Second order Field equations

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1 Metric and geometric quantities

In order to implement in the Gevolution we need to change Φ and Ψ Notation
.
We take the metric in ADM form as below,

$$ds^2 = -N(t,\vec{x})^2 dt^2 + h_{ij}(t,\vec{x})\Big(dx^i + N^i(t,\vec{x})dt\Big)\Big(dx^j + N^j(t,\vec{x})dt\Big) \tag{1}$$

where

$$N(t,\vec{x}) = \bar{N}(t)e^{\epsilon\delta N(t,\vec{x})} \tag{2}$$

$$N^i = \epsilon\sigma^{0i}(t,\vec{x}) \tag{3}$$

$$h_{ij} = a^2\Big(e^{2\zeta(t,\vec{x})\epsilon}\delta_{ij} + \epsilon\sigma_{ij}(t,\vec{x})\Big) \tag{4}$$

ϵ shows the order of terms in the scheme.
For the first order equations we can define $\sigma_{ij} = (\partial_i\partial_j - \frac{\nabla^2}{3}\delta_{ij})B(t,\vec{x})\epsilon$ and $N^i = \delta^{ij}\partial_j\psi(t,\vec{x})\epsilon$ since we can separate the scalar, vector and tensor equations.
On the other hand in second order equations we do observe the mixing of the scalar, vector and tensor equations according to $T^{\mu\nu}\frac{\delta g_{\mu\nu}}{\delta(scalars)}$, which cannot be written as a derivative of a scalar equation and suggest other definition of the metric. The calculation of $T^{\mu\nu}\frac{\delta g_{\mu\nu}}{\delta(scalars)}$ is presented in the appendix (?).

For the second order equations we should think of writing the metric in general way, for example σ^{ij} is a general function but How many? degrees of freedom in equations (scalar) or vector ...

2 Definitions

The inverse of the metric is defined by the inverse of the matrix.

$$g^{\cdot\cdot} = (g_{\cdot\cdot})^{-1} \tag{5}$$

$$N_i = h_{ij}N^j \tag{6}$$

Christoffel symbols:

$$\Gamma^{\mu}_{\zeta\rho} = \frac{g^{\mu\xi}}{2}\left(g_{\xi\zeta,\rho} + g_{\xi\rho,\zeta} - g_{\rho\zeta,\xi}\right) \tag{7}$$

$$K_{ij} = \frac{1}{2N(t,\vec{x})}\left[\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right] = \frac{1}{2N(t,\vec{x})}\left[\dot{h}_{ij} - \partial_i N_j - \partial_j N_i - 2\Gamma^l_{ij} N_l\right] \tag{8}$$

$$\delta K = K^i_i(t,\vec{x}) - \bar{K}^i_i(t) \tag{9}$$

Full metric,

$$g_{00} = -N^2(t,\vec{x}) + h_{ij}N^i N^j\,,\; g_{ij} = h_{ij}\,,\; g_{0i} = g_{i0} = h_{ij}N^j \tag{10}$$

Riemann tensor

$$R^\rho_{\sigma\mu\zeta} = \partial_\mu \Gamma^\rho_{\zeta\sigma} - \partial_\zeta \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\zeta\sigma} - \Gamma^\rho_{\zeta\lambda}\Gamma^\lambda_{\mu\sigma} \tag{11}$$

Ricci tensor;

$$R_{\mu\rho} = R^\eta_{\mu\eta\rho} \tag{12}$$

Ricci scalar;

$$R = g^{\mu\rho}R_{\mu\rho} \tag{13}$$

2.1 Stuckelberg trick

Full calculation in the appendix?!

$$f(t) \longrightarrow f(t) + \dot{f}(t)\pi + \frac{1}{2}\ddot{f}(t)\pi^2 + \frac{1}{6}\dddot{f}(t)\pi^3 \tag{14}$$

$$\Lambda(t) \longrightarrow \Lambda(t) + \dot{\Lambda}(t)\pi + \frac{1}{2}\ddot{\Lambda}(t)\pi^2 + \frac{1}{6}\dddot{\Lambda}(t)\pi^3 \tag{15}$$

$$M_2^4(t) \longrightarrow M_2^4(t) + \dot{M}_2^4(t)\pi + \frac{1}{2}\ddot{M}_2^4(t)\pi^2 + \frac{1}{6}\dddot{M}_2^4(t)\pi^3 \tag{16}$$

