1 Non-Linear equation study

TODO:
-Why we get different power after realization in Gevolution for low k? this need to be checked and is so wierd.
TODO:
- Check non-linear term not suppressed by c_s^2 in stress tensor? why it should be there?
- Write a Mathematica notebook for computing stress tensor in a systematic way,

Up to now we have studied some feature in Non-linear equation, but here we want to study the equation carefully,
Changing from linear to non-linear equations is done with a number in Gevolution which in case 1, turns on the non-linearities. It seems that the non-linear terms make the equation unstable, to see that we plot our best non-linear run,

$$\begin{aligned}
&\zeta' - 3w\mathcal{H}\zeta + 3c_s^2\Big(\mathcal{H}^2 - \mathcal{H}'\Big)\pi - 3c_s^2\Big(\Phi' + \mathcal{H}\Psi\Big) - c_s^2\nabla^2\pi \\
&- 2c_s^2\Phi\nabla^2\pi + (1-c_s^2)\Psi\nabla^2\pi + 3c_s^2\mathcal{H}(1+w)\pi\nabla^2\pi - (1-c_s^2)(\zeta+\Psi)\nabla^2\pi \\
&+ c_s^2\nabla\Phi.\nabla\pi - (2c_s^2-1)\nabla\Psi.\nabla\pi + \frac{\mathcal{H}}{2}\Big(2+3w+c_s^2\Big)\nabla\pi.\nabla\pi - 2(1-c_s^2)\nabla\pi.\nabla(\zeta+\Psi) = 0
\end{aligned} \tag{1}$$

After simplification we have,

$$\begin{aligned}
&\zeta' - 3w\mathcal{H}\zeta + 3c_s^2\Big(\mathcal{H}^2 - \mathcal{H}'\Big)\pi - 3c_s^2\Big(\Phi' + \mathcal{H}\Psi\Big) - c_s^2\nabla^2\pi \\
&- 2c_s^2\Phi\nabla^2\pi + 3c_s^2\mathcal{H}(1+w)\pi\nabla^2\pi - (1-c_s^2)\zeta\nabla^2\pi + c_s^2\nabla\Phi.\nabla\pi - \nabla\Psi.\nabla\pi + \frac{\mathcal{H}}{2}\Big(2+3w+c_s^2\Big)\nabla\pi.\nabla\pi \\
&- 2(1-c_s^2)\nabla\pi.\nabla\zeta = 0
\end{aligned} \tag{2}$$

The other equation is like before,

$$\pi' = \zeta + \Psi - \mathcal{H}\pi \tag{3}$$

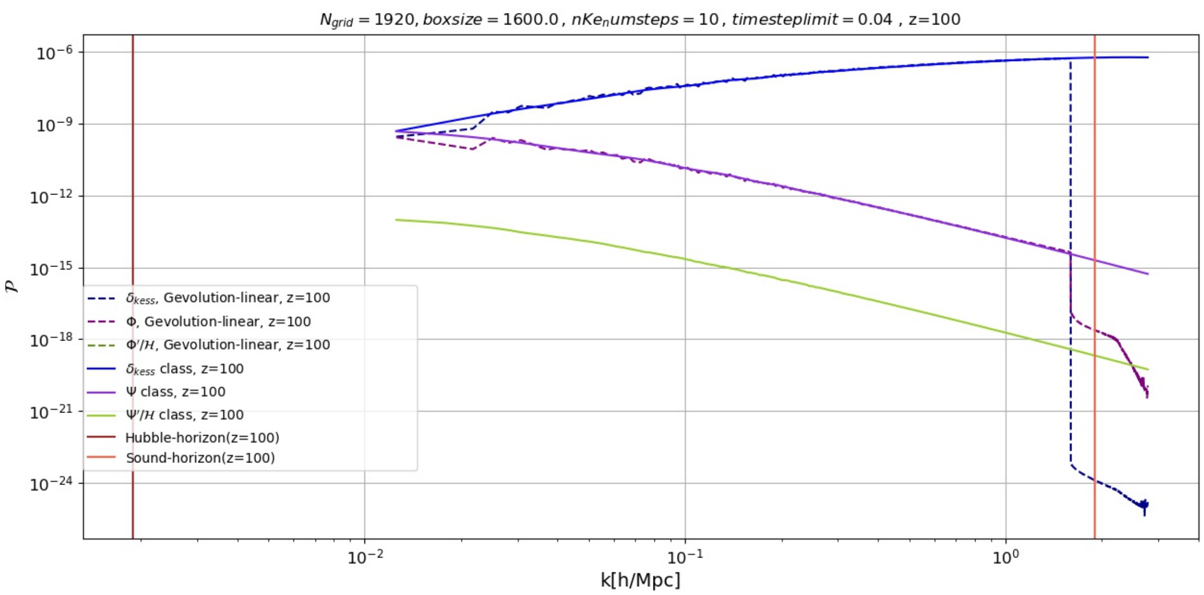
We actually did nothing except substituting the π' and in non-linear part the substitution is very straightforward.

