_\beta g_{\gamma\eta}\\ &- 6g_4\hat{X}^2 g^{\alpha\beta}g^{\gamma\eta}X\partial_\alpha\phi\partial_\beta g_{\gamma\eta} + 6g_4\hat{X}g^{\alpha\beta}g^{\gamma\eta}X^2\partial_\alpha\phi\partial_\beta g_{\gamma\eta} - 2g_2 g^{\alpha\beta}X\partial_\beta\partial_\alpha\phi\\ &- 2g_4 g^{\alpha\beta}g^{\gamma\eta}X^3\partial_\alpha\phi\partial_\beta g_{\gamma\eta} + 2g_2\hat{X}g^{\alpha\beta}\partial_\beta\partial_\alpha\phi + 4g_4\hat{X}^3 g^{\alpha\beta}\partial_\beta\partial_\alpha\phi\\ &- 12g_4\hat{X}^2 g^{\alpha\beta}X\partial_\beta\partial_\alpha\phi + 12g_4\hat{X}g^{\alpha\beta}X^2\partial_\beta\partial_\alpha\phi - 4g_4 g^{\alpha\beta}X^3\partial_\beta\partial_\alpha\phi\\ &- 2g_2\hat{X}g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\eta g_{\beta\gamma} - 4g_4\hat{X}^3 g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\eta g_{\beta\gamma} + 2g_2 g^{\alpha\beta}g^{\gamma\eta}X\partial_\alpha\phi\partial_\eta g_{\beta\gamma}\\ &+ 12g_4\hat{X}^2 g^{\alpha\beta}g^{\gamma\eta}X\partial_\alpha\phi\partial_\eta g_{\beta\gamma} - 12g_4\hat{X}g^{\alpha\beta}g^{\gamma\eta}X^2\partial_\alpha\phi\partial_\eta g_{\beta\gamma}\\ &+ 4g_4 g^{\alpha\beta}g^{\gamma\eta}X^3\partial_\alpha\phi\partial_\eta g_{\beta\gamma} + 2g_2 g^{\alpha\gamma}g^{\beta\eta}\partial_\alpha\phi\partial_\beta\phi\partial_\eta\partial_\gamma\phi\\ &+ 12g_4\hat{X}^2 g^{\alpha\gamma}g^{\beta\eta}\partial_\alpha\phi\partial_\beta\phi\partial_\eta\partial_\gamma\phi - 24g_4\hat{X}g^{\alpha\gamma}g^{\beta\eta}X\partial_\alpha\phi\partial_\beta\phi\partial_\eta\partial_\gamma\phi\\ &+ 12g_4 g^{\alpha\gamma}g^{\beta\eta}X^2\partial_\alpha\phi\partial_\beta\phi\partial_\eta\partial_\gamma\phi - g_2 g^{\alpha\eta}g^{\beta\lambda}g^{\gamma\mu}\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda}\\ &- 6g_4\hat{X}^2 g^{\alpha\eta}g^{\beta\lambda}g^{\gamma\mu}\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda} + 12g_4\hat{X}g^{\alpha\eta}g^{\beta\lambda}g^{\gamma\mu}X\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda}\\ &- 6g_4 g^{\alpha\eta}g^{\beta\lambda}g^{\gamma\mu}X^2\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda} = 0\end{aligned} \tag{12}$$

- This is after some massage and redefinition of variables

$$\begin{aligned}&-K_X g^{\alpha\beta}\partial_\beta\partial_\alpha\phi + K_{XX}g^{\alpha\gamma}g^{\beta\eta}\partial_\alpha\phi\partial_\beta\phi\partial_\eta\partial_\gamma\phi - \frac{1}{2}K_X g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\beta g_{\gamma\eta}\\ &+ K_X g^{\alpha\beta}g^{\gamma\eta}\partial_\alpha\phi\partial_\eta g_{\beta\gamma} - \frac{1}{2}K_{XX}g^{\alpha\eta}g^{\beta\lambda}g^{\gamma\mu}\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda} = 0\end{aligned} \tag{13}$$