$$m_3^3(t) \longrightarrow m_3^3(t) + \dot{m}_3^3(t)\pi + \frac{1}{2}\ddot{m}_3^3(t)\pi^2 + \frac{1}{6}\dddot{m}_3^3(t)\pi^3 \tag{17}$$

$$g^{00} \longrightarrow g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\rho\nu}\partial_\rho\partial_\nu\pi \tag{18}$$

$$\begin{aligned}\delta K \longrightarrow & \delta K - 3\left(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2\right) - (1-\dot{\pi})Nh^{ij}\partial_i\partial_j\pi + \frac{1}{2}\partial_i h^{ij}\partial_j\pi \\ & + \frac{H}{2a^2}\delta^{ij}\partial_i\pi\partial_j\pi + \frac{2}{a^2}\delta^{ij}\partial_i\pi\partial_j\dot{\pi} - \frac{2}{a^2}\delta^{ij}\partial_i N\partial_j\pi\end{aligned} \tag{19}$$

The followed Stuckelberg trick is not true for second order, so one should write the Stuckelberg using the mathematica! The EFT action;

$$S = \sqrt{-g}\left[\frac{M_*^2}{2}f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}\left(g^{00} + \frac{1}{\bar{N}^2}\right) - \frac{m_3^3(t)}{2}\delta K\left(g^{00} + \frac{1}{\bar{N}^2}\right)\right] \tag{20}$$

Then the scalar field equations obtained by varying the action with respect to the scalars for second order equations we need to decide for N^i and σ^{ij} instead of naively varying with respect to ψ and B

$$\frac{\delta S}{\delta\pi} = ... \tag{21}$$

2.2 Gauge transformation

Newtonian gauge: Appendix!

$$\delta N \to \Phi\,,\; \zeta \to -\Psi\,,\; \psi \to 0,\;\; B \to 0 \tag{22}$$

3 Field equations

By varying the action with respect to the scalars we get exactly the same first order equations of references (Ref!) and for the second order equations up to Gevolution's scheme we get,

3.1 π:

$$\frac{\delta S}{\delta \pi} = \int\int d^3k d^3k' e^{i(\vec{k}+\vec{k}').\vec{x}} \Bigg[-\frac{k^2}{a^2}C^{(2)}_{\Phi\Phi}\Phi\Phi - \frac{k^2}{a^2}C^{(2)}_{\Phi\Psi}\Phi\Psi - \frac{k^2}{a^2}C^{(2)}_{\Phi\pi}\Phi\pi - \frac{k^2}{a^2}C^{(2)}_{\Psi\Phi}\Psi\Phi - \frac{k^2}{a^2}C^{(2)}_{\Psi\Psi}\Psi\Psi - \frac{k^2}{a^2}C^{(2)}_{\Psi\pi}\Psi\pi$$
$$- \frac{k^2}{a^2}C^{(2)}_{\pi\Phi}\pi\Phi - \frac{k^2}{a^2}C^{(2)}_{\pi\Psi}\pi\Psi - \frac{k^2}{a^2}C^{(2)}_{\pi\pi}\pi\pi - \frac{k^2}{a^2}C^{(2)}_{\dot{\Phi}\Phi}\dot{\Phi}\Phi - \frac{k^2}{a^2}C^{(2)}_{\dot{\Phi}\Psi}\dot{\Phi}\Psi - \frac{k^2}{a^2}C^{(2)}_{\dot{\Phi}\pi}\dot{\Phi}\pi$$
$$- \frac{k^2}{a^2}C^{(2)}_{\dot{\Psi}\Phi}\dot{\Psi}\Phi - \frac{k^2}{a^2}C^{(2)}_{\dot{\Psi}\Psi}\dot{\Psi}\Psi - \frac{k^2}{a^2}C^{(2)}_{\dot{\Psi}\pi}\dot{\Psi}\pi + C_{\dot{\pi}\dot{\pi}}\dot{\pi}\dot{\pi} - \frac{k^2}{a^2}C^{(2)}_{\dot{\pi}\Phi}\dot{\pi}\Phi - \frac{k^2}{a^2}C^{(2)}_{\dot{\pi}\Psi}\dot{\pi}\Psi - \frac{k^2}{a^2}C^{(2)}_{\dot{\pi}\pi}\dot{\pi}\pi \ .$$
$$- \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\Phi\Psi}\Phi\Psi - \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\Phi\pi}\Phi\pi - \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\Psi\pi}\Psi\pi - \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\pi\pi}\pi\pi - \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\Phi\Phi}\Phi\Phi - \frac{\vec{k}.\vec{k}'}{a^2}C^{1,1}_{\Psi\Psi}\Psi\Psi$$
$$+ \vec{k}C^{(1)}_{\dot{\Phi}}(\Phi,\Psi,\pi)\dot{\Phi} + \vec{k}C^{(1)}_{\dot{\Psi}}(\Phi,\Psi,\pi)\dot{\Psi} + \vec{k}C^{(1)}_{\dot{\pi}}(\Phi,\Psi,\pi)\dot{\pi} + C_{\ddot{\Phi}}(\Phi,\Psi,\pi)\ddot{\Phi} + C_{\ddot{\Psi}}(\Phi,\Psi,\pi)\ddot{\Psi} + C_{\ddot{\pi}}(\Phi,\Psi,\pi)\ddot{\pi} \Bigg] \ . \tag{23}$$