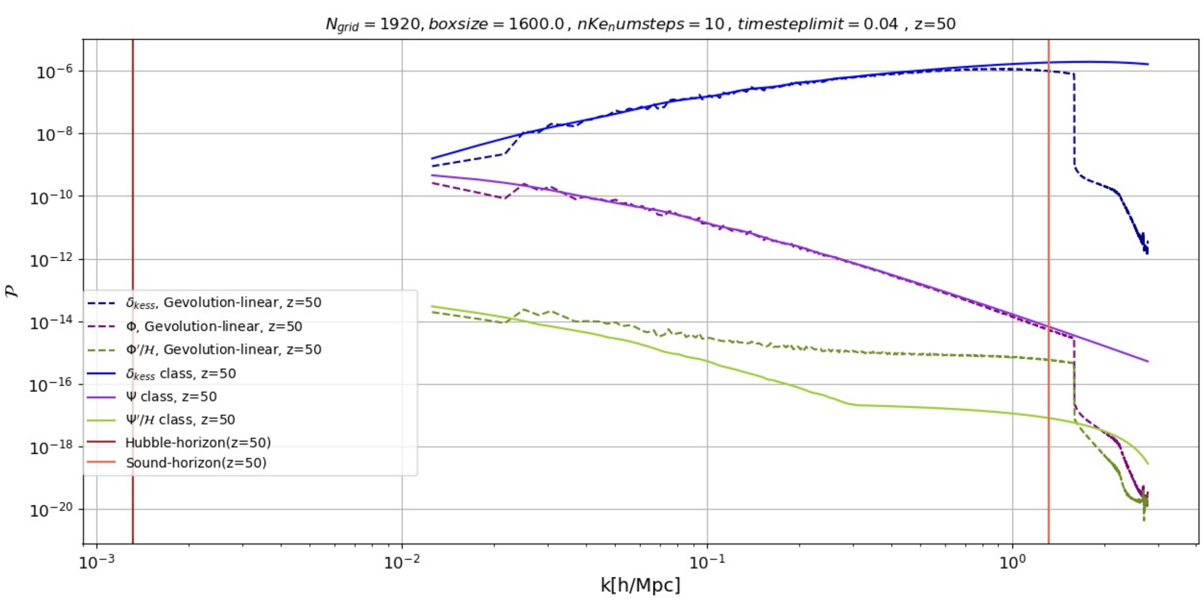
$$\begin{aligned}
T_0^0(Gev) &= \Omega_{kess}^0 a^{-3w}\left[1 + \frac{1+w}{c_s^2}\left(\zeta - 3\mathcal{H}c_s^2\pi - (1-2c_s^2)\frac{(\vec{\nabla}\pi)^2}{2}\right)\right] \\
T_0^i(Gev) &= -\Omega_{kess}^0 a^{-3w}(1+w)\Big[1 - (\frac{1}{c_s^2} - 1)\frac{(\vec{\nabla}\pi)^2}{2}\Big]\partial_i\pi \\
T_j^i(Gev) &= w\,\Omega_{kess}^0 a^{-3w}\left(1 + \frac{1+w}{w}\Big[-3\mathcal{H}w\pi + \zeta - \frac{(\vec{\nabla}\pi)^2}{2}\Big]\delta_j^i + \frac{1+w}{w}\delta^{ik}\partial_k\pi\partial_j\pi\right)
\end{aligned} \tag{4}$$