  where

$$K_X = 2\left(X - \hat{X}\right)\left[g_2 + 2g_4\left(X - \hat{X}\right)^2\right] \tag{14}$$

$$K_{XX} = 2\left[g_2 + 6g_4\left(X - \hat{X}\right)^2\right] \tag{15}$$

- This is after more massage (I changed the dummy indices by hand to collect together terms)

$$\left(K_X g^{\alpha\beta} - K_{XX} g^{\mu\alpha} g^{\nu\beta} \partial_\mu\phi \partial_\nu\phi\right) \partial_\alpha\partial_\beta\phi = K_X \left(\partial_\eta g_{\beta\gamma} - \frac{1}{2}\partial_\beta g_{\gamma\eta}\right) g^{\alpha\beta} g^{\gamma\eta} \partial_\alpha\phi - \frac{1}{2} K_{XX} g^{\alpha\eta} g^{\beta\lambda} g^{\gamma\mu} \partial_\alpha\phi \partial_\beta\phi \partial_\gamma\phi \partial_\mu g_{\eta\lambda} \,. \tag{16}$$

- This is without the exta definitions of K_X and K_{XX}

$$2\left(X-\hat{X}\right)\left[g_2 + 2g_4\left(X-\hat{X}\right)^2\right] g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - 2\left[g_2 + 6g_4\left(X-\hat{X}\right)^2\right] g^{\mu\alpha} g^{\nu\beta}\partial_\mu\phi\partial_\nu\phi\partial_\alpha\partial_\beta\phi = 2\left(X-\hat{X}\right)\left[g_2 + 2g_4\left(X-\hat{X}\right)^2\right]\left(\partial_\eta g_{\beta\gamma} - \frac{1}{2}\partial_\beta g_{\gamma\eta}\right) g^{\alpha\beta} g^{\gamma\eta}\partial_\alpha\phi - \left[g_2 + 6g_4\left(X-\hat{X}\right)^2\right] g^{\alpha\eta} g^{\beta\lambda} g^{\gamma\mu}\partial_\alpha\phi\partial_\beta\phi\partial_\gamma\phi\partial_\mu g_{\eta\lambda} \,. \tag{17}$$

All these equations should be equivalent. I would use either the last one or the previous one if you don't mind the extra definitions (there should not be mistakes), and the other ones in case to check intermediate steps.

2 Background solution

In a flat FRW metric the Friedmann equations read

$$3M_{\rm Pl}^2\mathcal{H}^2 = a^2\left(\rho_m + \mathcal{E}_S\right) \tag{18}$$
$$M_{\rm Pl}^2\left(2\mathcal{H}' + \mathcal{H}^2\right) = -a^2\left(p_m + \mathcal{P}_S\right) \,, \tag{19}$$

where ρ_m and p_m are the density and pressure of matter respectively, and

$$\mathcal{E}_S = g_0 + g_2\left(X-\hat{X}\right)\left(3X+\hat{X}\right) + g_4\left(X-\hat{X}\right)^3\left(7X+\hat{X}\right) \tag{20}$$
$$\mathcal{P}_S = -g_0 + g_2\left(X-\hat{X}\right)^2 + g_4\left(X-\hat{X}\right)^4 \,. \tag{21}$$

It is possible to see the contribution of each term of Eq. (20) for the models in Table 1 in Figure 1. The equation of motion of the scalar field is

$$\left[g_2\left(3X-\hat{X}\right) + 2g_4\left(7X-\hat{X}\right)\left(X-\hat{X}\right)^2\right]\phi'' = 2\left[g_2\hat{X} + 2g_4\left(X-\hat{X}\right)^2\left(2X+\hat{X}\right)\right]\mathcal{H}\phi' \tag{22}$$

