The Galileon model

Farbod Hassani

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1 The Galileon model

In Galileon model the modifications to GR arise through a Galilean-invariant scalar field, i.e., the scalar field is invariant under the transformation $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$, where b_μ is a constant vector.

The action of the cubic Galileon model, for when the scalar field does not have a direct coupling matter sector reads as,

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{2} c_2 \mathcal{L}_2 - \frac{1}{2} c_3 \mathcal{L}_3 - \mathcal{L}_m \right], \tag{1}$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, c_2, c_3 are dimensionless constants and $\mathcal{L}_m$ is the matter Lagrangian. The terms $\mathcal{L}_2$ and $\mathcal{L}_3$ in the Lagrangian are describing the Galileon field, and ∇ denotes the covariant derivative.

$$\mathcal{L}_2 = \nabla_\mu \phi \nabla^\mu \phi, \quad \mathcal{L}_3 = \frac{2}{M^3} \Box\phi \nabla_\mu \phi \nabla^\mu \phi, \tag{2}$$

where ϕ is the Galileon scalar field and $\mathcal{M}^3 \equiv M_{\mathrm{Pl}} H_0^2$ where H_0 is the value Hubble function today. The Einstein field equations read:

write some details

$$G_{\mu\nu} = \kappa \left[T^m_{\mu\nu} + T^{c_2}_{\mu\nu} + T^{c_3}_{\mu\nu} \right], \tag{3}$$

and the scalar field equation,

derive!

$$c_2 \Box\phi + \frac{2}{M^3} c_3 \left[(\Box\phi)^2 - \nabla^\alpha \nabla_\beta \phi \nabla^\beta \nabla_\alpha \phi \right] - R_{\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi = 0, \tag{4}$$

where the energy-momentum tensor reads as

derive the equations by hand and using mathematica!

$$T^{c_2}_{\mu\nu} = c_2 \left[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right], \tag{5}$$

$$T^{c_3}_{\mu\nu} = \frac{c_3}{M^3} \left[2\nabla_\mu \phi \nabla_\nu \phi \Box\phi + 2 g_{\mu\nu} \nabla^\alpha \nabla^\beta \phi \nabla_\alpha \nabla_\beta \phi - 4 \nabla^\lambda \phi \nabla_{(\mu} \phi \nabla_{\nu)} \nabla_\lambda \phi \right], \tag{6}$$

and $T^m_{\mu\nu}$ is the energy-momentum tensor of other species in the universe.

It is worth noting that the general Galileon action has two more Lagrangian densities $\mathcal{L}_4$ and $\mathcal{L}_5$, which are quartic and quintic in derivatives of scalar field. However, the higher order derivatives make the theory suffer from various theoretical problems. There is also a linear term in the general Galileon theory $\mathcal{L}_1$ which is often neglected if the scalar field plays the role of dark energy.

1.1 Background evolution

To derive the background equations we assume a FLRW metric,

$$ds^2 = dt^2 - a(t)^2\delta_{ij}dx^i dx^j \tag{7}$$

then the Friedmann equations read,

$$3H^2 = 8\pi G\left[\rho_m + \frac{c_2}{2}\dot{\bar{\phi}}^2 + \frac{6c_3}{M^3}\dot{\bar{\phi}}^3 H\right], \tag{8}$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}\left[\rho_m + 4c_2\dot{\bar{\phi}}^2\right] + \frac{6c_3}{M^3}\left(\dot{\bar{\phi}}^3 H - \dot{\bar{\phi}}\ddot{\bar{\phi}}\right), \tag{9}$$

where the derivatives and quantities are calculated with respect to physical time, ρ_m is the matter density and we have neglected the contribution from the radiation. The Galileon field equation of motion reads,

$$c_2[\ddot{\bar{\phi}} + 3\dot{\bar{\phi}}H] + 6\frac{c_3}{\mathcal{M}^3}\left[2\ddot{\bar{\phi}}\dot{\bar{\phi}}H + 3\dot{\bar{\phi}}^2H^2 + \dot{\bar{\phi}}^2\dot{H}\right] = 0 \tag{10}$$