Some Important notes:
In order to write the equation we must write all the terms, since its possible we have made a mistake somewhere! (So all the equations which have been written up to now must change).
Be sure that all the terms are written, for example in the last notes I forgot to write the terms $\nabla^2\Phi\dot{\pi}$, so write all the terms even if they are zero!
The red terms are where Filippo and I do not agree! so one of us has made a mistake! I wrote the Filippo's results in the red color terms!
This results are only variation with respect to field withour multiplying to $1/\sqrt{-g}$
By $C_{\ddot{\Psi}}$ I mean the coefficient of $\ddot{\Psi}$ and by $C^{(1)}_{\dot{\Phi}}$ I mean the coefficient of $\partial_i\dot{\Phi}$ and $C^{(2)}_{\dot{\Psi}\Phi}$ the coefficient of $\nabla^2\Psi\Phi$.
Here we dont write the coefficient in Fourier space, so we dont need to multiply by a^2 or a minus sign!
So by the below I mean the coefficient of $\nabla^2\Phi\Phi$

$$C^{(2)}_{\Phi\Phi} = -aM_*^2\dot{f} \ . \tag{24}$$

$$C^{(2)}_{\Phi\Psi} = -a(m_3^3 - M_*^2\dot{f}) \ . \tag{25}$$

$$C^{(2)}_{\Phi\pi} = a(\dot{m_3^3} - M_*^2\ddot{f}) \ . \tag{26}$$

$$C^{(2)}_{\Phi\dot{\Phi}} = 0 \ . \tag{27}$$

$$C^{(2)}_{\Phi\dot{\Psi}} = 0 \ . \tag{28}$$

$$C^{(2)}_{\Phi\dot{\pi}} = am_3^3 \ . \tag{29}$$

$$C^{(2)}_{\Psi\Phi} = a(2M_*^2\dot{f}) \ . \tag{30}$$

Something is wrong in my new calculation (Stuckelberg trick of K, I get (-1+a^2) but it should be 0.

$$C^{(2)}_{\Psi\Psi} = a(2M_*^2\dot{f}) \ . \tag{31}$$

$$C^{(2)}_{\Psi\pi} = a(2M_*^2\ddot{f}) \ . \tag{32}$$

$$C^{(2)}_{\Psi\dot{\Phi}} = 0 \ . \tag{33}$$

$$C^{(2)}_{\Psi\dot{\Psi}} = 0 \ . \tag{34}$$

$$C^{(2)}_{\Psi\dot{\pi}} = 0 \ . \tag{35}$$

Again in top the sam easy problem $(-1 + a^2)$

$$C^{(2)}_{\pi\Phi} = a(-3m_3^3 H + 2c - 4M_2^4 + \dot{m}_3^3) \ . \tag{36}$$

$$C^{(2)}_{\pi\Psi} = a(-m_3^3 H - 2c - \dot{m}_3^3) \ . \tag{37}$$

$$C^{(2)}_{\pi\pi} = a(\dot{m}_3^3 H + 2\dot{c} - 3m_3^3\dot{H} + \ddot{m}_3^3) \ . \tag{38}$$