Note that in Gevolution to compute $\nabla\pi.\nabla(\zeta+\Psi)$ we use the symmetric derivative as follwoing,

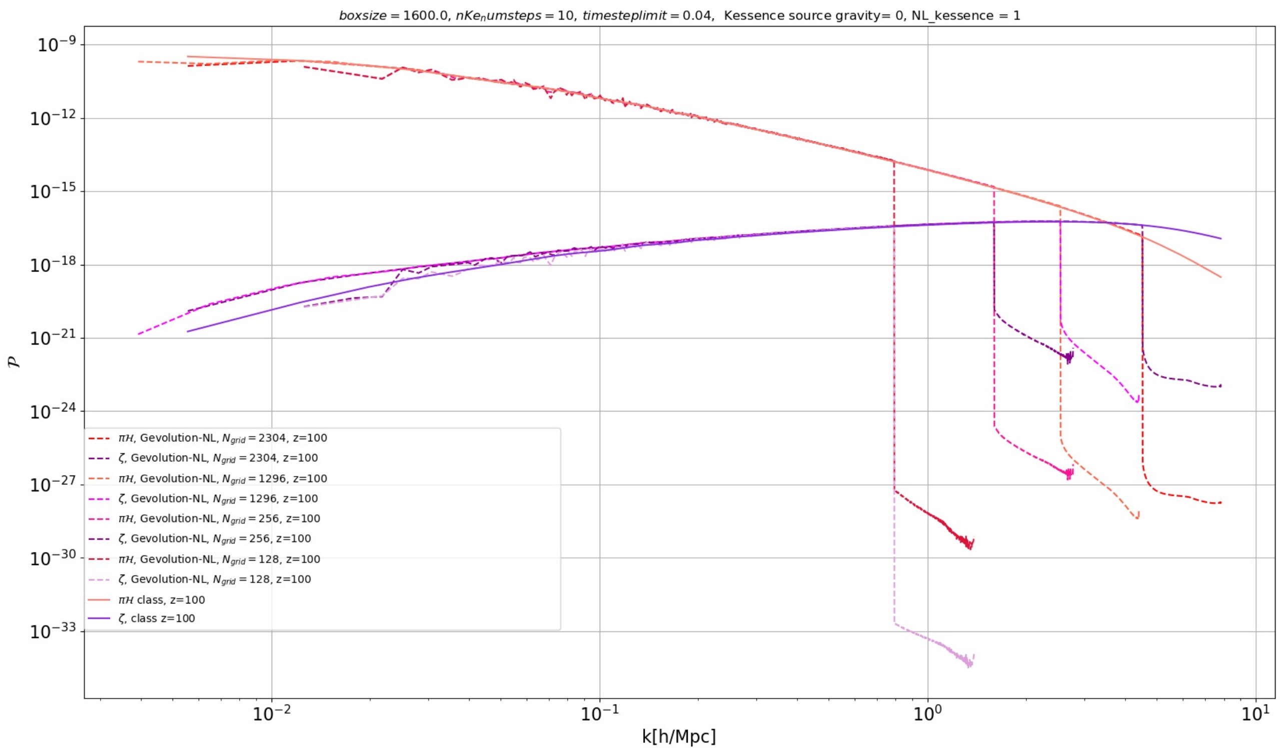
$$\nabla\pi.\nabla(\zeta+\Psi) = \frac{1}{4dx^2}\sum_{i=0}^{2}\Big[\pi(x_i+1) - \pi(x_i-1)\Big]\Big[\Big(\zeta(x_i+1)+\Psi(x_i+1)\Big) - \Big(\zeta(x_i-1)+\Psi(x_i-1)\Big)\Big] \tag{5}$$