1.2 Perturbation, quasi-static approximation

The derive the perturbation equations we assume a perturbed FLRW metric in Newtonian gauge,

$$ds^2 = (1 + 2\Psi)dt^2 - a(t)^2(1 - 2\Phi)\delta_{ij}dx^i dx^j, \tag{11}$$

and we assume that the gravitational potentials are small and $\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t)$. Moreover we assume the quasi-static limit, which seems to be a good approximation in the sub-horizon scales. In the quasi-static limit we neglect the time derivative of scalar field compared with the spatial derivatives. Moreover additionally, the Newtonian potentials Φ, Ψ and their first order spatial derivatives $\Phi_{,i}$ and $\Psi_{,i}$ can be neglected compared with the $\nabla^2\Phi$ as the Newtonian potential and their first order derivative are small in the scales of interest. So we have $(1 + 2\Psi)\partial_i\partial^i\phi \approx \partial_i\partial^i\phi$ and $\partial_i\partial^i\Phi\partial_j\Phi \ll \partial_i\partial^i\Phi$. Under the previously mentioned approximations the (0,0) component of the Einstein field equations (the Poisson equation) reads

$$\partial^2\Phi = 4\pi G a^2\delta\rho_m - \frac{\kappa c_3}{\mathcal{M}^3}\dot{\phi}^2\partial^2\phi. \tag{12}$$

The scalar field equations,

$$\frac{2c_3}{\mathcal{M}^3}\dot{\phi}^2\partial^2\Psi = \left[-c_2 - \frac{4c_3}{\mathcal{M}^3}(\ddot{\phi} + 2H\dot{\phi})\right]\partial^2\phi + \frac{2c_3}{a^2\mathcal{M}^3}\left[\left(\partial^2\phi\right)^2 - \left(\partial_i\partial_j\phi\right)^2\right]. \tag{13}$$

Where we have not used $\delta\phi$ and we write it as ϕ but from the order of equations we can guess whether $\bar{\phi}$ or $\delta\phi$ should be used.
Assuming $\Phi = \Psi$ we can combine the two equations,

$$\partial^2\phi + \frac{1}{3\beta_1 a^2\mathcal{M}^3}\left[\left(\partial^2\phi\right)^2 - \left(\partial_i\partial_j\phi\right)^2\right] = \frac{M_{\rm Pl}}{3\beta_2}8\pi G a^2\delta\rho_m \tag{14}$$

Here, $\partial^2 = \partial_i \partial^i$ is the spatial Laplacian, $(\partial_i\partial_j\phi)^2 = (\partial_i\partial_j\phi)\left(\partial^i\partial^j\phi\right).\delta\rho_m$ is the matter density perturbation, $\rho_m = \bar{\rho}_m(t) + \delta\rho_m(t,\vec{x})$. The dimensionless functions β_1 and β_2 are defined as

$$\beta_1 = \frac{1}{6c_3}\left[-c_2 - \frac{4c_3}{\mathcal{M}^3}(\ddot{\phi} + 2H\dot{\phi}) + 2\frac{\kappa c_3^2}{\mathcal{M}^6}\dot{\phi}^4\right]$$
$$\beta_2 = 2\frac{\mathcal{M}^3 M_{\rm Pl}}{\dot{\phi}^2}\beta_1.$$

1.3 Vainshtein screening

In order to observe the screening through the derivative couplings, it is instructive to use spherical symmetry. Where we can rewrite the equation as,

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\phi_{,r}\right] + \frac{2}{3}\frac{1}{\mathcal{M}^3 a^2 \beta_1}\frac{1}{r^2}\frac{d}{dr}\left[r\phi,_r^2\right] = \frac{M_{\rm Pl}}{3\beta_2}8\pi G a^2 \delta\rho \tag{15}$$