The conservation of the energy-momentum tensor for the can be written as a shift $\mathcal{P}_\phi$ plus a current n

$$n' + 3\mathcal{H}n = \mathcal{P}_\phi \,. \tag{23}$$

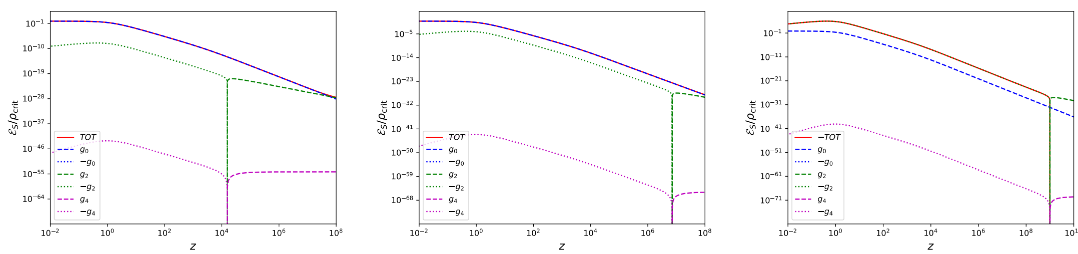

Figure 1: Density contributions

Since the shift is null for shift-symmetric theories the current is conserved on attractor solutions. It reads

$$n = 2\frac{\phi'}{a}\left(X-\hat{X}\right)\left[g_2+2g_4\left(X-\hat{X}\right)^2\right]. \tag{24}$$

It clear that there are at most two different attractors, i.e.

$$X = \hat{X} \tag{25}$$

$$X = \hat{X} + \sqrt{-\frac{g_2}{2g_4}} \tag{26}$$

The equation of state and the sound speed in this model look like

$$w_S = -\frac{g_0 - g_2\left(X-\hat{X}\right)^2 - g_4\left(X-\hat{X}\right)^4}{g_0 + g_2\left(X-\hat{X}\right)\left(3X+\hat{X}\right) + g_4\left(X-\hat{X}\right)^3\left(7X+\hat{X}\right)} \tag{27}$$

$$c_S^2 = \frac{\left(X-\hat{X}\right)\left[g_2+2g_4\left(X-\hat{X}\right)^2\right]}{g_2\left(3X-\hat{X}\right)+2g_4\left(X-\hat{X}\right)^2\left(7X-\hat{X}\right)}. \tag{28}$$

On both the attractors, Eqs. (25) and (26) we have

$$w_S = -1 \tag{29}$$

$$c_S^2 = 0\,, \tag{30}$$

as we wanted. At the end we just have to decide when we want the transition to the attractor happen. In Figure 2 we show the time evolution of the equation of state, while in Figure 3 the time evolution of the equation of state. The values used for these plots are shown in Table 1.

3 Non-linear evolution

The non-linear equations are (in red we highlight the differences w.r.t. [1])