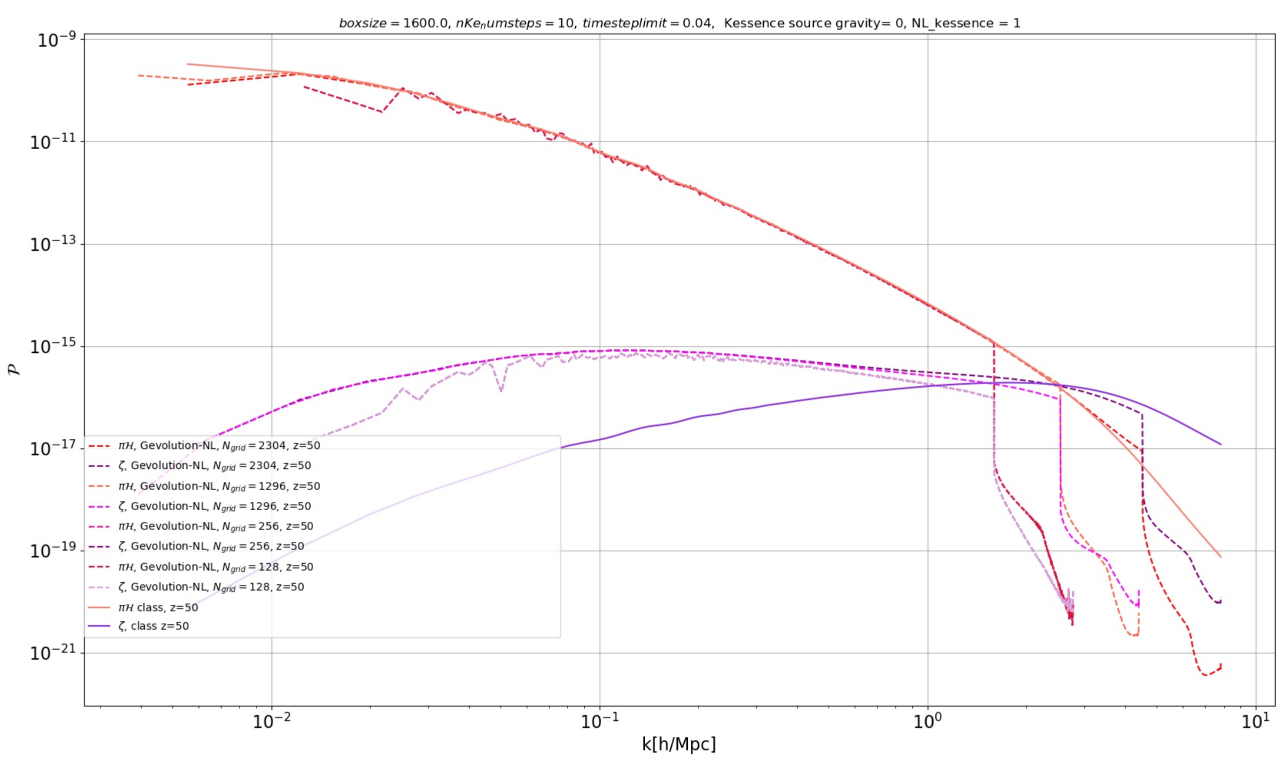
where "1/4" coefficient appear since we are using symmetric derivative and using points with distance two. "x_i" is the lattice coordinate.
As we have seen before the equation grows in time and causes some problems in the stress tensor part.
Running the equation in Gevolution will result the following figures