Integrating it once,

$$\phi_{,r} + \frac{2}{3}\frac{1}{\mathcal{M}^3 a^2\beta_1}\frac{1}{r}\phi,_r^2 = \frac{2M_{\rm Pl}}{3\beta_2}\frac{GM(r)}{r^2}a^2 \tag{16}$$

Now the equation becomes an algebraic equation for $\phi_{,r}$ which considering a mass profile can be solved. And $M(R) = 4\pi\int_0^R \delta\rho_m(r)r^2 dr$ is the matter contribution to the mass enclosed within a radius R. For simplicity considering a top-hat density distribution of radius R, the physical solutions of $\phi_{,r}$ are,

$$\phi_{,r} = \frac{4M_{\rm Pl}a^2 r^3}{3\beta r_V^3}\left[\sqrt{\left(\frac{r_V}{r}\right)^3 + 1} - 1\right]\frac{GM(R)}{r^2}, \quad r \geq R \tag{17}$$

$$\phi_{,r} = \frac{4M_{\rm Pl}a^2 R^3}{3\beta r_V^3}\left[\sqrt{\left(\frac{r_V}{R}\right)^3 + 1} - 1\right]\frac{GM(r)}{r^2}, \quad r < R \tag{18}$$

Where we have used the field redefinition

$$\delta\phi \to \frac{\beta}{\beta_2}\delta\phi$$

as discussed in section 2 of 1306.3219. The distance scale r_V is the Vainshtein radius and is given by,

$$r_V^3 = \frac{8M_{Pl}r_S}{9\mathcal{M}^3\beta_1\beta_2}, \tag{19}$$

where $r_S \equiv 2GM(R)$ is the Schwarzschild radius of the top-hat mass density.

Considering the last term in the modified Poisson equation, it represents a fifth force due to the presence of Galileon scalar field.

$$F_{5th} = -\frac{\kappa c_3}{\mathcal{M}^3}\frac{\beta}{\beta_2}\dot{\phi}^2\phi_{,r} \tag{20}$$

Taking the limits where $r \gg r_V$ and $r \ll r_V$ we haves

$$F_{5th} = -\frac{2c_3 a^2 \dot{\phi}^2}{3\mathcal{M}^3 M_{\rm Pl}\beta_2}\frac{GM(R)}{r^2}, \quad r \gg r_V, \tag{21}$$

$$F_{5th} \sim 0, \quad r \ll r_V. \tag{22}$$

2 MG-evolution implementation

In this section we discuss the approximations and framework we use to obtain the expression for implementing in MG-evolution.

2.1 Linear part

The effective gravitational potential in Cubic Galileon theory at linear regime reads as (Eq. 29 of 1306.3219)

$$G_{\rm eff} = G\left(1 - \frac{2}{3}\frac{c_3\dot{\phi}^2}{M_{\rm Pl}\mathcal{M}^3\beta_2}\right) \tag{23}$$

where $M_{\rm Pl}$ is the Planck mass, ϕ is the Galileon scalar field and $\mathcal{M}^3 \equiv M_{\rm Pl}H_0^2$, β_1 and β_2 are,

$$\beta_1 = \frac{1}{6c_3}\left[-c_2 - \frac{4c_3}{\mathcal{M}^3}(\ddot{\phi} + 2H\dot{\phi}) + 2\frac{\kappa c_3^2}{\mathcal{M}^6}\dot{\phi}^4\right],$$
$$\beta_2 = 2\frac{\mathcal{M}^3 M_{\rm Pl}}{\dot{\phi}^2}\beta_1.$$
$$\kappa = \frac{1}{M_{\rm Pl}^2} = 8\pi G$$