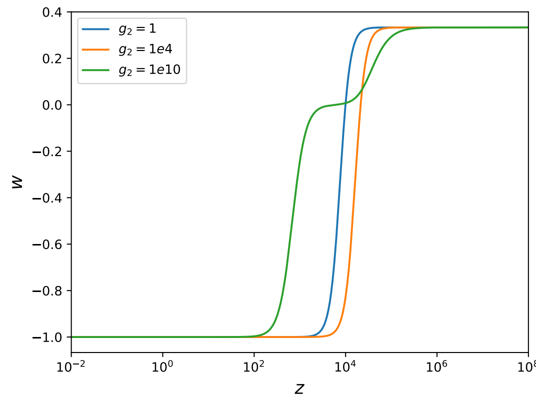

Figure 2: Power law model. w_S

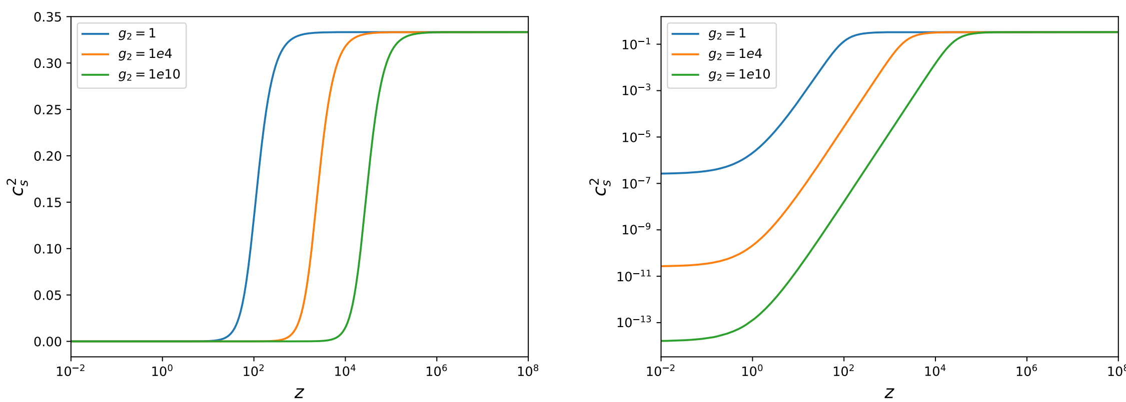

Figure 3: Power law model. c_S^2

$\hat{X}$	g_2	g_4	X_{ini}
1	1	$1.e-20$	$1.e20$
1	$1.e4$	$1.e-20$	$1.e20$
$1.e1$	$1.e10$	$1.e-20$	$1.e20$

Table 1: Parameter values

$$
\begin{aligned}
\pi' =& \xi - \mathcal{H}\pi + \Psi && (31)\\
\xi' =& \left(3c_S^2 + s\right)\mathcal{H}\xi - 3c_S^2\left(\mathcal{H}^2\pi - \mathcal{H}\Psi - \mathcal{H}'\pi - \Phi'\right) + c_S^2\nabla^2\pi \\
&- \partial_i\left[2\left(c_S^2 - 1\right)\xi + c_S^2\Phi - \Psi\right]\partial^i\pi - \left[\left(c_S^2 - 1\right)\xi - 2c_S^2\Phi + \dots\right]\nabla^2\pi \\
&- \frac{\mathcal{H}}{2}\left[\left(2 + 4c_S^2 - 6c_S^4 + s\right)\partial_i\pi\partial^i\pi + 6c_S^2\left(1 - c_S^2\right)\pi\nabla^2\pi\right] \\
&+ \frac{c_S^2 - 1}{2}\partial^i\left[\partial_i\pi\partial_j\pi\partial^j\pi\right] + \mathcal{O}\left(\varepsilon^2\right) , && (32)
\end{aligned}
$$

where

$$
\begin{aligned}
c_{SD}^2 =& g_2\left(3X - \hat{X}\right) + 2g_4\left(7X - \hat{X}\right)\left(X - \hat{X}\right)^2 && (33)\\
c_S^2 =& \frac{\left(X - \hat{X}\right)}{c_{SD}^2}\left[g_2 + 2g_4\left(X - \hat{X}\right)^2\right] && (34)\\
c_a^2 \doteq& \frac{p'}{\rho'} = \frac{c_S^2}{c_{SD}^2} && (35)\\
s \doteq& \frac{\left(c_S^2\right)'}{\mathcal{H}c_S^2} = 3\left(c_S^2 - 1\right) - 18X\left[g_2 + 2g_4\left(7X - 3\hat{X}\right)\left(X - \hat{X}\right)\right]c_a^2 . && (36)
\end{aligned}
$$

- The magic is that in this result there are no additional terms (more perturbations with more derivatives). After a quick inspection of Eq. (10) it makes sense. The only hope to add new terms is to add more second derivatives in the lagrangian (G_3, G_4, G_5)
- Some of these discrepancies could be caused by the loss of generality of this theory w.r.t. the full EFT of K-essence. Some others are substantial differences (missing a perturbation term). Check if it is a typo in the paper.
- In the paper you write you want to keep up to $\mathcal{O}\left(\varepsilon^2\right)$ terms. This makes sense to me, but I can see only up to $\mathcal{O}\left(\varepsilon^1\right)$ terms.
- If we keep $\mathcal{O}\left(\varepsilon^2\right)$ terms, to be fully consistent we should use metric perturbations up to second order