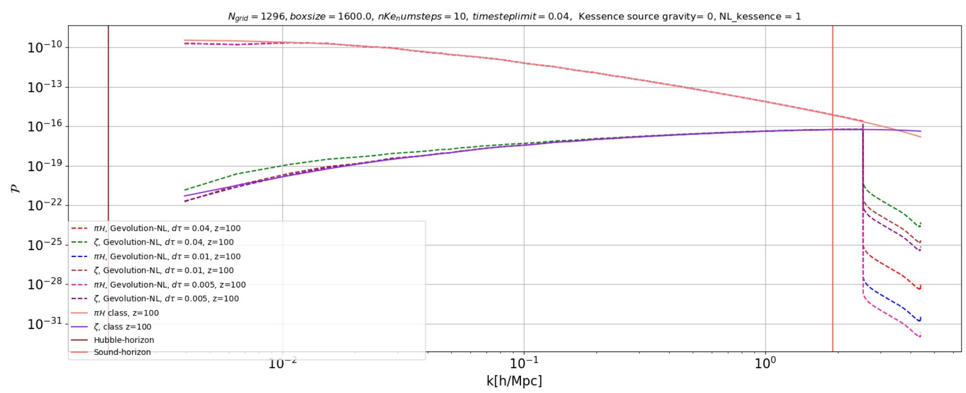


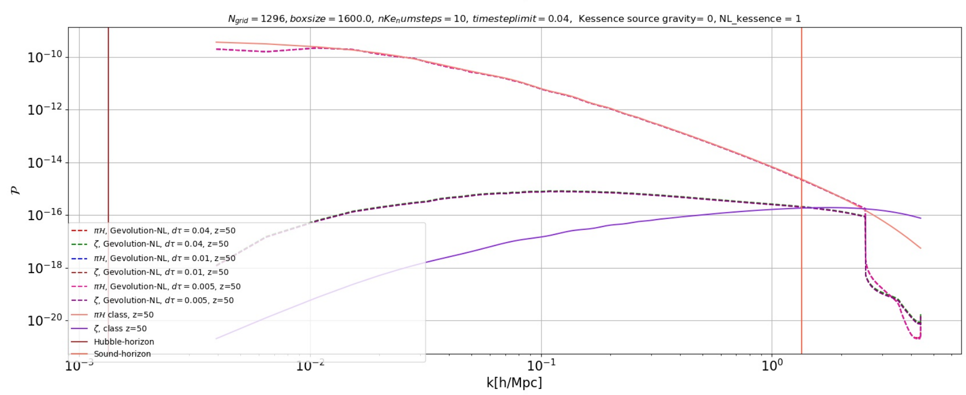
which is just evolution up to redshift z=50, now lets look at the non-linear terms exactly, for different number of grids run we get the following plot,