In our discussion we set $c_2 = -1$ (see discussion at page 5 of 1709.09135). We also use the tracker solution,

$$\xi \equiv \frac{\dot{\phi}H}{M_{\rm Pl}H_0^2} \tag{24}$$

where ξ is a constant. It is important to note that in cases where $c_2/c_3^{2/3}$ and c_3 is given similar to the first table in 1306.3219 we need to rescale c_2 to -1. So for example if,

$$c_2/c_3^{3/2} = -5.378, c_3 = 10 \tag{25}$$

which is equivalent to

$$c_2 = -24.9624, c_3 = 10 \tag{26}$$

If we convert these to our notation where $c_2 = -1$, then we obtain

$$c_3 = 10/(24.9624)^{3/2} = 0.080, \xi = -\frac{c_2}{6c_3} = 2.0786 \tag{27}$$

where it follows from the fact that to get $c_2 = -1$ and since we have $\mathcal{L}_2 \sim \nabla_\mu \phi \nabla^\mu \phi$. We need to redefine the field as

$$\phi \to \phi/(-c_2)^{1/2} \tag{28}$$

As $\mathcal{L}_3 \sim \phi^3$ ignoring the derivatives, we get

$$c_3 \to c_3/(-c_2)^{3/2} \tag{29}$$

Following the discussion which leads to E. 18 in 1709.09135 we can deduce that ξ is a constant and can be obtained given c_3,

$$\xi = -\frac{1}{6c_3} \tag{30}$$

As a result we can write,

$$\dot{\phi} = \frac{\xi M_{\mathrm{Pl}} H_0^2}{H} \tag{31}$$

$$\ddot{\phi} = -\frac{\xi M_{\mathrm{Pl}} H_0^2 \dot{H}}{H^2} \tag{32}$$

Note that $H = \frac{da}{dt} = \mathcal{H}/a$, where $\mathcal{H}$ is the conformal Hubble factor. Following the discussion presented in 1308.3699 for the background tracker solution we can derive the Hubble expansion rate as a function of a (Eq. 12 of 1308.3699)

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2}{2}\left[\left(\Omega_{m0}a^{-1} + \Omega_{r0}a^{-2}\right) + a^2\sqrt{\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4}\right)^2 + 4\left(1 - \Omega_{m0} - \Omega_{r0}\right)}\right]. \tag{33}$$

Where $H_0^2 = \frac{8\pi G}{3}$ in MG-evolution unit.Computing $\mathcal{H}'$ results in,

$$\begin{aligned} \mathcal{H}' = &- \frac{\mathcal{H}_0^2(a\Omega_m + 2\Omega_r)}{4a^2} - \frac{\mathcal{H}_0^2(a\Omega_m + \Omega_r)(3a\Omega_m + 4\Omega_r)}{4a^6\sqrt{4(1-\Omega_m-\Omega_r) + \frac{(a\Omega_m+\Omega_r)^2}{a^8}}} \\ &+ \frac{\mathcal{H}_0^2 a^2}{2}\sqrt{4(1-\Omega_m-\Omega_r) + \frac{(a\Omega_m+\Omega_r)^2}{a^8}} \end{aligned} \tag{34}$$

We also have,

$$\dot{H} = \frac{\mathcal{H}' - \mathcal{H}^2}{a^2} \tag{35}$$

With all the previous expressions we can obtain the linear $\Delta G/G$ for the Cubic Galileon model.

3 Screening part

Following the discussion which leads to Eq. 21 of 1306.3219 we can write,

$$\frac{\Delta G}{G}|_{\mathrm{tot}} = \frac{\Delta G}{G}|_{\mathrm{linear}} \times \frac{\Delta G}{G}|_{\mathrm{Vainshtein}} \tag{36}$$

where $\frac{\Delta G}{G}|_{\rm linear}$ is obtained following the discussion in the previous section and $\frac{\Delta G}{G}|_{\rm Vainshtein}$ can be written in Fourier space as 2003.05927,

$$\frac{\Delta G}{G}|_{\rm Vainshtein} = \frac{\sqrt{1+\epsilon}-1}{\epsilon} \tag{37}$$

where,

$$\epsilon \equiv (\frac{r_V}{r})^3 \rightarrow (\frac{k_*}{k})^3 \tag{38}$$

where r_V the Vainshtein radius and k_* is the corresponding wavenumber in Fourier space. As a result we have two free parameters, namely k_* in the screening and c_3 in the linear modification.