4 Non-linear evolution (iter 1)

The non-linear equations are (in red we highlight the differences w.r.t. [1])

$$
\begin{aligned}
\pi' =& \xi - \mathcal{H}\pi + \Psi && (37)\\
\xi' =& \left(3c_a^2 + s\right)\mathcal{H}\xi - 3c_S^2\left(\mathcal{H}^2\pi - \mathcal{H}\Psi - \mathcal{H}'\pi - \Phi'\right) + c_S^2\nabla^2\pi \\
& - \partial_i\left[2\left(c_S^2 - 1\right)\xi + c_S^2\Phi - \Psi\right]\partial^i\pi - \left[\left(c_S^2 - 1\right)\xi - 2c_S^2\Phi + \ldots\right]\nabla^2\pi \\
& - \frac{\mathcal{H}}{2}\left[\left(2 + 3c_a^2 + c_S^2 - 6c_S^4 + s\right)\partial_i\pi\partial^i\pi + 6c_S^2\left(1 - c_S^2\right)\pi\nabla^2\pi\right] \\
& + \frac{c_S^2 - 1}{2}\partial^i\left[\partial_i\pi\partial_j\pi\partial^j\pi\right] + \mathcal{O}\left(\varepsilon^2\right)\,, && (38)
\end{aligned}
$$

where

$$
\begin{aligned}
c_{SD}^2 =& g_2\left(3X - \hat{X}\right) + 2g_4\left(7X - \hat{X}\right)\left(X - \hat{X}\right)^2 && (39)\\
c_S^2 =& \frac{\left(X - \hat{X}\right)}{c_{SD}^2}\left[g_2 + 2g_4\left(X - \hat{X}\right)^2\right] && (40)\\
c_a^2 \doteq& \frac{p'}{\rho'} = \frac{c_S^2}{\cancel{c_{SD}^2}} = c_S^2 && (41)\\
s =& \frac{\left(c_S^2\right)'}{\mathcal{H}c_S^2} = 3\left(c_S^2 - 1\right) + 18X\left[g_2 + 2g_4\left(7X - 3\hat{X}\right)\left(X - \hat{X}\right)\right]\frac{c_S^2}{c_{SD}^2}\,. && (42)
\end{aligned}
$$

5 Second-order

The approach used to calculate the k-evolution equations assumes that some EFT operators are all we need to know to get them. While this assumption is pretty well motivated, the approach used is subsantially different from mine. Then, we need to find a bridge between them. Here we are looking at the second-order equations calculated with my code. This will not give us the third order operator (we do agree on that term), but it will give us all the linear and second-order ones. I calculated these equations starting from a covariant lagrangian (beyond-Horndeski) but then they were compressed using the α_i description (very close to the EFT language). The expansion we perform is

$$
\phi \equiv \bar{\phi} + \delta\phi + \frac{1}{2}\delta\phi^{(2)} + \ldots\,, \qquad (43)
$$

where $\bar{\phi}$ is a background quantity and the superscript $^{(2)}$ means second-order perturbations. For simplicity we keep first-order perturbations are without indices. While I did not used the Stuckelberg trick, I had to make a scalar field redefinition in order to get the expressions in the EFT language. They are

$$
\begin{aligned}
\delta\phi =& \phi'\pi && (44)\\
\partial_j\partial_i\delta\phi^{(2)} =& \phi'\partial_j\partial_i\pi^{(2)} + 2\phi'\left(\frac{\phi''}{\phi'}\pi + \pi' - \Psi\right)\partial_i\partial_j\pi && (45)\\
& + \phi'\partial_i\left(\frac{\phi''}{\phi'}\pi + \pi' - \Psi\right)\partial_j\pi + \phi'\partial_j\left(\frac{\phi''}{\phi'}\pi + \pi' - \Psi\right)\partial_i\pi\,.
\end{aligned}
$$