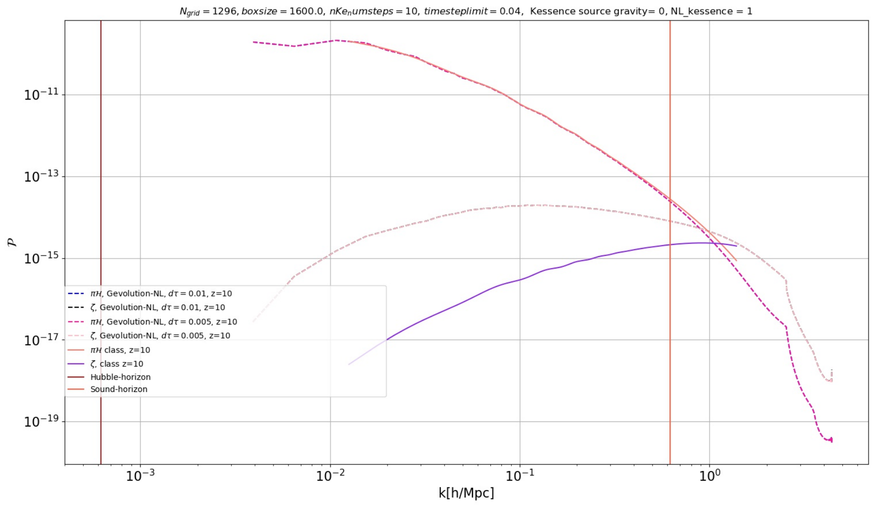

boxsize = 1600.0, nKe_numsteps = 10, timesteplimit = 0.04, Kessence source gravity= 0, NL_kessence = 1
πH, Gevolution-NL, N_grid = 2304, z=100
ζ, Gevolution-NL, N_grid = 2304, z=100
πH, Gevolution-NL, N_grid = 1296, z=100
ζ, Gevolution-NL, N_grid = 1296, z=100
πH, Gevolution-NL, N_grid = 256, z=100
ζ, Gevolution-NL, N_grid = 256, z=100
πH, Gevolution-NL, N_grid = 128, z=100
ζ, Gevolution-NL, N_grid = 128, z=100
πH class, z=100
ζ, class z=100
k[h/Mpc]


boxsize = 1600.0, nKe_numsteps = 10, timesteplimit = 0.04, Kessence source gravity= 0, NL_kessence = 1
πH, Gevolution-NL, N_grid = 2304, z=50
ζ, Gevolution-NL, N_grid = 2304, z=50
πH, Gevolution-NL, N_grid = 1296, z=50
ζ, Gevolution-NL, N_grid = 1296, z=50
πH, Gevolution-NL, N_grid = 256, z=50
ζ, Gevolution-NL, N_grid = 256, z=50
πH, Gevolution-NL, N_grid = 128, z=50
ζ, Gevolution-NL, N_grid = 128, z=50
πH class, z=50
ζ, class z=50
k[h/Mpc]

As it is clear the non-linearities source long wavemodes and it is the case for different number of grids! So it means that in principle we can work with small runs and solve the issue if there is any,
Moreover we see that when we increase the number of grids the solution grows more! which is already observed when we the particles blowup at z=0 for Ngrid=128 which they do not for Ngrid=64!
Now lets decrease the time stepping for the fixed Ngrid=1296 and check the solver,







So according to the figures the solution seems stable in both time and spatial resolution! Now what we can do?

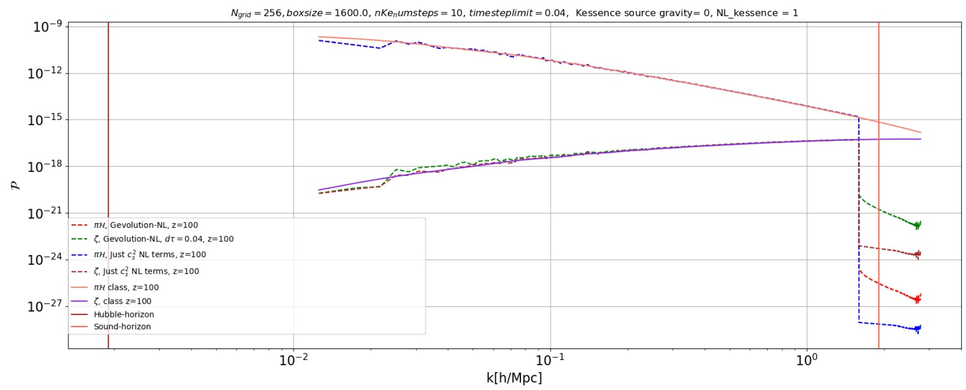
For the start we consider just non-linear terms which are suppressed by c_s^2,