$$C^{(2)}_{\pi\dot{\Phi}} = -am_3^3 \ . \tag{39}$$

$$C^{(2)}_{\pi\dot{\Psi}} = -4am_3^3 \ . \tag{40}$$

$$C^{(2)}_{\pi\dot{\pi}} = 4aM_2^4 \ . \tag{41}$$

$$C^{(2)}_{\dot{\Phi}\Phi} = 0 \ . \tag{42}$$

$$C^{(2)}_{\dot{\Phi}\Psi} = 0 \ . \tag{43}$$

$$C^{(2)}_{\dot{\Phi}\pi} = 0 \ . \tag{44}$$

$$C^{(2)}_{\dot{\Phi}\dot{\Phi}} = 0 \ . \tag{45}$$

$$C^{(2)}_{\dot{\Phi}\dot{\Psi}} = 0 \ . \tag{46}$$

$$C^{(2)}_{\dot{\Phi}\dot{\pi}} = 0 \ . \tag{47}$$

$$C^{(2)}_{\dot{\Psi}\Phi} = 0 \ . \tag{48}$$

$$C^{(2)}_{\dot{\Psi}\Psi} = 0 \ . \tag{49}$$

$$C^{(2)}_{\dot{\Psi}\pi} = 0 \ . \tag{50}$$

$$C^{(2)}_{\dot{\Psi}\dot{\Phi}} = 0 \ . \tag{51}$$

$$C^{(2)}_{\dot{\Psi}\dot{\Psi}} = 0 \ . \tag{52}$$

$$C^{(2)}_{\dot{\Psi}\dot{\pi}} = 0 \ . \tag{53}$$

$$C^{(2)}_{\dot{\pi}\Phi} = 0 \ . \tag{54}$$

$$C^{(2)}_{\dot{\pi}\Psi} = 0 \ . \tag{55}$$

$$C^{(2)}_{\dot{\pi}\pi} = 0 \ . \tag{56}$$

$$C^{(2)}_{\dot{\pi}\dot{\Phi}} = 0 \ . \tag{57}$$

$$C^{(2)}_{\dot{\pi}\dot{\Psi}} = 0 \ . \tag{58}$$

$$C^{(2)}_{\dot{\pi}\dot{\pi}} = 0 \ . \tag{59}$$

$$C^{1,1}_{\Phi\Phi} = -aM_*^2\dot{f} \ . \tag{60}$$

$$C^{1,1}_{\Phi\Psi} = a(-m_3^3 + M_*^2\dot{f}) \ . \tag{61}$$

$$C^{1,1}_{\Phi\pi} = 2a(c - 2M_2^4 - Hm_3^3 + \dot{m_3^3}). \ . \tag{62}$$

$$C^{1,1}_{\Psi\pi} = a(-2c - Hm_3^3 + \dot{m_3^3}) \ . \tag{63}$$

I see a difference with Filippo in the last equation,

$$C^{1,1}_{\Psi\Psi} = 3aM_*^2\dot{f} \ . \tag{64}$$

$$C^{1,1}_{\pi\pi} = \frac{a^3}{2}\Big(- m_3^3H^2 - 4m_3^3\dot{H} + 2\dot{c} + 4\dot{M_2^4} + 4M_2^4H - \dot{m_3^3}H\Big) \ . \tag{65}$$

There is a problem in the coefficient of m_3^3 which in my case is like $5H^2$ but in Filoppo's is different.

$$C^{(1)}_{\dot{\Phi}}(\Phi,\Psi,\pi) = 0 \tag{66}$$

$$C^{(1)}_{\dot{\Psi}}(\Phi,\Psi,\pi) = -4am_3^3\partial_i\pi \tag{67}$$

$$C^{(1)}_{\dot{\pi}}(\Phi,\Psi,\pi) = 2a(4M_2^4\partial_i\pi - Hm_3^3\partial_i\pi - \dot{m_3^3}\partial_i\pi + m_3^3\partial_i\Phi) \tag{68}$$

$$C_{\ddot{\Phi}}(\Phi,\Psi,\pi) = 0 \ . \tag{69}$$

$$C_{\ddot{\Psi}}(\Phi,\Psi,\pi) = 3a^3(m_3^3 - M_*^2\dot{f}) \ . \tag{70}$$

Again we get different.

$$C_{\ddot{\pi}}(\Phi,\Psi,\pi) = a^3(-2a^2(c + 2m_2^4) + m_3^3\nabla^2\pi) \ . \tag{71}$$

Check again the top equation.