To get an intuition on these relations, consider you have your equations of motion in a particular gauge (first order for now). Additionally these equations are fully specified by a set of EFT functions or alphas (plus clearly the perturbations themselves). Now, suppose you want to change gauge. A gauge transformation of the scalar field gives something like

$$\hat{\delta\phi} = \delta\phi + \phi'\alpha \,, \tag{46}$$

where α is the scalar perturbation associated with the 0 component of the gauge transformation. From this relation it is clear that, whenever we do a gauge transformation we are reintroducing terms like ϕ'. And we do not want them. This would mean that different gauges have different freedom, and this is clearly unphysical. At first order it is sufficient to absorb ϕ' and use π instead. At second order is a bit more complicated. But the combination presented in Eq. (45) does the job. Originally I derived it for a single derivative, i.e. $\partial_i\delta\phi^{(2)}$, but then I simmetrised the relation (at the end of the day spatial derivatives are always even in scalar perturbations). This means that when I calculate the second order equations I need to take two spatial derivatives of them (later it will be clear). Another way of calculating and understanding it is to read π as a scalar potential for a velocity field, i.e.

$$u_\mu \equiv \frac{\delta\phi}{\sqrt{2X}} \tag{47}$$

forms a natural four-velocity. Then

$$\delta u_\mu = (\ldots,\ \partial_i\pi + \ldots) \ . \tag{48}$$

Anyway, this could be a source of discrepancies, even if I think that the EFT scalar field perturbations should be π (or similar) by construction. Indeed, to get Eq. (38) I used only the linear redefinition Eq. (44) for all the perturbations.

The second order equation reads

$$\begin{aligned}\nabla^2\xi'^{(2)} = &\left(3c_a^2 + s\right)\mathcal{H}\nabla^2\xi^{(2)} + c_S^2\nabla^2\nabla^2\pi^{(2)} \\ &- 3c_S^2\nabla^2\left(\mathcal{H}^2\pi^{(2)} - \mathcal{H}\Psi^{(2)} - \mathcal{H}'\pi^{(2)} - \Phi'^{(2)}\right) + S_8^{(2)} \,,\end{aligned} \tag{49}$$

where

$$\pi^{(2)\prime} = \xi^{(2)} - \mathcal{H}\pi^{(2)} + \Psi^{(2)} \,, \tag{50}$$

(this redefinition could be wrong!) and $S_8^{(2)}$ is a source term containing all the first-order perturbations squared. The additional laplacian terms w.r.t. the fully non-linear equations are caused by Eq. (45). They could cause non-local operators if $S_8^{(2)}$ is not a total laplacian. The source reads

$$
\begin{aligned}
\frac{S_8^{(2)}}{2} = & \left[1 - c_S^2 + \frac{12\alpha_{KK}c_S^4\mathcal{H}^2}{a^2\left(\mathcal{E}_S + \mathcal{P}_S\right)}\right]\xi\nabla^2\nabla^2\pi + 2\left(2 - c_S^2\right)\partial_i\partial_j\xi\partial^i\partial^j\pi \\
& + 4\left[1 - c_S^2 + \frac{6\alpha_{KK}c_S^4\mathcal{H}^2}{a^2\left(\mathcal{E}_S + \mathcal{P}_S\right)}\right]\partial_i\xi\partial^i\nabla^2\pi + 2\left(1 - c_S^2\right)\partial_i\nabla^2\xi\partial^i\pi \\
& + \left[1 - 3c_S^2 + \frac{12\alpha_{KK}c_S^4\mathcal{H}^2}{a^2\left(\mathcal{E}_S + \mathcal{P}_S\right)}\right]\nabla^2\xi\nabla^2\pi + 2c_S^2\mathcal{H}\left(\nabla^2\pi\right)^2 \\
& - \mathcal{H}\left(2 + 3c_a^2 - c_S^2 + s - sc_S^2\right)\partial_i\pi\partial^i\nabla^2\pi + sc_S^2\mathcal{H}\pi\nabla^2\nabla^2\pi \\
& - \mathcal{H}\left(2 + 3c_a^2 + c_S^2 + s\right)\partial_i\partial_j\pi\partial^i\partial^j\pi + 2c_S^2\nabla^2\Phi\nabla^2\pi \\
& + 2c_S^2\Phi\nabla^2\nabla^2\pi - c_S^2\partial_i\nabla^2\Phi\partial^i\pi + 3c_S^2\partial_i\Phi\partial^i\nabla^2\pi \\
& - 2c_S^2\partial_i\partial_j\Phi\partial^i\partial^j\pi - c_S^2\nabla^2\Psi\nabla^2\pi + 2c_S^2\Psi\nabla^2\nabla^2\pi \\
& + 2\partial_i\partial_j\Psi\partial^i\partial^j\pi + \left(1 - c_S^2\right)\partial_i\pi\partial^i\nabla^2\Psi + \left(1 + 2c_S^2\right)\partial_i\Psi\partial^i\nabla^2\pi
\end{aligned}
\tag{51}
$$