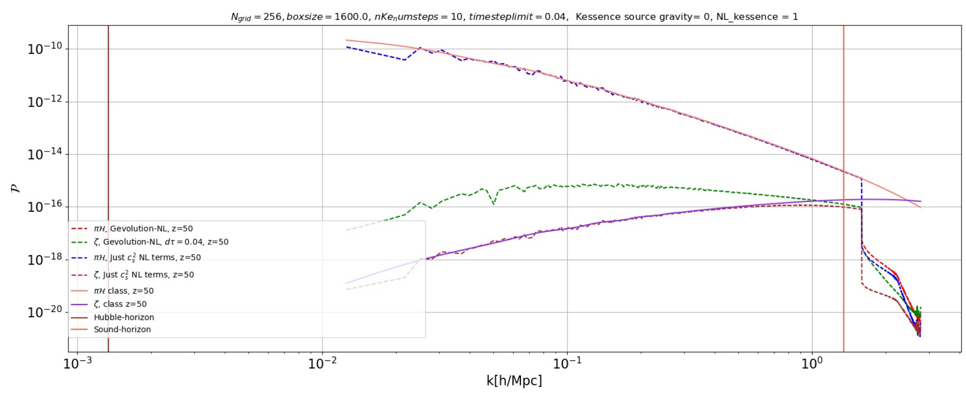
$$\zeta' - 3w\mathcal{H}\zeta + 3c_s^2\Big(\mathcal{H}^2 - \mathcal{H}'\Big)\pi - 3c_s^2\Big(\Phi' + \mathcal{H}\Psi\Big) - c_s^2\nabla^2\pi - 2c_s^2\Phi\nabla^2\pi - c_s^2\Psi\nabla^2\pi + 3c_s^2\mathcal{H}(1+w)\pi\nabla^2\pi + c_s^2(\zeta+\Psi)\nabla^2\pi$$
$$+ c_s^2\nabla\Phi.\nabla\pi - 2c_s^2\nabla\Psi.\nabla\pi + c_s^2\frac{\mathcal{H}}{2}\nabla\pi.\nabla\pi + 2c_s^2\nabla\pi.\nabla(\zeta+\Psi) = 0 \tag{6}$$

and,

$$\pi' = \zeta + \Psi - \mathcal{H}\pi \tag{7}$$

At the moment we just look at the evolution up to z=50





As it is clear and we could guess the problem comes from the fact that all linear terms are suppressed by c_s^2 while some of the non-linear terms are not, so for some range of c_s^2 and according to the variation of the field ζ in space we could have a strong force from non-linear terms which push the spectrum to the very high numbers and causes growing in the differential equation. What we conclude is that either the differential equation is wrong and all the non-linear terms should be suppressed by c_s^2 or k-essence model is really unstable at non-linear level....

1.1 Theoretical studies

TODO:
Martin:
At the limit $c_s^2 \to 0$ why the model does not exist?
- Look at the scaling at the level of action, and check the same in the equation.
- Look at the most important terms, and just solve them and compare with explicit theoretical solution,
-Starting the linear paper!
For $c_s^2 \to 1$ check you get what you expect.
Check some terms vanish, and you forgot to vanish them,
What is the dominant terms, take control over the terms.
Send an email t Filipo and Martin,
Study the terms in the differential equation in the continuum limit,
-Study some features from action to the level of equations like scaling and make sure at some level we are dealing with the right solution.
- Check the non-linear terms again, and make sure the equation is true,
-Simplify the equation as much as possible,
- Write a mathematica to calculate stress tensor and check everything at the level of stress tensor,
- why we don't get higher order contributions to the stress tensor and equations of motion (like $\nabla^2\pi\nabla^2\pi\nabla^2\pi$?

1.2 Studying the non-linear equation precisely: Just field equation

As we know that the spatial resolution does not so much matter in the case of non-linearities we run with small number of grids, The full equation n is written as , After simplification we have,

$$\begin{aligned}&\zeta' - 3w\mathcal{H}\zeta + 3c_s^2\Big(\mathcal{H}^2 - \mathcal{H}'\Big)\pi - 3c_s^2\Big(\Phi' + \mathcal{H}\Psi\Big) - c_s^2\nabla^2\pi \\ &- 2c_s^2\Phi\nabla^2\pi + 3c_s^2\mathcal{H}(1+w)\pi\nabla^2\pi - (1-c_s^2)\zeta\nabla^2\pi + c_s^2\nabla\Phi.\nabla\pi - \nabla\Psi.\nabla\pi + \frac{\mathcal{H}}{2}\Big(2+3w+c_s^2\Big)\nabla\pi.\nabla\pi \\ &- 2(1-c_s^2)\nabla\pi.\nabla\zeta = 0 \end{aligned} \tag{8}$$