In these expressions we used

$$
\alpha_K c_S^2 = \frac{a^2}{\mathcal{H}^2}\left(\mathcal{E}_S + \mathcal{P}_S\right) . \tag{52}
$$

which is valid for K-essence models. If I try to pull out laplacians the source can be written as

$$
\begin{aligned}
\frac{S_8^{(2)}}{2} = & - \nabla^2\left\{\partial_i\left[2\left(c_S^2 - 1\right)\xi + c_S^2\Phi - \Psi\right]\partial^i\pi\right\} \\
& - \nabla^2\left\{\left[\left(c_S^2 - 1\right)\xi - 2c_S^2\Phi + \ldots\right]\nabla^2\pi\right\} \\
& - \frac{\mathcal{H}}{2}\nabla^2\left[\left(2 + 3c_a^2 + c_S^2 + \ldots + s\right)\partial_i\pi\partial^i\pi + \ldots\right] \\
& + \frac{12\alpha_{KK}c_S^4\mathcal{H}^2}{a^2\left(\mathcal{E}_S + \mathcal{P}_S\right)}\nabla^2\left[\xi\nabla^2\pi\right] - 2c_S^2\left[\nabla^2\pi\nabla^2\xi - \partial_j\partial^i\pi\partial^j\partial_i\xi\right] \\
& + 2c_S^2\partial_i\left[\Psi\partial^i\nabla^2\pi\right] - c_S^2\partial_i\left[\nabla^2\Psi\partial^i\pi\right] \\
& + 2c_S^2\mathcal{H}\partial_i\left[\partial^i\pi\nabla^2\pi\right] + sc_S^2\mathcal{H}\partial_i\left[\pi\partial^i\nabla^2\pi\right] .
\end{aligned}
\tag{53}
$$

In blue terms that are different w.r.t. [1], in magenta terms that are different w.r.t. Eq. (38) and in red I wrote the terms that are different w.r.t. everyone. Things to notice:

- Not all the terms are laplacians, so we should expect non-local terms. This is in disagreement with what we found before;
- All the terms that are not total laplacians are are dependent on π, ξ or Ψ. These terms are the same terms that appear in both Eqs. (45) and (37). It could be that one of these definitions has to be changed. Maybe in the second-rder equivalent of Eq. (37) we should add first order quadratic terms.

- I have an extra operator, defined as

$$\alpha_{KK} = -\frac{a^2 X^2}{\mathcal{H}^2}\left(K_{XX} + \frac{2}{3} X K_{XXX}\right) .$$

  This could explain that I have terms $\propto c_S^4$ and in [1] there are not. It can be that in the specific example of the theory used to calculate the fully non-linear equations this term is not independent w.r.t. the others. But still, those terms are in the wrong place w.r.t. Eq. (38). But maybe a field redefinition adjust everyhting.

A Gauge transformations

The exponential map

$$\tilde{\boldsymbol{T}} = e^{\pounds_\xi} \boldsymbol{T} , \tag{54}$$

allows to write the gauge transformation of any tensor $\boldsymbol{T}$. Here, $\tilde{\boldsymbol{T}}$ is the gauge transformed tensor and $\pounds_\xi$ is the Lie derivative w.r.t. the generator of the gauge transformation ξ^μ. Expanding $\boldsymbol{T}$ as

$$\boldsymbol{T} = \boldsymbol{T}^{(0)} + \delta\boldsymbol{T}^{(1)} + \frac{1}{2}\delta\boldsymbol{T}^{(2)} , \tag{55}$$