The other equation is like before,

$$\pi' = \zeta + \Psi - \mathcal{H}\pi \tag{9}$$

First lets print all the values related here in the code's unit to have an idea in mind what we are exactly dealing with, these are just the order of the terms

```
c_s^2 =10^-6
w= -0.9
Pi_k= -9.92475e-07
zeta= -3.38927e-09
Hcon= 0.950102
psi=phi= -1.23213e-06
phi'= -4.27317e-05
Laplacian_pi= -0.00794965

Gradphi_Gradpi= 4.02666e-08

Gradpsi_Gradpi= 4.02666e-08

Gradpi_Gradpi= 2.82606e-08

GradZeta_Gradpi= -1.03946e-11
```

Comparing all the relevant terms, we can through out all the suppressed terms and just look at important terms,

```
The linear terms =  3. * Hcon * ( w * zeta/2. + cs2 * psi ) - C2 * pi_k(x)+ 3. * cs2 * phi_prime + cs2 *
    Laplacian_pi= 4.59506e-09

2. * cs2 * phi(x) * Laplacian_pi = -1.53897e-13

+(1. - cs2) * (zeta ) * Laplacian_pi=  1.43016e-11

Hcon * (1. + w) * pi_k(x) * Laplacian_pi  -5.21284e-09

zeta_half(x) * Laplacian_pi =   1.43016e-11
```

Simplifying the equation gives,

$$\begin{aligned}
&\zeta' - 3w\mathcal{H}\zeta + 3c_s^2\Big(\mathcal{H}^2 - \mathcal{H}'\Big)\pi - 3c_s^2\Big(\Phi' + \mathcal{H}\Psi\Big) - c_s^2\nabla^2\pi \\
&- \nabla\Psi.\nabla\pi + \frac{\mathcal{H}}{2}\Big(2 + 3w\Big)\nabla\pi.\nabla\pi = 0
\end{aligned} \tag{10}$$

Now lets again compare all the terms in the previous equation,

```
Linear terms= + 3. * Hcon * ( w * zeta_half(x)/2. + cs2 * psi ) - C2 * pi_k(x) + 3. * cs2 * phi_prime +
    cs2 * Laplacian_pi  =  5.55352e-09

Gradpsi_Gradpi = 1.51868e-08

(2. + 3. * w  ) *  Hcon/2. * Gradpi_Gradpi= -1.47696e-08
```

Using $\pi' = \zeta + \Psi - \mathcal{H}\pi$ and taking Ψ constant in time and just considering the non-linear terms,

$$\pi'' + \mathcal{H}'\pi + \mathcal{H}\pi' - \nabla\Psi.\nabla\pi + \frac{\mathcal{H}}{2}\Big(2 + 3w\Big)\nabla\pi.\nabla\pi = 0 \tag{11}$$

Defining $c_1 = \nabla\Psi \approx 5\times10^{-5}$, $c_2 = \Big(2+3w\Big) = -0.7$ and assuming matter dominated universe $\mathcal{H} = 2/\tau$ and $\mathcal{H}' = -2/\tau^2$ we get,

$$\pi''(\tau,\vec{x}) - \frac{2\pi(\tau,\vec{x})}{\tau^2} + \frac{2\pi'(\tau,\vec{x})}{\tau} - c_1\,\nabla\pi(\tau,\vec{x}) + c_2\,\frac{\nabla\pi(\tau,\vec{x}).\nabla\pi(\tau,\vec{x})}{\tau} = 0 \tag{12}$$

The time solution would be,

1.3 Mathematica script for calculating stress tensor

1.4 Check the non-linear terms in the ODE from the action

1.5 Time solution to the full differential equation by matrix method

1.6 Sensitivity to initial conditions

What would be error if we set the initial condition at z=100 to zero?

1.7 The effect of non linearities on gravitational potential at some different redshifts Ψ and matter power spectrum

Just if we get bad results, try to solve the terms separately...

1.8 Trace the average of the perturbation to be consistent

Lorenzo function

1.9 Vector elliptic and vector parabolic consistency check

Turn on the other equations, vector parabolic and see if we get the same results