Eq. (54) becomes

$$\tilde{\boldsymbol{T}}^{(0)} = \boldsymbol{T}^{(0)} \tag{56}$$

$$\delta\tilde{\boldsymbol{T}}^{(1)} = \delta\boldsymbol{T}^{(1)} + \pounds_{\xi^{(1)}} \boldsymbol{T}^{(0)} \tag{57}$$

$$\delta\tilde{\boldsymbol{T}}^{(2)} = \delta\boldsymbol{T}^{(2)} + \pounds_{\xi^{(2)}} \boldsymbol{T}^{(0)} + \pounds_{\xi^{(1)}} \pounds_{\xi^{(1)}} \boldsymbol{T}^{(0)} + 2\pounds_{\xi^{(1)}} \delta\boldsymbol{T}^{(1)} . \tag{58}$$

We can decompose ξ as

$$\xi^\mu = \left(\alpha,\, \partial^i\beta + \gamma^i\right) , \tag{59}$$

where γ_i is a divergence-free vector ($\partial_i\gamma^i = 0$). Now we want to turn this into gauge transformations of the metric $g_{\mu\nu}$ and the scalar field ϕ. The scalar field perturbations can be written as

$$\phi = \bar{\phi} + \delta\phi + \frac{1}{2}\delta\phi^{(2)} ,$$

where we omit the superscript $^{(1)}$ on first-order perturbations and a bar indicates background quantities (we will omit also this bar in the following without confusion). Instead, for the metric we consider a line element of this form

$$ds^2 = a^2\left[-\left(1 + \delta g_{00}\right) d\tau^2 + 2\delta g_{0i} dx^i d\tau + \left(\delta_{ij} + \delta g_{ij}\right) dx^i dx^j\right] , \tag{60}$$

where

$$\delta g_{00} = 2\Psi + \Psi^{(2)} \tag{61}$$

$$\delta g_{0i} = \partial_i B - S_i + \frac{1}{2}\left(\partial_i B^{(2)} - S_i^{(2)}\right) \tag{62}$$

$$\begin{aligned}\delta g_{ij} =& -2\Phi\delta_{ij} + 2\partial_i\partial_j E + \partial_i F_j + \partial_j F_i + h_{ij} \\ & - \Phi^{(2)}\delta_{ij} + \partial_i\partial_j E^{(2)} + \frac{1}{2}\left(\partial_i F_j^{(2)} + \partial_j F_i^{(2)} + h_{ij}^{(2)}\right) .\end{aligned} \tag{63}$$

Here $S_i^{(n)}$ and $F_i^{(n)}$ are divergence free vectors, while $h_{ij}^{(n)}$ are divergence-free and traceless tensors ($\gamma^{ij} h_{ij}^{(n)} = 0$). At linear-order the metric perturbations transform as (for now let me neglect all vector and tensor perturbations)

$$\begin{aligned}&\mathcal{H}\alpha + \Psi - \tilde{\Psi} + \alpha' \\ &\nabla^2 B - \nabla^2 \tilde{B} - \nabla^2\alpha + \nabla^2\beta' \\ &-\mathcal{H}\nabla^2\alpha + \nabla^2\Phi - \nabla^2\tilde{\Phi} \\ &\nabla^2\nabla^2 E - \nabla^2\nabla^2\tilde{E} + \nabla^2\nabla^2\beta\end{aligned}$$

$$\begin{aligned}\tilde{\Psi} =& \Psi + \alpha' + \mathcal{H}\alpha \\ \nabla^2\tilde{B} =& \nabla^2 B - \nabla^2\alpha + \nabla^2\beta' \\ \nabla^2\tilde{\Phi} =& \nabla^2\Phi - \mathcal{H}\nabla^2\alpha\end{aligned}$$

$$\frac{\tilde{\delta\phi}}{\phi'} = \frac{\delta\phi}{\phi'} + \alpha$$

References

[1] Farbod Hassani, Julian Adamek, Martin Kunz, and Filippo Vernizzi. k-evolution: a relativistic N-body code for clustering dark energy. *JCAP*, 12:011, 2019.